

The Choice to Rent or Own: Why Rents and Mortgages Rates Are Jointly Determined*

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Abstract

This paper proposes a unified framework of housing choice in the presence of a lenders' and landlords' market. I consider a finitely lived risk averse agent who is exposed to two sources of uncertainty: stochastic labor income and stochastic house price. The main contribution is to jointly derive the default risk premium, determined by expected losses due to default on the mortgage, and the rent risk premium, determined by the risk of house prices fluctuations. The agent, as an owner, optimally determines the time at which defaulting and moving into the rental market provides a greater expected continuation utility than continuing the mortgage payments. Consequently, the default and rent risk premia are systematically related. The model predicts that the default and rent risk premia are both increasing in the volatility of labor income and house prices. In addition, the default risk premium (the rent risk premium) is increasing (decreasing) with respect to the correlation between them.

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1 Introduction

The long-term durability of housing complicates the choice between owning or renting: ownership creates a risk with regard to the house value at the end of the planning horizon, whereas renting creates the risk of fluctuating costs during the planning horizon. In principle, of course, the risk of the future housing value for ownership could be hedged with an appropriate futures market, and the risk of fluctuating rents could be eliminated with long-term rental contracts, but as a practical matter these markets do not exist or they are very thin. In addition, the durability of structures makes their purchase price relatively high, with the result being that most purchases must be financed with a mortgage. These mortgages necessarily have a risk of default which must be priced in the coupon. There is also a parallel question as to how landlords price their rental contracts.

In this paper, I propose a unified framework of housing choice in the presence of a lenders' and landlords' market. The model is developed in continuous time and considers a finitely lived risk averse agent who is exposed to two sources of uncertainty: stochastic labor income and stochastic house price. I endogenously determine the default risk premium, associated with the lenders' market, and the rent risk premium, associated with the landlords' market. My analysis addresses the question of what the impact of a lenders' and landlords' market is on the agent's choice of owning versus renting.

There are two approaches to the owning versus renting choice in the literature: (1) the first is to analyze the factors affecting the tenure choice in a simple equilibrium framework; (2) the second is to focus on how owning and renting impact the agent's consumption and portfolio choices in a partial equilibrium framework. Both approaches lead to fundamental predictions but none of the models developed in this context endogenously incorporate the lender's and landlord's problem.

Sinai and Souleles (2005) develop a simple equilibrium model to study the circumstances under which an agent is better off owning rather than renting. They identify a hedging benefit to owning if the agent has a long horizon and is exposed to volatile rents. Their claim that owning is a natural hedge against rent fluctuations is intuitive, but is built on three restrictive assumptions. The first assumption is that the agent does not need to take a mortgage, because he is endowed with enough wealth to pay the full purchase price. The second assumption is that the agent does not earn any risky labor income which might be positively correlated with the rent fluctuations.¹ The third assumption is that they rule

¹In a partial equilibrium framework, Ortalo-Magné and Rady (2002) analyze an agent's tenure choice under uncertainty of income and house prices. They show that renting becomes relatively more attractive than homeownership as the covariance between labor income and house price increases. Davidoff (2006) shows that an agent with mean-variance preferences optimally purchases less housing as the covariance

out fixed-price, long-term, lease contracts (whereby the tenant could lock in fixed rental payments for the duration of his tenancy). The effect of these assumptions is that the model's result may be significantly biased in favor of owning over renting.

Yao and Zhang (2005a) and Van Hemert (2006) develop models that fall under the second approach. They study the agent's choice of owning with a mortgage versus renting. Under their conditions, if the agent can provide the downpayment and his labor income is sufficiently high to pay the future mortgage payments, then owning is preferred. However, these results depend on two assumptions. The first assumption is that renting includes an arbitrary risk premium that makes it more expensive than owning per unit of housing services. The second assumption is that the mortgage interest rate contains no default premium. Here too, the effect is that the model's results may be significantly biased against renting.

Campbell and Cocco (2003) model the choice between a fixed rate mortgage (FRM) and an adjustable rate mortgage (ARM); their model also belongs to the second approach.² They assume that buying a house with a mortgage is strictly preferred to renting the same house and that the agent can choose between a fixed rate mortgage (FRM) or an adjustable rate mortgage (ARM). Both mortgage contracts, however, have default risk premia that are not fairly priced. Furthermore, the agent must move into the rental market if he chooses to default on the mortgage and there he faces an exogenously determined rent level. The default timing is endogenously determined and is affected by the default risk premium and the rent risk premium. A lower (higher) rent risk premium increases (decreases) the probability of default, but the exogenous default risk premia on the FRM and ARM do not adjust accordingly.³ Because they assume that housing ownership is always preferred to renting and that the rental rate is exogenously determined, this model cannot shed significant light on the choice between owning and renting.

In this paper, I explicitly derive an equilibrium model of the choice between owning and renting in a dynamic setting, where the default and rent risk premia are determined by expected losses due to a default on the mortgage and by the risk of house price fluctuations. I consider a representative agent with limited wealth, who generally requires a mortgage to purchase a house. The mortgage interest rate is determined endogenously and incorporates a default risk premium. The agent can costlessly increase (decrease) the mortgage balance,

 between labor income and housing prices rises.

²Campbell and Cocco (2003) is the first paper of the current literature on household risk management that extends the previous literature of life-cycle consumption and saving in the presence of liquidity constraints, Zeldes (1989), Deaton (1991), Carroll (1997) and Gourinchas and Parker (2002), introducing housing consumption and mortgage contracts. In the next section, I will provide a complete discussion of Campbell and Cocco (2003), Cocco (2005), Yao and Zhang (2005a) and Van Hemert (2006) who contributed to the household risk management literature.

³This limitation is acknowledged by the two authors.

thereby decreasing (increasing) his equity position in the house, although the mortgage cannot exceed the current house value (a *margin requirement* constraint). Following Leland (1994), I assume that the agent optimally determines the time at which defaulting provides a greater expected continuation utility than continuing the mortgage payments. As a consequence of the default, the agent loses a fraction of his wealth because the house is seized and the agent is exposed to a dead weight loss proportional to the house value at default. Then the agent moves into the rental market. An implication of the model is that the agent does not assume a highly leveraged position. If he does, he is exposed to house price fluctuations through the *margin requirement* constraint. Decreases in house prices might increase the agent's loan-to-value position inducing the agent to default on the mortgage. The default also depends on the rent risk premium since the higher the rent risk premium, the lower the probability that the agent exercises the default option. The risk neutral lender *ex ante* charges the default risk premium, accounting for the agent's default strategy and the dead weight loss he may incur.⁴

If the agent chooses to rent, he pays a rent that is a fraction (the rental cost) of the current house value. The rental cost is the sum of two components: the user cost of the housing services and the rent risk premium. Assuming a risk neutral landlord, the user cost of housing is derived when the expected annual cost of owning a house equals that of renting. In my model, the landlord is assumed to be comparable to the agents and therefore is risk averse. Therefore, I determine the rent risk premium endogenously, such that the landlords (and all agents) are indifferent *ex ante* to owning with a mortgage or renting.

In the current mortgage pricing literature, the occurrence of default is driven by the house price process, and is triggered when the owner has a negative equity position in the house; labor income is not modeled.⁵ In the household risk management literature, in contrast, labor income is modeled as a stochastic process, but it does not affect the default risk premium, which is instead set exogenously. Campbell and Cocco (2003) have highlighted the limitations of the mortgage pricing literature, recognizing the importance of labor income and borrowing constraints in determining default. However, they have not incorporated the labor income effect into the mortgage price. A major contribution of my model is that the parameters of the labor income process directly affect the default risk premium, which is

⁴For convenience, I assume a risk neutral lender. If the lender is not risk neutral, then it would demand a higher coupon than the one I calculate to compensate for the default. Moreover, I assume that the lender does not have any outside investment options and does not face any systematic risk across the agents and the properties in providing the mortgage contracts.

⁵The structural approach treats mortgage termination as the optimal response of a rational risk neutral borrower who chooses when to prepay or default. The agent is exposed to two sources of uncertainty: interest rates and house prices. Under this approach, standard contingent claims techniques are used to calculate the agent's optimal exercise policy, Kau and Keenan (1995) and Downing et al. (2005).

endogenously determined. Despite the fact that my model is too stylized to capture all of the features that many mortgages offer in the market place,⁶ it captures some important aspects. The model predicts that the default risk premium is increasing with respect to the volatility of both labor income and house prices as well as to the correlation between the two. On the other hand, rent also comoves continuously with house prices, and a positive correlation between rent and income provides a natural hedge against rent fluctuations, allowing the renter to smooth numeraire consumption. Therefore, the model predicts that the rent risk premium is decreasing with respect to the correlation coefficient of labor income and house prices.⁷

In this paper, I abstract from considering some factors that might affect the choice between owning and renting, such as tax law and individuals' mobility. However, while tax laws may offer incentives to owning in the United States, in many countries the tax treatment of renting and owning is not financially relevant, or may favor renting. Therefore, I concentrate on the noninstitutional economic aspects of the problem.⁸ The choice between owning and renting might be influenced by timing and frequency of the agent's move. Furthermore, transactions costs associated with selling and buying the house would make renting more attractive than owning. On the other hand, people may desire housing attributes that are only available in owned units. These factors may be important determinants of the owning versus renting choice, but in this paper I will focus on the fundamental role of house and rental prices and mortgages rates on this decision.

In order to make the model numerically tractable, I make two simplifying assumptions. The first is that, as in Campbell and Cocco (2003), the owner can change his status, from owning to renting, by defaulting on the mortgage, while the renter can never change his status. The second major assumption is that while the default and rent risk premia are determined endogenously at the mortgage's origination and rent's signature, then they remain constant throughout the agent's horizon.⁹ The second assumption precludes the possibility that, after signing the mortgage contract, the agent or the lender might renegotiate the mortgage coupon varying the default risk premium. It also precludes the possibility that, after signing the rental contract, the landlord might charge a different rental cost varying the rent risk premium. While the assumption is restrictive, it may be realistic if the fixed costs of negotiating contracts limit the potential to renegotiate.

⁶Moreover, I do not include stochastic interest rates in my analysis.

⁷Because Campbell and Cocco (2003) assume that house prices and labor income are perfectly positively correlated, they cannot investigate how this correlation affects the agent's propensity to default.

⁸I leave tax considerations for further developments of the model.

⁹Campbell and Cocco (2003) assume an exogenous constant default and rent risk premia. Yao and Zhang (2005a) and Van Hemert (2006) assume an exogenous constant rent risk premium and a zero valued default risk premium.

The paper is organized as follows. Section 2 reviews the relevant literature. Section 3 describes the agent's optimization problem. Subsection 3.1 derives the renter's problem, showing that in the absence of a default risk premium but in the presence of a rent risk premium, the agent may prefer owning with a mortgage to renting. Subsection 3.2 derives the owner's problem in the presence of a mortgage, describing how default is endogenously determined. Subsection 3.2.2 provides the mortgage valuation. Section 4 provides a brief overview of the numerical schemes implemented to solve the agent's problem and the mortgage valuation. Section 5 describes the calibration of the model parameters, while Section 6 presents the numerical results of the model. Finally, Section 7 concludes and discusses possible directions of my analysis.

2 Review of the literature

The papers of Grossman and Laroque (1990), Campbell and Cocco (2003), Cocco (2005), Yao and Zhang (2005a) and Van Hemert (2006) analyze housing in the presence of a mortgage contract. I will refer to this literature as the Household Risk Management (HRM) literature. Much of this recent literature solves a finite horizon discrete-time intertemporal optimal housing consumption and investment problem in the presence of a mortgage; where one or more of the following features are modeled: the agent

1. may rent or own a house;
2. receives a stochastic labor income stream;
3. is the owner or renter of a house whose price evolves stochastically;
4. as owner, may sell the current house to buy a new one, smaller or bigger;
5. as owner, may be forced to sell the current house as a consequence of exogenous shocks such as divorce or unemployment; and
6. may invest in both a risk free asset and a risky asset.

In the following sections, I will highlight the main contributions of the HRM literature under these many respects as well as summarize the main contributions of my model to this field.

2.1 Housing consumption

In an early and fundamental contribution, Grossman and Laroque (1990) consider the impact of an agent's investment in a house on portfolio choice. The agent derives utility from owning

a house whose stock depreciates at a constant rate. Housing consumption is illiquid because the owner incurs a transaction cost when selling the current house to buy a bigger or smaller one. Cocco (2005), Yao and Zhang (2005a), Yao and Zhang (2005b) and Van Hemert (2006) all assume that the owner incurs transaction costs to sell the house and therefore that the housing consumption is lumpy. Moreover, the agent is exposed to shocks, such as divorce and unemployment, that force him to sell the house. In Campbell and Cocco (2003), the agent cannot adjust housing consumption by selling the current house and buying a new one. In the current HRM literature, the agent derives utility also from other goods (numeraire consumption).

Following the HRM literature, I assume that the agent maximizes a utility function that depends on housing services and other goods (numeraire consumption). The housing stock provides a constant flow of housing services to the agent. I assume that (i) the housing stock is given and does not depreciate; and (ii) the housing stock provides a constant flow of housing services to the agent.

Campbell and Cocco (2003) assume that buying a house is strictly preferred to renting and do not model this decision. However, they assume that the agent is forced into the rental market if default occurs. Yao and Zhang (2005a), Yao and Zhang (2005b) and Van Hemert (2006) introduce the house rental market and explicitly deal with the agent's choice of owning versus renting. The agent can change his status of owning by moving into the rental market and vice versa at any time.

In my model, the agent is unconstrained with respect to owning versus renting, and chooses the path that maximizes his expected utility at time zero. The agent remains in the same house size regardless of the realized *ex post* values for the path of his income and wealth. Following Campbell and Cocco (2003), I assume that the agent's status changes from owner to renter upon the agent defaulting, but the renter never changes his status.

2.2 Rent

The HRM literature does not explicitly solve the landlord's problem. These papers usually assume that the rent is a fraction (rental cost) of the current house value. The rental cost is defined as the sum of two components: the user cost (the cost of consuming the house) and an exogenous rent risk premium. The user cost is derived assuming a risk neutral landlord who is indifferent *ex ante* to owning or renting. Campbell and Cocco (2003) identify the rent risk premium as the remuneration that covers the average realized maintenance costs: tenants have no incentive to look after the property so home maintenance is expensive.¹⁰ In

¹⁰Henderson and Ioannides (1983) derive a full equilibrium model of the choice between owning and renting where the tenant may not properly care for the property. Because of limited information, landlords cannot

their paper, the rent risk premium affects default timing: the lower the rent risk premium, the lower the cost of renting, and the higher the probability of default. Because they work in partial equilibrium, the default risk premium does not account for the rent risk premium. Yao and Zhang (2005a), Yao and Zhang (2005b) and Van Hemert (2006) assume a positive rent risk premium but they do not provide any motivation for it. The rent risk premium crucially affects their model predictions with a higher risk premium causing the agent to prefer owning with a mortgage. However, the mortgage rate is the risk free rate, implying that there is no default risk premium. In my model, the landlord, as the agent, is risk averse with a finite horizon. Therefore, to compensate himself for the house price risk, he would charge a rent higher than the user cost.

Given an exogenous process for the rent, Sinai and Souleles (2005) determine the house price as a function of expected present value of future rents, rent risk premium and house price risk premium, which is defined as the risk that the agent faces when selling the house. The agent is indifferent *ex ante* to owning or renting. Their framework involves some simplifications: the agent has an initial wealth endowment high enough to make the house purchase affordable; the agent's preferences do not depend on other goods; the agent does not receive a stochastic labor income flow.

Following Sinai and Souleles (2005), I adopt a utility indifference pricing approach. The goal is to identify the rent risk premium, at time zero, in a dynamic setting where the agent is exposed to labor income and house price uncertainty. Moreover, the agent can decide to consume other goods but he can never borrow at the risk free rate. I identify the rent risk premium such that a representative agent is indifferent *ex ante* to owning with a mortgage or renting at time zero.¹¹ Hence, I assume the rent risk premium is constant after signing the rent contract implying that the landlord cannot renegotiate the rent contract resetting the rent risk premium.

Alternatively, the agent could rent the house with a long term rent contract, paying a constant rent for a specific time horizon. This alternative is not explored by the current HRM literature nor by Sinai and Souleles (2005). I do not model this alternative, but, in the next section, I provide intuition that suggests how this mechanism could be implemented.

distinguish *ex ante* good tenants from bad tenants. Under such circumstances, landlords charge rents which reflect average maintenance costs across potential tenants. It follows that tenants who have a predisposition to maintain their home pay rents which exceed the marginal costs they impose on landlords. These tenants have the incentive to own the house avoiding externalities in the rental market.

¹¹The utility indifference pricing approach has been recently applied to price securities in the presence of incomplete markets. Market frictions, for example transactions costs, non-traded assets and portfolio constraints, make perfect replication impossible. In such situations, many different option prices are consistent with no-arbitrage, each corresponding to a different martingale measure. The law of one price does not hold. See Henderson and Hobson (2007) for an overview of the application of this technique in finance.

2.3 Mortgage contract

Grossman and Laroque (1990) assume that the agent can borrow at the risk free rate, in order to finance both housing consumption and stock market investment. The agent can also adjust his short position in the risk free rate at no additional cost, which is equivalent to a mortgage contract with no default risk. If the agent defaults, he is forced to sell the house and can never buy a new one, which would imply an infinitely negative utility. Therefore, the agent never chooses to default and there is no default risk premium. Campbell and Cocco (2003) provide the first model in which the agent finances his housing consumption by a mortgage contract. The agent has to provide an exogenous downpayment in order to buy the house and can take either an ARM or a FRM. The authors assume that the mortgage rate is equal to the risk free rate, which evolves stochastically, plus an exogenous premium to compensate the lender for the exposure to default risk. Cocco (2005) and Yao and Zhang (2005a) assume that the agent can finance the house purchase using a mortgage that is a home equity line contract. The residual mortgage balance can be paid off in advance or increased up to the current house value minus the downpayment component. As in Campbell and Cocco (2003), they set the downpayment at 20 per cent. Cocco (2005) and Yao and Zhang (2005b) assume an exogenous default risk premium over the risk free rate. Because they work in partial equilibrium, there is no relation between default risk premium and the downpayment. Yao and Zhang (2005a) and Van Hemert (2006) set the mortgage rate equal to the risk free rate. The impact of this assumption crucially depends on the level of the downpayment. If the agent has the wealth to pay a significant downpayment and he can increase the outstanding balance up to the current house value minus the downpayment, the house value at default can be high enough to compensate the lenders for most of the default risk. However, if the downpayment is not significant, the borrowing rate should be higher than the risk free rate.

Following Cocco (2005) and Yao and Zhang (2005a), I assume that the agent can finance the house purchase at time zero, taking on a mortgage that is a fixed rate, interest only, home equity line. Differently from these papers, however, in my model the agent has incentive to provide his initial savings as a downpayment, because the lender *ex ante* charges a higher default risk premium, the lower the downpayment. Hence, the coupon rate is the sum of a constant risk free rate and a constant default premium that is endogenously determined at the contract's origination. The coupon rate is set equal to the full mortgage rate so that the mortgage is issued at par. I assume that the agent can costlessly readjust the outstanding balance, decreasing or increasing the home equity line balance up to the current house value. After signing the contract, I assume that neither the agent nor the lender can reset the default risk premium.

2.4 Mortgage default

The papers discussed so far provide insights on how the optimal housing choice depends on the mortgage contract. However, none of these models assumes a relation between the agent's probability of default and the recovery value at default on one side, and the mortgage rate on the other side. Campbell and Cocco (2003) assume that the agent can choose to default on the mortgage at each time period, when faced with negative labor income shocks that make the saving account low; or negative house price shocks that make the home equity negative. If the agent does default, he loses the house and is forced into the rental market for the remainder of his life. They consider how changes in the main parameters, that capture the agent's characteristics, affect the agent's potential for default. Because the default risk premium is exogenous, it does not account for these variations. Moreover, the default risk premium is not related to the rent risk premium that affects the default timing. In Cocco (2005), Yao and Zhang (2005a) and Van Hemert (2006), the agent does not have the option to default on the mortgage. Yao and Zhang (2005b) extend their own previous model to explore the bankruptcy issue. They set the default risk premium at 1.50 percent. The agent exercises his default option whenever the home equity position reaches a default threshold. If the agent defaults, he is forced to rent indefinitely, however he can still adjust his housing consumption.

In my model, I assume that the lenders' market: (i) has full information on the agent's income and wealth processes, and house price; (ii) anticipates the optimal agent default policy; and (iii) operates in a perfectly competitive market, which implies zero profit. Therefore, the default risk premium depends on

1. the current level of the agent's income;
2. the dollar amount the agent is able to provide as downpayment;
3. the probability of default, which depends on the initial conditions and path in terms of growth rate and volatility for income and house prices, on the agent's risk aversion, and on the agent's rate of time preference; and
4. the consequences of default, which depend on the house value at default and on the dead weight losses that both the agent and the lender incur.

As in Campbell and Cocco (2003), I assume that, after default, the agent will move into the rental market and continue to rent from that moment on.

3 Statement of the problem

Consider an agent whose preferences are represented by a constant relative risk aversion utility function (CRRA), and depend on the periodic flow of housing services, H , and other goods, C_t :

$$u(C_t, H) = \frac{(C_t^\beta H^{1-\beta})^{1-\gamma}}{1-\gamma}, \quad (1)$$

where β measures the relative importance of numeraire consumption versus housing services, $\beta \in (0, 1)$, and γ is the curvature parameter and the coefficient of relative risk aversion, $\gamma > 0$. The agent is unconstrained with respect to the owning versus renting choice. He chooses the status that maximizes his expected utility at time zero.

Housing services are provided as a proportion of the housing stock, δH , where $\delta = 1$ for convenience; I also make the assumption that the housing stock H does not depreciate. As a future extension of this paper, it would be interesting to derive H^* , the optimal house size at time zero.¹² Moreover, the agent cannot change the house size regardless of the path of his income and wealth. Thus, I ignore the possibility that the agent can adjust the level of housing services and move to a larger or smaller house.¹³ The agent also chooses consumption of all goods other than housing, C_t , at any time t .

The agent has a finite horizon and discounts the utility function at the constant rate of time preference ρ :

$$\int_0^T e^{-\rho t} \frac{(C_t^\beta H^{1-\beta})^{1-\gamma}}{1-\gamma} dt, \quad (2)$$

where T is the terminal date. The agent also derives utility from terminal wealth W_T , which can be interpreted as bequest motive:

$$V(W, T) = \frac{W_T^{1-\gamma}}{1-\gamma}. \quad (3)$$

The agent is exposed to two sources of uncertainty: (i) the stochastic flow of labor income, L_t ; and (ii) the stochastic house price per square foot, P_t . The agent earns a labor income whose process L_t is governed by a geometric Brownian motion subject to the initial labor

¹²Identifying H^* requires running the numerical algorithm for different levels of housing stock H and verifying at which level of the housing stock, H^* , the agent gets the highest indirect utility.

¹³I make this assumption to make the model numerically tractable. If the agent was allowed to adjust the housing services, the amount he would borrow to buy the bigger or smaller house, could vary. In my model, the default risk premium directly depends on the house size and the mortgage amount. Therefore, I would need to identify the default risk premium for different combinations of housing and mortgage amounts and to verify whether the agent could get a higher indirect utility with a different combination of housing and mortgage amount.

flow L_0 :

$$dL_t = \mu_L(t)L_t dt + \sigma_L L_t dZ_{1,t}, \quad (4)$$

where $\mu_L(t)$ is the drift rate of the income flow, σ_L is the volatility parameter and $Z_{1,t}$ is a standard Wiener process. I introduce a time dependent drift to capture the fact that the drift of an agent's labor income is a function of his age. This assumption is adopted by the HRM literature. Specifically, I set

$$\mu_L(t) = \mu_{L,0} + \mu_{L,1}t, \quad (5)$$

where $\mu_{L,0}$ and $\mu_{L,1}$ are calibrated to capture the hump shape of earnings over the life cycle.

The house price is given and is also governed by a geometric Brownian motion:

$$dP_t = \mu_P P_t dt + \rho_{PL} \sigma_P P_t dZ_{1,t} + \sigma_P \sqrt{1 - \rho_{PL}^2} P_t dZ_{2,t}, \quad (6)$$

where μ_P is the rate of real home price appreciation, σ_P is the volatility parameter, $Z_{1,t}$ and $Z_{2,t}$ are two uncorrelated Wiener processes, and ρ_{PL} is the correlation coefficient between the house price and the labor income processes.

Because there is always the possibility that the agent would default on any loan, my model has the feature that he can never borrow at the risk free rate. The wealth of the agent at time zero, W_0 , corresponds to the dollar amount in the risk free asset, B_0 , and does not take into account the present value of the future uncertain income stream. The risk free asset pays a constant interest rate r and its dynamic is governed by

$$dB_t = rB_t dt. \quad (7)$$

Following the HRM literature, if the agent chooses to rent, he continuously pays a rent that is a constant fraction α , the rental cost, of the current house value, HP_t , exposing himself to the rent fluctuations. The rental cost α is defined as the sum of the user cost, the cost of consuming the house, and the constant rent risk premium λ . The HRM literature assumes that the rent risk premium λ is exogenous and it may have different interpretations. In my model, the rent risk premium λ has to be determined endogenously. Instead of explicitly solving a landlord's problem, as in Sinai and Souleles (2005), I set the rent risk premium such that the agent is indifferent *ex ante* to owning with a mortgage or renting at time zero. This property determines the equilibrium rent when landlords are competitive and have the same horizon and risk aversion as all the agents.

If the agent chooses to own, he can finance the house purchase at time zero, taking a mortgage up to the house value HP_0 . The mortgage, F_t , is a fixed rate, interest only, home

equity line. The coupon rate, i , is the the sum of two components: (i) the risk free rate, r ; and (ii) the default risk premium, κ . In my model, the default risk premium, κ , is determined at the contract's origination. The coupon rate, i , is set equal to the full mortgage rate so that the mortgage is issued at par. Following Cocco (2005) and Yao and Zhang (2005a), I assume that the agent can costlessly readjust the outstanding balance at the same rate i , decreasing or increasing the home equity line balance up to the current house value; this assumption leads to the following *margin requirement* constraint:

$$F_t \leq HP_t. \quad (8)$$

The agent has to pay the coupon payment iF_t and this expense is not tax deductible.¹⁴ Hence, he chooses his optimal numeraire consumption C_t^{om} and can increase or decrease the leverage position. If the agent's net cash flow $L_t - C_t^{om} - iF_t$ at time t is positive, it is paid to decrease the outstanding mortgage balance. Otherwise, if the cash flow $L_t - C_t^{om} - iF_t$ is negative, the lender provides the cash flow to the agent and the outstanding balance increases by the same amount. After signing the contract, I assume that the agent or the lender cannot renegotiate the mortgage coupon, resetting the default risk premium κ .

In my model, the agent has no incentive to lend at the risk free rate r while borrowing at the higher mortgage coupon rate i that includes a default risk premium. The agent can save only after paying off the mortgage balance, $F_t = 0$. Moreover, the assumption simplifies the numerical solution for the problem since the risk free asset position B_t does not need to be treated as a separate state variable when the outstanding balance F_t is positive. This is a mild assumption. Cocco (2005) and Yao and Zhang (2005a) allow the agent to save at the risk free rate, r , while borrowing at coupon rate i at the same time. Nevertheless, their models do not generate simultaneous holdings of a mortgage and savings.

In my model, the owner's problem is divided into two separate problems: owning with a mortgage (a leveraged position) and owning without a mortgage (not a leveraged position). The main difference between the two problems is that, if the owner takes on a mortgage, he is long a default option. The mortgage default can be broken down into two parts: the occurrence of default and the consequences of default. In my model, the two aspects are related.

¹⁴I disregard tax considerations as they are too country-specific. In the United States, the homeownership is substantially subsidized. A owner can deduct the mortgage interest payment from the income tax decreasing the after-tax cost of owning with a mortgage. Then, the owner can deduct the property taxes and the points charged when the mortgage contract is signed. Moreover, the most fundamental subsidy is that homeowners are not taxed on the implicit rent they receive from their housing investment. Jaffee and Quigley (2006) provide a review and taxonomy of U.S. federal housing programs, including direct public expenditures on housing and indirect expenditures through the tax system.

To model the occurrence of default, I follow Leland (1994) by assuming that default is triggered (endogenously) at the stopping time τ . The agent optimally determines at which point to default based on whether defaulting or continuing to pay mortgage coupons provides him the greater expected continuation utility. After signing the contract, the agent can borrow more at the rate i , decreasing contemporaneously his equity position in the house. However, it is not optimal for him to increase the outstanding balance up to the current house value. If he does, he is exposed to the house price fluctuations due to the borrowing mechanism. This is because decreases in house prices may expose the agent to the risk of not satisfying the *margin requirement* constraint: $F_t \leq HP_t$. If the agent meets the *margin requirement* constraint with equality at some date t , he is left with zero wealth. Hence, he is forced to leave the house at that date, and is left with zero housing and numeraire consumption from then on. Therefore, an agent, whose preference are represented by a constant relative risk aversion utility function (CRRA), optimally defaults with a positive equity position in the house.

At default, the agent and the lenders face several consequences. The agent loses a fraction of his wealth: the house is seized and the loss is proportional to the house value at default, ϵHP_τ , due to transaction and legal costs.¹⁵ Then, the agent moves into the rental market with the wealth $W_\tau - \epsilon HP_\tau$.¹⁶ On the other side, the lender incurs a loss proportional to the house value at default ψHP_τ , due to missing coupon payments, delay and legal costs associated with the possession of the collateral. The default risk premium, κ , compensates the lenders for the potential consequences of default. In my model, lenders have rational expectations and take the optimal homeowner default policy as given. Hence, the default premium κ is set such that the lenders are indifferent to the return r on the risk free asset, B_t , or the return i on the mortgage, F_t .

In the next sections, I separately analyze the renter's and the owner's problem (Subsection 3.1 and 3.2 respectively). While the notation $V^r(W_t, L_t, P_t, t)$ is used to indicate the indirect utility when the agent rents, $V^{om}(W_t, L_t, P_t, t)$ and $V^o(W_t, L_t, P_t, t)$ are used to indicate the indirect utility when the agent owns with a mortgage and owns without a mortgage. The mortgage valuation is discussed in Subsection 3.2.2. The notation $M(W_t, L_t, P_t, t)$ is used to

¹⁵The dead weight loss associated with default is also a *stigma*: the agent is forced into the rental market and he loses access to the mortgage market.

¹⁶I do not allow the agent to sell the current house, possibly incurring a transaction cost, liquidating the mortgage balance, and to move into the rental market. Yao and Zhang (2005b) explicitly deal with the two options. In their model, the agent sells the house, incurring a transaction cost, and moves into the rental market when the house value-to-wealth ratio deviates too far from the optimal house value-to-wealth ratio that the agent, as a renter, with identical wealth-to-income ratio would have chosen. The house selling boundary is significantly affected by the mortgage balance. If the loan-to-house value ratio is high enough, the agent chooses to default paying a penalty cost.

indicate the mortgage fair value.

The goal is to solve the owning with a mortgage versus renting problem to identify the default risk premium, κ , and the rent risk premium, λ , such that (i) the agent is indifferent *ex ante* to owning with a mortgage or renting; (ii) the mortgage fair value equals the mortgage face value. The two risk premia depend on the level of wealth W_0 , labor income L_0 , and house price P_0 . Specifically, in order to identify the default risk premium κ and the rent risk premium λ , the following conditions have to be satisfied:

$$V^r(W_0, L_0, P_0, 0) = V^{om}(W_0, L_0, P_0, 0), \quad (9)$$

$$M(W_0, L_0, P_0, 0) = F_0. \quad (10)$$

3.1 Renting

Following the HRM literature, if the agent rents a house of size H , I assume that he continuously pays a fraction α , rental cost, of the current house value HP_t . The agent's wealth, W_t , corresponds to the risk free asset, B_t , and the associated dynamic is:

$$dW_t = dB_t + (L_t - C_t - \alpha HP_t)dt. \quad (11)$$

Given $W_0 \geq 0$, the renter's problem is to choose the numeraire consumption, C_t , to solve:

$$V^r(W_0, L_0, P_0, 0) = \sup_{C_t} \mathbb{E} \left[\int_0^T e^{-\rho t} \frac{(C_t^\beta H^{1-\beta})^{1-\gamma}}{1-\gamma} dt + e^{-\rho T} \frac{W_T^{1-\gamma}}{1-\gamma} \right]. \quad (12)$$

The associated Hamilton-Jacobi-Bellman equation is the following:

$$\begin{aligned} \rho V^r = \sup_{C_t} & \left\{ \frac{(C_t^\beta H^{1-\beta})^{1-\gamma}}{1-\gamma} + (rW_t + L_t - C_t - \alpha HP_t)V_W^r \right\} + \mu_P P_t V_P^r + \mu_L(t) L_t V_L^r + V_t^r \\ & + \frac{\sigma_P^2}{2} P_t^2 V_{P,P}^r + \frac{\sigma_L^2}{2} L_t^2 V_{L,L}^r + \rho_{PL} \sigma_P \sigma_L P_t L_t V_{P,L}^r. \end{aligned} \quad (13)$$

Solving for the first order condition, I obtain the optimal numeraire consumption:

$$C_t^r = \frac{(V_W^r)^{1/(\beta(1-\gamma)-1)}}{\beta}. \quad (14)$$

The optimal numeraire consumption, C_t^r , is subject to the constraint that the wealth position must be non negative at any time t : $W_t \geq 0$. If the agent meets the constraint with equality at some date t , he is forced to leave the house at that date, and is left with zero housing and

numeraire consumption from then on.

Using the homogeneity of $V^r(W_t, L_t, P_t, t)$, I make a change of variables which reduces the problem (12) from three state variables (W_t, L_t, P_t) to two state variables: $x_t = W_t/HP_t$, and $y_t = L_t/HP_t$. As in Grossman and Laroque (1990), the state variable x_t represents the evolution of the owner's total wealth, W_t , divided by the current house value, HP_t , while the state variable y_t represents the evolution of the labor income flow, L_t , divided by HP_t . Define the normalized indirect utility of renting as $v^r(x_t, y_t, t)$. A complete derivation of the problem is provided in Appendix A.

The nature and properties of the solution in the presence of the liquidity constraint need to be numerically investigated. This is also the case for the saving-consumption problem studied in the macroeconomics literature of *buffer-stock* models (see Zeldes (1989), Deaton (1991), Carroll (1997) and Gourinchas and Parker (2002)) and in the HRM literature. The details of the numerical algorithm are provided in Appendix E.

I assume that the rental cost has two components: $\alpha = (r - \mu_P) + \lambda$. The first component, $r - \mu_P$, is the user cost of the house. The second component, λ , is the rent risk premium. I derive the user cost, $r - \mu_P$, assuming that the landlord (i) is risk neutral, (ii) is endowed with enough wealth to afford the house purchase, (iii) is only exposed to the house price fluctuations. Hence, equilibrium in the housing market occurs when the expected cost of owning a house equals that of renting. Equating the *ex ante* utilities of renting and owning:

$$\underbrace{W_0 - \text{E} \left[\int_0^T e^{-r(T-s)} \tilde{\alpha} HP_s ds \right]}_{\text{Renting}} = \underbrace{W_0 - HP_0 + \text{E} [e^{-rT} HP_T]}_{\text{Owning}}, \quad (15)$$

I obtain that $\tilde{\alpha} = (r - \mu_P)^{17}$, where r is the cost of foregone interest that the landlord could have earned on the risk free asset, B_t . It is adjusted to include the offsetting benefit given by the expected capital gain, μ_P , on the house value, HP_t .

If the landlord is risk averse, he charges a rent higher than the user cost to compensate himself for the risk. Therefore, the rent risk premium, λ , has to be positive. In order to derive it, I could explicitly model the landlord problem, assuming that the landlord is a risk averse agent who receives the rent as cash flow, who is borrowing from the lender in order to be the owner of the house, who receives a stochastic labor income flow, who is consuming

¹⁷Alternatively, the user cost can be derived assuming that the current house value, HP_t , is equal to the expected present value of the future rents. The user cost is determined imposing that the expected value of capital gain on house value and rent is equal to the risk free rate over the interval dt :

$$\text{E} \left[\frac{d(HP_t)}{HP_t} + \frac{\alpha HP_t dt}{HP_t} \right] = r dt.$$

the housing services of another house and so on. This approach would add unnecessary complications to the model, requiring the solution of the landlord’s optimal control problem, while not adding to the generality of the problem.

Following Sinai and Souleles (2005), I set the rent risk premium, λ , such that the agent is indifferent *ex ante* to owning with a mortgage or renting. In this way, I endogenously determine the rent risk premium, λ , and the default risk premium, κ , that are intimately related. I assume the rent risk premium is constant after signing the rent contract, implying that the landlord cannot renegotiate the rent contract resetting the rent risk premium.

The HRM literature assumes that the rent risk premium, λ , is exogenous. However, setting the rent risk premium, λ , and the default risk premium, κ , crucially affects the model’s prediction with respect to the owning versus renting decision. To illustrate this point, I provide evidence representing the indirect utility of owning with a mortgage, $V^{om}(W_t, L_t, P_t, t)$, owning without a mortgage, $V^o(W_t, L_t, P_t, t)$, and renting, $V^r(W_t, L_t, P_t, t)$ at time zero. Figure 1 plots the indirect utilities as function of the wealth state variable, W_t .

Insert Figure 1 here

The parameters I assume are reported in Table 1 and will be discussed in Section 5. For convenience, I assume that the agent can own or rent a house which has a price, P_0 , of \$1, and a size, H , of 100 square feet. If the agent’s wealth is less than the house value at time zero, $W_0 < HP_0$ ($F_0 > 0$), the agent has to borrow to buy the house. The space is divided into two main areas: (I) owning without a mortgage ($W_0 \geq HP_0$) and (II) owning with a mortgage ($W_0 < HP_0$).

I start by following Yao and Zhang (2005a) and Van Hemert (2006) and I simply set the default risk premium, κ , to zero implying that the agent borrows at the risk free rate r : $i = r$, $F_t = B_t$. To illustrate the implications of the rent risk premium, λ , Figure 1 plots the renter’s indirect utility for a null rent risk premium, $\lambda = 0$ percent, and for a positive one, $\lambda = 4$ percent. As the plot highlights, the λ parameter is capable of generating large differences in terms of indirect utility. Renting dominates owning when the rent risk premium is zero and the agent has to borrow in order to buy the house. However, if the rent risk premium is equal to 4 per cent, the agent prefers owning to renting. It is not surprising that a risk averse agent prefers renting to owning when the rent is based only on the user cost of housing services: $\lambda = 0$ percent. In this case, the agent can keep the savings as a *buffer-stock* instead of investing it in the house purchase. As a renter, he is exposed to the rent fluctuations due to the house price uncertainty, $(r - \mu_P)HP_t$, but as an owner, he would also be exposed to the house price uncertainty and, because of the mortgage’s contract structure, he wouldn’t be allowed to borrow more than the current house price, $F_t \leq HP_t$.

As Yao and Zhang (2005a) and Van Hemert (2006), I need to increase the rent risk premium, λ , in order to induce the agent to become an owner. Then, the agent is indifferent *ex ante* to owning or renting when he is endowed with enough wealth to be able to buy the house without borrowing. Therefore, exogenously setting the rent risk premium, λ , and the default risk premium, κ , can crucially affect the model's predictions.

As a possible future extension of this paper, it would be interesting to introduce a rent that is constant for the entire agent's life horizon, T . This alternative is not explored by the current HRM literature. In this case, the renter wouldn't be exposed to the rent fluctuations, because he would pay a constant rent for a specific time horizon, and his wealth dynamic would be:

$$dW_t = dB_t + (L_t - C_t - R)dt, \quad (16)$$

where R would be the constant rent that the agent pays for T years. I would define the agent's indirect utility, $V^{rl}(W_t, L_t, t)$, as a function of the wealth, W_t , and of the labor income dynamic, L_t . The constant rent, R , could be determined endogenously and jointly with the rent risk premium, λ , and the default risk premium, κ , such that the agent was indifferent *ex ante* between owning with a mortgage, renting with a continuously adjusting rent, and renting with a constant long term rent. Specifically, in order to identify (i) the default risk premium κ , (ii) the rent risk premium λ , and (iii) the constant rent R , the following conditions would have to be satisfied:¹⁸

$$V^{om}(W_0, L_0, P_0, 0) = V^r(W_0, L_0, P_0, 0) = V^{rl}(W_0, L_0, 0), \quad (17)$$

$$M(W_0, L_0, P_0, 0) = F_0. \quad (18)$$

3.2 Owning

3.2.1 Owner's problem

The dollar wealth amount, W_t , is

$$W_t = HP_t + B_t - F_t \quad \text{where} \quad \begin{cases} F_t \geq 0 & \text{if } B_t = 0; \\ B_t \geq 0 & \text{if } F_t = 0. \end{cases} \quad (19)$$

If the agent takes leverage through the mortgage ($F_t \geq 0$), the net dollar amount wealth, W_t , is the current house value, HP_t , minus the outstanding balance of the mortgage, F_t . If

¹⁸Following Grenadier (1995), I could generalize the model introducing long term contracts with different maturities, obtaining a term structure for rent contracts. Achieving a numerical solution for this problem would be challenging and I am leaving this extension for future research.

he pays off the outstanding balance or he does not take leverage ($B_t \geq 0$), the net dollar amount, W_t , is the sum of the current house value, HP_t , and the risk free asset, B_t .

The associated dynamic of the wealth is:

$$\begin{aligned} dW_t &= HdP_t + dB_t\mathbf{1}_{\{B_t \geq 0\}} - dF_t\mathbf{1}_{\{F_t \geq 0\}} + (L_t - C_t)dt \\ &= [\mu_P HP_t + r(W_t - HP_t)\mathbf{1}_{\{B_t \geq 0\}} - i(HP_t - W_t)\mathbf{1}_{\{F_t \geq 0\}} + L_t - C_t]dt + HP_t d\tilde{Z}_t, \end{aligned} \quad (20)$$

where $d\tilde{Z}_t = \rho_{PL}\sigma_P dZ_{1,t} + \sigma_P\sqrt{1 - \rho_{PL}^2} dZ_{2,t}$. The wealth dynamic takes two different forms, depending on whether or not the agent is currently taking leverage. Therefore, the two equations differ by the terms iF_t , the coupon payment, and rB_t , the return on the risk free asset. The other terms represent respectively the return of housing investment, the labor income flow and the numeraire consumption. The diffusion component represents the house price uncertainty in the homeowner's wealth dynamic; the price uncertainty depends on the current house value, HP_t , on the house price volatility, σ_P , and on the correlation coefficient between house price and labor income, ρ_{PL} .¹⁹

If the agent takes leverage, $F_t \geq 0$, the owner's problem is:

$$\begin{aligned} V^{om}(W_0, L_0, P_0, 0) &= \sup_{C_t, \tau} \mathbb{E} \left[\int_0^T e^{-\rho t} \frac{(C_t^\beta H^{1-\beta})^{1-\gamma}}{1-\gamma} dt \right. \\ &\quad \left. + \underbrace{\mathbf{1}_{\{\tau \leq T\}} e^{-\rho \tau} V^r(W_\tau - \epsilon HP_\tau, L_\tau, P_\tau, \tau)}_{\text{Default}} + \mathbf{1}_{\{\tau > T\}} e^{-\rho T} \frac{W_T^{1-\gamma}}{1-\gamma} \right], \end{aligned} \quad (21)$$

where $V^r(W_\tau - \epsilon HP_\tau, L_\tau, P_\tau, \tau)$ is the indirect utility at default. Because the agent is forced into the rental market, the indirect utility at default is denoted by V^r .

The associated Hamilton-Jacobi-Bellman equation is:

$$\begin{aligned} \rho V^{om} &= \sup_{C_t, \tau} \left\{ \frac{1}{1-\gamma} (C_t^\beta H^{1-\beta})^{1-\gamma} + (\mu_P HP_t - i(HP_t - W_t) + L_t - C_t) V_W^{om} \right\} \\ &\quad + \mu_P P_t V_P^{om} + \mu_L(t) L_t V_L^{om} + V_t^{om} + \frac{\sigma_P^2}{2} H^2 P_t^2 V_{W,W}^{om} + \frac{\sigma_P^2}{2} P_t^2 V_{P,P}^{om} + \frac{\sigma_L^2}{2} L_t^2 V_{L,L}^{om} \\ &\quad + \sigma_P^2 H P_t^2 V_{W,P}^{om} + \rho_{PL} \sigma_P \sigma_L H P_t L_t V_{W,L}^{om} + \rho_{PL} \sigma_P \sigma_L P_t L_t V_{P,L}^{om}, \end{aligned} \quad (22)$$

over a region where the agent takes the mortgage in order to buy the house and he has not yet defaulted.

¹⁹The owner's wealth equation is similar to the one that arises in a portfolio choice problem where the agent is not allowed to trade the underlying stock, as in Kahl et al. (2003), or where the asset is not traded, as in Henderson and Hobson (2002).

The agent chooses the numeraire consumption, C_t , and the default time, τ , to maximize his expected utility at time zero. Solving for the first order condition, I obtain the optimal numeraire consumption C_t :

$$C_t^{om} = \frac{(V_W^{om})^{1/(\beta(1-\gamma)-1)}}{\beta}. \quad (23)$$

The optimal numeraire consumption, C_t^{om} , is subject to the constraint that the net wealth position must be positive at any time t :

$$W_t = HP_t - F_t \geq 0, \quad (24)$$

due to the assumption that the outstanding balance of the mortgage cannot exceed the current house value: $F_t \leq HP_t$. If the agent meets the constraint with equality at some date t , he is forced to sell his house at that date, and is left with a zero housing and numeraire consumption from then on.

The optimal default time τ crucially depends on the house price volatility σ_P . To understand this point, assume that the house price, P_t , is constant ($\mu_P = 0$, $\sigma_P = 0$). Rewrite the renter's and the owner's wealth dynamics as

$$dW_t = (rW_t + L_t - C_t - \underbrace{(r + \lambda)}_{\alpha} HP)dt, \quad (25)$$

$$dW_t = [r(W_t - HP)\mathbf{1}_{\{B_t \geq 0\}} - \underbrace{(r + \kappa)}_i (HP - W_t)\mathbf{1}_{\{F_t \geq 0\}} + L_t - C_t]dt. \quad (26)$$

In this case, the agent is not exposed to the house price fluctuations either as an owner or as a renter, therefore, the owner, who takes leverage to finance the house purchase, has no incentive to default. As a consequence, there are no rent and default risk premia: $\lambda = 0$, $\kappa = 0$. The wealth dynamics (25) and (26) are equivalent and therefore the owner's problem corresponds to the renter's problem.

Income uncertainty also affects the optimal default time τ . Specifically, the wealth value $W_\tau = HP_\tau - F_\tau$, at which default occurs, increases (i) as the riskiness of labor income, σ_L , increases, (ii) as the correlation coefficient, ρ_{PL} , between the house price, P_t , and the labor income, L_t , increases. An increase in the labor income volatility, σ_L , induces the agent to default with a higher level of net wealth and then to move into the rental market with a higher *buffer-stock* of savings. A positive correlation coefficient, ρ_{PL} , increases the riskiness of owning with a mortgage: in expectation, the labor income moves in the same direction as the house price. Therefore, the agent is more exposed to the risk of meeting the *margin requirement* constraint. At the same time, the renting status is more attractive,

because a positive correlation between rent and income provides a natural hedge against rent fluctuations, allowing the renter to smooth numeraire consumption.

When the agent defaults, he and the lender incur dead weight losses, ϵ and ψ respectively. An increase in dead weight loss, ϵ , induces the agent to move into the rental market *ex ante* with a higher level of wealth. On the other side, an increase in dead weight loss, ψ , raises the default risk premium, κ , that the lender *ex ante* charges.

The owner's problem with a mortgage is solved using the homogeneity property of $V^{om}(W_t, L_t, P_t, t)$ and making a change of variables which reduces the problem (21) from three state variables (W_t, L_t, P_t) to two state variables. I adopt the same change of variables as for the renter's problem: $x_t = W_t/HP_t$, and $y_t = L_t/HP_t$. Define the normalized indirect utility of owning with a mortgage as $v^{om}(x_t, y_t, t)$. A complete derivation of the problem with the associated boundary conditions is provided in Appendix C. The owner's problem without a mortgage is discussed in Appendix B.

3.2.2 Contingent claims evaluation of the mortgage

In this section, I derive the mortgage fair value $M(W_t^*, L_t, P_t, t)$. The mortgage is a fixed rate, interest only, home equity line, where the coupon rate i is the sum of two components: (i) the risk free rate, r , and (ii) the default risk premium, κ . I assume that the agent can costlessly readjust the outstanding balance decreasing or increasing the home equity line balance up to the current house value. After signing the contract, I assume that the agent or the lender cannot renegotiate the mortgage. Specifically, the default risk premium is determined at the contract's origination and it is constant.

I assume that the lenders' market has rational expectations and takes the optimal default strategy of the owner as given. The market is also competitive, implying that the coupon rate, i , is set equal to the full mortgage rate so that the mortgage is issued at par. Moreover, the lender adopts a contingent claims approach to fairly price the mortgage contract. The solution approach is based on option pricing analysis. Traditionally, in the option pricing literature, prices of contingent claims are based on arbitrage arguments. However, such an approach requires assumptions about the liquidity of underlying assets. In the case of mortgage contracts, such arbitrage arguments are questionable. One underlying asset is the house that is subject to substantial transaction costs and the inability to be sold short. The second one is labor income for which contingent claims contracts are not observed in reality for reasons of moral hazard and adverse selection. Alternatively, an equilibrium approach relaxes the tradability assumptions needed for arbitrage pricing, although an appropriate model must be chosen. For simplicity, I assume the risk neutrality, so that all assets are

priced to yield an expected rate of return equal to the risk free rate, r .²⁰ This restrictive assumption can be relaxed by adjusting the drift rates to account for a risk premium in the manner of Cox and Ross (1976).

Consider the instantaneous return on $M(W_t^*, L_t, P_t, t)$ over a region in which the agent is taking leverage and the mortgage has not yet defaulted. The value of the mortgage depends on W_t^* , which is the owner's wealth accounting for the optimal numeraire consumption C_t^{om} , the owner's labor income, L_t , and the house price, P_t . The lender (a) receives a cash inflow due to the coupon payment of iF_t over the interval dt ; (b) receives the dollar amount $(L_t - C_t^{om} - iF_t) > 0$ when the agent decreases the outstanding balance, or provides the dollar amount $(L_t - C_t^{om} - iF_t) < 0$ when the agent increases the outstanding balance. The net instantaneous cash flow from the owner to the lender is the labor income minus the optimal numeraire consumption

$$\left(\underbrace{iF_t}_{\text{Coupon}} + \underbrace{L_t - C_t^{om} - iF_t}_{\Delta \text{Balance}} \right) dt = (L_t - C_t^{om}) dt,$$

which can be positive or negative. The total expected return on M_t per unit of time, μ_M , is defined as

$$\mathbb{E} \left[\frac{dM_t}{M_t} + \frac{(L_t - C_t^{om}) dt}{M_t} \right] = \mu_M dt. \quad (27)$$

Setting the expected return, μ_M , to the risk free rate, r , and simplifying, yields the following equilibrium partial differential equation:

$$\begin{aligned} rM &= L_t - C_t^{om} + (\mu_P H P_t - i(H P_t - W_t^*)) + L_t - C_t^{om}) M_{W^*} + \mu_P P_t M_P \\ &+ \mu_L(t) L_t M_L + M_t + \frac{\sigma_P^2}{2} H^2 P_t^2 M_{W^*, W^*} + \frac{\sigma_P^2}{2} P_t^2 M_{P, P} + \frac{\sigma_L^2}{2} L_t^2 M_{L, L} \\ &+ H P_t^2 \sigma_P^2 M_{W^*, P} + \rho_{PL} \sigma_P \sigma_L H P_t L_t M_{W^*, L} + \rho_{PL} \sigma_P \sigma_L P_t L_t M_{P, L}. \end{aligned} \quad (28)$$

Using the homogeneity property of $M(W_t^*, L_t, P_t, t)$, I make a change of variables which enables me to reduce the partial differential equation (28) from three state variables (W_t^*, L_t, P_t) to two state variables. I adopt the same change of variables as for the renter's and owner's problem: $x_t^* = W_t^*/H P_t$, and $y_t = L_t/H P_t$. Define the mortgage fair value as $m(x_t^*, y_t, t)$. A complete derivation of the mortgage evaluation with the associated boundary conditions is provided in Appendix D.

In the next section, I briefly present the numerical scheme adopted to solve the problems of both the owner and the renter as well as the mortgage valuation.

²⁰The assumption is the same made by Grenadier (1996) who proposes a unified framework for determining the equilibrium credit spread on commercial leases subject to default risk.

4 Numerical approach

The problems of owner and renter as well as the mortgage valuation cannot be solved analytically. I adopt a finite difference scheme following Kushner and Dupuis (2002).²¹ Appendix E provides an exposition of the numerical approach.

The owning with a mortgage (or without a mortgage) versus renting problem is the solution of a system of four partial differential equations: (22) (owning with a mortgage), (A-8) (owning without a mortgage), (13) (renting) and (28) (mortgage valuation). For the equations (22), (A-8) and (13), I obtain the optimal numeraire consumption of the owner (with and without a mortgage) and the renter respectively. To obtain the mortgage fair value $M(W_t^*, L_t, P_t, t)$, I need to consider the default strategy of the owner and the value of the collateral at default. At the terminal date, T , the mortgage face value is $F_T = HP_T - W_T$: the agent sells the house and liquidates the outstanding balance. Prior to maturity, the mortgage value satisfies the partial differential equation (28). The default strategy is derived solving simultaneously the partial differential equations (22) (owning with a mortgage) and (13) (renting), at any time step Δt , and verifying whether the agent does or does not exercise the default option. Conditional on the optimal exercise policy, the lender receives (or provides) the flow $L_t - C_t^{om}$ or recovers the outstanding balance at net of the dead weight loss: $F_\tau - \psi HP_\tau = HP_\tau - W_\tau - \psi HP_\tau$. Hence, the partial differential equations (22) and (28) have to satisfy the associated default boundary conditions.

I normalize the problem and solve the numerical scheme with respect to the state variables $x_t (W_t/HP_t)$ and $y_t (L_t/HP_t)$.

5 Model calibration

The following numerical analysis is based on the parameters listed in Table 1. Following Campbell and Cocco (2003), I assume a relative risk aversion, γ , of 3 and a rate of time preference, ρ , of 2 percent. The parameter β measures how much the agent values housing consumption relative to other goods. Housing preference, β , is set at 0.3 consistent with the average proportion of household housing expenditure in the United States.²² I assume that the risk free rate, r , is equal to 3 percent. The labor income flow evolves according to the dynamics described in equation (4). The parameter $\mu_L(t)$ captures the drift of the income process. I calibrate the coefficients of the drift to generate the typical income pattern due

²¹The approach has been adopted in portfolio choice problems by Fitzpatrick and Fleming (1991), Hindy et al. (1997), Munk (2000), Øksendal and Sulem (2007) and Budhiraja and Ross (2007); in capital structure problems by Titman et al. (2004).

²²While Cocco (2005) sets β at 0.1, Yao and Zhang (2005a) assume that β equals 0.2.

to the predictable growth component described in the HRM literature. I consider a twenty-year-old agent at $t = 0$, who will work until retirement at age 65. I set $\mu_{L,0}$ at 6 percent and $\mu_{L,1}$ at -0.24 percent, implying a deterministic labor income profile.

Insert Figure 2 here

Figure 2 depicts the deterministic life cycle's profile along with two samples of simulated paths for the labor income as defined in equation (4). I set the volatility parameter, σ_L , at 11.30 percent, close to the value the HRM literature typically assumes.²³ As Campbell and Cocco (2003), I set the house price appreciation parameter, μ_P , at 1.65 percent.

For the baseline analysis, I set the correlation between income and house price process, ρ_{PL} , at 10 percent.²⁴ I assume a standard deviation of house price, σ_P , of 10 per cent.²⁵ I set both the owner's and the lender's default costs, ϵ and ψ , at 10 percent for the baseline analysis.

6 Results

To derive the optimal housing choice in terms of renting versus owning, along with the default policy, I solve the model assuming a certain default risk premium, κ , and a certain rent risk premium, λ , and I identify the points x_0^* and y_0^* such that (i) the agent is indifferent *ex ante* to owning with a mortgage or renting; and (ii) the mortgage fair value is equal to the mortgage face value. An alternative approach would be to pin down the default risk premium, κ , and the rent risk premium, λ , given the levels of state variables x_t (W_t/HP_t) and y_t (L_t/HP_t) at time zero. However, this second approach would imply a longer and more complex numerical iteration of the algorithm.

For the following results, I set the default risk premium, κ , at 0.5 percent (50 basis points) and the rent risk premium, λ , at 3 percent. Specifically, in order to identify x_0^* and y_0^* , the

²³Campbell and Cocco (2003) decompose the labor shocks into a persistent and idiosyncratic component; they estimate a standard deviation of 2 and 14.10 percent for the persistent and idiosyncratic component. Cocco (2005) splits his sample into groups according to the level of education and he estimates the standard deviation of the idiosyncratic component for each group. The standard deviation varies from 13.10 to 13.60 per cent. Yao and Zhang (2005a) assume a standard deviation of the labor income shocks at 13 per cent.

²⁴Campbell and Cocco (2003) and Cocco (2005) assume that house price and labor income are perfectly positive correlated to economize on state variables. Yao and Zhang (2005a) set the correlation at 20 per cent.

²⁵Campbell and Cocco (2003) estimate a standard deviation parameter of the house price of 11.5 per cent; Cocco (2005) estimates a standard deviation of 6.2 per cent. Yao and Zhang (2005a) set the standard deviation at 10 per cent.

following conditions have to be satisfied:

$$v^r(x_0^*, y_0^*, 0) = v^{om}(x_0^*, y_0^*, 0), \quad (29)$$

$$m(x_0^*, y_0^*, 0) = (1 - x_0^*), \quad (30)$$

where $(1 - x_0^*)$ corresponds to the mortgage face value, F_0 .

Insert Figure 3 here

Figure 3 graphically represents the numerical solution of the model. The space (x_0, y_0) span all the possible combinations of values of x_t and y_t at time zero. The space is divided into three areas: (I) owning without a mortgage ($x_0 \geq 1$); (II) owning with a mortgage ($x_0 < 1$); and (III) renting. The existence and location of all these three areas depend on my parameters assumptions; the case shown involves all three regions and is therefore the most interesting. An alternative could be a case in which renting and owning without a mortgage are the only possible scenarios; however, this case does not offer interesting insights.

Because I assume that the rent risk premium is substantial, $\lambda = 3$ per cent, owning without a mortgage dominates renting for every combination of x_0 and y_0 , $v^o > v^r$. This prediction is reflected in Figure 3, where the owning without a mortgage area (I) only borders the owning with a mortgage area (II), indicating that renting is not a valuable alternative to owning without a mortgage.

The owning with a mortgage area (II) borders the renting area (III) in which renting dominates owning with a mortgage, $v^r > v^{om}$. Area (III) is delimited by the renting frontier, which is the locus of points of x_0 and y_0 for which the agent is indifferent to owning with a mortgage or renting. The frontier is decreasing in x_0 , making an agent endowed with a relevant wealth, x_0 , and low income, y_0 , (or, vice versa, low wealth x_0 and high income y_0) indifferent to owning with a mortgage or renting.

For each default and rent risk premia pair, I identify the unique initial level of wealth, labor income, and house price. Given the house value, the level of wealth determines the mortgage amount and also the agent's equity share in the house. In Figure 3, the star (*) corresponds to (x_0^*, y_0^*) such that these are $x_0^* = 0.375$ and $y_0^* = 0.101$, implying that the agent is borrowing 62.5 per cent of the house value ($1 - x_0^* = 0.625$). The agent has to provide a substantial downpayment in order to be indifferent *ex ante* to owning a house with a mortgage or renting the same house. With a high level of income relative to the house value and a significant rent risk premium, the agent is not willing to take a highly leveraged position. In fact, the agent provides a substantial downpayment in order to reduce the probability of meeting the *margin requirement* constraint.

Insert Figure 4 here

Figure 4 graphically represents the numerical solution to the optimal default problem, at a certain time t , during the life of the agent. In this case, the agent is an owner with a mortgage (area (II)), who, for certain combinations of x_t and y_t , is able to pay off the mortgage completely and move into area (I) of the graph. Area (I) and (II) are delimited by the locus of points of x_t and y_t for which the agent's wealth is the house.

For certain other combinations of x_t and y_t , the agent optimally defaults and moves into the renting area (III). Area (II) and (III) are delimited by the default boundary, which is the locus of points x_t^d and y_t^d for which the agent is indifferent to owning with a mortgage or defaulting. The default boundary is decreasing in x_t^d , making an agent endowed with a relevant wealth, x_t^d , and low income, y_t^d , (or, vice versa, low wealth x_t^d and high income y_t^d) indifferent to owning with a mortgage or defaulting. I obtain an interesting, limiting, case if the agent's dead weight loss is set to zero: $\epsilon = 0$. In this case, the default boundary coincides with the renting frontier identified at time zero. This is an intuitive result, as the cost associated with default delays the exercise of the option, moving the default boundary to the left of the renting frontier.

I consider the effect of the house price volatility, σ_P , labor income volatility, σ_L , and correlation between labor income and house price, ρ_{PL} , on the default and rent risk premia. I increase (or decrease) one of these parameters and identify the rent risk premium, λ , and the default risk premium, κ , such that x_0^* and y_0^* are still those of the baseline analysis ($x_0^* = 0.375$ and $y_0^* = 0.101$). If I set the house price volatility, σ_P , at 8 percent (instead of 10 percent), I obtain the following pair: $\kappa = 0.42$ per cent and $\lambda = 2.83$ per cent. Because a lower house price volatility, σ_P , decreases the agent's exposure to house prices fluctuations through the *margin requirement* constraint, his propensity to default decreases. On the other side, the agent, as a renter, also benefits of a decrease in the house price volatility becoming less exposed to rent fluctuations. Therefore, the owner's default option becomes more valuable. The overall effect is that (i) the lender *ex ante* charges a lower default risk premium of 0.42 per cent, instead of 0.50 per cent, and (ii) the landlord demands a lower rent risk premium of 2.83 per cent, instead of 3 per cent.

Labor income volatility, σ_L , also affects the default and rent risk premia. If I decrease the labor income volatility from 11.30 per cent to 9 per cent, the agent's probability of default decreases. In fact, the lender *ex ante* charges a lower default risk premium of 0.43 per cent instead of 0.50 per cent. The rent risk premium slightly decreases from 3 per cent to 2.96 per cent, suggesting that the labor income volatility does not affect the rent risk premium the landlord charges. The result is consistent with Campbell and Cocco (2003). Although their default risk premium does not account for variations in the labor income

volatility, they find that less risky income significantly decreases the probability of default. Their model predicts that the default option becomes less valuable for an adjustable rate mortgage (ARM) than a fixed rate mortgage (FRM), because the risk of an ARM is the income risk of short-term variability in the payments that are required each month. In my model, the effect of labor income volatility is softened by the borrowing mechanism. If the agent faces negative income shocks, he can costlessly increase the outstanding balance of the mortgage making the borrowing constraints less severe. However, the impact of negative income shocks depends on the house value and the correlation between labor income and house prices. In fact, borrowing constraints bind in states of the world with low income and low house prices.

Finally, I consider the effect of the correlation between house prices and labor income on the two premia. Because Campbell and Cocco (2003) assume that house prices and labor income are perfectly positively correlated, they cannot investigate how this correlation affects the agent's propensity to default. First, I increase the correlation from 10 per cent to 20 per cent. From the landlord's prospective, I obtain a lower rent risk premium, λ , of 2.87 per cent (instead of 3 per cent). The higher the correlation between labor income and house prices, the more attractive the status of renting is: in expectation, the rent co-moves more with the labor income smoothing the numeraire consumption. From the lender's prospective, I obtain a higher default risk premium, κ , of 0.61 per cent (instead of 0.50 per cent). The higher the correlation between labor income and house prices, the riskier the status of owning with a mortgage is. Moreover, the lower the rent risk premium (from 3 per cent to 2.87 per cent), the higher the probability of default. Therefore, the lender *ex ante* sets a higher default risk premium (from 0.50 per cent to 0.61 per cent). If I decrease the correlation coefficient from 10 per cent to 0 per cent, I identify a higher rent risk premium, λ , and a lower default risk premium, κ . Specifically, I obtain a rent risk premium, λ , of 3.12 per cent (instead of 3 per cent) and a default risk premium, κ , of 0.37 per cent (instead of 0.50 per cent).

7 Conclusions

This paper provides a unified framework of housing choice in the presence of a lenders' and landlords' market. A main contribution of this paper is to derive jointly the default and the rent risk premia in a dynamic setting where the agent is exposed to both stochastic labor income and stochastic house prices. The default and rent risk premia are jointly determined based on expected losses, due to default on the mortgage, and by the risk of house price fluctuations on the rental market.

An important implication of the model is that default and rent risk premia are intimately

related. Because the agent moves into the rental market after defaulting, the rent risk premium directly affects the agent's propensity to default. On the other side, the lender *ex ante* charges a default risk premium that accounts for the agent's default strategy and the dead weight loss he may incur. A novel implication is that labor income and borrowing constraints directly affect the default risk premium. The model predicts that the default risk premium is increasing with respect to volatility of both labor income and house prices as well as with the correlation between them.

Another prediction of the model is that the agent has to be able to provide a substantial downpayment in order for owning with a mortgage to dominate renting. Because the outstanding balance of the mortgage can not exceed the current house value (the *margin requirement* constraint), the agent, being exposed to the house price fluctuations, is not willing to take a highly leveraged position.²⁶ As a result, a lower house price volatility induces the agent to provide less downpayment, increasing his leveraged position and paying the same default risk premium to the lender. In my model, the default risk premium and the downpayment are intimately and inversely related. The higher the downpayment, the lower the default risk premium the lender charges *ex ante*.

Finally, a future contribution will be to analyze how the labor income and house price characteristics determine the housing the agent is willing to consume. In my model, the level of housing services would endogenously determine the equity share (the level of leverage) the agent is willing to take.

²⁶Because a mortgage contract, in reality, has no *margin requirement* constraint, a future extension of this work aims to relax this assumption.

A Renting

I use the homogeneity property to reduce the problem with three state variables (W_t, L_t, P_t) to a problem with two state variables: $x_t = W_t/HP_t$ and $y_t = L_t/HP_t$. I obtain

$$V^r(W_t, L_t, P_t, t) = H^{1-\gamma} P_t^{\beta(1-\gamma)} v^r \left(\frac{W_t}{HP_t}, \frac{L_t}{HP_t}, t \right) = H^{1-\gamma} P_t^{\beta(1-\gamma)} v^r(x_t, y_t, t). \quad (\text{A-1})$$

I introduce the normalized consumption rate $c_t = C_t/HP_t$. The rescaled Hamilton-Bellman-Jacobi equation is:

$$\begin{aligned} \tilde{\rho} v^r = \sup_{c_t} & \left\{ \frac{c_t^{\beta(1-\gamma)}}{1-\gamma} + (-c_t + (r - \mu_P)x_t + y_t - \alpha + x_t(\beta(\gamma - 1) + 1)\sigma_P^2) v_x^r \right\} \\ & + (\mu_L(t) - \mu_P + (\beta(\gamma - 1) + 1)(\sigma_P - \rho_{PL}\sigma_L)\sigma_P) y_t v_y^r + v_t^r \\ & + \frac{\sigma_P^2}{2} x_t^2 v_{x,x}^r + \frac{(\sigma_P^2 - 2\rho_{PL}\sigma_P\sigma_L + \sigma_L^2)}{2} y_t^2 v_{y,y}^r + (\sigma_P - \rho_{PL}\sigma_L)\sigma_P y_t x_t v_{x,y}^r, \end{aligned} \quad (\text{A-2})$$

where $\tilde{\rho} = \frac{1}{2} (\beta(\gamma - 1) ((\beta(\gamma - 1) + 1)\sigma_P^2 - 2\mu_P) - 2\rho)$.

Solving for the first order condition, I obtain

$$c_t^r = \frac{(v_x^r)^{1/(\beta(1-\gamma)-1)}}{\beta}. \quad (\text{A-3})$$

The consumption rate c_t^r is subject to the constraint that the wealth position must be non negative at any time t : $x_t \geq 0$. The boundary condition of the indirect utility at time T is

$$v^r(x_T, T) = \frac{x_T^{1-\gamma}}{1-\gamma}. \quad (\text{A-4})$$

B Owning without a mortgage

Given $W_0 \geq 0$, the owner's problem is to choose, C_t , to solve:

$$V^o(W_0, L_0, P_0, 0) = \sup_{C_t} \mathbb{E} \left[\int_0^T e^{-\rho t} \frac{(C_t^\beta H^{1-\beta})^{1-\gamma}}{1-\gamma} dt + e^{-\rho T} \frac{W_T^{1-\gamma}}{1-\gamma} \right]. \quad (\text{A-5})$$

The associated Hamilton-Jacobi-Bellman equation is:

$$\begin{aligned} \rho V^o = \sup_{C_t} & \left\{ \frac{1}{1-\gamma} (C_t^\beta H^{1-\beta})^{1-\gamma} + (rW_t + (\mu_P - r)HP_t + L_t - C_t) V_W^o \right\} + \mu_P P_t V_P^o + \mu_L(t) L_t V_L^o + V_t^o \\ & + \frac{\sigma_P^2}{2} H^2 P_t^2 V_{W,W}^o + \frac{\sigma_P^2}{2} P_t^2 V_{P,P}^o + \frac{\sigma_L^2}{2} L_t^2 V_{L,L}^o \\ & + \sigma_P^2 H P_t^2 V_{W,P}^o + \rho_{PL}\sigma_P\sigma_L H P_t L_t V_{W,L}^o + \rho_{PL}\sigma_P\sigma_L P_t L_t V_{P,L}^o. \end{aligned} \quad (\text{A-6})$$

I use the homogeneity property to reduce the problem with three state variables (W_t, L_t, P_t) to a problem with two state variables: $x_t = W_t/HP_t$, and $y_t = L_t/HP_t$. I obtain

$$V^o(W_t, L_t, P_t, t) = H^{1-\gamma} P_t^{\beta(1-\gamma)} v^o \left(\frac{W_t}{HP_t}, \frac{L_t}{HP_t}, t \right) = H^{1-\gamma} P_t^{\beta(1-\gamma)} v^o(x_t, y_t, t). \quad (\text{A-7})$$

I introduce the normalized numeraire consumption rate $c_t = C_t/HP_t$. The rescaled Hamilton-Bellman-Jacobi equation is

$$\begin{aligned} \tilde{\rho}v^o = \sup_{c_t} & \left\{ \frac{c_t^{\beta(1-\gamma)}}{1-\gamma} + (-c_t + (r - \mu_P)(x_t - 1) + y_t + (x_t - 1)((\beta(\gamma - 1) + 1)\sigma_P^2)) v_x^o \right\} \\ & + (\mu_L(t) - \mu_P + (\beta(\gamma - 1) + 1)(\sigma_P - \rho_{PL}\sigma_L)\sigma_P)y_t v_y^o + v_t^o \\ & + \frac{\sigma_P^2}{2}(x_t - 1)^2 v_{x,x}^o + \frac{(\sigma_P^2 - 2\rho_{PL}\sigma_P\sigma_L + \sigma_L^2)}{2}y_t^2 v_{y,y}^o + (\sigma_P - \rho_{PL}\sigma_L)\sigma_P y_t(x_t - 1)v_{x,y}^o, \end{aligned} \quad (\text{A-8})$$

where $\tilde{\rho} = \frac{1}{2}(\beta(\gamma - 1)((\beta(\gamma - 1) + 1)\sigma_P^2 - 2\mu_P) - 2\rho)$.

Solving for the first order condition, I obtain

$$c_t^o = \frac{(v_x^o)^{1/(\beta(1-\gamma)-1)}}{\beta}. \quad (\text{A-9})$$

The boundary condition for the agent's indirect utility at time T is normalized to

$$v^o(x_T, T) = \frac{x_T^{1-\gamma}}{1-\gamma}. \quad (\text{A-10})$$

C Owning with a mortgage

I use the homogeneity property to reduce the problem with three state variables (W_t, L_t, P_t) to a problem with two state variables: $x_t = W_t/HP_t$, and $y_t = L_t/HP_t$. I obtain

$$V^{om}(W_t, L_t, P_t, t) = H^{1-\gamma}P_t^{\beta(1-\gamma)}v^{om}\left(\frac{W_t}{HP_t}, \frac{L_t}{HP_t}, t\right) = H^{1-\gamma}P_t^{\beta(1-\gamma)}v^{om}(x_t, y_t, t). \quad (\text{A-11})$$

I introduce the normalized numeraire consumption rate $c_t = C_t/HP_t$. The rescaled Hamilton-Bellman-Jacobi equation is

$$\begin{aligned} \tilde{\rho}v^{om} = \sup_{c_t, \tau} & \left\{ \frac{c_t^{\beta(1-\gamma)}}{1-\gamma} + (-c_t + (i - \mu_P)(x_t - 1) + y_t + (x_t - 1)((\beta(\gamma - 1) + 1)\sigma_P^2)) v_x^{om} \right\} \\ & + (\mu_L(t) - \mu_P + (\beta(\gamma - 1) + 1)(\sigma_P - \rho_{PL}\sigma_L)\sigma_P)y_t v_y^{om} + v_t^{om} + \frac{\sigma_P^2}{2}(x_t - 1)^2 v_{x,x}^{om} \\ & + \frac{(\sigma_P^2 - 2\rho_{PL}\sigma_P\sigma_L + \sigma_L^2)}{2}y_t^2 v_{y,y}^{om} + (\sigma_P - \rho_{PL}\sigma_L)\sigma_P y_t(x_t - 1)v_{x,y}^{om} \quad \text{if } x_t \leq 1, \end{aligned} \quad (\text{A-12})$$

where $\tilde{\rho} = \frac{1}{2}(\beta(\gamma - 1)((\beta(\gamma - 1) + 1)\sigma_P^2 - 2\mu_P) - 2\rho)$. Solving for the first order condition, I obtain the optimal numeraire consumption rate:

$$c_t^{om} = \frac{(v_x^{om})^{1/(\beta(1-\gamma)-1)}}{\beta}. \quad (\text{A-13})$$

The consumption rate c_t^{om} is subject to the constraint that the net wealth position must be positive at any time t : $x_t \geq 0$.

The boundary condition for the agent's indirect utility at time T , becomes:

$$v^{om}(x_T, T) = \frac{x_T^{1-\gamma}}{1-\gamma}. \quad (\text{A-14})$$

At $x_t = 1$ ($W_t = HP_t$), the boundary condition (value matching) is:

$$v^{om}(1, y_t, t) = v^o(1, y_t, t). \quad (\text{A-15})$$

The agent pays off the mortgage balance becoming the full owner of the house. Hence, the indirect utilities values of v^{om} and v^o must be matched. The associated smooth pasting conditions, with respect to x_t and y_t , are:

$$v_x^{om}(1, y_t, t) = v_x^o(1, y_t, t), \quad (\text{A-16})$$

$$v_y^{om}(1, y_t, t) = v_y^o(1, y_t, t). \quad (\text{A-17})$$

At default, the owner's boundary condition (value matching) is:

$$v^{om}(x_\tau, y_\tau, \tau) = v^r(x_\tau - \epsilon, y_\tau, \tau). \quad (\text{A-18})$$

At default, the agent incurs a dead weight loss that is proportional to the house value at default, where $x_\tau - \epsilon$ corresponds to $W_\tau - \epsilon HP_\tau$. Because the agent is forced into the rental market, the indirect utility at default is denoted by v^r . The associated smooth pasting conditions, with respect to x_t and y_t , are:

$$v_x^{om}(x_\tau, y_\tau, \tau) = v_x^r(x_\tau - \epsilon, y_\tau, \tau), \quad (\text{A-19})$$

$$v_y^{om}(x_\tau, y_\tau, \tau) = v_y^r(x_\tau - \epsilon, y_\tau, \tau). \quad (\text{A-20})$$

D Mortgage valuation

In this section, I derive the mortgage fair value $M(W_t^*, L_t, P_t, t)$. I use the homogeneity properties of the value function to reduce the problem with three state variables (W_t^*, L_t, P_t) to one with two state variables $x_t^* = W_t^*/HP_t$, $y_t = L_t/HP_t$ since

$$M(W_t^*, L_t, P_t, t) = HP_t \times m\left(\frac{W_t^*}{HP_t}, \frac{L_t}{HP_t}, t\right) = HP_t \times m(x_t^*, y_t, t). \quad (\text{A-21})$$

Simplifying, I obtain the following partial differential equation:

$$\begin{aligned} (r - \mu_P)m &= y_t - c_t^{om} + ((i - \mu_P)(x_t^* - 1) + y_t - c_t^{om})m_{x^*} + (\mu_L(t) - \mu_P)y_t m_y + m_t + \\ &+ \frac{\sigma_P^2}{2}(x_t^* - 1)^2 m_{x^*, x^*} + \frac{(\sigma_P^2 - 2\rho_{PL}\sigma_P\sigma_L + \sigma_L^2)}{2} y_t^2 m_{y, y} \\ &+ (\sigma_P - \rho_{PL}\sigma_L)\sigma_P y_t (x_t^* - 1) m_{x^*, y} \quad \text{if } x_t^* \leq 1. \end{aligned} \quad (\text{A-22})$$

The boundary conditions are:

$$m(x_T^*, T) = 1 - x_T^*, \quad (\text{A-23})$$

$$m(1, y_t, t) = 0. \quad (\text{A-24})$$

The boundary condition (A-23) is the terminal condition, reflecting the residual balance due at the terminal date T , where $1 - x_T^*$ corresponds to $HP_T - W_T^*$. The boundary condition (A-24) arises when the agent pays off the mortgage balance, that corresponds to $M_t = 0$ ($F_t = 0$).

At default, the boundary condition is:

$$m(x_\tau^*, y_\tau, t) = 1 - x_\tau^* - \psi. \quad (\text{A-25})$$

The lender recovers the outstanding balance $1 - x_\tau^*$ at net of the loss ψ , where $1 - x_\tau^* - \psi$ corresponds to $HP_\tau - W_\tau^* - \psi HP_\tau$.

E Numerical algorithm

The covariance matrix associated with the partial differential equations (A-8) (owning without a mortgage), (A-12) (owning with a mortgage) and (A-22) (mortgage valuation) depends on the state variables x_t and y_t :

$$\frac{1}{2} \begin{pmatrix} \sigma_P^2 (x_t - 1)^2 & (\sigma_P - \rho_{PL}\sigma_L)\sigma_P y_t (x_t - 1) \\ (\sigma_P - \rho_{PL}\sigma_L)\sigma_P y_t (x_t - 1) & (\sigma_P^2 - 2\rho_{PL}\sigma_P\sigma_L + \sigma_L^2)y_t^2 \end{pmatrix}. \quad (\text{A-26})$$

The covariance structure lacks a key property required by the numerical approximation scheme of Kushner and Dupuis (2002). The property required is that the diagonal terms of the matrix are bigger in absolute value than off diagonal terms for all values of x_t and y_t :

$$\sigma_P^2 (x_t - 1)^2 - |(\sigma_P - \rho_{PL}\sigma_L)\sigma_P y_t (x_t - 1)| \geq 0, \quad (\text{A-27})$$

$$(\sigma_P^2 - 2\rho_{PL}\sigma_P\sigma_L + \sigma_L^2)y_t^2 - |(\sigma_P - \rho_{PL}\sigma_L)\sigma_P y_t (x_t - 1)| \geq 0. \quad (\text{A-28})$$

Moreover, the degeneracy is strong because the variance matrix is not positive definite for some x_t and y_t . Instead, the degeneracy is less severe for the renting case (equation (A-2)). The strong degeneracy of the variance matrix is due to the model set up: the house price value, P_t , enters in the wealth state variable, W_t , and is its self a state variable of the problem. This can be remedied by a more careful choice of the approximation. I adopt the following strategy to get rid of the strong degeneracy. I make the following transformation $\xi(x_t, y_t) = \theta_{11}x_t + \theta_{12} \log y_t$ and $\eta(x_t, y_t) = \theta_{21}x_t + \theta_{22} \log y_t$.²⁷ The log transformation of the state variable y_t makes the variance matrix dependent only on x_t . The value functions of the problems have the relation $v^o(x_t, y_t, t) = \mathcal{V}^o(\xi_t, \eta_t, t)$ (owning without a mortgage), $v^{om}(x_t, y_t, t) = \mathcal{V}^{om}(\xi_t, \eta_t, t)$ (owning with a mortgage), $v^r(x_t, y_t, t) = \mathcal{V}^r(\xi_t, \eta_t, t)$ (renting) and $m(x_t^*, y_t, t) = \mathcal{M}(\xi_t^*, \eta_t^*, t)$ (mortgage valuation). I use the following substitutions for renting:

$$v_x^r = \mathcal{V}_\xi^r \frac{d\xi}{dx} + \mathcal{V}_\eta^r \frac{d\eta}{dx}, \quad v_y^r = \mathcal{V}_\xi^r \frac{d\xi}{dy} + \mathcal{V}_\eta^r \frac{d\eta}{dy}, \quad (\text{A-29})$$

$$v_{xx}^r = \mathcal{V}_{\xi\xi}^r \left(\frac{d\xi}{dx} \right)^2 + \mathcal{V}_\xi^r \frac{d^2\xi}{dx^2} + \mathcal{V}_{\eta\eta}^r \left(\frac{d\eta}{dx} \right)^2 + \mathcal{V}_\eta^r \frac{d^2\eta}{dx^2} + 2\mathcal{V}_{\eta\xi}^r \frac{d\xi}{dx} \frac{d\eta}{dx}, \quad (\text{A-30})$$

²⁷See Tavella and Randall (2000) and Knupp and Steinberg (1993).

$$v_{yy}^r = \mathcal{V}_{\xi\xi}^r \left(\frac{d\xi}{dy} \right)^2 + \mathcal{V}_{\xi}^r \frac{d^2\xi}{dy^2} + \mathcal{V}_{\eta\eta}^r \left(\frac{d\eta}{dy} \right)^2 + \mathcal{V}_{\eta}^r \frac{d^2\eta}{dy^2} + 2\mathcal{V}_{\eta\xi}^r \frac{d\xi}{dy} \frac{d\eta}{dy}, \quad (\text{A-31})$$

$$v_{xy}^r = \mathcal{V}_{\xi\xi}^r \frac{d\xi}{dx} \frac{d\xi}{dy} + \mathcal{V}_{\eta\eta}^r \frac{d\eta}{dx} \frac{d\eta}{dy} + \mathcal{V}_{\xi\eta}^r \frac{d\xi}{dx} \frac{d\eta}{dy} + \mathcal{V}_{\xi\eta}^r \frac{d\eta}{dx} \frac{d\xi}{dy} + \mathcal{V}_{\xi}^r \frac{d^2\xi}{dxdy} + \mathcal{V}_{\eta}^r \frac{d^2\eta}{dxdy}. \quad (\text{A-32})$$

The same substitutions apply for the owning with a mortgage, \mathcal{V}^{om} , owning without a mortgage, \mathcal{V}^o , and the mortgage valuation, \mathcal{M} . Moreover, I pick values of $\theta_{11}, \theta_{12}, \theta_{21}, \theta_{22}$ to (1) obtain an increasing and monotone grid in the original space, $x(\xi_t, \eta_t)$ and $y(\xi_t, \eta_t)$; (2) make the variance covariance matrix positive definite for all $x(\xi_t, \eta_t)$. If the diagonal property lacks at the node (i, j, k) , the Markov probabilities are adjusted to derive a local consistent approximation (see Kushner and Dupuis (2002), Kushner (1999)). The new and original coordinates are related by a set of equations

$$x_t = \frac{\eta_t \theta_{12} - \xi_t \theta_{22}}{\theta_{12} \theta_{21} - \theta_{11} \theta_{22}} \quad \text{and} \quad y_t = \exp \left(-\frac{\eta_t \theta_{11} - \xi_t \theta_{21}}{\theta_{12} \theta_{21} - \theta_{11} \theta_{22}} \right). \quad (\text{A-33})$$

I introduce a sequence of grid for the state variable ξ_t, η_t and t with $\Delta\xi, \Delta\eta$ and Δt as step sizes respectively. Let $\mathcal{V}^r(\xi_t, \eta_t, t) = \mathcal{V}^r(i\Delta\xi, j\Delta\eta, k\Delta t) = \mathcal{V}_{i,j,k}^r$, where $0 < i \leq N_i, 0 < j \leq N_j$ and $0 \leq k \leq N_k$. I work backward in time to solve the partial differential equation (A-2). I use the boundary condition at date T

$$\mathcal{V}_{i,j,N_k}^r = \frac{x(i\Delta\xi, j\Delta\eta)^{1-\gamma}}{1-\gamma}$$

to initialize the recursion a time $T - \Delta t$ adopting a Jacobi value iteration algorithm (see Kushner and Dupuis (2002)). The finite difference approximation, together with the boundary specifications, can be expressed in terms of Markov probabilities. The chain can possibly move from the current state (i, j, k) only to one of the seven neighboring states: $(i, j, k+1), (i+1, j, k), (i-1, j, k), (i, j+1, k), (i, j-1, k)$ and $(i+1, j+1, k), (i-1, j-1, k)$ if the non diagonal covariance term is positive or $(i+1, j-1, k), (i-1, j+1, k)$ if the non diagonal covariance term is negative. The numerical scheme combines a value iteration and policy iteration algorithm. Values of \mathcal{V}^r are determined using a guess of the optimal consumption policy and using the equations provided by Kushner and Dupuis (2002) to obtain the transition probabilities (value iteration). Hence, I adopt a policy iteration algorithm updating the numeraire consumption calculating the first order condition. Then, the algorithm recurses again through value iteration and policy iteration until $\|\mathcal{V}_{i,j,k}^{r,h+1} - \mathcal{V}_{i,j,k}^{r,h}\| < \text{tolerance}$. I adopt the same strategy to solve the owning with a mortgage, \mathcal{V}^{om} , and owning without a mortgage, \mathcal{V}^o , problems with the appropriate boundary conditions. The mortgage valuation, \mathcal{M} , requires only the value iteration algorithm and the appropriate boundary conditions.

References

- Budhiraja, A. and Ross, K. (2007). Convergent numerical scheme for singular stochastic control with state constraints in a portfolio selection problem. *SIAM Journal on Control and Optimization*, 45(6):2169–2206.
- Campbell, J. Y. and Cocco, J. F. (2003). Household risk management and optimal mortgage choice. *The Quarterly Journal of Economics*, 118(4):1449–1494.
- Carroll, C. (1997). Buffer-stock saving and the life cycle/permanent income hypothesis. *The Quarterly Journal of Economics*, 112(1):1–55.
- Cocco, J. (2005). Portfolio choice in the presence of housing. *The Review of Financial Studies*, 18(2):535–567.
- Cox, J. C. and Ross, S. A. (1976). The valuation of options for alternative stochastic processes. *The Journal of Financial Economics*, 3(1-2):145–166.
- Davidoff, T. (2006). Labor income, housing prices and homeownership. *Journal of Urban Economics*, 59(2):209–235.
- Deaton, A. (1991). Saving and liquidity constraints. *Econometrica*, 59(5):1221–1248.
- Downing, C., Stanton, R., and Wallace, N. (2005). An empirical test of a two-factor mortgage valuation model: How much do house prices matter? *Real Estate Economics*, 33(4):681–710.
- Fitzpatrick, B. and Fleming, W. (1991). Numerical methods for optimal investment-consumption. *Mathematics of Operations Research*, 16(4):823–841.
- Gourinchas, P. and Parker, J. (2002). Consumption over the life cycle. *Econometrica*, 70(1):47–89.
- Grenadier, S. (1995). Valuing lease contracts: a real-option approach. *The Journal of Financial Economics*, 38(3):297–331.
- Grenadier, S. (1996). Leasing and credit risk. *The Journal of Financial Economics*, 42(3):333–364.
- Grossman, S. and Laroque, G. (1990). Asset pricing and optimal portfolio choice in the presence of illiquid durable consumption goods. *Econometrica*, 58(1):25–51.

- Henderson, J. V. and Ioannides, Y. M. (1983). A model of housing tenure choice. *American Economic Review*, 73(1):89–113.
- Henderson, V. and Hobson, D. G. (2002). Real options with constant relative risk aversion. *Journal of Economic Dynamics and Control*, 27(2-3):329–355.
- Henderson, V. and Hobson, D. G. (2007). Utility indifference pricing - an overview. Indifference pricing. Princeton University Press, New Jersey.
- Hindy, A., Huang, C. F., and Zhu, S. (1997). Numerical analysis of a free-boundary singular control problem in financial economics. *Journal of Economic Dynamics and Control*, 21(2-3):297–327.
- Jaffee, D. M. and Quigley, J. M. (2006). Housing subsidies and homeowners: What role for government-sponsored enterprises? *Brookings Papers on Urban Affairs*, 6(1):103–150.
- Kahl, M., Liu, J., and Longstaff, F. (2003). Paper millionaires: how valuable is stock to a stockholder who is restricted from selling it? *The Journal of Financial Economics*, 11(67):385–410.
- Kau, J. and Keenan, D. (1995). An overview of the option-theoretic pricing of mortgages. *Journal of Housing Research*, 6(2):217–244.
- Knupp, P. and Steinberg, S. (1993). *Fundamentals of Grid Generation*. Boca Raton, CRC Press.
- Kushner, H. (1999). Consistency issues for numerical methods for variance control with applications to optimization in finance. *IEEE Transactions on Automatic Control*, 44(12):2283–2296.
- Kushner, J. K. and Dupuis, P. (2002). *Numerical Methods for Stochastic Control Problems in Continuous Time*. Springer Verlag.
- Leland, H. (1994). Corporate debt value, bond covenants, and optimal capital structure. *The Journal of Finance*, 49(4):1213–1252.
- Munk, C. (2000). Optimal consumption/investment policies with undiversifiable income risk and liquidity constraints. *Journal of Economic Dynamics and Control*, 24(9):1315–1343.
- Øksendal, B. and Sulem, A. (2007). *Applied Stochastic Control of Jump Diffusions*. Springer-Verlag, Berlin, second edition.

- Ortalo-Magné, F. and Rady, S. (2002). Tenure choice and the riskiness of non-housing consumption. *Journal of Housing Economics*, 11:266–279.
- Sinai, T. and Souleles, N. (2005). Owner occupied housing as a hedge against rent risk. *The Quarterly Journal of Economics*, 120(2):763–789.
- Tavella, D. and Randall, C. (2000). *Pricing Financial Instruments: The Finite Difference Method*. New York: Wiley.
- Titman, S., Tompaidis, S., and Tsyplakov, S. (2004). Market imperfections, investment flexibility, and default spreads. *The Journal of Finance*, 59(1):165–205.
- Van Hemert, O. (2006). Life-cycle housing and portfolio choice with bond markets. Working paper, New York University, Stern Business School.
- Yao, R. and Zhang, H. H. (2005a). Optimal consumption and portfolio choices with risky housing and borrowing constraints. *The Review of Financial Studies*, 18(1):197–239.
- Yao, R. and Zhang, H. H. (2005b). Optimal life-cycle asset allocation with housing as collateral. Working paper, Baruch College, Zicklin School of Business.
- Zeldes, S. (1989). Optimal consumption with stochastic income: deviations from certainty equivalence. *The Quarterly Journal of Economics*, 104(2):275–298.

Variable	Symbol	Value
Relative risk aversion	γ	3
House flow services	β	0.30
Time preference	ρ	0.02
Risk free rate	r	0.03
Income drift (<i>I</i>)	$\mu_{L,0}$	0.06
Income drift (<i>II</i>)	$\mu_{L,1}$	-0.0024
Income volatility	σ_L	0.113
House price drift	μ_P	0.0165
House price volatility	σ_P	0.10
House price - income correlation	ρ_{PL}	0.10
Owner's deadweight loss	ϵ	0.10
Lender's deadweight loss	ψ	0.10
Time horizon (years)	T	10

Table 1: Parameter values.

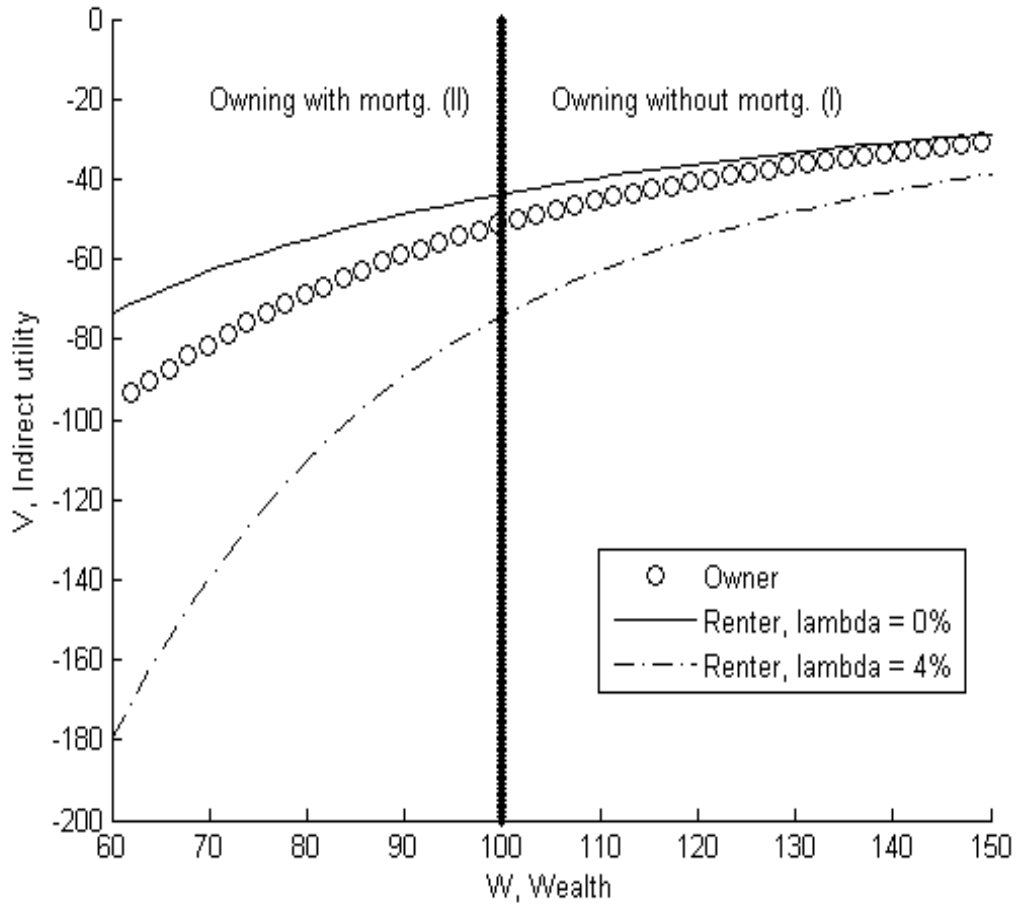


Figure 1: The figure displays the indirect utility of owning with a mortgage, V^{om} , owning without a mortgage, V^o , and renting, V^r , as function of the wealth state variable W , at time 0. The parameters are reported in Table 1. The agent can own or rent a house whose price P_0 , is \$1 and size H is 100 square feet. The space is divided into two main areas: (I) owning without a mortgage ($B \geq 0$) and (II) owning with a mortgage ($F \geq 0$). The renting indirect utility, V^r , is represented with two different rent risk premium: $\lambda = 0$ percent (continuous line) and $\lambda = 4$ percent (dotted line).

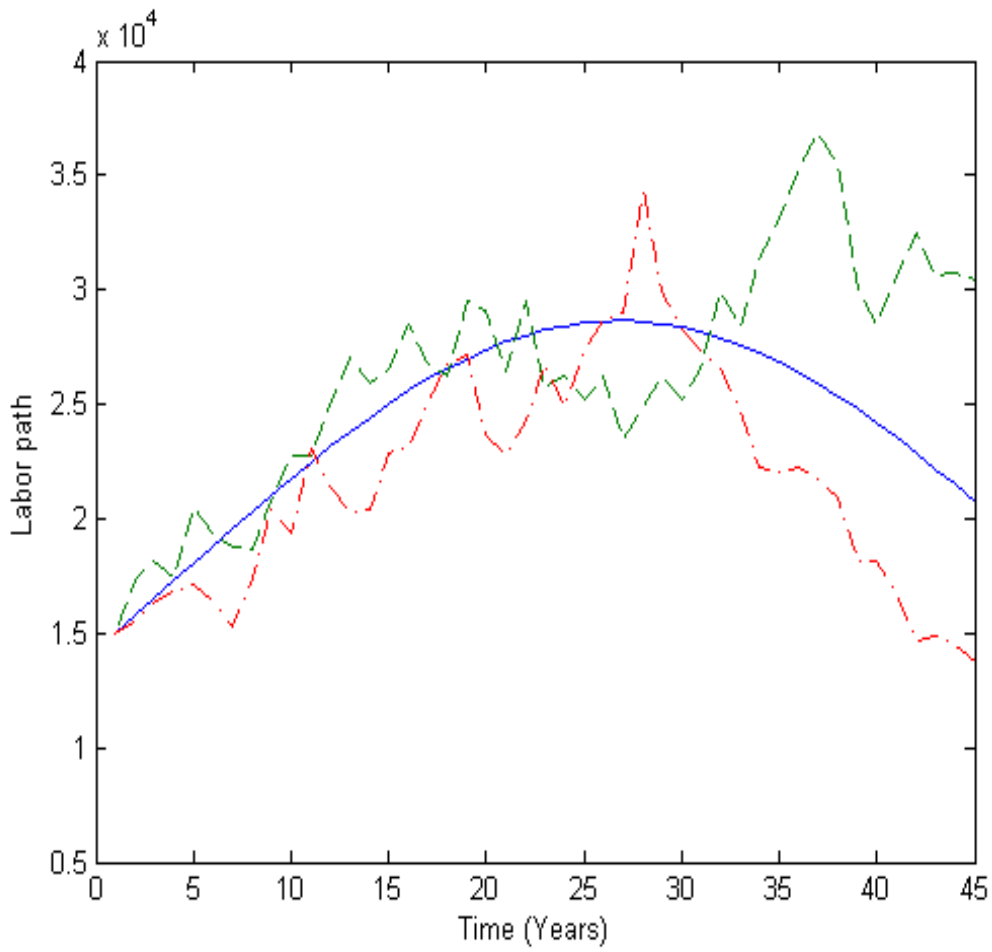


Figure 2: The figure depicts the deterministic life cycle's profile along with two simulated paths for the labor income as defined in equation (4). The agent enters in the job market at age 20, earning an annual income of \$15,000, and retires at age 65.

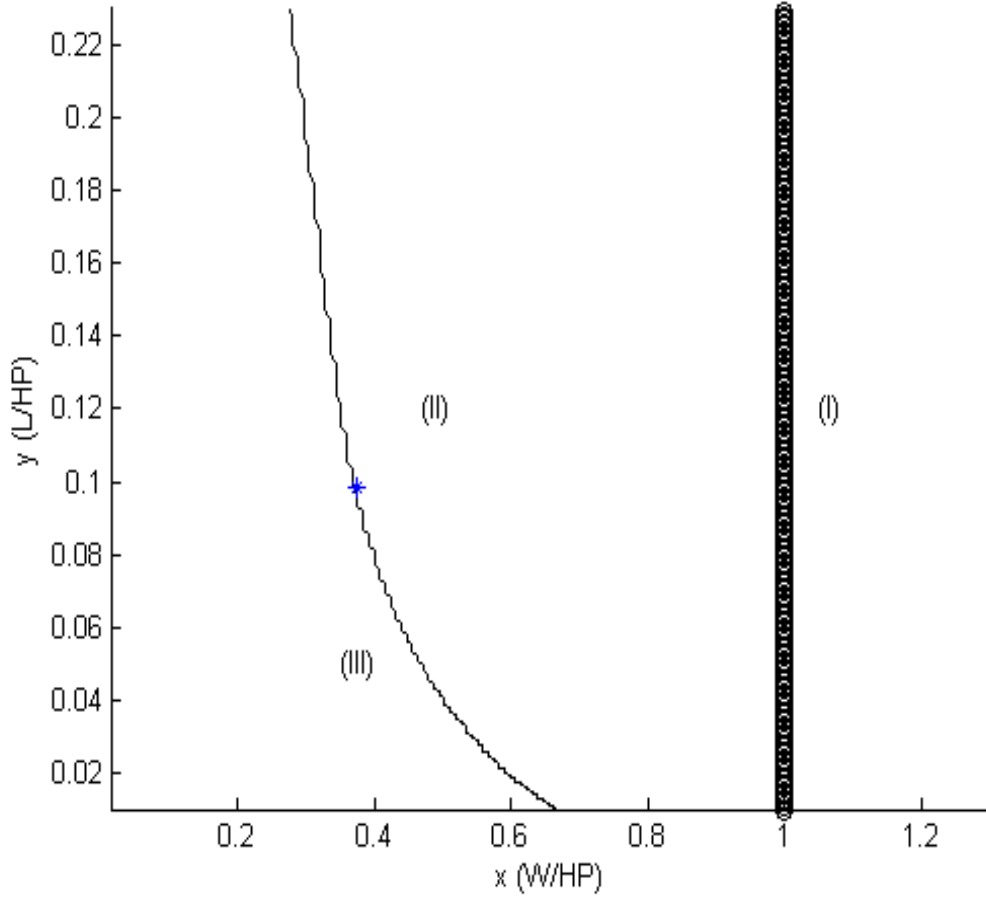


Figure 3: The figure graphically represents the numerical solution of the model. The space (x_0, y_0) span all the possible combinations of values of x and y at time zero. The space is divided into three areas: (I) owning without a mortgage ($x_0 \geq 1$); (II) owning with a mortgage ($x_0 < 1$); and (III) renting. The existence of all these three areas depends on my parameters assumptions. Because I assume that the rent risk premium is substantial, $\lambda = 3$ per cent, owning without a mortgage dominates renting for every combination of x_0 and y_0 , $v^o > v^r$: the owning without a mortgage area (I) only borders the owning with a mortgage area (II), indicating that renting is not a valuable alternative to owning without a mortgage. The owning with a mortgage area (II) borders the renting area (III) in which renting dominates owning with a mortgage, $v^r > v^{om}$. Area (III) is delimited by the renting frontier, which is the locus of points of x_0 and y_0 for which the agent is indifferent *ex ante* to owning with a mortgage or renting. Given a default risk premium, κ , set at 0.50 percent, the pair (x_0^*, y_0^*) identifies where the agent is indifferent *ex ante* to owning with a mortgage or renting; and the mortgage fair value is equal the mortgage face value. The points x_0^* and y_0^* are identified by the * point. Specifically, these points are $x_0^* = 0.375$ and $y_0^* = 0.101$ implying that the agent is borrowing $1 - x_0^* = 0.625$.

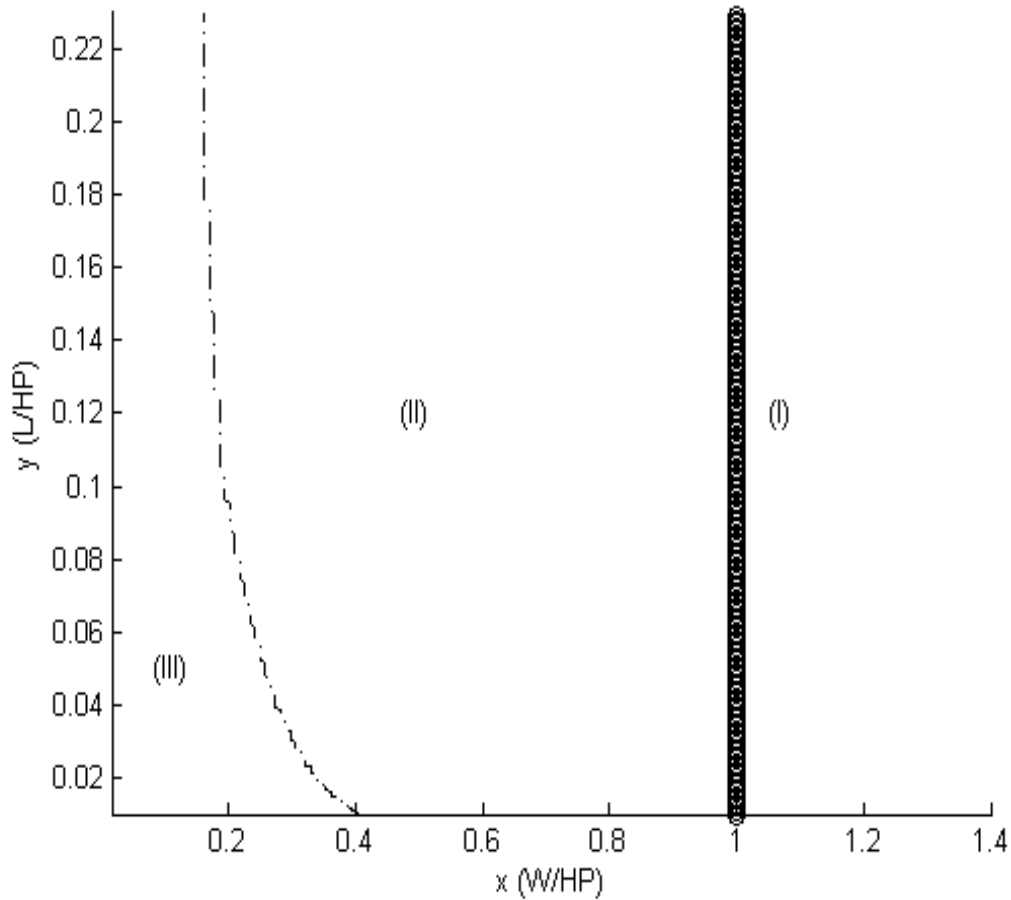


Figure 4: The figure represents the numerical solution to the optimal default problem. The agent is an owner with a mortgage (area (II)), who, for certain combinations of x_t and y_t , is able to pay off the mortgage completely and move into area (I) of the graph. Area (I) and (II) are delimited by the locus of points of x_t and y_t for which the agent's wealth is the house. For certain other combinations of x_t and y_t , the agent optimally defaults incurring a dead weight loss ϵ and moves into the renting area (III). Area (II) and (III) are delimited by the default boundary, which is the locus of points x_t^d and y_t^d for which the agent is indifferent to owning with a mortgage or defaulting. The default boundary is decreasing in x_t^d , making an agent endowed with a relevant wealth x_t^d and low income y_t^d (or, vice versa, low wealth x_t^d and high income y_t^d) indifferent to owning with a mortgage or defaulting.