Pricing American-style Basket Options By Implied Binomial Tree

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Abstract

It is known that the most difficult problem of pricing and hedging multi-asset basket options are those with both high dimensionality and early exercise. This article proposes a numerical algorithm by reducing multivariate distributions of a portfolio into a single variable and modeling that as a univariate stochastic process in the form of an implied binomial tree. It is demonstrated that the method provides a fast and flexible way of pricing and hedging high dimensional multi-asset basket options with early exercise.

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Introduction

A basket option is an option on a portfolio of several underlying assets. With growing diversification in investor's portfolio, options on such portfolios are increasingly demanded. One example of the demand is purchasing a protective put option on a portfolio such that investor's down side risk is limited.

However, American-style basket options continue to be challenging in terms of both algorithm complexity and computational burden. The modeling and mathematics of pricing basket options are similar to those of options on a single asset except that there are correlated random walks and multi-variate Ito's lemma that need to be applied. In very few special cases, such as exchange options, closed form solutions can be found. More often, numerical methods such as numerical integration, finite-difference methods and Monte Carlo simulations are necessary for low or medium size problems. When dimension is higher, it is relatively cheaper to use Monte Carlo simulations since its computational cost does not increase exponentially as other methods. The ease exists only for European-style options. It is however known that the most difficult problems of pricing and hedging multi-asset basket options are those with both high dimensionality, for which we would like to use Monte Carlo simulation, and with early exercise, for which we would like to use either binomial tree or finite difference methods. "There is currently no numerical method that copes well with such a problem" (Wilmott, 1998).

There have been many articles published on topics of multi-dimensional American-style options. They can be categorized by either lattice-based or simulation-based approaches. Lattice-based approaches, such as binomial trees, trinomial trees and finite difference methods, are widely used for options on a single asset. When dimension is higher, usually up to four, extensions of binomial and trinomial trees from the univariate binomial tree (Cox, Ross and Rubinstein 1979) can be applied. Rubinstein (1994a) models two-asset rainbow options using "binomial pyramids". It generalizes one-dimension up and down trees into two-dimensional squares. The similar approach can be further extended into higher dimensions. Appendex A generalizes this approach into four dimensions. The finite difference methods can also be extended into multiple dimension cases by using Alternating Directions Implicit (ADI) method (Clewlow and Strickland, 1998). Although these lattice-based approaches are generally easy to deal with early exercise by a backward algorithm, their memory and computation requirements explode exponentially as the dimension of problem increases.

On the other hand, without exponentially growing computation effort as most lattice-based methods do for higher dimension options, Monte Carlo simulation enjoys high flexibility and modest computational cost independent of dimensions of a problem. The embedded forward simulation algorithm, however, underlies its difficulty in pricing options with early exercise features, such as American options. These options generally require a backward algorithm to determine the optimal exercise policy. Since the claim written by Hull (1997) that "A limitation of the Monte Carlo simulation approach is that it can be used only for European-style derivatives", many papers have devoted to overcome this challenge.

As the reviewing work done by Broadie and Glasserman (1997b), there are three main techniques in applying Monte Carlo simulation into multi-asset American options. In many cases, multiple of such techniques are applied into an algorithm. One category of techniques is based on parametrization of exercise boundary and uses simulation to maximize the expected payoff within the parametric family. Different optimization methods and families of curve fitting functions have been proposed. The second approach is based on finding both upper and lower bounds for option price and giving valid confidence intervals for the true value. The simulated trees method and the stochastic mesh algorithm proposed by Broadie and Glasserman (1997a, 1997) and the recent primal-dual simulation algorithm by Andersen and Broadie (2001) belong to this category. In the simulated tree method, each single node generates its own non-combining independent subtree from samples. The pricing is then carried out by a backward recursive procedure. Although the method is linear in the problem dimension, it is exponential in terms of the number of steps. The stochastic mesh method has the advantage of linear in the number of exercise opportunities, but interlocks among nodes in the mesh create considerably complication in the algorithm. Recently there are developments of applying duality to compute an upper bound of true price from the specification of some arbitrary martingale process. The tightness of the upper bound depends critically on the choice of the martingale process. Andersen and Broadie (2001) significantly improve the performance of the calculation. The third category of techniques, which is also the most widely used one in combining with other approaches, is the dynamic programming style backward recursion in determining the optimal exercise policy. Tilley (1993) used backward algorithm and bundling technique for approximating the price and the optimal exercise decision into the Monte Carlo simulation. It is efficient and easy in pricing one-dimension American options. Since then articles were published on the improvement of this basic concept. Longstaff and Schwartz (2001) used least-squares regression on polynomials to approximate the holding value and optimum to exercise. Barraquand and Martineau (1995) applied a state aggregation technique and reduced a high-dimensional problem into a single dimension one since the exercise decision only depends on the single variable, the intrinsic value. The resulting one-dimension problem can be solved quickly by standard backward dynamic programming and provides significant computation saving for high-dimensional problems. Although it is extremely fast in calculating an approximation of price, the accuracy primarily depends on how well the optimal exercise policy can be represented by the single payoff variable. In some multi-dimensional cases, the option payoff is not sufficient to determine the optimum to exercise. As Broadie and Glasserman (1997) indicated, "the stratification algorithm will not converge to the correct value."

The approach to pricing basket options in the article is similar in spirit to the concept of Barraquand and Martineau (1995) by reducing a high-dimensional problem into a single variable one. In the case of basket options, the basket itself is the variable in determining its holding and exercise values. Instead of using simulated histogram, our method uses a set of European-style standard options to infer the risk-neutral probabilities and hence the stochastic process of the basket represented by an implied binomial tree. We will show the results of this approach converge to the correct value with modestly increasing the number of steps.

The Approach

The method we use to price a basket option involves four steps. First, we value European-style standard call and put options with different strikes, whose underlying is the multi-asset basket, by Monte Carlo simulations. Second, we infer the risk-neutral probabilities of the basket from the set of European-style options on the basket, the associated market price of the basket, and the risk-less interest rate bond. Third, we recover the fully specified stochastic process of the basket, in the form of an implied binomial tree, from its risk-neutral probabilities. Finally, with the implied tree, we can calculate the value and hedging parameters of any derivative instruments on the basket maturing with or before the European options.

Monte Carlo simulation in valuing European-style standard basket options

The first step is valuing European-style standard call and put options of the basket. As we know, an N-variate binomial tree becomes exponentially complicated and computationally expensive when both the number of assets and the number of moves are getting larger. For example, when a tree has 100 moves and deals with only 4 assets simultaneously, the number of nodes at the end will be $(100+1)^4$, or about 100 million. Such numerous computations and memory requirement prevents us from using the lattice-based method to value high-dimensional Europen-style basket options.

Another set of approaches to value multi-asset basket options is by Monte Carlo simulations. Since the value of a European-style option is the risk-neutral expectation of its discounted payoff at the time of maturity, Monte Carlo simulations obtain an estimate of such a risk-neutral expectation by computing the average of a large number of discounted payoffs. Because the computation effort of Monte Carlo simulation is independent of the number of random factors, it is cheap to value European-style multi-asset basket options with high dimensionality in terms of computation and memory requirements. The simulation is also flexible to incorporate other processes and factors, such as random volatility, random interest rates, jumps in asset prices, and other more realistic market conditions.

Our approach, however, is not valuing American-style basket options by Monte Carlo simulations. The purpose of using simulation is to utilize its flexibility in modeling multiple random factors or more realistic random processes, as a tool to obtain the risk-neutral distribution of the underlying basket. The approach involves three steps in applying Monte Carlo simulations[1]:

(1) Simulate the ending asset prices in a basket under the risk-neutral measure.

A way of simulating multi-dimensional stochastic process is described in Appendix B, where a correlation matrix among different process is assumed given. In general, a European option can be valued as the expected risk-neutral payoff at expiry.

$$\hat{C} = \frac{1}{M}\sum_{s=1}^{M} \exp(-rT) Payoff_T^{(s)} \tag{1}$$

where M is the number samples.

(2) Estimate the mean and standard deviation of the return on the basket.

Assuming there are M simulation samples, each simulation ends up with a set of asset prices at the expiration T. The asset prices, therefore, determine a value of the basket and hence the return on the basket from the time zero. With M samples of basket returns, one can calculate the unbiased estimations on both mean and standard deviation of basket returns under the risk-neutral measure:

$$\hat{\mu} = \frac{1}{M}\sum_{s=1}^{M}\left[R^{(s)}\right] \qquad \hat{\sigma} = \sqrt{\frac{1}{M-1}\sum_{s=1}^{M}\left(R^{(s)} - \hat{\mu}\right)^2}$$

$$R^{(s)} \equiv \log\left[n_1 S_{1,T}^{(s)} + n_2 S_{2,T}^{(s)} + \cdots + n_N S_{N,T}^{(s)}\right] - \log\left[S_0\right]$$

$$S_0 \equiv n_1 S_{1,0} + n_2 S_{2,0} + \cdots + n_N S_{N,0}$$

(3) Value the European-style standard call and put options based on a set of equally log-spaced strike prices.

Consider we want to value m+1 options with different strikes, the strikes we choose are:

$$K_j = S_0 \exp\left(\hat{\mu} + x_j \hat{\sigma}\right) \qquad x_j \equiv \frac{2j - m}{\sqrt{m}} \quad , \quad j = 0, 1, 2, \cdots, m$$

The options we choose are all out-of-money options. That is, we value the European put options for the strikes $K_j < S_0 \exp(\hat{\mu})$, and the European call options for those strikes $K_j \geq S_0 \exp(\hat{\mu})$.

The valuation for the set of calls and puts are fast because we only need to proceed the time-consuming simulation in step (1) once. After the M outcomes have been simulated, the valuation is simply applying Equation (1) in computing expected payoffs.

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[1] The efficiency of all Monte Carlo simulation can be improved by using variance reduction techniques. In the interest of this article, the description of these techniques is not emphasized.

Implied Ending Risk-Neutral Probabilities

The second step of the approach is to infer the implied ending risk-neutral probabilities of the univariate basket from the set of European-style calls and puts on the basket.

Consider a state security, or Arrow-Debreu security, which pays one dollar if a specific future state is realized and zero otherwise, the price of such a security can be used to build more complicated payoffs at the expiration. For example, if a security pays X_1, X_2, …, and X_n if the states 1, 2, …, and N are realized respectively. The price of this security at time zero will be the sum of those payoffs weighted by state prices associated with the corresponding states.

$$C = (X_1 * q_1 + X_2 * q_2 + \ldots + X_n * q_n) / r$$

Now, if we discretize the continuous price space into discrete states similar to a binomial tree at the end of the expiration date, the payoffs of a standard European call option with strike price K under those states will be:

$$C = \max\{ 0,\ ST_j - K \} \qquad j = 0, 1, 2, \ldots, m$$

The price of a European call option is:

$$C = \sum_j \pi_j \max\{0,\ ST_j - K\}$$

Similarly, the price of a European put option will be:

$$P = \sum_j \pi_j \max\{0,\ K - ST_j\}$$

As we mentioned, we have obtained a set of European-style call and put options on the basket through the Monte Carlo simulation. We can use them to imply the state prices and hence the risk-neutral probabilities of the univariate basket.

Assuming there are m+1 discrete states at the expiration. We want to find out the state prices for them. The state prices can be computed by the prices of standard European options, the current value of the basket, and the price of zero-coupon risk-free bond. If the states are chosen to be the set of strikes we choose for the standard options in the previous section, where $K_0 < K_1 < \ldots < K_m$, we can create a payoff table as shown below for the standard calls, puts, the underlying asset and the risk-free bond.

			States									
Security	Prices	Strike	K_0	K_1	K_2		K_{j-1}	K_j		K_{m-2}	K_{m-1}	K_m
Put	$P(K_1)$	K_1	K_1-K_0									
Put	$P(K_2)$	K_2	K_2-K_0	K_2-K_1								
Put	$P(K_3)$	K_3	K_3-K_0	K_3-K_1	K_3-K_2							
…		…										
Put	$P(K_{j-1})$	K_{j-1}	$K_{j-1}-K_0$	$K_{j-1}-K_1$	$K_{j-1}-K_2$	…	0					
Call	$C(K_j)$	K_j						0	…	$K_{m-2}-K_j$	$K_{m-1}-K_j$	K_m-K_j
…		…										
Call	$C(K_{m-3})$	K_{m-3}								$K_{m-2}-K_{m-3}$	$K_{m-1}-K_{m-3}$	K_m-K_{m-3}
Call	$C(K_{m-2})$	K_{m-2}									$K_{m-1}-K_{m-2}$	K_m-K_{m-2}
Call	$C(K_{m-1})$	K_{m-1}										K_m-K_{m-1}
Asset	S_0	0	$e^{d*T}K_0$	$e^{d*T}K_1$	$e^{d*T}K_2$	…	$e^{d*T}K_{j-1}$	$e^{d*T}K_j$	…	$e^{d*T}K_{m-2}$	$e^{d*T}K_{m-1}$	$e^{d*T}K_m$
Bond	e^{-r*T}		1	1	1	…	1	1	…	1	1	1

Table 1: Payoff table of standard calls and puts

Consider the lowest strike of the put option $P(K_1)$, it pays $K_1–K_0$ when the state is K_0 and zero for all the states above K_0. Therefore, the state price for K_0 is nothing but $P(K_1)/(K_1–K_0)$.

$$P(K_1) = (K_2–K_0)\, \pi_0 \quad \Rightarrow \quad \pi_0 = P(K_1) / (K_2–K_0)$$

Now consider the put options with strike K_2, its price is given by:

$$P(K_2) = (K_2–K_0)\, \pi_0 + (K_2–K_1)\, \pi_1$$

There is only one unknown π_1, since π_0 has been solved previously. We can continue this process forward by applying the price of put option and previously solved state prices π_k, k = 0, 1, …, i-2 to solve for the state price π_{i-1}:

$$P(K_i) = \sum_{k=0,\dots,i-2}(K_i–K_k)\, \pi_k + (K_i–K_{i-1})\, \pi_{i-1}$$

To avoid building up numerical errors in states with high prices, we use call options and start with the high end of strikes. For example, the state price for K_m can be computed by $C(K_{m-1})/(K_m–K_{m-1})$ since the call option $C(K_{m-1})$ only pays $K_m–K_{m-1}$ when the state is K_m and zero elsewhere. Other state prices can be computed progressively by applying the following:

$$C(K_i) = (K_{i+1}–K_i)\, \pi_{i+1} + \sum_{k=i+2,\dots,m}(K_k–K_i)\, \pi_k$$

The above procedure can be used to compute most of the state prices except two states adjacent to the forward prices of the underlying asset. In our case, they are strikes K_{j-1} and K_j where the puts and calls are met in our payoff table. To solve for state prices for these two states, another two equations are introduced to satisfy no-arbitrage conditions for the forward price of underlying asset and the price of risk-free bond. According to the payoffs of these two securities, we have,

$$S_0 \exp(-dT) = \sum_{k=0,\dots,j-2} K_k\, \pi_k + K_{j-1}\, \pi_{j-1} + K_j\, \pi_j + \sum_{k=j+1,\dots,m}K_k\, \pi_k \qquad \text{, and}$$

$$\exp(-rT) = \sum_{k=0,\dots,j-2} \pi_k + \pi_{j-1} + \pi_j + \sum_{k=j+1,\dots,m} \pi_k$$

Two equations and two unknowns, we can solve for both π_{j-1} and π_j.

After we obtain all the state prices, the risk-neutral probabilities are nothing but the state prices inflated by interest rate.

$$q_i = \pi_i \exp(rT), \qquad \text{for all } i = 0, 1, \dots, m$$

Implied Binomial Tree of the basket

After retrieving the ending risk-neutral probabilities, the next step of our approach is to recover the implied stochastic process of the basket, in the form of its implied binomial tree. The idea behind this is recognizing that the prices of standard European options of the basket embed information about the underlying basket portfolio. If there exists a diffusion process to achieve its values of European options, the American-style or other exotic options should follow the same process. Here, the standard European-style options are treated as fundamental securities, whose prices are given. In our case, the option prices are through Monte Carlo simulations.

Rubinstein (1994, 1998) presents a method in constructing an implied binomial tree by using the ending risk-neutral distributions. Other approaches in constructing implied trees, for example implied trinomial trees, can be found from the overview literature by Jackwerth (1999). Implied trees are not unique without making specific assumptions on how the underlying asset price reaches the terminal distribution. The main assumption of Rubinstein's method is that the path probabilities for all paths reaching the same terminal state are equal. Here is a summary of the method. Assuming that an asset may either move up or down at any given time, and assuming two paths with up-then-down and down-then-up lead to the same outcome of the asset in the end. These assumptions lead a combining binomial tree as shown in Figure 1. At the end of n-th period, there are total n+1 possible outcomes or nodes on the tree. Each has a node probability $P_{node}(n,j)$ which is the discrete probability mass of the outcome at node (n,j). Figure 1 also shows a common bell-shaped node probabilities. With the assumption of combining binomial tree, there are “n-choose-j” leading paths that cause the same outcome at node (n,j). We further assume all the paths leading to the same outcome (n,j) have the same path probability $p_{path}(n,j)$. Therefore,

$$p_{path}(n,j) = \frac{P_{node}(n,j)}{C_j^n} = P_{node}(n,j) \Big/ \frac{n!}{j!(n-j)!}$$

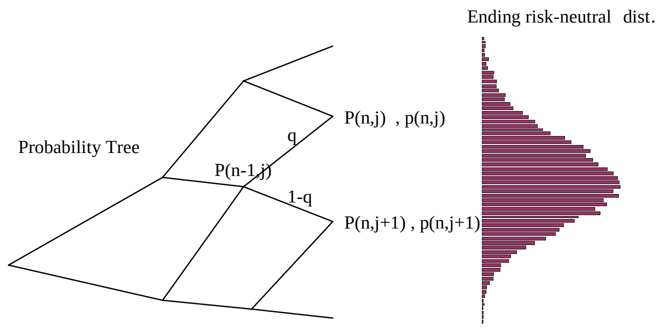


Figure 1: Implied Binomial Tree

Moving backward from the n-th period into "n-1"-th period, for a node (n-1,j), it has two paths to move ahead, either up to the node (n, j) or down to the node (n,j+1). Each path has the path probability of $p_{path}(n,j)$ and $p_{path}(n,j+1)$. So the node probability of this node (n-1,j), which is the probability of getting to there from the root of the tree, is the sum of two path probabilities leading to the next period.

$$P_{node}(n-1,j) = p_{path}(n,j) + p_{path}(n,j+1)$$

The local up-move probability q is the probability of the movement to (n,j) given the condition that the movement is started from node (n-1,j). Therefore, the conditional down-move probability is 1-q.

$$q = \frac{p_{path}(n,j)}{P_{node}(n-1,j)} = \frac{p_{path}(n,j)}{p_{path}(n,j) + p_{path}(n,j+1)}$$

If we are given a risk-neutral probability distribution at the end of expiry, the implied probability tree can be constructed, by working backward from the period N to 0 by applying the above equations recursively. At the same time, trees for both asset and the option are also constructed backward.

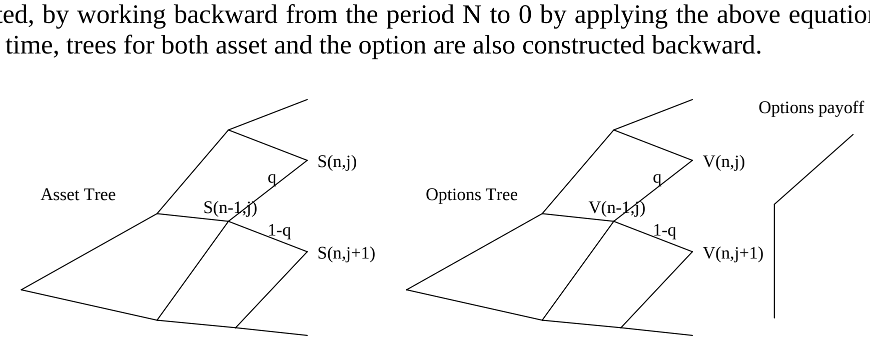


Figure 2: Implied Asset and Option Trees

In the case of an American put option on a basket, whose ending risk-neutral probabilities are given, the recursive formula for both the value of basket and the put option are as following:

$S(n-1,j) = [q * S(n,j) + (1-q) * S(n, j+1)] * \exp[-(r-d)h]$

$P(n-1,j) = \max \{ K - S(n-1,j), [q * P(n,j) + (1-q) * P(n,j+1)] * \exp(-rh) \}$

American call options can be proceeded similarly.

With the implied tree on the basket, we are able to both price and hedge the basket options by standard techniques used in binomial tree. The following section demonstrates numerical results of implementing our approach in comparing to other methods.

Results

Single asset options

Our first example is a special case of single asset options. We will show how to back out the implied binomial tree and obtain the exact same results as those calculated by the ordinary binomial tree for both European and American-style single asset options.

Consider options on single underlying stock:

Time to Expiration:	$T = 1.0$ year
Interest Rate:	$r = 10\%$
Dividend Yield:	$\delta = 5\%$
Volatility:	$\sigma = 20\%$
Spot Price:	$S_0 = 100$
Strike Prices:	$K = 60, 70, 80, 90, 100, 110, 120, 130, 140.$

We first construct a simple two-step binomial tree to illustrate our approach. In the example, we have two moves, i.e., $m = 2$, $h = 0.5$. After computing the multiplicative factors for both upward and downward moves, where

$u = \exp[(r-\delta-½\sigma^2)h + \sigma\sqrt{h}] = 1.1693$, and
$d = \exp[(r-\delta-½\sigma^2)h - \sigma\sqrt{h}] = 0.8812$,

We have the binomial tree as shown in Exhibit 1. The tree is constructed such that there is an equal probability of one-half for the asset price to either move up or move down.

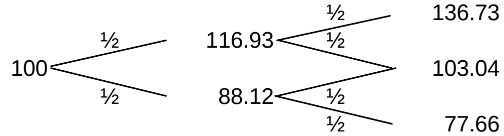


Exhibit 1: Standard Two-step Binomial Tree

This tree can be used to price both European and American-style options. The computation results for various strikes, ranging from 60 to 140, are shown in the 4th and 5th columns in Exhibit 5 for call options and in the 10th and 11th columns for put options. In comparison, we also list the Black-Scholes' prices for

European-style options. It is believed that binomial tree is the discretization of the continuous-time Black-Scholes model.

Now, we want to show that we can recover the above binomial tree by using a set of European-style standard call and puts, the current asset price and the risk-free bond. Consider the three ending outcomes on the tree. Their node probabilities are ¼, ½, and ¼ respectively. Therefore, the mean and standard deviation of their log-returns are 0.03 and 0.2.

$$0.03 = \frac{1}{4} * \log(136.73/100) + \frac{1}{2} * \log(103.04/100) + \frac{1}{4} * \log(77.66/100)$$
$$0.2 = \{\frac{1}{4} * [\log(136.73/100)-0.03]^2 + \frac{1}{2} * [\log(103.04/100)-0.03]^2 + \frac{1}{4} * [\log(77.66/100)-0.03]^2\}^{1/2}$$

As we indicate previously, we sample the ending state space by using the formula

$$S_0*\exp[\text{mean} + \text{std} * (2j - m)/\sqrt{m}],\ j=0, 1, \ldots, m.$$

In our case, this formula leads to sampling states at 77.66, 103.04, and 136.73. Next, we show we are able to recover the entire binomial tree by given European-style option prices struck at these sampling states. For example, a European call option struck at 103.05, is valued at 7.62 by using the above two-step binomial tree. The valuation tree is shown in Exhibit 2.

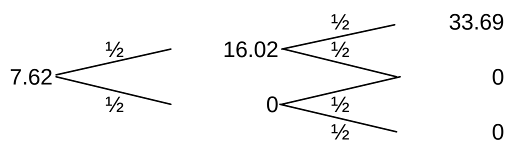


Exhibit 2: Binomial Tree for a Call Option

By using this European call option, together with the underlying asset and risk-free bond, we can compute the risk-neutral probabilities of the three states. The payoff table for the three securities under three different states and the calculated ending risk-neutral probabilities are listed in Exhibit 3.

			States		
Security	Prices	Strike	77.66	103.05	136.73
Call	7.62	103.05			33.69
Asset	100	0	$e^{0.05}$*77.66	$e^{0.05}$*103.05	$e^{0.05}$*136.73
Bond	90.48		100	100	100

states	probability
136.73	0.25
103.04	0.50
77.66	0.25

Exhibit 3: Payoff table and Risk-neutral probabilities

By using the ending risk-neutral probabilities, we recover the implied binomial tree as shown in Exhibit 4. In this trivial case, the implied tree is exactly the same as the original binomial tree. Therefore, the pricing results by using binomial tree and the implied tree should be the same for both European and American-style options. Exhibit 5 shows such comparison between results by standard binomial tree and those by implied tree with the same number of moving steps.

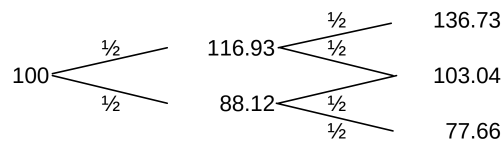


Exhibit 4: Implied Binomial Tree (One-Asset)

	Call Options					
Strike	Black-Scholes European	Binomial Tree (2 steps) European	Binomial Tree (2 steps) American	Monte Carlo (500000) European	Implied Tree (2 steps) European	Implied Tree (2 steps) American
60	40.8434	40.8327	40.8327	40.8469	40.8327	40.8327
70	31.9043	31.7843	31.7843	31.9066	31.7843	31.7843
80	23.3896	23.2653	23.2653	23.3924	23.2663	23.2663
90	15.8850	16.4779	16.4779	15.8971	16.4785	16.4785
100	9.9409	9.6906	9.6906	9.9587	9.6907	9.6907
110	5.7455	6.0496	6.0496	5.7597	6.0516	6.0516
120	3.0891	3.7864	3.7864	3.0988	3.7869	3.7869
130	1.5596	1.5233	1.5233	1.5579	1.5222	1.5222
140	0.7467	0.0000	0.0000	0.7432	0.0000	0.0000

	Put Options					
Strike	Black-Scholes European	Binomial Tree (2 steps) European	Binomial Tree (2 steps) American	Monte Carlo (500000) European	Implied Tree (2 steps) European	Implied Tree (2 steps) American
60	0.0107	0.0000	0.0000	0.0107	0.0000	0.0000
70	0.1200	0.0000	0.0000	0.1187	0.0000	0.0000
80	0.6537	0.5293	0.5293	0.6529	0.5304	0.5304
90	2.1974	2.7903	2.7903	2.2060	2.7909	2.7909
100	5.3017	5.0514	5.6469	5.3159	5.0515	5.6486
110	10.1547	10.4588	11.9751	10.1653	10.4608	11.9769
120	16.5466	17.2440	20.0000	16.5528	17.2445	20.0000
130	24.0655	24.0292	30.0000	24.0603	24.0281	30.0000
140	32.3010	31.5543	40.0000	32.2940	31.5543	40.0000

Differences to the results from 1-D Binomial Tree

	Call Options					
Strike	Black-Scholes European	Binomial Tree (2 steps) European	Binomial Tree (2 steps) American	Monte Carlo (500000) European	Implied Tree (2 steps) European	Implied Tree (2 steps) American
60	0.0107	-	-	0.0142	0.0000	0.0000
70	0.1200	-	-	0.1222	0.0000	0.0000
80	0.1244	-	-	0.1271	0.0011	0.0011
90	-0.5929	-	-	-0.5808	0.0006	0.0006
100	0.2503	-	-	0.2681	0.0001	0.0001
110	-0.3041	-	-	-0.2899	0.0020	0.0020
120	-0.6973	-	-	-0.6876	0.0005	0.0005
130	0.0364	-	-	0.0347	-0.0010	-0.0010
140	0.7467	-	-	0.7432	0.0000	0.0000

Differences to the results from 1-D Binomial Tree

	Put Options					
Strike	Black-Scholes European	Binomial Tree (2 steps) European	Binomial Tree (2 steps) American	Monte Carlo (500000) European	Implied Tree (2 steps) European	Implied Tree (2 steps) American
60	0.0107	-	-	0.0107	0.0000	0.0000
70	0.1200	-	-	0.1187	0.0000	0.0000
80	0.1244	-	-	0.1236	0.0011	0.0011
90	-0.5929	-	-	-0.5843	0.0006	0.0006
100	0.2503	-	-	0.2646	0.0001	0.0016
110	-0.3041	-	-	-0.2935	0.0020	0.0018
120	-0.6973	-	-	-0.6912	0.0005	0.0000
130	0.0364	-	-	0.0311	-0.0010	0.0000
140	0.7467	-	-	0.7397	0.0000	0.0000

Exhibit 5: Price Comparison (2-step trees)

Exhibit 5 compares the results of the implied tree to those of Black-Scholes' model and Monte Carlo simulations. Although both binomial tree and implied tree value options slightly different with those of continuous model, the difference will be corrected by increasing number of steps as shown in Exhibit 6, where 100 steps of trees are used.

	Call Options					
Strike	Black-Scholes European	Binomial Tree (100 steps) European	Binomial Tree (100 steps) American	Monte Carlo (500000) European	Implied Tree (100 steps) European	Implied Tree (100 steps) American
60	40.8434	40.8431	40.8916	40.8588	40.8431	40.8916
70	31.9043	31.9038	31.9113	31.9222	31.9038	31.9113
80	23.3896	23.3912	23.3923	23.4100	23.3912	23.3923
90	15.8850	15.8947	15.8948	15.9102	15.8947	15.8948
100	9.9409	9.9499	9.9499	9.9717	9.9499	9.9499
110	5.7455	5.7605	5.7605	5.7745	5.7605	5.7605
120	3.0891	3.0894	3.0894	3.1119	3.0894	3.0894
130	1.5596	1.5566	1.5566	1.5768	1.5566	1.5566
140	0.7467	0.7463	0.7463	0.7611	0.7463	0.7463

	Put Options					
Strike	Black-Scholes European	Binomial Tree (100 steps) European	Binomial Tree (100 steps) American	Monte Carlo (500000) European	Implied Tree (100 steps) European	Implied Tree (100 steps) American
60	0.0107	0.0104	0.0107	0.0126	0.0104	0.0107
70	0.1200	0.1195	0.1250	0.1244	0.1195	0.1250
80	0.6537	0.6553	0.6959	0.6606	0.6553	0.6959
90	2.1974	2.2071	2.3953	2.2091	2.2071	2.3953
100	5.3017	5.3107	5.9359	5.3190	5.3107	5.9359
110	10.1547	10.1697	11.7766	10.1701	10.1697	11.7766
120	16.5466	16.5469	20.0496	16.5559	16.5469	20.0496
130	24.0655	24.0625	30.0000	24.0692	24.0625	30.0000
140	32.3010	32.3006	40.0000	32.3019	32.3006	40.0000

Differences to the results from 1-D Binomial Tree

	Call Options					
Strike	Black-Scholes European	Binomial Tree (100 steps) European	Binomial Tree (100 steps) American	Monte Carlo (500000) European	Implied Tree (100 steps) European	Implied Tree (100 steps) American
60	0.0004	-	-	0.0157	0.0000	0.0000
70	0.0005	-	-	0.0184	0.0000	0.0000
80	-0.0016	-	-	0.0188	0.0000	0.0000
90	-0.0097	-	-	0.0155	0.0000	0.0000
100	-0.0090	-	-	0.0218	0.0000	0.0000
110	-0.0150	-	-	0.0139	0.0000	0.0000
120	-0.0003	-	-	0.0225	0.0000	0.0000
130	0.0030	-	-	0.0201	0.0000	0.0000
140	0.0004	-	-	0.0149	0.0000	0.0000

Differences to the results from 1-D Binomial Tree

	Put Options					
Strike	Black-Scholes European	Binomial Tree (100 steps) European	Binomial Tree (100 steps) American	Monte Carlo (500000) European	Implied Tree (100 steps) European	Implied Tree (100 steps) American
60	0.0004	-	-	0.0022	0.0000	0.0000
70	0.0005	-	-	0.0049	0.0000	0.0000
80	-0.0016	-	-	0.0053	0.0000	0.0000
90	-0.0097	-	-	0.0020	0.0000	0.0000
100	-0.0090	-	-	0.0083	0.0000	0.0000
110	-0.0150	-	-	0.0004	0.0000	0.0000
120	-0.0003	-	-	0.0090	0.0000	0.0000
130	0.0030	-	-	0.0066	0.0000	0.0000
140	0.0004	-	-	0.0014	0.0000	0.0000

Exhibit 6: Price Comparison (100-step trees)

Two-asset options

Assuming the options payoff depend on the value of basket which is defined as $n_1S_1 + n_2S_2$. Information about the basket options and the underlying assets are listed as below.

Time to Expiration:	$T = 1.0$ year
Interest Rate:	$r = 5\%$
Dividend Yield:	$\delta_1 = 5\%$, $\delta_2 = 5\%$
Volatilities:	$\sigma_1 = 20\%$, $\sigma_2 = 20\%$
Correlation:	$\rho = 0.5$
Spot Prices:	$S_{1,0} = 50$, $S_{2,0} = 50$
# of shares:	$n_1 = 1$, $n_2 = 1$
Strike Prices:	K = 60, 70, 80, 90, 100, 110, 120, 130, 140.

The benchmark valuation model is the two-dimensional binomial tree, or the so called the "binomial pyramids." As a simple example, we start with a two-step two-dimensional binomial tree. For each node, there are four possible moves in the 2-dimensional price space. The four multiplicative moving vectors can be calculated as: (u, A), (u, B), (d, C) and (d, D), where

$$u = \exp[(r-\delta_1-\tfrac{1}{2}\sigma_1^2)h + \sigma_1\sqrt{h}] = 1.1404$$
$$d = \exp[(r-\delta_1-\tfrac{1}{2}\sigma_1^2)h - \sigma_1\sqrt{h}] = 0.8595$$
$$A = \exp[(r-\delta_2-\tfrac{1}{2}\sigma_2^2)h + \sigma_2\sqrt{h}\,(\rho + \sqrt{1-\rho^2})] = 1.2010$$
$$B = \exp[(r-\delta_2-\tfrac{1}{2}\sigma_2^2)h + \sigma_2\sqrt{h}\,(\rho - \sqrt{1-\rho^2})] = 0.9401$$
$$C = \exp[(r-\delta_2-\tfrac{1}{2}\sigma_2^2)h - \sigma_2\sqrt{h}\,(\rho - \sqrt{1-\rho^2})] = 1.0426$$
$$D = \exp[(r-\delta_2-\tfrac{1}{2}\sigma_2^2)h - \sigma_2\sqrt{h}\,(\rho + \sqrt{1-\rho^2})] = 0.8161$$

By the end of two steps, there are 3-by-3 nodes in the two dimensional space. Exhibit 7 gives all the nine nodes and their associated node probabilities. The underscored numbers are values of the basket.

<u>137.16</u> (65.03, 72.12)	<u>121.49</u> (65.03, 56.46)	<u>109.22</u> (65.03, 44.19)
<u>111.62</u> (49.01, 62.61)	<u>98.02</u> (49.01, 49.01)	<u>87.37</u> (49.01, 38.36)
<u>91.29</u> (36.94, 54.36)	<u>79.48</u> (36.94, 42.55)	<u>70.24</u> (36.94, 33.30)

Ending Nodes

0.0625	0.1250	0.0625
0.1250	0.2500	0.1250
0.0625	0.1250	0.0625

Nodal Probabilities

Exhibit 7: Ending Nodes and Node Probabilities on a Two-Dimension Tree

As we see, the two-step bivariate binomial model effectively creates nine different states for the basket, and basket options are nothing but options on the single-variate basekt. We will show that a single-variate implied tree recovered by the risk-neutral probabilities of the basket can be used to price and hedge such options.

Our first step is to sample out three discrete states from the 3-by-3 possible ending values of the basket in order to build a two-steps implied tree. This is done by the equal log-spaced sampling method described in the previous example of single-asset options. We start with calculating the mean and standard deviation of the basket's log-returns. Knowing the 3-by-3 possible ending values of the basket and the corresponding nodal probabilities, we can compute the mean and standard deviation of log-returns on the basket. In this case, they are -0.0150 and 0.1733, respectively. We sample three states out for constructing a two-steps implied tree, based on the formula: S_0*exp[mean + std * (2j - m)/√m], j=0, 1, …, m, where m = 2. The states are 77.10, 98.51 and 125.87.

The second step is to retrieve risk-neutral probabilities of the three states. They are obtained by creating a no-arbitrage payoff table using the prices of standard European options, current price of the basket and the risk-free bond. The process is similar to that of in the single asset example. Exhibit 8 shows the "binomial pyramids" in pricing a two-asset call option struck at 98.51. Its price is 7.23. The resultant payoff table across states as well as the risk-neutral probabilities is in Exhibit 9.

38.64	22.98	10.71
13.11	0.00	0.00
0.00	0.00	0.00

18.22	8.21
3.20	0.00

7.23

Exhibit 8: Two-Dimension Binomial Tree for a Call Option

			States		
Security	Prices	Strike	77.10	98.51	125.87
Call	7.23	98.51			27.36
Asset	100	0	$e^{0.05}$*77.10	$e^{0.05}$*98.51	$e^{0.05}$*125.87
Bond	95.12		100	100	100

states	probability
125.87	0.2776
98.51	0.4372
77.10	0.2851

Exhibit 9: Payoff table and Risk-neutral Probabilities

Our third step is using the risk-neutral probabilities to construct implied binomial tree. The single-variate implied tree for the basket is obtained in Exhibit 10. The option tree for pricing an American call option with strike at 80 is also shown in Exhibit 11, where the underscored 33.82 indicate an early exercise at that node.

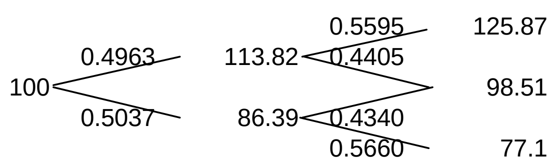


Exhibit 10: Implied Binomial Tree (Two-Asset basket)

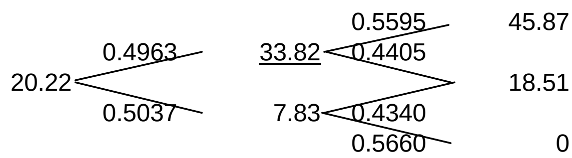


Exhibit 11: Option Tree for American Call with Strike at 80

In Appendix C, various cases are tested with increasing number of steps in demonstrating the convergence between the results from implied tree and those obtained from multi-dimension binomial tree. In comparison, we also implement three other methods, which are:

(1) the standard Black Scholes' formula for European options using the return volatility obtained from the ending risk-neutral probability,
(2) Monte Carlo simulation for European options, and
(3) the Least-squares Monte Carlo simulation for American options.

In most cases, we keep all five methods with the same steps except some of the three- and four-asset cases where neither the multi-dimensional binomial trees nor Monte Carlo simulations are able to handle 100 steps or more.

Conclusion

This article contributes the ongoing challenge of pricing American-style multi-asset basket options using a combination of Monte Carlo simulation and implied binomial tree. The method provides a quick valuation and further hedging parameters via the generated implied tree. At the mean time, the application of Monte Carlo simulation brings the possibility of incorporation of other factors such as random volatility and correlation, random interest rate, and jumps.

Appendix C:

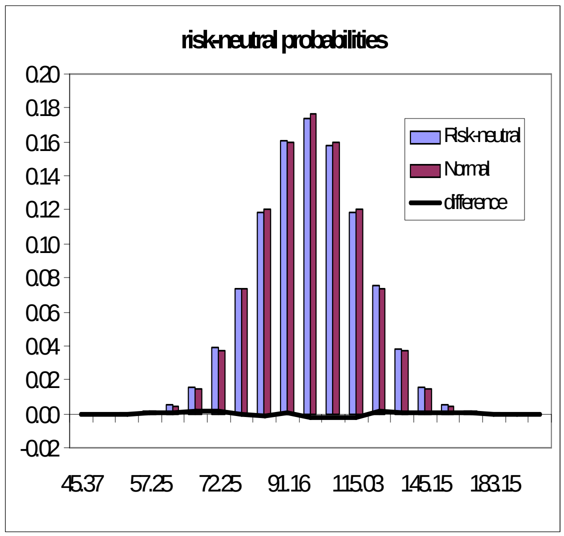
Test Results

Two-asset options (Case 1)

Time to Expiration:	T = 1.0 year
Interest Rate:	r = 5%
Dividend Yield:	δ_1 = 5%, δ_2 = 5%
Volatilities:	σ_1 = 20%, σ_2 = 20%
Correlation:	ρ = 0.5
Spot Prices:	$S_{1,0}$ = 50, $S_{2,0}$ = 50
# of shares:	n_1 = 1, n_2 = 1
Strike Prices:	K = 60, 70, 80, 90, 100, 110, 120, 130, 140.

Risk-Neutral Probabilities

Strike	Put/Call	Option Prices	Probabilities Risk-neutra	Normal	Diffs
45.374	P	0.0000	0.000006	0.000001	0.000005
49.031	P	0.0000	0.000046	0.000019	0.000027
52.983	P	0.0002	0.000296	0.000181	0.000115
57.254	P	0.0016	0.001400	0.001087	0.000313
61.869	P	0.0093	0.005262	0.004621	0.000642
66.856	P	0.0426	0.015925	0.014786	0.001139
72.245	P	0.1601	0.038520	0.036964	0.001555
78.068	P	0.5005	0.073920	0.073929	-0.000009
84.361	P	1.3109	0.118734	0.120134	-0.001401
91.161	P	2.9545	0.160426	0.160179	0.000247
98.509	C	7.2704	0.173841	0.176197	-0.002356
106.449	C	4.1614	0.158319	0.160179	-0.001860
115.029	C	2.0939	0.118283	0.120134	-0.001851
124.301	C	0.9031	0.075083	0.073929	0.001154
134.321	C	0.3318	0.037622	0.036964	0.000658
145.148	C	0.1019	0.015430	0.014786	0.000644
156.847	C	0.0253	0.005195	0.004621	0.000575
169.490	C	0.0049	0.001369	0.001087	0.000282
183.152	C	0.0007	0.000279	0.000181	0.000097
197.914	C	0.0001	0.000040	0.000019	0.000021
213.867	C	0.0000	0.000005	0.000001	0.000004



Two-Asset Case(1) Steps: 10

Call Options

Strike	Black-Scholes European	Monte Carlo (10000) European	LSMC (10000) American	Binomial Tree European	Binomial Tree American	Implied Tree European	Implied Tree American
60	38.0548	38.0838	40.0000	38.0516	39.9989	38.0556	40.0000
70	28.6346	28.6681	30.0000	28.6265	29.9989	28.6407	30.0000
80	19.7036	19.7583	20.2853	19.7062	20.2898	19.7712	20.3661
90	12.0933	12.1895	12.3530	12.1395	12.3464	12.2862	12.5069
100	6.5451	6.6550	6.7265	6.6109	6.6811	6.7430	6.8184
110	3.1330	3.2245	3.2624	3.1643	3.1863	3.1714	3.2049
120	1.3424	1.3939	1.4071	1.3355	1.3419	1.4995	1.5112
130	0.5229	0.5549	0.5667	0.5056	0.5073	0.6338	0.6364
140	0.1882	0.2016	0.2097	0.1671	0.1675	0.1990	0.1994

Put Options

Strike	Black-Scholes European	Monte Carlo (10000) European	LSMC (10000) American	Binomial Tree European	Binomial Tree American	Implied Tree European	Implied Tree American
60	0.0056	0.0070	0.0076	0.0035	0.0035	0.0064	0.0064
70	0.0977	0.1036	0.1068	0.0906	0.0909	0.1038	0.1040
80	0.6790	0.7061	0.7134	0.6826	0.6854	0.7467	0.7528
90	2.5810	2.6496	2.6731	2.6282	2.6480	2.7739	2.8037
100	6.5451	6.6274	6.7020	6.6119	6.6872	6.7430	6.8443
110	12.6453	12.7092	12.8693	12.6776	12.8968	12.6837	12.9551
120	20.3669	20.3909	20.7954	20.3611	20.8665	20.5241	21.0382
130	29.0598	29.0642	30.0509	29.0435	30.0446	29.1707	30.1717
140	38.2373	38.2231	40.0000	38.2173	40.0011	38.2482	40.0000

Differences to the results from 2-D Binomial Tree
Call Options

Strike	Black-Scholes European	Monte Carlo European	LSMC American	Binomial Tree European	Binomial Tree American	Implied Tree European	Implied Tree American
60	0.0032	0.0322	0.0011	-	-	0.0040	0.0011
70	0.0081	0.0416	0.0011	-	-	0.0142	0.0011
80	-0.0026	0.0521	-0.0045	-	-	0.0650	0.0763
90	-0.0462	0.0500	0.0066	-	-	0.1467	0.1605
100	-0.0658	0.0441	0.0454	-	-	0.1321	0.1373
110	-0.0313	0.0602	0.0761	-	-	0.0071	0.0186
120	0.0069	0.0584	0.0652	-	-	0.1640	0.1693
130	0.0173	0.0493	0.0594	-	-	0.1282	0.1291
140	0.0211	0.0345	0.0422	-	-	0.0319	0.0319

Differences to the results from 2-D Binomial Tree
Put Options

Strike	Black-Scholes European	Monte Carlo European	LSMC American	Binomial Tree European	Binomial Tree American	Implied Tree European	Implied Tree American
60	0.0021	0.0035	0.0041	-	-	0.0029	0.0029
70	0.0071	0.0130	0.0159	-	-	0.0132	0.0131
80	-0.0036	0.0235	0.0280	-	-	0.0641	0.0674
90	-0.0472	0.0214	0.0251	-	-	0.1457	0.1557
100	-0.0668	0.0155	0.0148	-	-	0.1311	0.1571
110	-0.0323	0.0316	-0.0275	-	-	0.0061	0.0583
120	0.0058	0.0298	-0.0711	-	-	0.1630	0.1717
130	0.0163	0.0207	0.0063	-	-	0.1272	0.1271
140	0.0200	0.0058	-0.0011	-	-	0.0309	-0.0011

Two-Asset Case(1) Steps: 20

Call Options

Strike	Black-Scholes European	Monte Carlo (10000) European	LSMC (10000) American	Binomial Tree European	Binomial Tree American	Implied Tree European	Implied Tree American
60	38.0548	38.0707	40.0000	38.0534	39.9995	38.0554	40.0000
70	28.6346	28.6530	30.0000	28.6329	29.9995	28.6480	30.0000
80	19.7037	19.7486	20.3362	19.7116	20.2899	19.7739	20.3535
90	12.0934	12.1377	12.2812	12.1278	12.3405	12.1862	12.4160
100	6.5453	6.5852	6.6563	6.5896	6.6627	6.6865	6.7629
110	3.1331	3.1798	3.2064	3.1651	3.1885	3.3058	3.3317
120	1.3425	1.3928	1.4225	1.3480	1.3553	1.4555	1.4642
130	0.5230	0.5448	0.5516	0.5179	0.5198	0.5781	0.5808
140	0.1882	0.1984	0.2023	0.1812	0.1817	0.2112	0.2120

Put Options

Strike	Black-Scholes European	Monte Carlo (10000) European	LSMC (10000) American	Binomial Tree European	Binomial Tree American	Implied Tree European	Implied Tree American
60	0.0056	0.0068	0.0073	0.0047	0.0047	0.0062	0.0062
70	0.0977	0.1014	0.1020	0.0966	0.0968	0.1111	0.1115
80	0.6791	0.7093	0.7189	0.6875	0.6907	0.7493	0.7537
90	2.5811	2.6107	2.6341	2.6160	2.6352	2.6739	2.6944
100	6.5453	6.5706	6.6262	6.5901	6.6651	6.6865	6.7731
110	12.6454	12.6775	12.8411	12.6779	12.8942	12.8181	13.0436
120	20.3670	20.4027	20.9134	20.3731	20.8815	20.4801	20.9904
130	29.0599	29.0670	30.1083	29.0553	30.0721	29.1150	30.1276
140	38.2374	38.2329	40.0000	38.2309	40.0005	38.2604	40.0000

Differences to the results from 2-D Binomial Tree
Call Options

Strike	Black-Scholes European	Monte Carlo European	LSMC American	Binomial Tree European	Binomial Tree American	Implied Tree European	Implied Tree American
60	0.0014	0.0173	0.0005	-	-	0.0020	0.0005
70	0.0017	0.0201	0.0005	-	-	0.0151	0.0005
80	-0.0079	0.0370	0.0463	-	-	0.0623	0.0636
90	-0.0344	0.0099	-0.0593	-	-	0.0584	0.0755
100	-0.0443	-0.0044	-0.0064	-	-	0.0969	0.1002
110	-0.0320	0.0147	0.0179	-	-	0.1407	0.1432
120	-0.0055	0.0448	0.0672	-	-	0.1075	0.1089
130	0.0051	0.0269	0.0318	-	-	0.0602	0.0610
140	0.0070	0.0172	0.0206	-	-	0.0300	0.0303

Differences to the results from 2-D Binomial Tree
Put Options

Strike	Black-Scholes European	Monte Carlo European	LSMC American	Binomial Tree European	Binomial Tree American	Implied Tree European	Implied Tree American
60	0.0009	0.0021	0.0026	-	-	0.0015	0.0015
70	0.0011	0.0048	0.0052	-	-	0.0145	0.0147
80	-0.0084	0.0218	0.0282	-	-	0.0618	0.0630
90	-0.0349	-0.0053	-0.0011	-	-	0.0579	0.0592
100	-0.0448	-0.0195	-0.0389	-	-	0.0964	0.1080
110	-0.0325	-0.0004	-0.0531	-	-	0.1402	0.1494
120	-0.0061	0.0296	0.0319	-	-	0.1070	0.1089
130	0.0046	0.0117	0.0362	-	-	0.0597	0.0555
140	0.0065	0.0020	-0.0005	-	-	0.0295	-0.0005

Two-Asset Case(1) Steps: 50

Call Options

Strike	Black-Scholes European	Monte Carlo (10000) European	LSMC (10000) American	Binomial Tree European	Binomial Tree American	Implied Tree European	Implied Tree American
60	38.0548	38.0677	40.0000	38.0544	39.9998	38.0548	40.0000
70	28.6346	28.6521	30.0000	28.6351	29.9998	28.6363	30.0000
80	19.7037	19.7422	20.3558	19.7117	20.2951	19.7308	20.3168
90	12.0935	12.1623	12.2931	12.1163	12.3309	12.1442	12.3623
100	6.5454	6.6260	6.6822	6.5741	6.6482	6.6331	6.7084
110	3.1332	3.2055	3.2258	3.1590	3.1829	3.1998	3.2245
120	1.3425	1.3902	1.3955	1.3560	1.3633	1.3591	1.3672
130	0.5230	0.5412	0.5571	0.5265	0.5286	0.5503	0.5528
140	0.1882	0.1943	0.2013	0.1886	0.1892	0.1938	0.1944

Put Options

Strike	Black-Scholes European	Monte Carlo (10000) European	LSMC (10000) American	Binomial Tree European	Binomial Tree American	Implied Tree European	Implied Tree American
60	0.0057	0.0046	0.0056	0.0054	0.0054	0.0056	0.0056
70	0.0977	0.1013	0.1022	0.0984	0.0987	0.0995	0.0999
80	0.6791	0.7036	0.7038	0.6873	0.6906	0.7062	0.7097
90	2.5812	2.6360	2.6335	2.6042	2.6235	2.6319	2.6532
100	6.5454	6.6120	6.6656	6.5744	6.6492	6.6331	6.7110
110	12.6455	12.7038	12.8478	12.6715	12.8877	12.7121	12.9337
120	20.3671	20.4008	20.8416	20.3808	20.8880	20.3837	20.9027
130	29.0599	29.0641	30.1320	29.0636	30.0854	29.0872	30.1005
140	38.2374	38.2295	40.0000	38.2380	40.0002	38.2430	40.0000

Differences to the results from 2-D Binomial Tree
Call Options

Strike	Black-Scholes European	Monte Carlo European	LSMC American	Binomial Tree European	Binomial Tree American	Implied Tree European	Implied Tree American
60	0.0004	0.0133	0.0002	-	-	0.0004	0.0002
70	-0.0005	0.0170	0.0002	-	-	0.0012	0.0002
80	-0.0080	0.0305	0.0607	-	-	0.0191	0.0217
90	-0.0228	0.0460	-0.0378	-	-	0.0279	0.0314
100	-0.0287	0.0519	0.0340	-	-	0.0590	0.0602
110	-0.0258	0.0465	0.0429	-	-	0.0408	0.0416
120	-0.0135	0.0342	0.0322	-	-	0.0031	0.0039
130	-0.0035	0.0147	0.0285	-	-	0.0238	0.0242
140	-0.0004	0.0057	0.0121	-	-	0.0052	0.0052

Differences to the results from 2-D Binomial Tree
Put Options

Strike	Black-Scholes European	Monte Carlo European	LSMC American	Binomial Tree European	Binomial Tree American	Implied Tree European	Implied Tree American
60	0.0003	-0.0008	0.0002	-	-	0.0002	0.0002
70	-0.0007	0.0029	0.0035	-	-	0.0011	0.0012
80	-0.0082	0.0163	0.0132	-	-	0.0189	0.0191
90	-0.0230	0.0318	0.0100	-	-	0.0277	0.0297
100	-0.0290	0.0376	0.0164	-	-	0.0587	0.0618
110	-0.0260	0.0323	-0.0399	-	-	0.0406	0.0460
120	-0.0137	0.0200	-0.0464	-	-	0.0029	0.0147
130	-0.0037	0.0005	0.0466	-	-	0.0236	0.0151
140	-0.0006	-0.0085	-0.0002	-	-	0.0050	-0.0002

Two-Asset Case(1) Steps:[2] 100

Call Options

Strike	Black-Scholes European	Monte Carlo (10000) European	LSMC (10000) American	Binomial Tree European	Binomial Tree American	Implied Tree European	Implied Tree American
60	38.0548	38.0677	40.0000	38.0547	39.9999	38.0551	40.0000
70	28.6346	28.6521	30.0000	28.6360	29.9999	28.6379	30.0000
80	19.7037	19.7422	20.3558	19.7123	20.2962	19.7126	20.3013
90	12.0935	12.1623	12.2931	12.1148	12.3297	12.1389	12.3548
100	6.5454	6.6260	6.6822	6.5740	6.6480	6.6055	6.6804
110	3.1332	3.2055	3.2258	3.1578	3.1818	3.1744	3.1989
120	1.3425	1.3902	1.3955	1.3575	1.3648	1.3752	1.3830
130	0.5230	0.5412	0.5571	0.5303	0.5324	0.5303	0.5327
140	0.1882	0.1943	0.2013	0.1908	0.1914	0.1932	0.1938

Put Options

Strike	Black-Scholes European	Monte Carlo (10000) European	LSMC (10000) American	Binomial Tree European	Binomial Tree American	Implied Tree European	Implied Tree American
60	0.0057	0.0046	0.0056	0.0057	0.0057	0.0059	0.0059
70	0.0978	0.1013	0.1022	0.0992	0.0995	0.1010	0.1013
80	0.6791	0.7036	0.7038	0.6878	0.6911	0.6880	0.6915
90	2.5812	2.6360	2.6335	2.6026	2.6217	2.6266	2.6463
100	6.5454	6.6120	6.6656	6.5741	6.6485	6.6055	6.6810
110	12.6455	12.7038	12.8478	12.6702	12.8860	12.6867	12.9057
120	20.3671	20.4008	20.8416	20.3822	20.8890	20.3998	20.9077
130	29.0599	29.0641	30.1320	29.0673	30.0870	29.0672	30.0923
140	38.2374	38.2295	40.0000	38.2400	40.0001	38.2424	40.0000

Differences to the results from 2-D Binomial Tree
Call Options

Strike	Black-Scholes European	Monte Carlo European	LSMC American	Binomial Tree European	Binomial Tree American	Implied Tree European	Implied Tree American
60	0.0001	0.0130	0.0001	-	-	0.0004	0.0001
70	-0.0014	0.0161	0.0001	-	-	0.0019	0.0001
80	-0.0086	0.0299	0.0596	-	-	0.0003	0.0051
90	-0.0213	0.0475	-0.0366	-	-	0.0241	0.0251
100	-0.0286	0.0520	0.0342	-	-	0.0315	0.0324
110	-0.0246	0.0477	0.0440	-	-	0.0166	0.0171
120	-0.0150	0.0327	0.0307	-	-	0.0177	0.0182
130	-0.0073	0.0109	0.0247	-	-	0.0000	0.0003
140	-0.0026	0.0035	0.0099	-	-	0.0024	0.0024

Differences to the results from 2-D Binomial Tree
Put Options

Strike	Black-Scholes European	Monte Carlo European	LSMC American	Binomial Tree European	Binomial Tree American	Implied Tree European	Implied Tree American
60	0.0000	-0.0011	-0.0001	-	-	0.0002	0.0002
70	-0.0014	0.0021	0.0027	-	-	0.0018	0.0018
80	-0.0087	0.0158	0.0127	-	-	0.0002	0.0004
90	-0.0214	0.0334	0.0118	-	-	0.0240	0.0246
100	-0.0287	0.0379	0.0171	-	-	0.0314	0.0325
110	-0.0247	0.0336	-0.0382	-	-	0.0165	0.0197
120	-0.0151	0.0186	-0.0474	-	-	0.0176	0.0187
130	-0.0074	-0.0032	0.0450	-	-	-0.0001	0.0053
140	-0.0026	-0.0105	-0.0001	-	-	0.0024	-0.0001

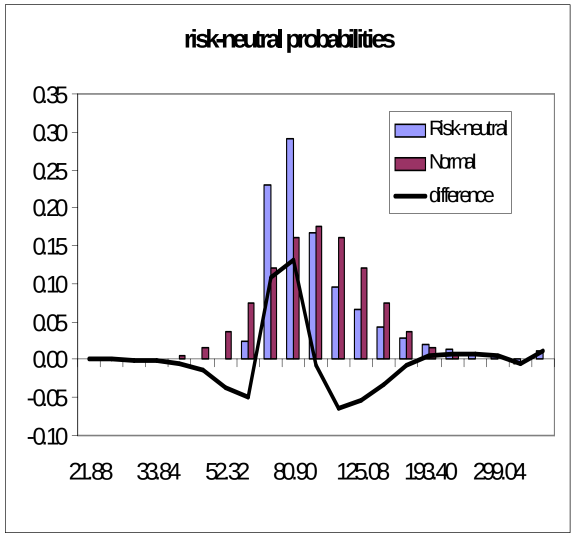
[2] 50 steps are applied to both Monte Carlo simulation and the Least Squares Monte Carlo.

Two-asset options (Case 2)

Time to Expiration: $T = 1.0$ year
Interest Rate: $r = 5\%$
Dividend Yield: $\delta_1 = 5\%$, $\delta_2 = 5\%$
Volatilities: $\sigma_1 = 20\%$, $\sigma_2 = 90\%$
Correlation: $\rho = -0.9$
Spot Prices: $S_{1,0} = 50$, $S_{2,0} = 50$
\# of shares: $n_1 = 1$, $n_2 = 1$
Strike Prices: K = 60, 70, 80, 90, 100, 110, 120, 130, 140.

Risk-Neutral Probabilities

		Option	Probabilities		
Strike	Put/Call	Prices	Risk-neutra	Normal	Diffs
21.884	P	0.0000	0.000000	0.000001	-0.000001
25.306	P	0.0000	0.000000	0.000019	-0.000019
29.262	P	0.0000	0.000000	0.000181	-0.000181
33.837	P	0.0000	0.000000	0.001087	-0.001087
39.128	P	0.0000	0.000000	0.004621	-0.004621
45.246	P	0.0000	0.000000	0.014786	-0.014786
52.320	P	0.0000	0.000123	0.036964	-0.036841
60.500	P	0.0010	0.025021	0.073929	-0.048908
69.959	P	0.2272	0.229349	0.120134	0.109215
80.897	P	2.8751	0.291802	0.160179	0.131623
93.545	C	15.5879	0.167917	0.176197	-0.008280
108.171	C	11.6119	0.096203	0.160179	-0.063977
125.083	C	8.5619	0.065523	0.120134	-0.054611
144.640	C	6.2540	0.042088	0.073929	-0.031841
167.254	C	4.4907	0.028615	0.036964	-0.008350
193.404	C	3.1634	0.019361	0.014786	0.004575
223.643	C	2.1854	0.012758	0.004621	0.008137
258.609	C	1.4789	0.008220	0.001087	0.007133
299.043	C	0.9782	0.005197	0.000181	0.005016
345.798	C	0.6302	-0.004431	0.000019	-0.004450
399.863	C	0.3939	0.012254	0.000001	0.012254



Two-Asset Case(2) Steps: 10

Call Options

Strike	Black-Scholes European	Monte Carlo (10000) European	LSMC (10000) American	Binomial Tree European	Binomial Tree American	Implied Tree European	Implied Tree American
60	38.5961	37.9077	40.0000	37.8723	39.8136	38.8215	40.8092
70	30.2163	28.6341	30.0000	28.5876	29.8136	30.0214	30.9306
80	22.8939	21.4239	21.9614	21.3935	21.8086	22.8136	23.3034
90	16.8366	16.8052	17.3085	16.7687	16.9846	17.9410	18.3225
100	12.0665	13.5824	13.9667	13.5230	13.6573	14.5383	14.7980
110	8.4634	11.1848	11.5379	11.1016	11.1955	11.9509	12.1742
120	5.8329	9.3422	9.6220	9.2592	9.3227	9.9571	10.1153
130	3.9641	7.8862	8.1123	7.7166	7.7683	8.5140	8.6479
140	2.6646	6.7407	6.9911	6.5877	6.6222	7.0709	7.1850

Put Options

Strike	Black-Scholes European	Monte Carlo (10000) European	LSMC (10000) American	Binomial Tree European	Binomial Tree American	Implied Tree European	Implied Tree American
60	0.5470	0.0014	0.0014	0.0004	0.0004	0.0026	0.0026
70	1.6794	0.2400	0.2423	0.2281	0.2288	0.7148	0.7162
80	3.8693	2.5421	2.5397	2.5463	2.5601	3.0193	3.0376
90	7.3243	7.4358	7.4456	7.4337	7.5016	7.6590	7.6967
100	12.0665	13.7252	13.8238	13.7004	13.8716	13.7686	13.9295
110	17.9757	20.8399	21.0792	20.7912	21.0937	20.6935	20.9912
120	24.8575	28.5096	28.8986	28.4611	28.9349	28.2120	28.7971
130	32.5009	36.5659	37.1190	36.4308	37.1321	36.2812	37.1421
140	40.7137	44.9327	45.7550	44.8142	45.7414	44.3504	45.7046

Differences to the results from 2-D Binomial Tree
Call Options

Strike	Black-Scholes European	Monte Carlo European	LSMC American	Binomial Tree European	Binomial Tree American	Implied Tree European	Implied Tree American
60	0.7238	0.0354	0.1864	-	-	0.9492	0.9956
70	1.6287	0.0465	0.1864	-	-	1.4338	1.1170
80	1.5004	0.0304	0.1528	-	-	1.4201	1.4948
90	0.0679	0.0365	0.3239	-	-	1.1723	1.3379
100	-1.4565	0.0594	0.3094	-	-	1.0153	1.1407
110	-2.6382	0.0832	0.3424	-	-	0.8493	0.9787
120	-3.4263	0.0830	0.2993	-	-	0.6979	0.7926
130	-3.7525	0.1696	0.3440	-	-	0.7974	0.8796
140	-3.9231	0.1530	0.3689	-	-	0.4832	0.5628

Differences to the results from 2-D Binomial Tree
Put Options

Strike	Black-Scholes European	Monte Carlo European	LSMC American	Binomial Tree European	Binomial Tree American	Implied Tree European	Implied Tree American
60	0.5466	0.0010	0.0010	-	-	0.0022	0.0022
70	1.4513	0.0119	0.0135	-	-	0.4867	0.4874
80	1.3230	-0.0042	-0.0204	-	-	0.4730	0.4775
90	-0.1094	0.0021	-0.0560	-	-	0.2253	0.1951
100	-1.6339	0.0248	-0.0478	-	-	0.0682	0.0579
110	-2.8155	0.0487	-0.0145	-	-	-0.0977	-0.1025
120	-3.6036	0.0485	-0.0363	-	-	-0.2491	-0.1378
130	-3.9299	0.1351	-0.0131	-	-	-0.1496	0.0100
140	-4.1005	0.1185	0.0136	-	-	-0.4638	-0.0368

Two-Asset Case(2) Steps: 20

Call Options

Strike	Black-Scholes European	Monte Carlo (10000) European	LSMC (10000) American	Binomial Tree European	Binomial Tree American	Implied Tree European	Implied Tree American
60	38.6064	38.8562	40.0289	37.9604	39.9059	39.0816	41.0844
70	30.2375	29.5706	30.1782	28.6785	29.9059	29.8045	31.0844
80	22.9269	22.3005	22.4821	21.4695	21.8971	22.7023	23.2447
90	16.8788	17.6760	17.6119	16.8502	17.0779	18.1158	18.4949
100	12.1134	14.4648	14.6292	13.6274	13.7736	14.8036	15.0945
110	8.5104	12.0520	12.1744	11.2076	11.3091	12.2218	12.4580
120	5.8765	10.1882	10.3079	9.3527	9.4273	10.3844	10.5815
130	4.0021	8.6936	8.8717	7.9032	7.9583	8.8521	9.0144
140	2.6963	7.4864	7.7061	6.6953	6.7372	7.6352	7.7749

Put Options

Strike	Black-Scholes European	Monte Carlo (10000) European	LSMC (10000) American	Binomial Tree European	Binomial Tree American	Implied Tree European	Implied Tree American
60	0.5572	0.0005	0.0007	0.0007	0.0007	0.0009	0.0009
70	1.7006	0.2272	0.2302	0.2311	0.2318	0.2361	0.2373
80	3.9023	2.4694	2.4843	2.5344	2.5478	2.6463	2.6520
90	7.3665	7.3572	7.4382	7.4274	7.4931	7.5720	7.6105
100	12.1134	13.6583	13.8031	13.7169	13.8788	13.7721	13.9084
110	18.0227	20.7578	21.0248	20.8094	21.1071	20.7026	21.0271
120	24.9010	28.4063	28.8076	28.4668	28.9334	28.3775	28.9174
130	32.5389	36.4240	37.0295	36.5295	37.2041	36.3575	37.2168
140	40.7454	44.7290	45.5608	44.8339	45.7603	44.6529	45.8534

Differences to the results from 2-D Binomial Tree
Call Options

Strike	Black-Scholes European	Monte Carlo European	LSMC American	Binomial Tree European	Binomial Tree American	Implied Tree European	Implied Tree American
60	0.6460	0.8958	0.1230	-	-	1.1212	1.1785
70	1.5590	0.8921	0.2723	-	-	1.1260	1.1785
80	1.4574	0.8310	0.5850	-	-	1.2328	1.3476
90	0.0286	0.8258	0.5340	-	-	1.2656	1.4170
100	-1.5140	0.8374	0.8556	-	-	1.1762	1.3209
110	-2.6972	0.8444	0.8653	-	-	1.0142	1.1489
120	-3.4762	0.8355	0.8806	-	-	1.0317	1.1542
130	-3.9011	0.7904	0.9134	-	-	0.9489	1.0561
140	-3.9990	0.7911	0.9689	-	-	0.9399	1.0377

Differences to the results from 2-D Binomial Tree
Put Options

Strike	Black-Scholes European	Monte Carlo European	LSMC American	Binomial Tree European	Binomial Tree American	Implied Tree European	Implied Tree American
60	0.5565	-0.0002	0.0000	-	-	0.0002	0.0002
70	1.4695	-0.0039	-0.0016	-	-	0.0050	0.0055
80	1.3679	-0.0650	-0.0635	-	-	0.1119	0.1042
90	-0.0609	-0.0702	-0.0549	-	-	0.1446	0.1174
100	-1.6035	-0.0586	-0.0757	-	-	0.0552	0.0296
110	-2.7867	-0.0516	-0.0823	-	-	-0.1068	-0.0800
120	-3.5658	-0.0605	-0.1258	-	-	-0.0893	-0.0160
130	-3.9906	-0.1055	-0.1746	-	-	-0.1720	0.0127
140	-4.0885	-0.1049	-0.1995	-	-	-0.1810	0.0931

Two-Asset Case(2) Steps: 50

Call Options

Strike	Black-Scholes European	Monte Carlo (10000) European	LSMC (10000) American	Binomial Tree European	Binomial Tree American	Implied Tree European	Implied Tree American
60	38.6125	38.1226	40.0000	38.0140	39.9622	38.2001	40.1543
70	30.2500	28.8574	30.0514	28.7350	29.9622	28.9624	30.1543
80	22.9463	21.6275	21.6520	21.5129	21.9479	21.8443	22.3232
90	16.9036	17.0169	17.4222	16.9016	17.1386	17.1734	17.5148
100	12.1410	13.8171	14.0481	13.6859	13.8384	13.8978	14.1658
110	8.5380	11.4360	11.7531	11.2836	11.3899	11.4784	11.6952
120	5.9021	9.5986	9.8586	9.4312	9.5089	9.6205	9.7989
130	4.0245	8.1287	8.3996	7.9678	8.0267	8.1572	8.3053
140	2.7149	6.9352	7.1374	6.7951	6.8409	6.9757	7.1002

Put Options

Strike	Black-Scholes European	Monte Carlo (10000) European	LSMC (10000) American	Binomial Tree European	Binomial Tree American	Implied Tree European	Implied Tree American
60	0.5633	0.0006	0.0008	0.0009	0.0009	0.0041	0.0041
70	1.7132	0.2477	0.2470	0.2341	0.2346	0.2788	0.2791
80	3.9217	2.5301	2.5348	2.5243	2.5369	2.6730	2.6803
90	7.3913	7.4317	7.4566	7.4253	7.4878	7.5144	7.5507
100	12.1410	13.7443	13.8219	13.7219	13.8790	13.7511	13.8795
110	18.0503	20.8755	21.0542	20.8319	21.1214	20.8440	21.1415
120	24.9267	28.5504	28.9479	28.4917	28.9506	28.4983	29.0335
130	32.5613	36.5927	37.1226	36.5406	37.2062	36.5474	37.3779
140	40.7641	44.9115	45.6813	44.8802	45.7881	44.8782	46.0534

Differences to the results from 2-D Binomial Tree
Call Options

Strike	Black-Scholes European	Monte Carlo European	LSMC American	Binomial Tree European	Binomial Tree American	Implied Tree European	Implied Tree American
60	0.5985	0.1086	0.0378	-	-	0.1861	0.1921
70	1.5150	0.1224	0.0892	-	-	0.2274	0.1921
80	1.4334	0.1146	-0.2959	-	-	0.3314	0.3753
90	0.0020	0.1153	0.2836	-	-	0.2718	0.3762
100	-1.5449	0.1312	0.2097	-	-	0.2119	0.3274
110	-2.7456	0.1524	0.3632	-	-	0.1948	0.3053
120	-3.5291	0.1674	0.3497	-	-	0.1893	0.2900
130	-3.9433	0.1609	0.3729	-	-	0.1894	0.2786
140	-4.0802	0.1401	0.2965	-	-	0.1806	0.2593

Differences to the results from 2-D Binomial Tree
Put Options

Strike	Black-Scholes European	Monte Carlo European	LSMC American	Binomial Tree European	Binomial Tree American	Implied Tree European	Implied Tree American
60	0.5624	-0.0003	-0.0001	-	-	0.0032	0.0032
70	1.4791	0.0136	0.0124	-	-	0.0447	0.0445
80	1.3974	0.0058	-0.0021	-	-	0.1487	0.1434
90	-0.0340	0.0064	-0.0312	-	-	0.0891	0.0629
100	-1.5809	0.0224	-0.0571	-	-	0.0292	0.0005
110	-2.7816	0.0436	-0.0672	-	-	0.0121	0.0201
120	-3.5650	0.0587	-0.0027	-	-	0.0066	0.0829
130	-3.9793	0.0521	-0.0836	-	-	0.0068	0.1717
140	-4.1161	0.0313	-0.1068	-	-	-0.0020	0.2653

Two-Asset Case(2) Steps:[3] 100

Call Options

Strike	Black-Scholes European	Monte Carlo (10000) European	LSMC (10000) American	Binomial Tree European	Binomial Tree American	Implied Tree European	Implied Tree American
60	38.6145	38.1226	40.0000	38.0321	39.9810	38.0553	40.0041
70	30.2542	28.8574	30.0514	28.7545	29.9810	28.8327	30.0041
80	22.9527	21.6275	21.6520	21.5287	21.9681	21.6414	22.1147
90	16.9118	17.0169	17.4222	16.9171	17.1569	16.9861	17.3200
100	12.1501	13.8171	14.0481	13.7043	13.8588	13.7125	13.9741
110	8.5472	11.4360	11.7531	11.3052	11.4130	11.3436	11.5541
120	5.9105	9.5986	9.8586	9.4542	9.5333	9.4759	9.6493
130	4.0319	8.1287	8.3996	7.9951	8.0550	8.0039	8.1481
140	2.7211	6.9352	7.1374	6.8227	6.8693	6.8411	6.9619

Put Options

Strike	Black-Scholes European	Monte Carlo (10000) European	LSMC (10000) American	Binomial Tree European	Binomial Tree American	Implied Tree European	Implied Tree American
60	0.5653	0.0006	0.0008	0.0009	0.0009	0.0022	0.0022
70	1.7173	0.2477	0.2470	0.2357	0.2362	0.2919	0.2922
80	3.9281	2.5301	2.5348	2.5221	2.5343	2.6129	2.6176
90	7.3995	7.4317	7.4566	7.4228	7.4843	7.4699	7.5049
100	12.1501	13.7443	13.8219	13.7223	13.8777	13.7086	13.8408
110	18.0595	20.8755	21.0542	20.8355	21.1223	20.8520	21.1473
120	24.9351	28.5504	28.9479	28.4969	28.9525	28.4966	29.0302
130	32.5687	36.5927	37.1226	36.5500	37.2112	36.5369	37.3672
140	40.7703	44.9115	45.6813	44.8900	45.7929	44.8864	46.0523

Differences to the results from 2-D Binomial Tree
Call Options

Strike	Black-Scholes European	Monte Carlo European	LSMC American	Binomial Tree European	Binomial Tree American	Implied Tree European	Implied Tree American
60	0.5824	0.0905	0.0190	-	-	0.0232	0.0231
70	1.4997	0.1029	0.0704	-	-	0.0782	0.0231
80	1.4240	0.0988	-0.3161	-	-	0.1127	0.1466
90	-0.0053	0.0998	0.2653	-	-	0.0690	0.1631
100	-1.5542	0.1128	0.1893	-	-	0.0082	0.1153
110	-2.7580	0.1308	0.3401	-	-	0.0384	0.1411
120	-3.5437	0.1444	0.3253	-	-	0.0217	0.1160
130	-3.9632	0.1336	0.3446	-	-	0.0088	0.0931
140	-4.1016	0.1125	0.2681	-	-	0.0184	0.0926

Differences to the results from 2-D Binomial Tree
Put Options

Strike	Black-Scholes European	Monte Carlo European	LSMC American	Binomial Tree European	Binomial Tree American	Implied Tree European	Implied Tree American
60	0.5644	-0.0003	-0.0001	-	-	0.0013	0.0013
70	1.4816	0.0120	0.0108	-	-	0.0562	0.0560
80	1.4060	0.0080	0.0005	-	-	0.0908	0.0833
90	-0.0233	0.0089	-0.0277	-	-	0.0471	0.0206
100	-1.5722	0.0220	-0.0558	-	-	-0.0137	-0.0369
110	-2.7760	0.0400	-0.0681	-	-	0.0165	0.0250
120	-3.5618	0.0535	-0.0046	-	-	-0.0003	0.0777
130	-3.9813	0.0427	-0.0886	-	-	-0.0131	0.1560
140	-4.1197	0.0215	-0.1116	-	-	-0.0036	0.2594

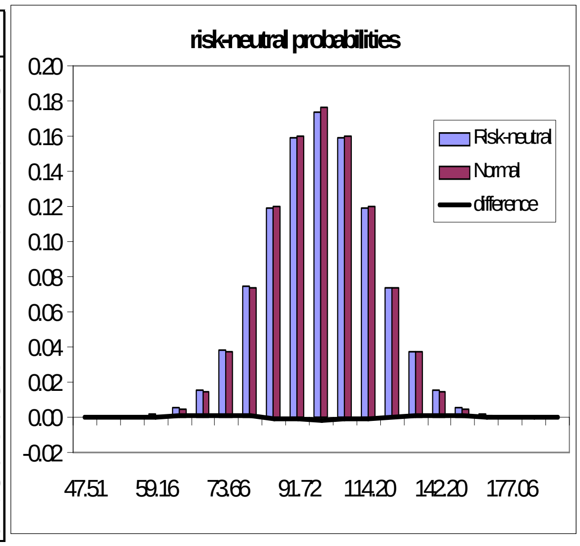
[3] 50 steps are applied to both Monte Carlo simulation and the Least Squares Monte Carlo.

Three-asset options (Case 1)

Time to Expiration:	$T = 1.0$ year
Interest Rate:	$r = 5\%$
Dividend Yield:	$\delta_1 = 5\%, \delta_2 = 5\%, \delta_3 = 5\%$
Volatilities:	$\sigma_1 = 20\%, \sigma_2 = 20\%, \sigma_2 = 20\%$
Correlation:	$\begin{bmatrix} 1 & 0.5 & 0.5 \\ 0.5 & 1 & 0.5 \\ 0.5 & 0.5 & 1 \end{bmatrix}$
Spot Prices:	$S_{1,0} = 33.33, S_{2,0} = 33.33, S_{3,0} = 33.33$
# of shares:	$n_1 = 1, n_2 = 1, n_3 = 1$
Strike Prices:	K = 60, 70, 80, 90, 100, 110, 120, 130, 140.

Risk-Neutral Probabilities

Strike	Put/Call	Option Prices	Probabilities Risk-neutra	Normal	Diffs
47.511	P	0.0000	0.000007	0.000001	0.000006
51.113	P	0.0000	0.000049	0.000019	0.000030
54.989	P	0.0002	0.000308	0.000181	0.000127
59.158	P	0.0017	0.001439	0.001087	0.000352
63.644	P	0.0094	0.005353	0.004621	0.000733
68.469	P	0.0422	0.015798	0.014786	0.001012
73.661	P	0.1556	0.037933	0.036964	0.000969
79.246	P	0.4791	0.074482	0.073929	0.000553
85.255	P	1.2528	0.119255	0.120134	-0.000879
91.719	P	2.8185	0.159259	0.160179	-0.000920
98.673	C	6.8183	0.174020	0.176197	-0.002177
106.155	C	3.8855	0.158922	0.160179	-0.001257
114.204	C	1.9471	0.119019	0.120134	-0.001115
122.863	C	0.8421	0.074068	0.073929	0.000139
132.179	C	0.3096	0.037595	0.036964	0.000630
142.201	C	0.0952	0.015544	0.014786	0.000758
152.984	C	0.0239	0.005220	0.004621	0.000599
164.583	C	0.0048	0.001385	0.001087	0.000298
177.062	C	0.0008	0.000291	0.000181	0.000110
190.488	C	0.0001	0.000046	0.000019	0.000027
204.931	C	0.0000	0.000007	0.000001	0.000006



Three-Asset Case(1) Steps: 10

Call Options								Put Options							
Strike	Black-Scholes European	Monte Carlo (10000) European	LSMC (10000) American	Binomial Tree European	Binomial Tree American	Implied Tree European	Implied Tree American	Strike	Black-Scholes European	Monte Carlo (10000) European	LSMC (10000) American	Binomial Tree European	Binomial Tree American	Implied Tree European	Implied Tree American
60	38.0520	38.0586	40.0000	38.0502	39.9991	38.0518	40.0000	60	0.0028	0.0027	0.0032	0.0019	0.0019	0.0026	0.0026
70	28.6021	28.6086	30.0000	28.5968	29.9991	28.6163	30.0000	70	0.0652	0.0650	0.0668	0.0608	0.0609	0.0794	0.0796
80	19.5664	19.5821	20.1410	19.5705	20.1736	19.5812	20.2386	80	0.5418	0.5508	0.5542	0.5468	0.5488	0.5566	0.5590
90	11.8041	11.8433	11.9634	11.8340	12.0422	11.9351	12.1601	90	2.2918	2.3243	2.3437	2.3226	2.3381	2.4228	2.4497
100	6.1739	6.2221	6.2533	6.2193	6.2850	6.3537	6.4250	100	6.1739	6.2153	6.2609	6.2202	6.2884	6.3537	6.4495
110	2.8018	2.8433	2.8766	2.8255	2.8442	2.8736	2.8962	110	12.3141	12.3489	12.4909	12.3387	12.5506	12.3859	12.6498
120	1.1173	1.1353	1.1511	1.1134	1.1183	1.1968	1.2068	120	20.1419	20.1532	20.6092	20.1389	20.6604	20.2214	20.7700
130	0.3986	0.4050	0.4095	0.3839	0.3850	0.4837	0.4859	130	28.9354	28.9351	30.0000	28.9217	30.0009	29.0206	30.0683
140	0.1295	0.1299	0.1346	0.1160	0.1162	0.1551	0.1554	140	38.1787	38.1724	40.0000	38.1661	40.0009	38.2042	40.0000

Differences to the results from 3-D Binomial Tree (Call Options) / Differences to the results from 3-D Binomial Tree (Put Options)

Call Options								Put Options							
Strike	Black-Scholes European	Monte Carlo European	LSMC American	Binomial Tree European	Binomial Tree American	Implied Tree European	Implied Tree American	Strike	Black-Scholes European	Monte Carlo European	LSMC American	Binomial Tree European	Binomial Tree American	Implied Tree European	Implied Tree American
60	0.0018	0.0084	0.0009	-	-	0.0016	0.0009	60	0.0009	0.0008	0.0013	-	-	0.0007	0.0007
70	0.0053	0.0118	0.0009	-	-	0.0195	0.0009	70	0.0044	0.0042	0.0059	-	-	0.0186	0.0187
80	-0.0041	0.0116	-0.0326	-	-	0.0107	0.0650	80	-0.0050	0.0040	0.0054	-	-	0.0098	0.0102
90	-0.0299	0.0093	-0.0788	-	-	0.1011	0.1179	90	-0.0308	0.0017	0.0056	-	-	0.1002	0.1116
100	-0.0454	0.0028	-0.0317	-	-	0.1344	0.1400	100	-0.0463	-0.0049	-0.0275	-	-	0.1335	0.1611
110	-0.0237	0.0178	0.0324	-	-	0.0481	0.0520	110	-0.0246	0.0102	-0.0597	-	-	0.0472	0.0992
120	0.0039	0.0219	0.0328	-	-	0.0834	0.0885	120	0.0030	0.0143	-0.0512	-	-	0.0825	0.1096
130	0.0147	0.0211	0.0245	-	-	0.0998	0.1009	130	0.0137	0.0134	-0.0009	-	-	0.0989	0.0674
140	0.0135	0.0139	0.0184	-	-	0.0391	0.0392	140	0.0126	0.0063	-0.0009	-	-	0.0381	-0.0009

Three-Asset Case(1) Steps: 20

Call Options

Strike	Black-Scholes European	Monte Carlo (10000) European	LSMC (10000) American	Binomial Tree European	Binomial Tree American	Implied Tree European	Implied Tree American
60	38.0520	38.0352	40.0000	38.0511	39.9995	38.0523	40.0000
70	28.6021	28.5853	30.0000	28.6003	29.9995	28.6125	30.0000
80	19.5665	19.5432	20.2169	19.5722	20.1809	19.6007	20.2151
90	11.8043	11.7616	12.0321	11.8287	12.0411	11.9144	12.1409
100	6.1742	6.1366	6.2579	6.2062	6.2743	6.2983	6.3704
110	2.8021	2.7660	2.8128	2.8237	2.8437	2.9596	2.9829
120	1.1175	1.1010	1.1098	1.1221	1.1275	1.2075	1.2150
130	0.3987	0.4001	0.4111	0.3950	0.3963	0.4342	0.4364
140	0.1296	0.1291	0.1407	0.1242	0.1245	0.1423	0.1429

Put Options

Strike	Black-Scholes European	Monte Carlo (10000) European	LSMC (10000) American	Binomial Tree European	Binomial Tree American	Implied Tree European	Implied Tree American
60	0.0028	0.0021	0.0026	0.0024	0.0024	0.0031	0.0031
70	0.0652	0.0645	0.0686	0.0639	0.0640	0.0757	0.0759
80	0.5419	0.5347	0.5374	0.5480	0.5503	0.5762	0.5799
90	2.2920	2.2655	2.3085	2.3168	2.3330	2.4021	2.4198
100	6.1742	6.1527	6.2535	6.2066	6.2761	6.2983	6.3801
110	12.3144	12.2944	12.5441	12.3364	12.5506	12.4719	12.6951
120	20.1421	20.1417	20.7009	20.1471	20.6705	20.2321	20.7593
130	28.9355	28.9531	30.0050	28.9323	30.0040	28.9711	30.0416
140	38.1787	38.1944	40.0000	38.1739	40.0005	38.1915	40.0000

Differences to the results from 3-D Binomial Tree

Call Options

Strike	Black-Scholes European	Monte Carlo European	LSMC American	Binomial Tree European	Binomial Tree American	Implied Tree European	Implied Tree American
60	0.0009	-0.0159	0.0005	-	-	0.0012	0.0005
70	0.0018	-0.0150	0.0005	-	-	0.0122	0.0005
80	-0.0057	-0.0290	0.0360	-	-	0.0285	0.0342
90	-0.0244	-0.0671	-0.0090	-	-	0.0857	0.0998
100	-0.0320	-0.0696	-0.0164	-	-	0.0921	0.0961
110	-0.0216	-0.0577	-0.0309	-	-	0.1359	0.1392
120	-0.0046	-0.0211	-0.0177	-	-	0.0854	0.0875
130	0.0037	0.0051	0.0148	-	-	0.0392	0.0401
140	0.0054	0.0049	0.0162	-	-	0.0181	0.0184

Differences to the results from 3-D Binomial Tree

Put Options

Strike	Black-Scholes European	Monte Carlo European	LSMC American	Binomial Tree European	Binomial Tree American	Implied Tree European	Implied Tree American
60	0.0004	-0.0003	0.0002	-	-	0.0007	0.0007
70	0.0013	0.0006	0.0046	-	-	0.0118	0.0119
80	-0.0061	-0.0133	-0.0129	-	-	0.0282	0.0296
90	-0.0248	-0.0513	-0.0245	-	-	0.0853	0.0868
100	-0.0324	-0.0539	-0.0226	-	-	0.0917	0.1040
110	-0.0220	-0.0420	-0.0065	-	-	0.1355	0.1445
120	-0.0050	-0.0054	0.0304	-	-	0.0850	0.0888
130	0.0032	0.0208	0.0010	-	-	0.0388	0.0376
140	0.0048	0.0205	-0.0005	-	-	0.0176	-0.0005

Three-Asset Case(1) Steps: 30

Call Options

Strike	Black-Scholes European	Monte Carlo (10000) European	LSMC (10000) American	Binomial Tree European	Binomial Tree American	Implied Tree European	Implied Tree American
60	38.0520	38.0492	40.0000	38.0514	39.9997	38.0522	40.0000
70	28.6021	28.6038	30.0000	28.6014	29.9997	28.6086	30.0000
80	19.5665	19.5688	20.2416	19.5722	20.1838	19.6092	20.2201
90	11.8044	11.7979	12.0296	11.8249	12.0387	11.9046	12.1225
100	6.1744	6.1488	6.2112	6.2023	6.2711	6.2738	6.3452
110	2.8022	2.8028	2.8100	2.8226	2.8431	2.8786	2.9032
120	1.1176	1.1293	1.1443	1.1255	1.1311	1.1715	1.1776
130	0.3987	0.4119	0.4191	0.3985	0.4000	0.4310	0.4329
140	0.1296	0.1387	0.1490	0.1272	0.1275	0.1357	0.1363

Put Options

Strike	Black-Scholes European	Monte Carlo (10000) European	LSMC (10000) American	Binomial Tree European	Binomial Tree American	Implied Tree European	Implied Tree American
60	0.0028	0.0028	0.0034	0.0026	0.0026	0.0030	0.0030
70	0.0652	0.0697	0.0751	0.0648	0.0650	0.0717	0.0720
80	0.5420	0.5470	0.5475	0.5479	0.5503	0.5846	0.5875
90	2.2921	2.2884	2.3026	2.3129	2.3293	2.3923	2.4108
100	6.1744	6.1516	6.2085	6.2026	6.2724	6.2738	6.3508
110	12.3145	12.3179	12.5577	12.3352	12.5499	12.3909	12.6114
120	20.1422	20.1567	20.7089	20.1504	20.6744	20.1960	20.7302
130	28.9356	28.9516	30.0756	28.9357	30.0143	28.9679	30.0417
140	38.1788	38.1907	40.0000	38.1767	40.0003	38.1849	40.0000

Differences to the results from 3-D Binomial Tree

Call Options

Strike	Black-Scholes European	Monte Carlo European	LSMC American	Binomial Tree European	Binomial Tree American	Implied Tree European	Implied Tree American
60	0.0006	-0.0022	0.0003	-	-	0.0008	0.0003
70	0.0007	0.0024	0.0003	-	-	0.0072	0.0003
80	-0.0057	-0.0034	0.0578	-	-	0.0370	0.0363
90	-0.0205	-0.0270	-0.0091	-	-	0.0797	0.0838
100	-0.0279	-0.0535	-0.0599	-	-	0.0715	0.0741
110	-0.0204	-0.0198	-0.0331	-	-	0.0560	0.0601
120	-0.0079	0.0038	0.0132	-	-	0.0460	0.0465
130	0.0002	0.0134	0.0191	-	-	0.0325	0.0329
140	0.0024	0.0115	0.0215	-	-	0.0085	0.0088

Differences to the results from 3-D Binomial Tree

Put Options

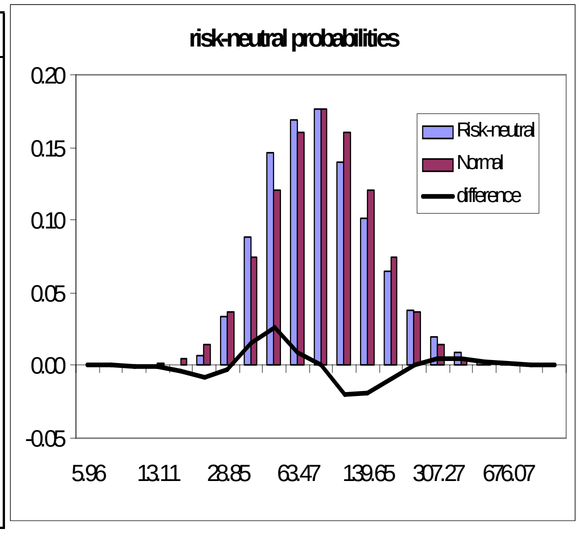
Strike	Black-Scholes European	Monte Carlo European	LSMC American	Binomial Tree European	Binomial Tree American	Implied Tree European	Implied Tree American
60	0.0002	0.0002	0.0008	-	-	0.0004	0.0004
70	0.0004	0.0049	0.0101	-	-	0.0069	0.0070
80	-0.0059	-0.0009	-0.0028	-	-	0.0367	0.0372
90	-0.0208	-0.0245	-0.0267	-	-	0.0794	0.0815
100	-0.0282	-0.0510	-0.0639	-	-	0.0712	0.0784
110	-0.0207	-0.0173	0.0078	-	-	0.0557	0.0615
120	-0.0082	0.0063	0.0345	-	-	0.0456	0.0558
130	-0.0001	0.0159	0.0613	-	-	0.0322	0.0274
140	0.0021	0.0140	-0.0003	-	-	0.0082	-0.0003

Three-asset options (Case 2)

Time to Expiration:	T = 1.0 year
Interest Rate:	r = 5%
Dividend Yield:	$\delta_1 = 5\%, \delta_2 = 5\%, \delta_3 = 5\%$
Volatilities:	$\sigma_1 = 20\%, \sigma_2 = 90\%, \sigma_2 = 90\%$
Correlation:	$\begin{bmatrix} 1 & 0.9 & 0.9 \\ 0.9 & 1 & 0.9 \\ 0.9 & 0.9 & 1 \end{bmatrix}$
Spot Prices:	$S_{1,0} = 33.33, S_{2,0} = 33.33, S_{3,0} = 33.33$
# of shares:	$n_1 = 1, n_2 = 1, n_3 = 1$
Strike Prices:	K = 60, 70, 80, 90, 100, 110, 120, 130, 140.

Risk-Neutral Probabilities

Strike	Put/Call	Option Prices	Probabilities Risk-neutral	Normal	Diffs
5.958	P	0.0000	0.000000	0.000001	-0.000001
7.750	P	0.0000	0.000000	0.000019	-0.000019
10.080	P	0.0000	0.000000	0.000181	-0.000181
13.110	P	0.0000	0.000004	0.001087	-0.001083
17.051	P	0.0000	0.000404	0.004621	-0.004217
22.178	P	0.0020	0.006475	0.014786	-0.008311
28.846	P	0.0457	0.034150	0.036964	-0.002814
37.518	P	0.3841	0.088915	0.073929	0.014986
48.798	P	1.7784	0.145935	0.120134	0.025801
63.468	P	5.6285	0.169199	0.160179	0.009020
82.550	C	30.3060	0.176776	0.176197	0.000579
107.368	C	21.3789	0.140165	0.160179	-0.020014
139.648	C	14.0716	0.101042	0.120134	-0.019092
181.633	C	8.6028	0.064885	0.073929	-0.009044
236.240	C	4.8602	0.037476	0.036964	0.000512
307.265	C	2.5243	0.019449	0.014786	0.004664
399.643	C	1.1952	0.009149	0.004621	0.004528
519.794	C	0.5121	0.003853	0.001087	0.002766
676.068	C	0.1964	0.001452	0.000181	0.001271
879.326	C	0.0666	0.000407	0.000019	0.000388
1143.692	C	0.0196	0.000265	0.000001	0.000264



Three-Asset Case(2) Steps: 10

Call Options

Strike	Black-Scholes European	Monte Carlo (10000) European	LSMC (10000) American	Binomial Tree European	Binomial Tree American	Implied Tree European	Implied Tree American
60	42.1253	42.3854	43.5336	42.3578	43.2765	42.8980	43.8341
70	35.5998	36.4341	37.2958	36.4068	37.0412	37.2793	37.9827
80	29.9635	31.4083	32.0445	31.3842	31.8401	31.6606	32.2618
90	25.1532	27.1717	27.7100	27.1455	27.4823	27.7775	28.1712
100	21.0826	23.5982	23.9735	23.5611	23.8155	24.4899	24.8220
110	17.6580	20.5784	20.8475	20.5269	20.7239	21.2022	21.4985
120	14.7879	18.0208	18.2536	17.9400	18.0939	17.9609	18.2031
130	12.3886	15.8479	16.0769	15.7438	15.8656	16.3294	16.4912
140	10.3855	13.9940	14.1747	13.8608	13.9582	14.6979	14.8419

Put Options

Strike	Black-Scholes European	Monte Carlo (10000) European	LSMC (10000) American	Binomial Tree European	Binomial Tree American	Implied Tree European	Implied Tree American
60	4.0761	4.4725	4.4941	4.5425	4.5742	4.8488	4.8943
70	7.0630	8.0336	8.0810	8.1038	8.1730	8.7424	8.8115
80	10.9389	12.5200	12.6296	12.5935	12.7184	12.6360	12.7485
90	15.6410	17.7957	17.9728	17.8671	18.0707	18.2652	18.5087
100	21.0826	23.7345	23.9997	23.7950	24.1011	24.4899	24.8088
110	27.1702	30.2269	30.5597	30.2730	30.7026	30.7145	31.1416
120	33.8125	37.1817	37.6735	37.1984	37.7806	36.9855	37.7320
130	40.9254	44.5211	45.1841	44.5145	45.2700	44.8663	45.7575
140	48.4347	52.1795	53.0657	52.1439	53.0982	52.7470	53.8236

Differences to the results from 3-D Binomial Tree
Call Options

Strike	Black-Scholes European	Monte Carlo European	LSMC American	Binomial Tree European	Binomial Tree American	Implied Tree European	Implied Tree American
60	-0.2325	0.0276	0.2571	-	-	0.5402	0.5576
70	-0.8070	0.0273	0.2546	-	-	0.8725	0.9415
80	-1.4207	0.0241	0.2044	-	-	0.2764	0.4217
90	-1.9923	0.0262	0.2277	-	-	0.6320	0.6889
100	-2.4785	0.0371	0.1580	-	-	0.9288	1.0065
110	-2.8689	0.0515	0.1236	-	-	0.6753	0.7746
120	-3.1521	0.0808	0.1597	-	-	0.0209	0.1092
130	-3.3552	0.1041	0.2113	-	-	0.5856	0.6256
140	-3.4753	0.1332	0.2165	-	-	0.8371	0.8837

Differences to the results from 3-D Binomial Tree
Put Options

Strike	Black-Scholes European	Monte Carlo European	LSMC American	Binomial Tree European	Binomial Tree American	Implied Tree European	Implied Tree American
60	-0.4664	-0.0700	-0.0801	-	-	0.3063	0.3201
70	-1.0408	-0.0702	-0.0920	-	-	0.6386	0.6385
80	-1.6546	-0.0735	-0.0888	-	-	0.0425	0.0301
90	-2.2261	-0.0714	-0.0979	-	-	0.3981	0.4380
100	-2.7124	-0.0605	-0.1014	-	-	0.6949	0.7077
110	-3.1028	-0.0461	-0.1429	-	-	0.4415	0.4390
120	-3.3859	-0.0167	-0.1071	-	-	-0.2129	-0.0486
130	-3.5891	0.0066	-0.0859	-	-	0.3518	0.4875
140	-3.7092	0.0356	-0.0325	-	-	0.6031	0.7254

Three-Asset Case(2) Steps: 20

	Call Options								Put Options						
Strike	Black-Scholes European	Monte Carlo (10000) European	LSMC (10000) American	Binomial Tree European	Binomial Tree American	Implied Tree European	Implied Tree American	Strike	Black-Scholes European	Monte Carlo (10000) European	LSMC (10000) American	Binomial Tree European	Binomial Tree American	Implied Tree European	Implied Tree American
60	42.1264	42.4116	43.2733	42.4413	43.3799	42.7674	43.7414	60	4.0772	4.4651	4.4998	4.5102	4.5415	4.7182	4.7426
70	35.6014	36.4336	36.9001	36.4811	37.1347	36.9306	37.6279	70	7.0645	7.9995	8.0663	8.0622	8.1294	8.3937	8.4594
80	29.9653	31.4035	31.9624	31.4538	31.9268	31.6521	32.2138	80	10.9407	12.4817	12.5673	12.5472	12.6691	12.6275	12.7376
90	25.1553	27.1906	27.5857	27.2199	27.5728	27.6263	28.0398	90	15.6430	17.7811	17.9299	17.8257	18.0232	18.1140	18.3168
100	21.0848	23.6465	23.9352	23.6483	23.9182	24.0293	24.3759	100	21.0848	23.7492	24.0152	23.7663	24.0624	24.0293	24.3173
110	17.6602	20.6539	21.0811	20.6297	20.8398	20.7831	21.0530	110	27.1725	30.2689	30.6638	30.2600	30.6778	30.2954	30.7650
120	14.7902	18.1331	18.6204	18.0644	18.2310	18.5194	18.7391	120	33.8147	37.2604	37.7259	37.2070	37.7714	37.5440	38.1426
130	12.3908	15.9822	16.4837	15.8796	16.0135	16.2557	16.4474	130	40.9277	44.6219	45.2044	44.5345	45.2695	44.7926	45.5691
140	10.3876	14.1474	14.5403	14.0111	14.1195	14.0258	14.1887	140	48.4368	52.2993	53.2031	52.1783	53.1087	52.0750	53.1522

Differences to the results from 3-D Binomial Tree — Call Options; Differences to the results from 3-D Binomial Tree — Put Options

	Call Options								Put Options						
Strike	Black-Scholes European	Monte Carlo European	LSMC American	Binomial Tree European	Binomial Tree American	Implied Tree European	Implied Tree American	Strike	Black-Scholes European	Monte Carlo European	LSMC American	Binomial Tree European	Binomial Tree American	Implied Tree European	Implied Tree American
60	-0.3149	-0.0297	-0.1066	-	-	0.3261	0.3615	60	-0.4330	-0.0451	-0.0417	-	-	0.2080	0.2011
70	-0.8797	-0.0475	-0.2346	-	-	0.4495	0.4932	70	-0.9977	-0.0627	-0.0631	-	-	0.3315	0.3300
80	-1.4885	-0.0503	0.0356	-	-	0.1983	0.2870	80	-1.6065	-0.0655	-0.1018	-	-	0.0803	0.0685
90	-2.0646	-0.0293	0.0129	-	-	0.4064	0.4670	90	-2.1827	-0.0446	-0.0933	-	-	0.2883	0.2936
100	-2.5635	-0.0018	0.0170	-	-	0.3810	0.4577	100	-2.6815	-0.0171	-0.0472	-	-	0.2630	0.2549
110	-2.9695	0.0242	0.2413	-	-	0.1534	0.2132	110	-3.0875	0.0089	-0.0140	-	-	0.0354	0.0872
120	-3.2742	0.0687	0.3894	-	-	0.4550	0.5081	120	-3.3923	0.0534	-0.0455	-	-	0.3370	0.3712
130	-3.4888	0.1026	0.4702	-	-	0.3761	0.4339	130	-3.6068	0.0874	-0.0651	-	-	0.2581	0.2996
140	-3.6235	0.1363	0.4208	-	-	0.0147	0.0692	140	-3.7415	0.1210	0.0944	-	-	-0.1033	0.0435

Three-Asset Case(2) Steps: 30

	Call Options								Put Options						
Strike	Black-Scholes European	Monte Carlo (10000) European	LSMC (10000) American	Binomial Tree European	Binomial Tree American	Implied Tree European	Implied Tree American	Strike	Black-Scholes European	Monte Carlo (10000) European	LSMC (10000) American	Binomial Tree European	Binomial Tree American	Implied Tree European	Implied Tree American
60	42.1268	43.1906	43.8169	42.4700	43.4150	42.7667	43.7161	60	4.0776	4.5300	4.5583	4.4997	4.5304	4.7175	4.7433
70	35.6019	37.2232	37.6774	36.5070	37.1662	36.7593	37.4612	70	7.0650	8.0748	8.1049	8.0491	8.1150	8.2224	8.2851
80	29.9659	32.1865	32.6565	31.4791	31.9572	31.6355	32.1901	80	10.9414	12.5505	12.6822	12.5334	12.6531	12.6109	12.7180
90	25.1560	27.9715	28.3053	27.2477	27.6053	27.5424	27.9596	90	15.6437	17.8477	18.0899	17.8144	18.0087	18.0301	18.2200
100	21.0855	24.4029	24.6605	23.6813	23.9552	23.8018	24.1500	100	21.0855	23.7914	24.0560	23.7603	24.0518	23.8018	24.0937
110	17.6610	21.3696	21.8232	20.6640	20.8781	20.9433	21.2125	110	27.1732	30.2704	30.6549	30.2553	30.6677	30.4556	30.8852
120	14.7909	18.7877	19.1986	18.1078	18.2778	18.3500	18.5785	120	33.8155	37.2008	37.7398	37.2113	37.7684	37.3746	37.9541
130	12.3915	16.5892	16.9005	15.9278	16.0647	16.0444	16.2283	130	40.9284	44.5146	45.1451	44.5436	45.2701	44.5813	45.3832
140	10.3883	14.7125	15.0450	14.0619	14.1735	14.3518	14.5058	140	48.4375	52.1502	52.9376	52.1900	53.1107	52.4010	53.3973

Differences to the results from 3-D Binomial Tree — Call Options; Differences to the results from 3-D Binomial Tree — Put Options

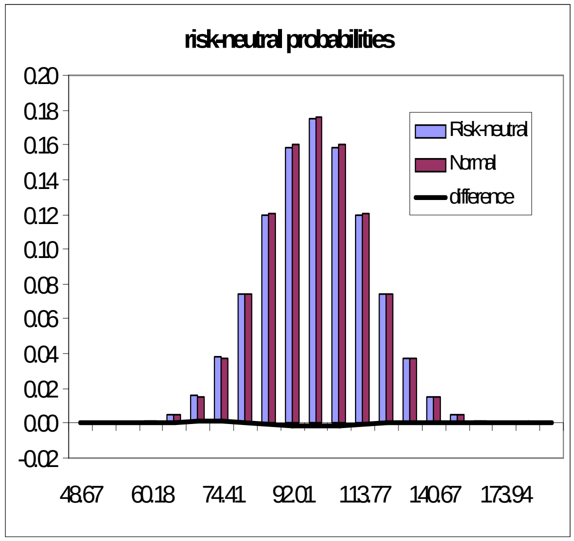
	Call Options								Put Options						
Strike	Black-Scholes European	Monte Carlo European	LSMC American	Binomial Tree European	Binomial Tree American	Implied Tree European	Implied Tree American	Strike	Black-Scholes European	Monte Carlo European	LSMC American	Binomial Tree European	Binomial Tree American	Implied Tree European	Implied Tree American
60	-0.3432	0.7206	0.4019	-	-	0.2967	0.3011	60	-0.4221	0.0303	0.0279	-	-	0.2178	0.2129
70	-0.9051	0.7162	0.5112	-	-	0.2523	0.2950	70	-0.9841	0.0257	-0.0101	-	-	0.1733	0.1701
80	-1.5132	0.7074	0.6993	-	-	0.1564	0.2329	80	-1.5920	0.0171	0.0291	-	-	0.0775	0.0649
90	-2.0917	0.7238	0.7000	-	-	0.2947	0.3543	90	-2.1707	0.0333	0.0812	-	-	0.2157	0.2113
100	-2.5958	0.7216	0.7053	-	-	0.1205	0.1948	100	-2.6748	0.0311	0.0042	-	-	0.0415	0.0419
110	-3.0030	0.7056	0.9451	-	-	0.2793	0.3344	110	-3.0821	0.0151	-0.0128	-	-	0.2003	0.2175
120	-3.3169	0.6799	0.9208	-	-	0.2422	0.3007	120	-3.3958	-0.0105	-0.0286	-	-	0.1633	0.1857
130	-3.5363	0.6614	0.8358	-	-	0.1166	0.1636	130	-3.6152	-0.0290	-0.1250	-	-	0.0377	0.1131
140	-3.6736	0.6506	0.8715	-	-	0.2899	0.3323	140	-3.7525	-0.0398	-0.1731	-	-	0.2110	0.2866

Four-asset options (Case 1)

Time to Expiration:	T = 1.0 year
Interest Rate:	r = 5%
Dividend Yield:	$\delta_1 = 5\%, \delta_2 = 5\%, \delta_3 = 5\%, \delta_4 = 5\%$
Volatilities:	$\sigma_1 = 20\%, \sigma_2 = 20\%, \sigma_3 = 20\%, \sigma_4 = 20\%$
Correlation:	$\begin{bmatrix} 1 & 0.5 & 0.5 & 0.5 \\ 0.5 & 1 & 0.5 & 0.5 \\ 0.5 & 0.5 & 1 & 0.5 \\ 0.5 & 0.5 & 0.5 & 1 \end{bmatrix}$
Spot Prices:	$S_{1,0} = 25, S_{2,0} = 25, S_{3,0} = 25, S_{4,0} = 25$
# of shares:	$n_1 = 1, n_2 = 1, n_3 = 1, n_4 = 1$
Strike Prices:	K = 60, 70, 80, 90, 100, 110, 120, 130, 140.

Risk-Neutral Probabilities

Strike	Put/Call	Option Prices	Probabilities Risk-neutra	Normal	Diffs
48.6700	P	0.0000	8.04E-06	9.5E-07	7.08E-06
52.2390	P	0.0000	5.18E-05	1.91E-05	3.27E-05
56.0690	P	0.0002	0.000313	0.000181	0.000131
60.1800	P	0.0017	0.001453	0.001087	0.000366
64.5930	P	0.0094	0.005349	0.004621	0.000729
69.3290	P	0.0417	0.015813	0.014786	0.001027
74.4120	P	0.1528	0.03789	0.036964	0.000926
79.8680	P	0.4688	0.074235	0.073929	0.000306
85.7240	P	1.2214	0.11957	0.120134	-0.000564
92.0100	P	2.7442	0.158822	0.160179	-0.001357
98.7560	C	6.5812	0.174664	0.176197	-0.001533
105.9970	C	3.7445	0.158662	0.160179	-0.001518
113.7690	C	1.8729	0.119223	0.120134	-0.000911
122.1110	C	0.8100	0.073828	0.073929	-0.000101
131.0640	C	0.2980	0.037559	0.036964	0.000595
140.6740	C	0.0918	0.015573	0.014786	0.000787
150.9890	C	0.0233	0.005229	0.004621	0.000609
162.0590	C	0.0048	0.001404	0.001087	0.000316
173.9420	C	0.0008	0.000298	0.000181	0.000117
186.6960	C	0.0001	4.82E-05	1.91E-05	2.92E-05
200.3850	C	0.0000	7.4E-06	9.5E-07	6.45E-06



Four-Asset Case(1) Steps: 10

Call Options

Strike	Black-Scholes European	Monte Carlo (10000) European	LSMC (10000) American	Binomial Tree European	Binomial Tree American	Implied Tree European	Implied Tree American
60	38.0510	38.0734	40.0000	38.0496	39.9991	38.0495	40.0000
70	28.5883	28.6192	30.0000	28.5841	29.9991	28.6041	30.0000
80	19.5001	19.5440	20.1473	19.5026	20.1190	19.5333	20.1976
90	11.6550	11.6924	11.8907	11.6821	11.8899	11.7496	11.9794
100	5.9789	6.0128	6.0379	6.0177	6.0803	6.1454	6.2144
110	2.6305	2.6712	2.6762	2.6516	2.6683	2.7184	2.7380
120	1.0056	1.0525	1.0529	1.0029	1.0068	1.0445	1.0537
130	0.3404	0.3825	0.3794	0.3278	0.3286	0.4075	0.4095
140	0.1041	0.1303	0.1320	0.0931	0.0932	0.1308	0.1311

Put Options

Strike	Black-Scholes European	Monte Carlo (10000) European	LSMC (10000) American	Binomial Tree European	Binomial Tree American	Implied Tree European	Implied Tree American
60	0.0019	0.0036	0.0032	0.0012	0.0012	0.0003	0.0003
70	0.0514	0.0617	0.0600	0.0480	0.0481	0.0672	0.0673
80	0.4756	0.4988	0.4984	0.4789	0.4805	0.5087	0.5105
90	2.1427	2.1595	2.1661	2.1707	2.1844	2.2373	2.2628
100	5.9789	5.9922	6.0638	6.0186	6.0834	6.1454	6.2381
110	12.1428	12.1628	12.3709	12.1647	12.3748	12.2307	12.4907
120	20.0302	20.0565	20.5912	20.0283	20.5583	20.0691	20.6392
130	28.8773	28.8988	30.0000	28.8655	30.0009	28.9444	30.0159
140	38.1533	38.1588	40.0000	38.1431	40.0009	38.1800	40.0000

Differences to the results from 4-D Binomial Tree
Call Options

Strike	Black-Scholes European	Monte Carlo European	LSMC American	Binomial Tree European	Binomial Tree American	Implied Tree European	Implied Tree American
60	0.0014	0.0238	0.0009	-	-	-0.0001	0.0009
70	0.0042	0.0351	0.0009	-	-	0.0200	0.0009
80	-0.0025	0.0414	0.0283	-	-	0.0307	0.0786
90	-0.0271	0.0103	0.0008	-	-	0.0675	0.0895
100	-0.0388	-0.0049	-0.0424	-	-	0.1277	0.1341
110	-0.0211	0.0196	0.0079	-	-	0.0668	0.0697
120	0.0027	0.0496	0.0461	-	-	0.0416	0.0469
130	0.0126	0.0547	0.0508	-	-	0.0797	0.0809
140	0.0110	0.0372	0.0388	-	-	0.0377	0.0379

Differences to the results from 4-D Binomial Tree
Put Options

Strike	Black-Scholes European	Monte Carlo European	LSMC American	Binomial Tree European	Binomial Tree American	Implied Tree European	Implied Tree American
60	0.0007	0.0024	0.0020	-	-	-0.0009	-0.0009
70	0.0034	0.0137	0.0119	-	-	0.0192	0.0192
80	-0.0033	0.0199	0.0179	-	-	0.0298	0.0300
90	-0.0280	-0.0112	-0.0183	-	-	0.0666	0.0784
100	-0.0397	-0.0264	-0.0196	-	-	0.1268	0.1547
110	-0.0219	-0.0019	-0.0039	-	-	0.0660	0.1159
120	0.0019	0.0282	0.0329	-	-	0.0408	0.0809
130	0.0118	0.0333	-0.0009	-	-	0.0789	0.0150
140	0.0102	0.0157	-0.0009	-	-	0.0369	-0.0009

Four-Asset Case(1) Steps: 20

Strike	Call Options: Black-Scholes European	Monte Carlo (10000) European	LSMC (10000) American	Binomial Tree European	Binomial Tree American	Implied Tree European	Implied Tree American
60	38.0510	38.0379	40.0000	38.0503	39.9996	38.0508	40.0000
70	28.5883	28.5698	30.0000	28.5868	29.9996	28.5933	30.0000
80	19.5003	19.4867	20.2131	19.5044	20.1320	19.5104	20.1545
90	11.6553	11.6288	11.9001	11.6757	11.8884	11.7696	11.9948
100	5.9793	5.9436	6.0146	6.0082	6.0737	6.0938	6.1637
110	2.6309	2.6097	2.6607	2.6497	2.6680	2.7805	2.8025
120	1.0058	1.0004	1.0327	1.0102	1.0149	1.0790	1.0858
130	0.3405	0.3270	0.3420	0.3374	0.3384	0.3589	0.3609
140	0.1042	0.0926	0.1020	0.1000	0.1003	0.1063	0.1068

Strike	Put Options: Black-Scholes European	Monte Carlo (10000) European	LSMC (10000) American	Binomial Tree European	Binomial Tree American	Implied Tree European	Implied Tree American
60	0.0019	0.0010	0.0015	0.0016	0.0016	0.0016	0.0016
70	0.0514	0.0452	0.0482	0.0503	0.0504	0.0564	0.0566
80	0.4757	0.4743	0.4882	0.4803	0.4822	0.4858	0.4891
90	2.1430	2.1287	2.1573	2.1639	2.1785	2.2573	2.2738
100	5.9793	5.9559	6.0213	6.0086	6.0753	6.0938	6.1731
110	12.1431	12.1342	12.3805	12.1624	12.3757	12.2928	12.5150
120	20.0304	20.0372	20.5592	20.0352	20.5684	20.1036	20.6439
130	28.8774	28.8761	30.0000	28.8747	30.0004	28.8958	30.0068
140	38.1533	38.1540	40.0000	38.1496	40.0004	38.1555	40.0000

Differences to the results from 4-D Binomial Tree
Call Options

Strike	Black-Scholes European	Monte Carlo European	LSMC American	Binomial Tree European	Binomial Tree American	Implied Tree European	Implied Tree American
60	0.0007	-0.0124	0.0004	-	-	0.0005	0.0004
70	0.0015	-0.0170	0.0004	-	-	0.0065	0.0004
80	-0.0041	-0.0177	0.0811	-	-	0.0060	0.0225
90	-0.0204	-0.0469	0.0117	-	-	0.0939	0.1064
100	-0.0289	-0.0646	-0.0591	-	-	0.0856	0.0900
110	-0.0188	-0.0400	-0.0073	-	-	0.1308	0.1345
120	-0.0044	-0.0098	0.0178	-	-	0.0688	0.0709
130	0.0031	-0.0104	0.0036	-	-	0.0215	0.0225
140	0.0042	-0.0074	0.0017	-	-	0.0063	0.0065

Differences to the results from 4-D Binomial Tree
Put Options

Strike	Black-Scholes European	Monte Carlo European	LSMC American	Binomial Tree European	Binomial Tree American	Implied Tree European	Implied Tree American
60	0.0003	-0.0006	-0.0001	-	-	0.0000	0.0000
70	0.0011	-0.0051	-0.0022	-	-	0.0061	0.0062
80	-0.0046	-0.0060	0.0060	-	-	0.0055	0.0069
90	-0.0209	-0.0352	-0.0212	-	-	0.0934	0.0953
100	-0.0293	-0.0527	-0.0540	-	-	0.0852	0.0978
110	-0.0193	-0.0282	0.0048	-	-	0.1304	0.1393
120	-0.0048	0.0020	-0.0092	-	-	0.0684	0.0755
130	0.0027	0.0014	-0.0004	-	-	0.0211	0.0064
140	0.0037	0.0044	-0.0004	-	-	0.0059	-0.0004

Four-Asset Case(1) Steps:[4] 100

20 steps are applied to the 4-Dimensional Binomial Tree, Monte Carlo simulation, and Least-squares Monte Carlo simulation. 100 steps are applied to the implied tree.

Strike	Call Options: Black-Scholes European	Monte Carlo (10000) European	LSMC (10000) American	Binomial Tree European	Binomial Tree American	Implied Tree European	Implied Tree American
60	38.0510	38.0379	40.0000	38.0503	39.9996	38.0509	40.0000
70	28.5883	28.5698	30.0000	28.5868	29.9996	28.5882	30.0000
80	19.5003	19.4867	20.2131	19.5044	20.1320	19.5141	20.1461
90	11.6553	11.6288	11.9001	11.6757	11.8884	11.6819	11.9012
100	5.9793	5.9436	6.0146	6.0082	6.0737	6.0363	6.1047
110	2.6309	2.6097	2.6607	2.6497	2.6680	2.6748	2.6944
120	1.0058	1.0004	1.0327	1.0102	1.0149	1.0189	1.0240
130	0.3405	0.3270	0.3420	0.3374	0.3384	0.3452	0.3466
140	0.1042	0.0926	0.1020	0.1000	0.1003	0.1004	0.1007

Strike	Put Options: Black-Scholes European	Monte Carlo (10000) European	LSMC (10000) American	Binomial Tree European	Binomial Tree American	Implied Tree European	Implied Tree American
60	0.0019	0.0010	0.0015	0.0016	0.0016	0.0017	0.0017
70	0.0514	0.0452	0.0482	0.0503	0.0504	0.0513	0.0514
80	0.4757	0.4743	0.4882	0.4803	0.4822	0.4895	0.4917
90	2.1430	2.1287	2.1573	2.1639	2.1785	2.1696	2.1858
100	5.9793	5.9559	6.0213	6.0086	6.0753	6.0363	6.1055
110	12.1431	12.1342	12.3805	12.1624	12.3757	12.1871	12.4040
120	20.0304	20.0372	20.5592	20.0352	20.5684	20.0435	20.5834
130	28.8774	28.8761	30.0000	28.8747	30.0004	28.8821	30.0085
140	38.1533	38.1540	40.0000	38.1496	40.0004	38.1496	40.0000

Differences to the results from 4-D Binomial Tree
Call Options

Strike	Black-Scholes European	Monte Carlo European	LSMC American	Binomial Tree European	Binomial Tree American	Implied Tree European	Implied Tree American
60	0.0007	-0.0124	0.0004	-	-	0.0006	0.0004
70	0.0015	-0.0170	0.0004	-	-	0.0014	0.0004
80	-0.0041	-0.0177	0.0811	-	-	0.0097	0.0141
90	-0.0204	-0.0469	0.0117	-	-	0.0062	0.0128
100	-0.0289	-0.0646	-0.0591	-	-	0.0281	0.0310
110	-0.0188	-0.0400	-0.0073	-	-	0.0251	0.0264
120	-0.0044	-0.0098	0.0178	-	-	0.0087	0.0091
130	0.0031	-0.0104	0.0036	-	-	0.0078	0.0082
140	0.0042	-0.0074	0.0017	-	-	0.0004	0.0004

Differences to the results from 4-D Binomial Tree
Put Options

Strike	Black-Scholes European	Monte Carlo European	LSMC American	Binomial Tree European	Binomial Tree American	Implied Tree European	Implied Tree American
60	0.0003	-0.0006	-0.0001	-	-	0.0001	0.0001
70	0.0011	-0.0051	-0.0022	-	-	0.0010	0.0010
80	-0.0046	-0.0060	0.0060	-	-	0.0092	0.0095
90	-0.0209	-0.0352	-0.0212	-	-	0.0057	0.0073
100	-0.0293	-0.0527	-0.0540	-	-	0.0277	0.0302
110	-0.0193	-0.0282	0.0048	-	-	0.0247	0.0283
120	-0.0048	0.0020	-0.0092	-	-	0.0083	0.0150
130	0.0027	0.0014	-0.0004	-	-	0.0074	0.0081
140	0.0037	0.0044	-0.0004	-	-	0.0000	-0.0004

[4] 20 steps are applied to the 4-Dimensional Binomial Tree, Monte Carlo simulation, and Least-squares Monte Carlo simulation. 100 steps are applied to the implied tree.

Four-asset options (Case 2)

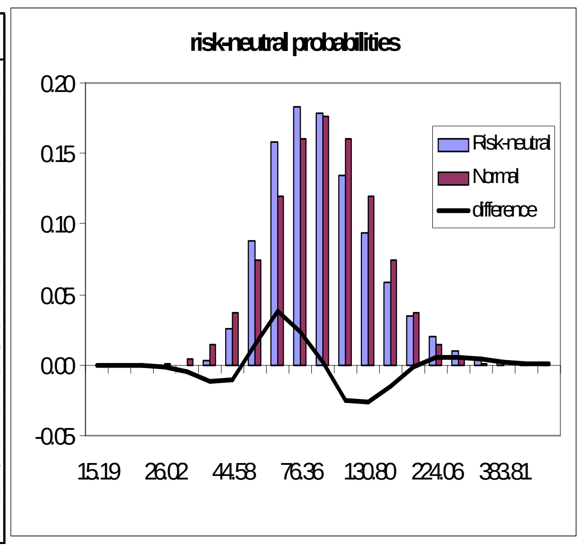
Time to Expiration: $T = 1.0$ year
Interest Rate: $r = 5\%$
Dividend Yield: $\delta_1 = 5\%, \delta_2 = 5\%, \delta_3 = 5\%, \delta_4 = 5\%$
Volatilities: $\sigma_1 = 20\%, \sigma_2 = 90\%, \sigma_3 = 90\%, \sigma_4 = 10\%$

Correlation:
$$\begin{bmatrix} 1 & 0.5 & 0.5 & 0.5 \\ 0.5 & 1 & 0.5 & 0.5 \\ 0.5 & 0.5 & 1 & 0.5 \\ 0.5 & 0.5 & 0.5 & 1 \end{bmatrix}$$

Spot Prices: $S_{1,0} = 25, S_{2,0} = 25, S_{3,0} = 25, S_{4,0} = 25$
\# of shares: $n_1 = 1, n_2 = 1, n_3 = 1, n_4 = 1$
Strike Prices: $K = 60, 70, 80, 90, 100, 110, 120, 130, 140.$

Risk-Neutral Probabilities

Strike	Put/Call	Option Prices	Probabilities Risk-neutra	Normal	Diffs
15.191	P	0.0000	0.000000	0.000001	-0.000001
18.176	P	0.0000	0.000000	0.000019	-0.000019
21.748	P	0.0000	0.000000	0.000181	-0.000181
26.022	P	0.0000	0.000000	0.001087	-0.001087
31.136	P	0.0000	0.000086	0.004621	-0.004535
37.254	P	0.0005	0.002987	0.014786	-0.011799
44.575	P	0.0219	0.026039	0.036964	-0.010925
53.335	P	0.2645	0.088264	0.073929	0.014335
63.816	P	1.4347	0.158476	0.120134	0.038341
76.357	P	4.7254	0.183447	0.160179	0.023268
91.362	C	19.4977	0.178405	0.176197	0.002208
109.316	C	13.3103	0.134565	0.160179	-0.025615
130.799	C	8.6566	0.093824	0.120134	-0.026310
156.503	C	5.3825	0.059003	0.073929	-0.014926
187.258	C	3.1911	0.035233	0.036964	-0.001732
224.057	C	1.8024	0.019842	0.014786	0.005057
268.088	C	0.9718	0.010413	0.004621	0.005792
320.771	C	0.4999	0.005161	0.001087	0.004074
383.807	C	0.2446	0.002429	0.000181	0.002248
459.232	C	0.1135	0.000506	0.000019	0.000487
549.478	C	0.0498	0.001322	0.000001	0.001321



Four-Asset Case(2) Steps: 10

Call Options

Strike	Black-Scholes European	Monte Carlo (10000) European	LSMC (10000) American	Binomial Tree European	Binomial Tree American	Implied Tree European	Implied Tree American
60	39.3576	39.1209	40.3079	38.7645	40.0220	39.2532	40.3900
70	31.6036	31.5524	32.0958	31.1838	31.8708	31.4220	32.1934
80	24.9045	25.3539	25.7533	24.9724	25.3769	25.6936	26.1593
90	19.3193	20.4029	20.6683	20.0172	20.2687	20.1800	20.5651
100	14.7972	16.4895	16.7674	16.1168	16.2794	16.7031	16.9341
110	11.2208	13.4208	13.6958	13.0248	13.1354	13.5482	13.7432
120	8.4436	10.9813	11.2089	10.6211	10.6967	10.7565	10.8817
130	6.3172	9.0364	9.2077	8.6718	8.7272	9.1763	9.2760
140	4.7067	7.4896	7.5673	7.1528	7.1913	7.5961	7.6776

Put Options

Strike	Black-Scholes European	Monte Carlo (10000) European	LSMC (10000) American	Binomial Tree European	Binomial Tree American	Implied Tree European	Implied Tree American
60	1.3084	0.8678	0.8774	0.8615	0.8648	1.2040	1.2092
70	3.0668	2.8115	2.8295	2.7931	2.8088	2.8851	2.8958
80	5.8800	6.1253	6.1450	6.0939	6.1401	6.6690	6.7205
90	9.8070	10.6867	10.7546	10.6510	10.7524	10.6677	10.7665
100	14.7972	16.2856	16.4456	16.2629	16.4483	16.7031	16.9224
110	20.7331	22.7292	22.9688	22.6832	22.9911	23.0605	23.3799
120	27.4682	29.8019	30.0989	29.7918	30.2518	29.7811	30.3875
130	34.8541	37.3693	37.7925	37.3548	38.0141	37.7132	38.5033
140	42.7558	45.3348	45.9888	45.3481	46.2362	45.6452	46.7075

Differences to the results from 4-D Binomial Tree
Call Options

Strike	Black-Scholes European	Monte Carlo European	LSMC American	Binomial Tree European	Binomial Tree American	Implied Tree European	Implied Tree American
60	0.5931	0.3564	0.2859	-	-	0.4887	0.3680
70	0.4198	0.3686	0.2250	-	-	0.2382	0.3226
80	-0.0679	0.3815	0.3764	-	-	0.7212	0.7824
90	-0.6979	0.3857	0.3996	-	-	0.1628	0.2964
100	-1.3196	0.3727	0.4880	-	-	0.5863	0.6547
110	-1.8040	0.3960	0.5604	-	-	0.5234	0.6078
120	-2.1775	0.3602	0.5122	-	-	0.1354	0.1850
130	-2.3546	0.3646	0.4805	-	-	0.5045	0.5488
140	-2.4461	0.3368	0.3760	-	-	0.4433	0.4863

Differences to the results from 4-D Binomial Tree
Put Options

Strike	Black-Scholes European	Monte Carlo European	LSMC American	Binomial Tree European	Binomial Tree American	Implied Tree European	Implied Tree American
60	0.4469	0.0063	0.0126	-	-	0.3425	0.3444
70	0.2737	0.0184	0.0207	-	-	0.0920	0.0870
80	-0.2139	0.0314	0.0049	-	-	0.5751	0.5804
90	-0.8440	0.0357	0.0022	-	-	0.0167	0.0141
100	-1.4657	0.0227	-0.0027	-	-	0.4402	0.4741
110	-1.9501	0.0460	-0.0223	-	-	0.3773	0.3888
120	-2.3236	0.0101	-0.1529	-	-	-0.0107	0.1357
130	-2.5007	0.0145	-0.2216	-	-	0.3584	0.4892
140	-2.5923	-0.0133	-0.2474	-	-	0.2971	0.4713

Four-Asset Case(2) Steps: 20

Call Options

Strike	Black-Scholes European	Monte Carlo (10000) European	LSMC (10000) American	Binomial Tree European	Binomial Tree American	Implied Tree European	Implied Tree American
60	39.3672	38.7803	40.3251	38.8348	40.1233	39.0578	40.2466
70	31.6195	31.1542	32.1446	31.2476	31.9577	31.5943	32.2983
80	24.9262	24.9224	25.6184	25.0341	25.4568	25.3417	25.8114
90	19.3452	19.9858	20.4219	20.0855	20.3525	20.1984	20.5528
100	14.8255	16.1194	16.3547	16.1867	16.3632	16.5209	16.7634
110	11.2496	13.1004	13.1441	13.1225	13.2435	13.1622	13.3470
120	8.4713	10.7234	10.8402	10.7092	10.7946	10.9959	11.1348
130	6.3429	8.8651	8.8757	8.7906	8.8524	8.8296	8.9450
140	4.7296	7.4124	7.5243	7.2697	7.3154	7.4846	7.5669

Put Options

Strike	Black-Scholes European	Monte Carlo (10000) European	LSMC (10000) American	Binomial Tree European	Binomial Tree American	Implied Tree European	Implied Tree American
60	1.3180	0.8521	0.8627	0.8593	0.8629	1.0086	1.0116
70	3.0826	2.7383	2.7676	2.7844	2.8009	3.0574	3.0720
80	5.9016	6.0187	6.0873	6.0831	6.1303	6.3171	6.3649
90	9.8329	10.5944	10.7683	10.6469	10.7497	10.6861	10.7806
100	14.8255	16.2403	16.4739	16.2604	16.4492	16.5209	16.7100
110	20.7619	22.7336	23.0882	22.7084	23.0167	22.6744	23.0341
120	27.4959	29.8689	30.3944	29.8074	30.2716	30.0205	30.5325
130	34.8798	37.5229	38.2540	37.4011	38.0604	37.3665	38.1477
140	42.7788	45.5825	46.5738	45.3925	46.2834	45.5338	46.5786

Differences to the results from 4-D Binomial Tree
Call Options

Strike	Black-Scholes European	Monte Carlo European	LSMC American	Binomial Tree European	Binomial Tree American	Implied Tree European	Implied Tree American
60	0.5324	-0.0545	0.2018	-	-	0.2230	0.1233
70	0.3719	-0.0934	0.1869	-	-	0.3467	0.3406
80	-0.1079	-0.1117	0.1616	-	-	0.3076	0.3546
90	-0.7403	-0.0997	0.0694	-	-	0.1129	0.2003
100	-1.3612	-0.0673	-0.0085	-	-	0.3342	0.4002
110	-1.8729	-0.0221	-0.0994	-	-	0.0397	0.1035
120	-2.2379	0.0142	0.0456	-	-	0.2867	0.3402
130	-2.4477	0.0745	0.0233	-	-	0.0390	0.0926
140	-2.5401	0.1427	0.2089	-	-	0.2149	0.2515

Differences to the results from 4-D Binomial Tree
Put Options

Strike	Black-Scholes European	Monte Carlo European	LSMC American	Binomial Tree European	Binomial Tree American	Implied Tree European	Implied Tree American
60	0.4587	-0.0072	-0.0002	-	-	0.1493	0.1487
70	0.2982	-0.0461	-0.0333	-	-	0.2730	0.2711
80	-0.1815	-0.0644	-0.0430	-	-	0.2340	0.2346
90	-0.8140	-0.0525	0.0186	-	-	0.0392	0.0309
100	-1.4349	-0.0201	0.0247	-	-	0.2605	0.2608
110	-1.9465	0.0252	0.0715	-	-	-0.0340	0.0174
120	-2.3115	0.0615	0.1228	-	-	0.2131	0.2609
130	-2.5213	0.1218	0.1936	-	-	-0.0346	0.0873
140	-2.6137	0.1900	0.2904	-	-	0.1413	0.2952

Four-Asset Case(2) Steps:[5] 100

Call Options

Strike	Black-Scholes European	Monte Carlo (10000) European	LSMC (10000) American	Binomial Tree European	Binomial Tree American	Implied Tree European	Implied Tree American
60	39.3672	38.7803	40.3251	38.8348	40.1233	38.9332	40.1687
70	31.6195	31.1542	32.1446	31.2476	31.9577	31.3686	32.0787
80	24.9262	24.9224	25.6184	25.0341	25.4568	25.1671	25.6338
90	19.3452	19.9858	20.4219	20.0855	20.3525	20.1417	20.4732
100	14.8255	16.1194	16.3547	16.1867	16.3632	16.2154	16.4545
110	11.2496	13.1004	13.1441	13.1225	13.2435	13.1775	13.3555
120	8.4713	10.7234	10.8402	10.7092	10.7946	10.7623	10.8979
130	6.3429	8.8651	8.8757	8.7906	8.8524	8.8436	8.9487
140	4.7296	7.4124	7.5243	7.2697	7.3154	7.3094	7.3920

Put Options

Strike	Black-Scholes European	Monte Carlo (10000) European	LSMC (10000) American	Binomial Tree European	Binomial Tree American	Implied Tree European	Implied Tree American
60	1.3180	0.8521	0.8627	0.8593	0.8629	0.8840	0.8858
70	3.0826	2.7383	2.7676	2.7844	2.8009	2.8317	2.8418
80	5.9016	6.0187	6.0873	6.0831	6.1303	6.1425	6.1778
90	9.8329	10.5944	10.7683	10.6469	10.7497	10.6294	10.7154
100	14.8255	16.2403	16.4739	16.2604	16.4492	16.2154	16.3953
110	20.7619	22.7336	23.0882	22.7084	23.0167	22.6898	23.0026
120	27.4959	29.8689	30.3944	29.8074	30.2716	29.7869	30.2822
130	34.8798	37.5229	38.2540	37.4011	38.0604	37.3805	38.1080
140	42.7788	45.5825	46.5738	45.3925	46.2834	45.3586	46.3657

Differences to the results from 4-D Binomial Tree
Call Options

Strike	Black-Scholes European	Monte Carlo European	LSMC American	Binomial Tree European	Binomial Tree American	Implied Tree European	Implied Tree American
60	0.5324	-0.0545	0.2018	-	-	0.0984	0.0454
70	0.3719	-0.0934	0.1869	-	-	0.1210	0.1210
80	-0.1079	-0.1117	0.1616	-	-	0.1330	0.1770
90	-0.7403	-0.0997	0.0694	-	-	0.0562	0.1207
100	-1.3612	-0.0673	-0.0085	-	-	0.0287	0.0913
110	-1.8729	-0.0221	-0.0994	-	-	0.0550	0.1120
120	-2.2379	0.0142	0.0456	-	-	0.0531	0.1033
130	-2.4477	0.0745	0.0233	-	-	0.0530	0.0963
140	-2.5401	0.1427	0.2089	-	-	0.0397	0.0766

Differences to the results from 4-D Binomial Tree
Put Options

Strike	Black-Scholes European	Monte Carlo European	LSMC American	Binomial Tree European	Binomial Tree American	Implied Tree European	Implied Tree American
60	0.4587	-0.0072	-0.0002	-	-	0.0247	0.0229
70	0.2982	-0.0461	-0.0333	-	-	0.0473	0.0409
80	-0.1815	-0.0644	-0.0430	-	-	0.0594	0.0475
90	-0.8140	-0.0525	0.0186	-	-	-0.0175	-0.0343
100	-1.4349	-0.0201	0.0247	-	-	-0.0450	-0.0539
110	-1.9465	0.0252	0.0715	-	-	-0.0186	-0.0141
120	-2.3115	0.0615	0.1228	-	-	-0.0205	0.0106
130	-2.5213	0.1218	0.1936	-	-	-0.0206	0.0476
140	-2.6137	0.1900	0.2904	-	-	-0.0339	0.0823

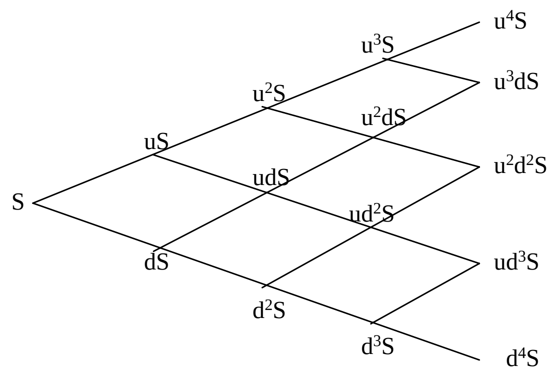
[5] 20 steps are applied to the 4-Dimensional Binomial Tree, Monte Carlo simulation, and Least-squares Monte Carlo simulation. 100 steps are applied to the implied tree.

Appendix A:

Valuing American-style options using binomial trees

Univariate Binomial Tree

Cox, Ross and Rubinstein (1979) developed the binomial option pricing model which converges to the Black-Scholes formula in the continuous limit and demonstrates the advantage in valuing American-style options. The model approximates the behavior of an asset price by the upward and downward changes in the asset price over a particular interval of time.



As shown in Figure 1, an asset with a current price of S follows a multiplicative binomial process in which the asset price can either go up to uS or down to dS, during the interval h = T/m, where m is the total number of moves over time. The figure 1 demonstrates a four moves binomial tree. Cox, Ross and Rubinstein (1979) has shown that the upward and downward movements, i.e., u and d, can be chosen such that the probabilities of each movements to be equally one-half. In addition, the mean and variance of the discrete binomial process matches those of the risk-neutral process of the asset price over the time step of the binomial tree. The structure leads to:

$$u = \exp\left(vh + \sigma\sqrt{h}\right) \qquad d = \exp\left(vh - \sigma\sqrt{h}\right)$$

$$p = \tfrac{1}{2}$$

$$v \equiv r - d - \tfrac{1}{2}\sigma^2$$

$$h \equiv T / m$$

After constructing the tree of asset prices, the valuation of options can be worked backwards by applying the concept that the value of a security is its risk-neutral expectation of discounted payoffs. In our setup, since both up and down moves have the equal probability of one-half, the holding value of security is the discounted average of its value in the next interval. If it is an American-style option, the value of the option at a particular time, is its intrinsic value or the holding value, whichever is greater.

Consider an American put option with two moves, m =2, for example. At expiration, depending on the asset prices, the value of option will be:

$C(u^2S) = \max\{ 0,\ K - u^2S \}$
$C(udS) = \max\{ 0,\ K - udS \}$

$C(d^2S) = \max\{ 0,\ K - d^2S \}$

Working backwards by one move, we have,

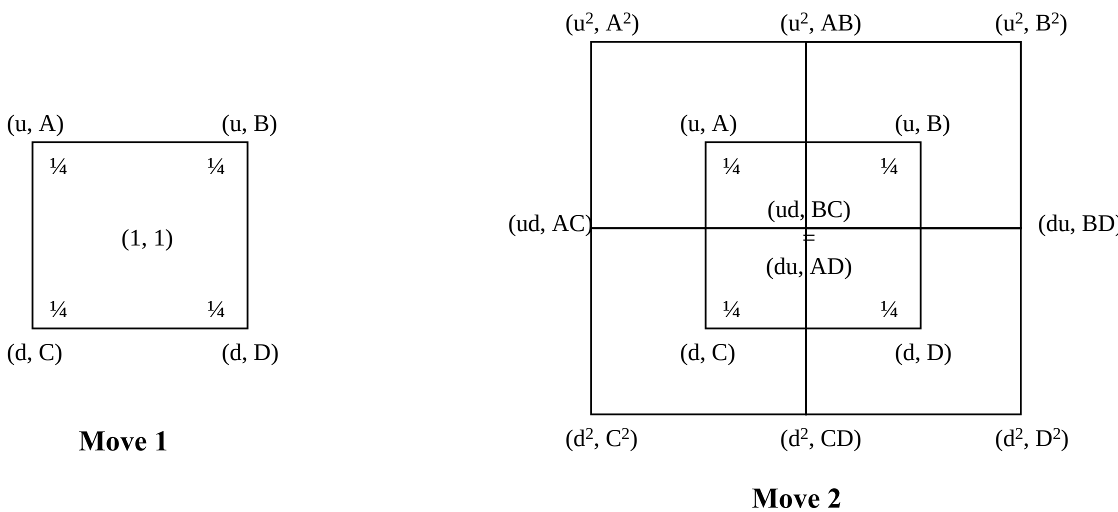
$C(uS) = \max\{ K - uS,\ ½[C(u^2S) + C(udS)] * \log(-r*h) \}$
$C(dS) = \max\{ K - dS,\ ½[C(udS) + C(d^2S)] * \log(-r*h) \}$

And, finally, working backwards one more move to the beginning of the tree:

$C(S) = \max\{ K - S,\ ½[C(uS) + C(dS)] * \log(-r*h) \}$

Bivariate Binomial Tree

Rubinstein (1995) shows an approach to value options with two underlying assets by approximating continuous bivariate normal density function with a discrete bivariate binomial density. The structure, or called "binomial pyramids", is shown in Figure 1.



The returns of the two underlying assets are written in pairs, and they start from (1, 1) at the beginning. During the first move, the first asset return is assumed to be either u or d, with equal probability. Simultaneously, if the first asset moves up by u, the second asset return may be either A or B with equal probability, depending on the correlation between two assets. Or, if the first asset moves down by d, the second asset return may be C or D. It structures total four possible outcomes, which are (u,A), (u,B), (d,C) and (d,D), with equal probability 1/4, which emanated from the pair (1,1).

Figure 1 also shows the nodes after the second move. The multiplicative returns over two moves will be (u^2, A^2), (u^2, AB), (ud, AC) and (ud, AD), if the return of first move was (u,A).

The value of moves can be obtained as following:

$$u = \exp\left(v_1 h + \sigma_1 \sqrt{h}\right) \qquad d = \exp\left(v_1 h - \sigma_1 \sqrt{h}\right)$$
$$A = \exp\left(v_2 h + \sigma_2 \sqrt{h}\left[\rho + \sqrt{1-\rho^2}\right]\right) \qquad B = \exp\left(v_2 h + \sigma_2 \sqrt{h}\left[\rho - \sqrt{1-\rho^2}\right]\right)$$
$$C = \exp\left(v_2 h - \sigma_2 \sqrt{h}\left[\rho - \sqrt{1-\rho^2}\right]\right) \qquad D = \exp\left(v_2 h - \sigma_2 \sqrt{h}\left[\rho + \sqrt{1-\rho^2}\right]\right)$$
$$p = \tfrac{1}{4}$$

$$v_1 \equiv r - d_1 - \tfrac{1}{2}\sigma_1^2$$
$$v_2 \equiv r - d_2 - \tfrac{1}{2}\sigma_2^2$$
$$h \equiv T / m$$

where m is the number of moves over the period T, r is the continuous compounding risk free rate, d_i are continuous dividend yields, σ_i are assets' volatilities, and ρ is the correlation coefficient between two assets.

These parameters can be used to construct the forward moving structure of asset returns. At the end of the tree, values of the basket option can be evaluated with the asset pair at each node. Then, we work backwards by discounting four nodes into one node in each move, using the same probability of 1/4 for each nodes.

For example, consider an American put option on a two-asset basket, the payoff function at expiration is max(0, K - (n_1S_1 + n_2S_2)). For a two moves bivariate binomial tree, where m = 2, values of the option at expiration are:

$C(u^2, A^2) = \max[\ 0,\ K - (n_1S_1u^2 + n_2S_2A^2)\ ]$
$C(u^2, AB) = \max[\ 0,\ K - (n_1S_1u^2 + n_2S_2AB)\ ]$
$C(u^2, B^2) = \max[\ 0,\ K - (n_1S_1u^2 + n_2S_2B^2)\ ]$
$C(ud, AC) = \max[\ 0,\ K - (n_1S_1ud + n_2S_2AC)\ ]$
$C(ud, BC) = C(du, AD) = \max[\ 0,\ K - (n_1S_1ud + n_2S_2BC)\ ]$
$C(ud, BD) = \max[\ 0,\ K - (n_1S_1ud + n_2S_2BD)\ ]$
$C(d^2, C^2) = \max[\ 0,\ K - (n_1S_1d^2 + n_2S_2C^2)\ ]$
$C(d^2, CD) = \max[\ 0,\ K - (n_1S_1d^2 + n_2S_2CD)\ ]$
$C(d^2, D^2) = \max[\ 0,\ K - (n_1S_1d^2 + n_2S_2D^2)\ ]$

Working backwards one move:

$C(u,A) = \max\{K - (n_1S_1u + n_2S_2A),\ ¼\ [C(u^2, A^2) + C(u^2, AB) + C(ud, AC) + C(ud, AD)]*\log(-r*h)\}$
$C(u,B) = \max\{K - (n_1S_1u + n_2S_2B),\ ¼\ [C(u^2, AB) + C(u^2, B^2) + C(ud, BC) + C(ud, BD)]*\log(-r*h)\}$
$C(d,C) = \max\{K - (n_1S_1d + n_2S_2C),\ ¼\ [C(ud, AC) + C(ud, BC) + C(d^2, C^2) + C(d^2, CD)\ ]*\log(-r*h)\}$
$C(d,C) = \max\{K - (n_1S_1d + n_2S_2C),\ ¼\ [C(ud, AD) + C(ud, BD) + C(d^2, CD) + C(d^2, D^2)\ ]*\log(-r*h)\}$

And, finally, working from backwards one more move to the beginning of the tree:

$C(0, 0) = \max\{K - (n_1S_1 + n_2S_2),\ ¼\ [C(u, A) + C(u, B) + C(d, C) + C(d, D)]*\log(-r*h)\}$

N-variate Binomial Tree

The bivariate binomial tree can be generalized into N-variate case. In the N-variate environment, each node is represented by an N-tuplet (S_1, S_2, …, S_N)'. There are 2^N possible moves from each node. Each has equal probability of 2^{-N}. It is easier if we transform the asset prices into its logarithm returns. Then, the multiplicative price movements become additive in terms of their logarithm. We denote an N elements vector $(dx_1, dx_2, \ldots, dx_N)'$ as the logarithm of a movement starting from a node. The multiplicative movements on asset prices can be calculated by exponentiating this vector. Obviously, in the N-dimension world, for each move, there are 2^N such vectors, representing both the direction and distance of possible movements emanated from a node.

Similar to the technique used in multi-dimensional Monte Carlo simulation, the logarithm movements can be obtained by the Cholesky decomposition of the correlation matrix.

For example, if there are four assets in the basket. The sixteen equal probability moves are:

$$\left[\begin{pmatrix} dx_1 \\ dx_2 \\ dx_3 \\ dx_4 \end{pmatrix}^{(1)} \begin{pmatrix} dx_1 \\ dx_2 \\ dx_3 \\ dx_4 \end{pmatrix}^{(2)} \cdots \begin{pmatrix} dx_1 \\ dx_2 \\ dx_3 \\ dx_4 \end{pmatrix}^{(16)}\right] = h\mathbf{v} + \sqrt{h}\begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \sigma_3 & \\ & & & \sigma_4 \end{bmatrix} \times \mathbf{w}$$

where

$$\mathbf{v} = \begin{bmatrix} r - d_1 - \frac{1}{2}\sigma_1^2 \\ r - d_2 - \frac{1}{2}\sigma_2^2 \\ r - d_3 - \frac{1}{2}\sigma_3^2 \\ r - d_4 - \frac{1}{2}\sigma_4^2 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\mathbf{w} = \begin{bmatrix} 1 & & & \\ c_{21} & 1 & & \\ c_{31} & c_{32} & 1 & \\ c_{41} & c_{42} & c_{43} & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \end{bmatrix}$$

$$\mathbf{\Omega} = \underbrace{\begin{bmatrix} 1 & \rho_{12} & \rho_{13} & \rho_{14} \\ \rho_{12} & 1 & \rho_{23} & \rho_{24} \\ \rho_{13} & \rho_{23} & 1 & \rho_{34} \\ \rho_{14} & \rho_{24} & \rho_{34} & 1 \end{bmatrix}}_{Correlation\ Matrix} = \underbrace{\begin{bmatrix} 1 & & & \\ c_{21} & 1 & & \\ c_{31} & c_{32} & 1 & \\ c_{41} & c_{42} & c_{43} & 1 \end{bmatrix}}_{Cholesky\ Decomposition} \times \begin{bmatrix} 1 & & & \\ c_{21} & 1 & & \\ c_{31} & c_{32} & 1 & \\ c_{41} & c_{42} & c_{43} & 1 \end{bmatrix}^T$$

Using the log-moving vector $(dx_1, dx_2, \ldots, dx_N)'$, one can construct an N-variate m-period binomial tree for N-asset basket. For each move, a price tuplet $(S_1, S_2, \ldots, S_N)'$ has 2^N possible outcomes to $(S_1\exp(dx_1^{(i)}), S_2\exp(dx_2^{(i)}), \ldots, S_N\exp(dx_N^{(i)}))'$, $i = 1, 2, \ldots, 2^N$. Each has a probability of 2^{-N}. By progressively working forwards using the logarithm moving vectors as shown in (1), there will be $(m+1)^N$ nodes after m moves. For each node, one can evaluate the payoff of the basket options. For example, at expiration, a put option on the N-asset basket has payoff:

$$\max\{0, K - (n_1S_1 + n_2S_2 + \ldots + n_NS_N)\}$$

where $n_1, n_2, \ldots, n_N$ are number of shares of each asset contained in the portfolio.

Appendix B:

Valuing European-style multi-asset basket options using Monte Carlo simulation

Assuming all the underlying assets of a basket option follows geometric Brownian motion (GBM) processes.

$$\begin{aligned} dS_1 &= (r - d_1)S_1 dt + \sigma_1 S_1 dw_1 \\ dS_2 &= (r - d_2)S_2 dt + \sigma_2 S_2 dw_2 \\ &\vdots \\ dS_N &= (r - d_N)S_N dt + \sigma_N S_N dw_N \end{aligned}$$

where r is the continuous compounding risk-less interest rate, d_i's are dividend yield for each asset, σ_i's are return volatilities, and the two assets (i and j) have correlation ρ_{ij}, i.e., $dw_i dw_j = \rho_{ij} dt$.

The best way to simulate a GBM process is through a transformation to the natural logarithm of asset prices. Let $x_i = \ln(S_i)$, via Ito's Lemma, we have,

$$\begin{aligned} dx_1 &= v_1 dt + \sigma_1 dw_1 \\ dx_2 &= v_2 dt + \sigma_2 dw_2 \\ &\vdots \\ dx_N &= v_N dt + \sigma_N dw_N \end{aligned}$$

where $v_i = r - d_i - \sigma_i^2/2$.

Next, we transform to a space where variables are uncorrelated. This can be conducted by decomposing the covariance matrix of original GBM process into its eigenvalues and eigenvectors.

$$\underbrace{\begin{bmatrix} \begin{pmatrix} e_{11} \\ e_{12} \\ \vdots \\ e_{1N} \end{pmatrix} \begin{pmatrix} e_{21} \\ e_{22} \\ \vdots \\ e_{2N} \end{pmatrix} \cdots \begin{pmatrix} e_{N1} \\ e_{N2} \\ \vdots \\ e_{NN} \end{pmatrix} \end{bmatrix}}_{eigenvectors} \underbrace{\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_N \end{bmatrix}}_{eigenvalues} \begin{bmatrix} e_{11} & e_{12} & & e_{1N} \\ e_{21} & e_{22} & & e_{2N} \\ & & \ddots & \\ e_{N1} & e_{N2} & & e_{NN} \end{bmatrix} = \underbrace{\begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 & \cdots & \rho\sigma_1\sigma_N \\ \rho\sigma_2\sigma_1 & \sigma_2^2 & & \rho\sigma_2\sigma_N \\ \vdots & & \ddots & \\ \rho\sigma_N\sigma_1 & \rho\sigma_N\sigma_2 & & \sigma_N^2 \end{bmatrix}}_{covariance matrix}$$

The eigenvectors lead the linear combination of x_1, x_2, …, and x_N be uncorrelated:

$$\begin{aligned} dy_1 &= \alpha_1 dt + \sqrt{\lambda_1} dz_1 \\ &\vdots \\ dy_N &= \alpha_N dt + \sqrt{\lambda_N} dz_N \end{aligned} \qquad where \qquad \begin{aligned} y_1 &= e_{11}x_1 + e_{12}x_2 + \cdots + e_{1N}x_N \\ &\vdots \\ y_N &= e_{N1}x_1 + e_{N2}x_2 + \cdots + e_{NN}x_N \end{aligned}$$

where $\alpha_i = e_{i1}v_1+e_{i2}v_2+\ldots+e_{iN}v_N$, for all i = 1, …, N, and dz_i, dz_j are uncorrelated Brownian motions.

Now, we can discretise the N uncorrelated arithmetic Brownian motions by replacing the infinitesimals dy_i, dt and dz_I with small changes Δy_i, h, and random samples of $\varepsilon_i\sqrt{h}$. Those ε_i's are independent draws from standard normal distributions. h = T/m is the time step, m is the number of time periods over which we want to simulate. If we have t_i = i h, i = 1, …,m, then the processes of y_i's can be simulated by the following:

$$
\begin{aligned}
y_{1,t_i} &= y_{1,t_{i-1}} + \alpha_1 h + \varepsilon_{1,i}\sqrt{\lambda_1 h} \\
&\vdots \\
y_{N,t_i} &= y_{N,t_{i-1}} + \alpha_N h + \varepsilon_{N,i}\sqrt{\lambda_N h}
\end{aligned}
$$

The asset prices at each time interval can be obtained by transforming the y_i's back:

$$
\begin{aligned}
x_{1,t_i} &= e_{11} y_{1,t_i} + e_{21} y_{2,t_i} + \cdots + e_{N1} y_{N,t_i} \\
&\vdots \\
x_{N,t_i} &= e_{1N} y_{1,t_i} + e_{2N} y_{2,t_i} + \cdots + e_{NN} y_{N,t_i} \\
\Rightarrow& \\
S_{j,t_i} &= \exp\left(x_{j,t_i}\right) \qquad \forall\ j = 1,\cdots,N
\end{aligned}
$$

where $S_{j,ti}$ is the price of the j-th asset at the end of the i-th period.
At the end of maturity, the payoffs of a European-style option can be determined by knowing the underlying asset prices from the simulation. If we repeat the process, and simulate large number of outcomes of asset prices at maturity, the value of the European-style option at time zero, can be estimated unbiasedly through its average of all the discounted payoff outcomes.

$$\hat{C} = \frac{1}{M}\sum_{s=1}^{M}\exp(-rT)\,Payoff_T^{(s)}$$

where M is the total number of simulations, and it is typically a large number where 10,000 is not unusual in practice, $Payoff_T^{(s)}$ is the option's payoff at maturity T under s-th simulation path.

Consider a European-style put option on an N-asset basket. After the s-th simulation, the underlying asset prices become $S_{1,T}^{(s)}$, $S_{2,T}^{(s)}$, …, $S_{N,T}^{(s)}$ at time T. Therefore, the payoff of the basket option is:

$$Payoff_T^{(s)} = \max\{\, 0, K - [n_1 S_{1,T}^{(s)} + n_2 S_{2,T}^{(s)} + \ldots + n_N S_{N,T}^{(s)}] \,\}$$

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