

BENDHEIM LECTURES IN FINANCE
PRINCETON UNIVERSITY

STRUCTURAL MODELS IN CORPORATE FINANCE

LECTURE 2:

A New Structural Model

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September 2006
Revision 3 December 31, 2006

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LECTURE 1 noted some **empirical shortcomings of traditional structural models:**

—*Underestimation of credit spreads*

- >> Quite severe for investment grade debt
- >> Particularly for short-maturity debt (of any rating)

—*Underestimation of short-term default probabilities*

- >> This can't be explained by a nondefault (or “liquidity”) factor

—Thus, these models are likely to offer poor advice on capital structure.

- **A Problem:** *assuming a pure diffusion process for firm value!*
 - Spreads *and* default rates $\rightarrow 0$ as $t \rightarrow 0$. (see Lando (2004), pp. 14-15).
- This has led to *alternatives* for traditional (diffusion) structural models
 - >> Relating default probabilities to “Distance to default” measures via proprietary data (Moody’s KMV)
 - >> Using reduced form models (“pure” jumps)
- These provide credit analysis, but don’t offer any advice on capital structure

IN THIS LECTURE, we introduce the a *structural model* that includes a *jump-diffusion* process of firm value, and a *debt liquidity premium**. . . .both elements are required to fit credit spread and default data.

The model

- Provides closed form solutions for bond, equity, and firm values, and for the (endogenously determined) optimal default boundary
- Can determine default probabilities as well as credit spreads
- Can be used to determine optimal capital structure
- Can be implemented with a simple Excel program (soon available)
- Provides a good fit to historical data for short- and long-term credit spreads and default probabilities, for both high yield and investment-grade debt.

*Perhaps this is better termed a debt “illiquidity premium.” Formally, the premium reflects the *non-default* component of spreads over the riskless rate (Treasury bonds). It may have a tax-related component (see Elton & Gruber (2001)) as well as a liquidity component.

- This is certainly not the first model to consider jumps in the value process
...previous models with jumps have been developed to study

Options (Merton (1976), Cox and Ross (1976), others)

Credit risk (Zhou (2001), Duffie and Lando (2001), Hilberink & Rogers (HR 2002),
Giesecke & Goldberg (2003), Chen & Kou (CK 2005))

Regime changes (Hackbarth, Miao & Morellec (HMM 2006))

- But these jump models typically require ***numerical analysis*** for logarithmic or single/double exponential jumps, and rarely offer closed form solutions.*
- No previous structural approaches (to my knowledge) have jointly considered jumps and a liquidity premium.

* HMM and CK are exceptions; see a further discussion in Appendix A.

- We consider a very simple mixed jump-diffusion process for firm value :

$$\begin{aligned} \frac{dV}{V} &= (r - \delta + \lambda k)dt + \sigma dW \text{ with probability } (1 - \lambda dt) \\ &= -k \text{ with probability } \lambda dt, \quad 0 \leq k \leq 1 \end{aligned}$$

This process has expectation

$$\begin{aligned} E\left(\frac{dV}{V}\right) &= (r - \delta + \lambda k)dt - k\lambda dt \\ &= (r - \delta)dt \end{aligned}$$

i.e. the *total expected return (including payouts) equals the riskfree rate.*

- But the *diffusion part* now has drift $g \equiv (r - \delta + \lambda k)$ to compensate for the possible jump with intensity λ to value $(1 - k)V$.
- Volatility of diffusion $\sigma = (\sigma_L^2 - \lambda k^2)^{0.5}$
(keeping long-horizon total volatility σ_L constant)

- We allow both λ and k to be parameters.
 - They will be chosen to match *observed default probabilities at short maturities*, and (jointly with diffusion default costs α) to be consistent with *observed recovery rates*.
- The jump here represents a relatively rare “disaster”,
 - The firm suddenly loses a large fraction of its value and *liquidates* (Enron, Refco?)
 - k includes default costs* (firm is always liquidated because of jump; liquidation would occur whether firm has debt or not—contrast CK)
 - Note that unlike pure diffusion models, the recovery rate is random, since V is random when a jump occurs
- Are jumps “rare”? Collin-Dufresne, Goldstein, Helwege (CGH, 2003):

“In practice, very few firms ‘jump’ to default. Indeed, since 1937, we are aware of only four firms that have defaulted on a bond which had an investment grade rating from Moody’s.”

* This is a modeling choice, not a necessity. Extensions are straightforward.

- Discriminating jump rates and diffusion parameters empirically is possible (Ait-Sahalia, (2002), (2004), survey (2006)).
 - >> Index options may provide insights (Cremers et al., 2005), but jumps in an aggregated index are likely to be much more frequent than jumps in a single component's value.
- We don't estimate the firm value process—just look at *consequences if there were a rare jump* on debt values, default rates.
 - >> Observed spreads and default rates can be explained by an assumption of such jumps—*similar to “Dark matter”*??

Valuation with Rare Jumps to Liquidation

- Again using risk-neutral expectations, and recalling that
 - The probability at any future time t of a jump to $(1 - k)V(t)$ is λdt , and
 - The cumulative probability of no jump before time t is $e^{-\lambda t}$

- We assume (without loss of generality) the current time $t = 0$ and asset value is V .
- For the exponentially-declining debt model in Lecture 1, we approximate the value of cash flows to debt at $t = 0$ by

$$\begin{aligned}
 D = & \int_0^{\infty} e^{-rt} \{e^{-mt} (C + mP)\} (1 - F) e^{-\lambda t} ds \\
 & + \int_0^{\infty} e^{-rt} \{e^{-mt} (1 - \alpha) V_B\} f e^{-\lambda t} ds \\
 & + \int_0^{\infty} e^{-rt} \{e^{-mt} (1 - k) e^{gt} V\} (1 - F) \lambda e^{-\lambda t} ds
 \end{aligned} \tag{1}$$

- The first line in (1) is discounted promised cash flows to debtholders given no default (diffusion or jump), recalling debt is retired at rate m .
- The second line is the expectation of discounted recovery when the value diffuses to default (with no prior jump)
- The last line is the (approximate) discounted value of debt's claim to liquidation value when the firm first jumps to default (see Appendix B).

—When $\lambda = 0$ that this reduces to the pure diffusion model in Lecture 1.

Recalling the result (see also equation (3) in Lecture 1) that

$$\int_0^{\infty} e^{-zt} f(t; V, V_B) dt = \left(\frac{V}{V_B} \right)^{-y(z)} \quad \text{where} \quad (2)$$

$$y(z) = \frac{(g - .5\sigma^2) + ((g - .5\sigma^2)^2 + 2z\sigma^2)^{0.5}}{\sigma^2}$$

and integrating (1) by parts gives

Debt Value:

$$D = \frac{C + mP}{z_1} \left(1 - \left(\frac{V}{V_B} \right)^{-y(z_1)} \right) + (1 - \alpha)V_B \left(\frac{V}{V_B} \right)^{-y(z_1)} + \frac{\lambda(1-k)V}{z_2} \left(1 - \left(\frac{V}{V_B} \right)^{-y(z_2)} \right) \quad (3)$$

$$z_1 = r + m + \lambda$$

where

$$z_2 = z_1 - g \quad (4)$$

Total Firm Value:

The value v of total cash flows is (unlevered) firm value V , plus the value of tax savings less diffusion default costs:*

$$v = V + TS - DC \quad (5)$$

where tax savings TS and default costs DC are given by

$$TS = \int_t^{\infty} e^{-rt} \tau C (1 - F) e^{-\lambda t} ds = \frac{\tau C}{r + \lambda} \left(1 - \left(\frac{V}{V_B} \right)^{-y(z_3)} \right)$$

$$DC = \alpha V_B \int_0^{\infty} e^{-rt} f e^{-\lambda t} dt = \alpha V_B \left(\frac{V}{V_B} \right)^{-y(z_3)}$$

where $z_3 = r + \lambda$ (6)

*Recall that k , the fraction of firm value V lost in a jump to liquidation, includes default costs.

We assume that these costs will be incurred whether the firm is levered or not. The alternative that additional default costs are incurred if the firm is levered has a negligible effect on credit spreads for the parameters considered, but may reduce optimal leverage.

Equity value:

The value of equity is the total firm value less the value of debt:

$$E = v - D \quad (7)$$

Optimal default boundary:

As discussed in Lecture 1, we assume that default is chosen optimally by equity holders. This implies that the *optimal default value* V_B satisfies the smooth-pasting condition $\partial E / \partial \mathcal{N} |_{V=V_B} = 0$, which in turn implies

$$V_B = \frac{\frac{(C + mP)y(z_1)}{z_1} + \frac{\lambda(1-k)V y(z_2)}{z_2} - \frac{\tau C y(z_3)}{z_3}}{1 + (1 - \alpha)y(z_1) + \alpha y(z_3)} \quad (8)$$

- V_B can be substituted into equations (3), (5), and (6) to give *closed form expressions* for ***bond value, equity value, and total firm value*** as functions of
 - > Debt coupon C , principal P , and maturity $T = 1/m$
 - > Risk (diffusion σ and jump intensity λ)
 - > Expected growth rate firm value $g (= r - \delta + \lambda k)$
 - > Default costs and jump-loss fraction (α, k)
 - > The riskfree rate of interest r
- The coupon C is set so that the bonds initially sell at par value ($D = P$ at $t = 0$)
- Formulas reduce to the exponential model in Lecture 1 when there is no jump risk ($\lambda = 0$).
- To predict ***default probabilities*** (but *not* credit spreads), we need to know:

Is there a jump risk premium?

—i.e., is there a difference between the risk neutral jump intensity λ , and the “real” (under the physical measure) intensity γ of a jump?

—**Yes**, if jump risk is imperfectly diversifiable.

—Measure by ratio $H = \gamma/\lambda$: smaller ratio \rightarrow larger jump risk premium.

—Given λ , the risk premium doesn't affect *pricing* (spreads), but it must be known to determine the *probability of default* γ .

- CGH (2003) show that ***jump risk will command a risk premium*** if:

—Multiple firms can default simultaneously, or

—Default of one firm can increase default intensities of others.

—We assume a jump risk premium, but don't know to need to know cause

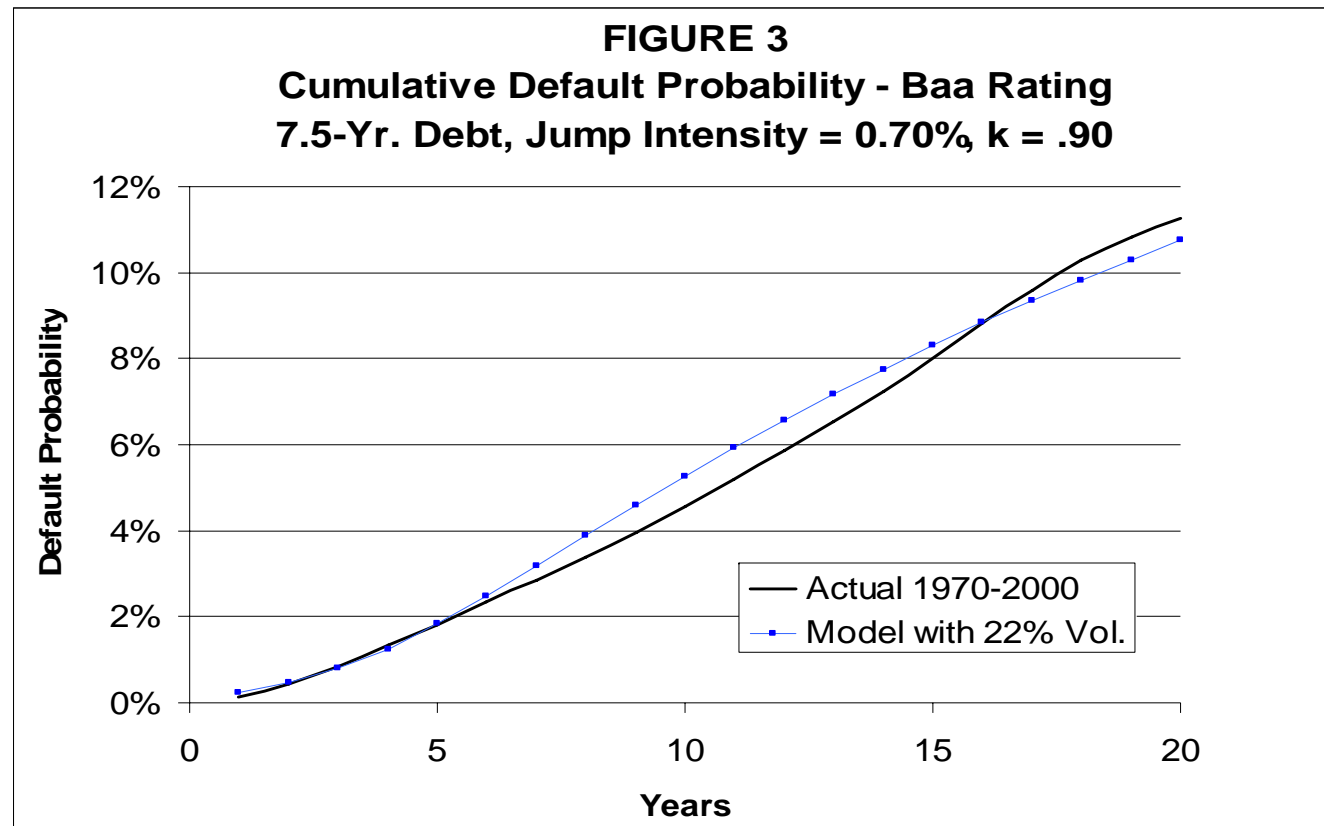
Our approach: (alternative jump risk premia approaches are possible!)

- A jump to default is at least “as bad as” a diffusion to default, in that it should command at least as high a risk premium.

- We assume the *jump risk premium H* is the same as the *default risk premium J for the pure diffusion part of the asset value process*
- Let η be the cumulative default probability of the pure diffusion process at debt maturity using the *risk neutral* drift g , and ζ be the cumulative default probability of the pure diffusion process at debt maturity using the actual (physical) drift $(g + \pi)$, where π is the asset risk premium. Then the diffusion risk premium is

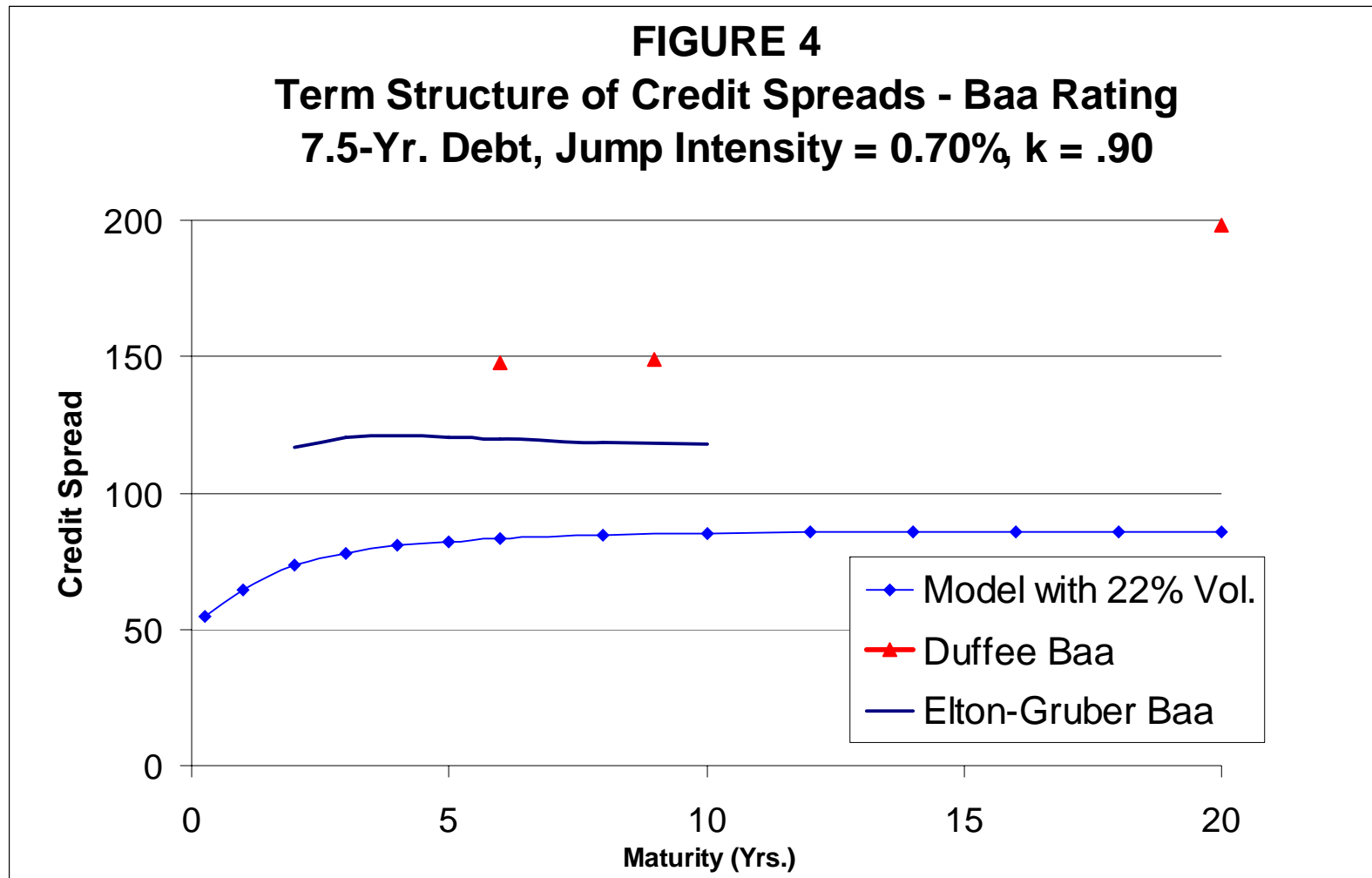
$$J = \zeta / \eta < 1.$$
- For **Baa** debt, $\lambda = 0.70\%$ and $\pi = 4\%/ \text{ yr.}$ (see Lec.1 Table 2). After 10 yrs.,
 - $\zeta = 1.84\%$, $\eta = 5.60\%$ $\rightarrow J = .329$
 - Assuming $H = J$: Predicted real jump intensity $\gamma = \lambda * J$
 \rightarrow Real jump intensity $\gamma = 0.7\% \times .329 = 0.23\%$
- For **B**-rated debt, $\lambda = 1.2\%$. At 5 yr. debt maturity, $J = 25.6\%/35.1\% = .729$
 - \rightarrow Real jump intensity $\gamma = 0.88\%$
 - If the jump risk premium is larger, default probabilities will be *lower*.

- For example, the choice of $\lambda = .007$ and $k = 0.90$ fits the Baa default and recovery rate data quite well. In contrast with Figure 2 of Lecture 1, *short-term default spreads are now well explained.*



Recall: Other studies are used to calibrate σ , α , τ , r , m , and *leverage*. (see Tables 1 and 2, Lec. 1)
 I've allowed a free choice λ and k , the jump process parameters. Predicted recovery rate = 49.9%.

But what about credit spreads? Not great news...Baa debt spreads are 90 - 95 bps.



- We could “pump up” predicted spreads by *increasing asset volatility*— but then *default probabilities would be far too high* in Figure 3.
- We could also increase spreads by assuming *higher default costs* (k or α)— but then *recovery rates would be too low*.
—Thus, *adding jump risk alone is insufficient to explain credit spreads*.

The missing factor: a liquidity premium to compensate for bond illiquidity

- *Different from a risk premium*, which is *already included* in structural models, but (as we’ve seen) is *insufficient to explain full spread*.
- Longstaff (1995) and Ericsson & Renault (2005) develop theoretical models of liquidity premiums, based on imperfect marketability.
—We don’t need to know *why* a nondefault spread exists, just that bond investors *do* require a higher rate of return. We term the higher required return a “liquidity premium,” regardless of its source

- **Huang & Huang (HH, 2004): Study residual spread from structural models**
 - Estimate several structural models, including L&T (perpetual debt)
 - Calibrate asset volatility 1985-95 of each model to match default data...
so each model assumes a *different* underlying firm volatility. Fair?
 - Since *physical* probabilities of default are *required* to be equal by HH,
not too surprising that *risk-neutral probabilities (and therefore spreads) are nearly equal*.
 - Residual termed a “liquidity spread” (or “illiquidity spread”)
Averages about **70%** of total Baa credit spread, **25%** of B spread
Liquidity fraction even greater for shorter term & higher quality debt.
- Elton & Gruber (2001), Delianedis & Geske (2001) find similarly large effects
- Collin-Dufresne, Goldstein, & Martin (2001) don't find residual spreads are fully explained by liquidity proxies, in contrast to Ericsson, Reneby & Wang (2005)

- **Longstaff, Mithal, Neis (LMN, 2004): Compares spreads for CDS (Credit Default Swaps) with observed credit spreads**
 - CDS appear to contain “pure” (risk neutral) default risk only
 - Compare with observed credit spreads: *Residual = “liquidity” premium*
- *Using a reduced-form model on data 3/2001 – 10/2002, LMN find that*
 - Nondefault risk* explains 44% of A-rated, 29% of Baa, 17% of Ba spreads
 - The nondefault component ranges from 50 to 72 basis points per year, and “is nearly constant across rating categories.”****

* Future empirical research may reveal systematic differences in liquidity premia across different ratings and maturities. (For example, Ericsson & Renault (2005) find preliminary evidence that liquidity premia decrease with maturity). Here, we assume a constant h across firm characteristics, but this assumption could be relaxed given further empirical findings.

- *Therefore we now analyze the case where*

Bond investors require a nondefault (“liquidity”) premium rate h

- This implies that debt cash flows are discounted at rate $r + h$, rather than r . *

It then follows directly from our previous arguments that

$$D(h) = \frac{C + mP}{z_1} \left(1 - \left(\frac{V}{V_B}\right)^{-y(z_1)}\right) + (1 - \alpha)V_B \left(\frac{V}{V_B}\right)^{-y(z_1)} + \frac{\lambda(1 - k)V}{z_2} \left(1 - \left(\frac{V}{V_B}\right)^{-y(z_2)}\right) \quad (9)$$

where $y(z)$ is given in equation (2), and now rather than (4) we have

$$z_1 = r + m + \lambda + h$$

$$z_2 = z_1 - g \quad (10)$$

* An alternative analysis would discount asset cash flows at a rate r that exceeds the riskfree (“Treasury”) rate r_f , reflecting a *liquidity premium for an all-equity firm*. In this case, the (incremental) bond nondefault premium h could be positive or negative, reflecting the relative liquidity of bonds vs. equity. Even with $h < 0$, the discount rate for bonds would reflect a credit spread over Treasuries, i.e. $r + h > r_f$. If $r + h - r_f$ is fixed at 60 bps, credit spreads and default probabilities decline slightly as the equity liquidity premium rises to 400 bps (implying $h = -340$ bps). Optimal leverage becomes greater.

- Define $C(h)$ as the coupon required for a given bond to sell at par, given liquidity premium h : $C(h)$ is chosen so that debt value $D(h)$ in (9) equals principal P . *
 - When discounted at r (i.e. $h = 0$), the present value $D(0)$ of the payments to bondholders is *greater than* when they are discounted at rate $r + h$, i.e. $D(0) - D(h) > 0$. Thus the cost to equity holders of providing payments to bondholders exceeds their value to bondholders.

- Net cash flows *to equity* are discounted at rate r . The value of equity is

$$E = v - D(0), \quad (11)$$

where v is given by (5), with z_3 given by (6), and V_B is determined by (8), with z_1 and z_2 given by equation (4).

- The value of the firm with $h > 0$ is the sum of debt and equity:

$$\begin{aligned} v(h) &= D(h) + E \\ &= v - (D(0) - D(h)) \end{aligned} \quad (12)$$

Note $v(h)$ is declining in the debt liquidity premium h .

* For given P , the endogenous default barrier V_B increases with h for newly-issued debt selling at par because the coupon $C(h)$ increases with h . The equilibrium yield spread will rise by slightly more than h . Thus, the impact of h on spreads exceeds that of simply adding h to spreads calculated when $h = 0$.

RESULTS OF THE COMPLETE MODEL

- **Following LMN, we assume $h = 60$ bp/yr.**
- We calibrate model parameters using the targets in Tables 1 and 2 of Lecture 1. These parameters are specified below for each of the debt ratings **A**, **Baa**, and **B**.
- For each debt rating, we choose the jump parameters (λ , k) and then use the model to predict *recovery rates* and *cumulative default rates* for periods 1-20 years. These are compared with actual recovery rates and default rates observed by Moody's over the period 1970-2000 in **Figures 5A-5C** below.
- Finally, we use the model to predict the *term structure of credit spreads* for each rating class, calibrating model parameters for each rating class and assuming a liquidity premium of 60 bps (consistent with Longstaff, Mithal, Neis). These results are presented in **Figures 6A-6C** below.

***Predictions of Default Risk and Spreads:* A-Rated Debt (Figures 5A, 6A).**

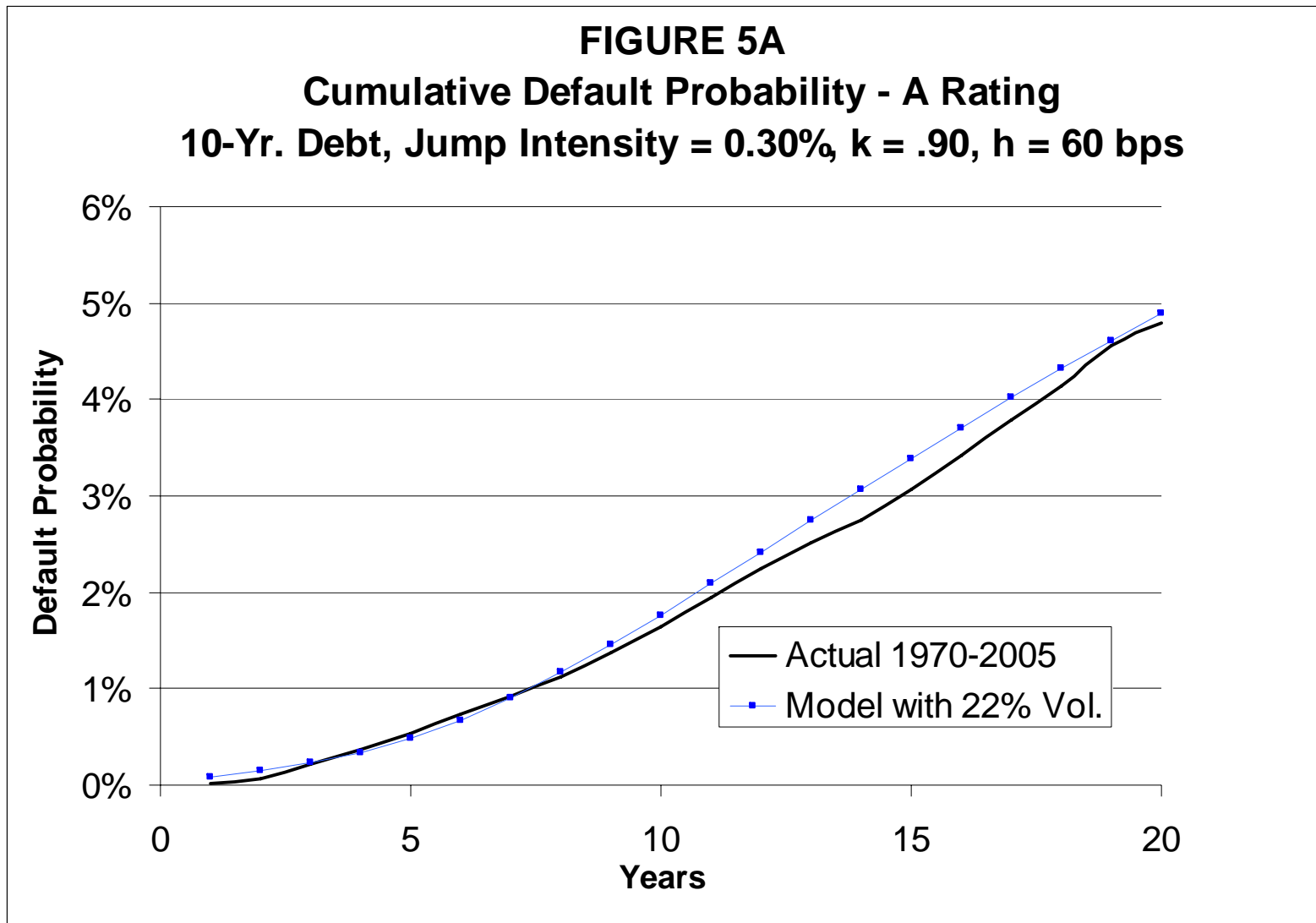
From Table 2 of Lecture 1, model parameters are

Leverage D/v	32.0%
Average Debt Maturity T	10 yrs.
Asset Volatility σ	22%
Payout Rate δ	6%
Tax Advantage to Debt τ	15%
Default Costs α	30%
Asset Risk Premium	4%

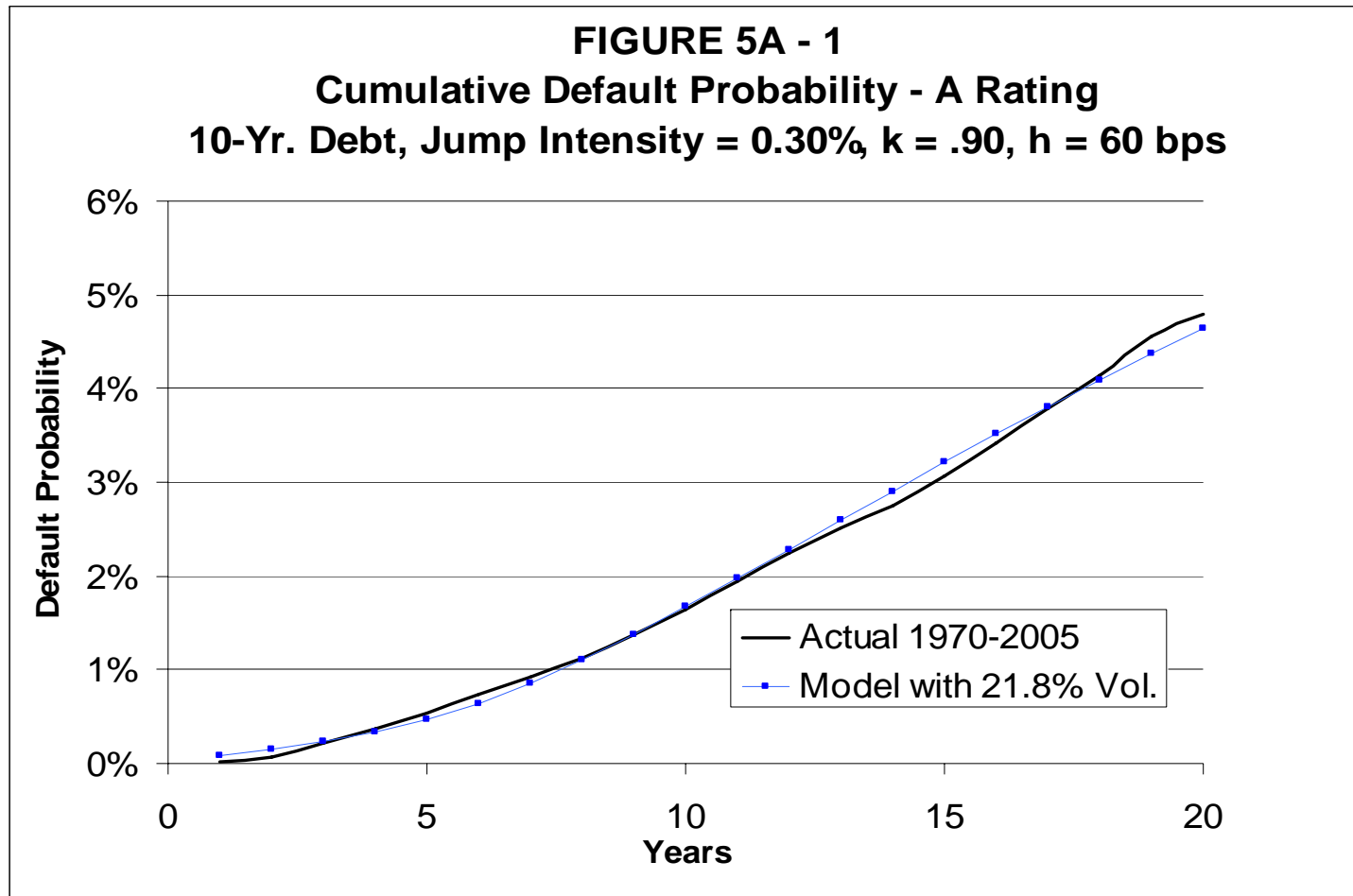
To match the target recovery rate and short-term default rates, the assumed (risk-neutral) jump intensity is $\lambda = 0.30\%$, with fractional value loss $k = 90\%$ if a jump occurs.

For *A-rated* debt, the model-predicted cumulative default rate (Figure 5A) at the target 22% asset volatility is quite close to Moody's data for A-rated debt, 1970-2000. The model predicts shorter-term default rates quite well, and predicts a recovery rate of 55.3%, vs. the target of 55%.

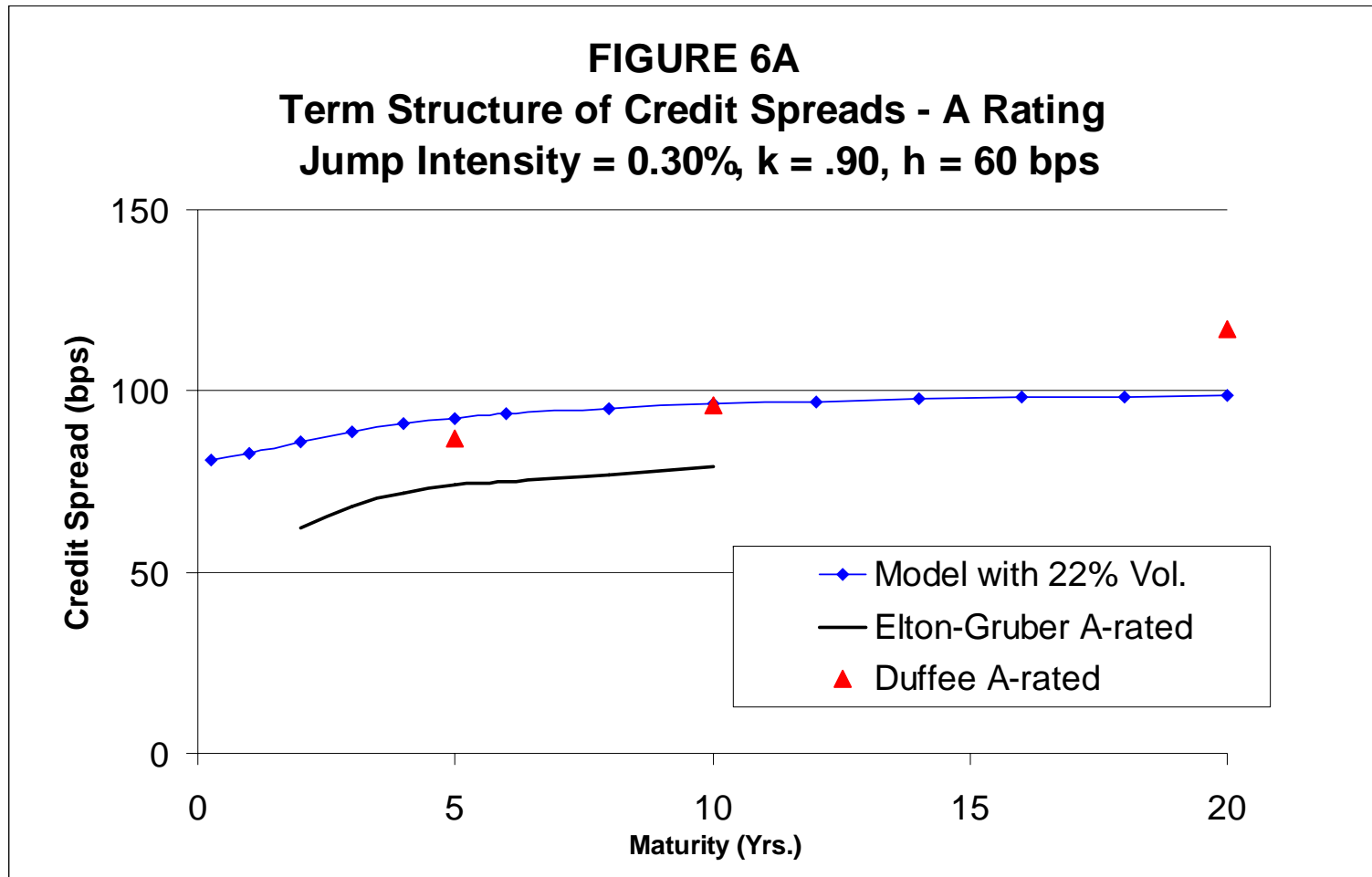
The term structure of A-rated credit spreads predicted by the model is given in Figure 6A. Spreads increase from 81 bps for 3-month debt to 99 bps for 20-year debt.



The model fits the default data even better if we assume volatility $\sigma = 21.8\%$:



The Term Structure of Credit Spreads: A-rated Debt



***Predictions of Default Risk and Spreads:* Baa-Rated Debt (Figures 5B, 6B).**

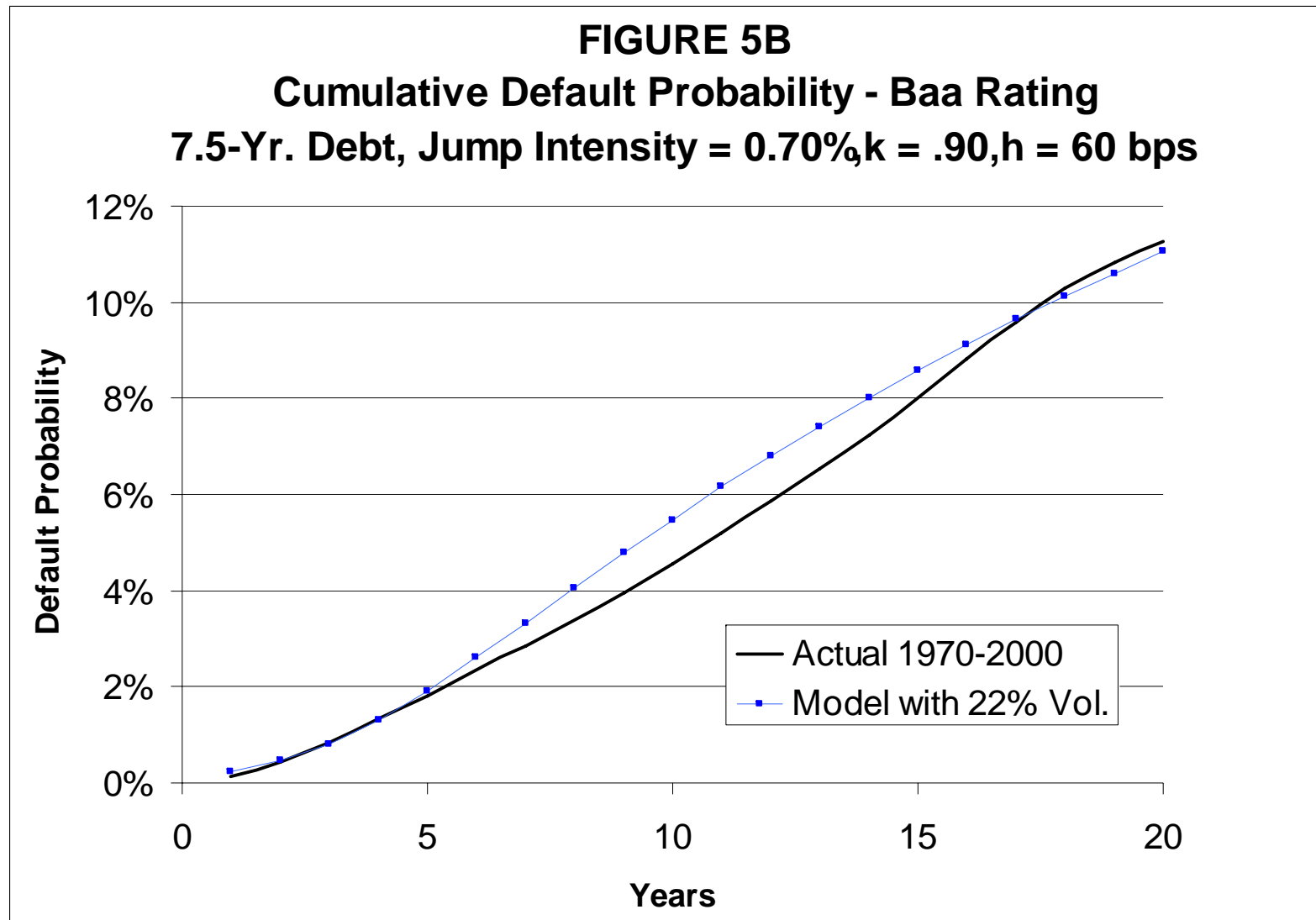
From Table 2 of Lecture 1, model parameters are

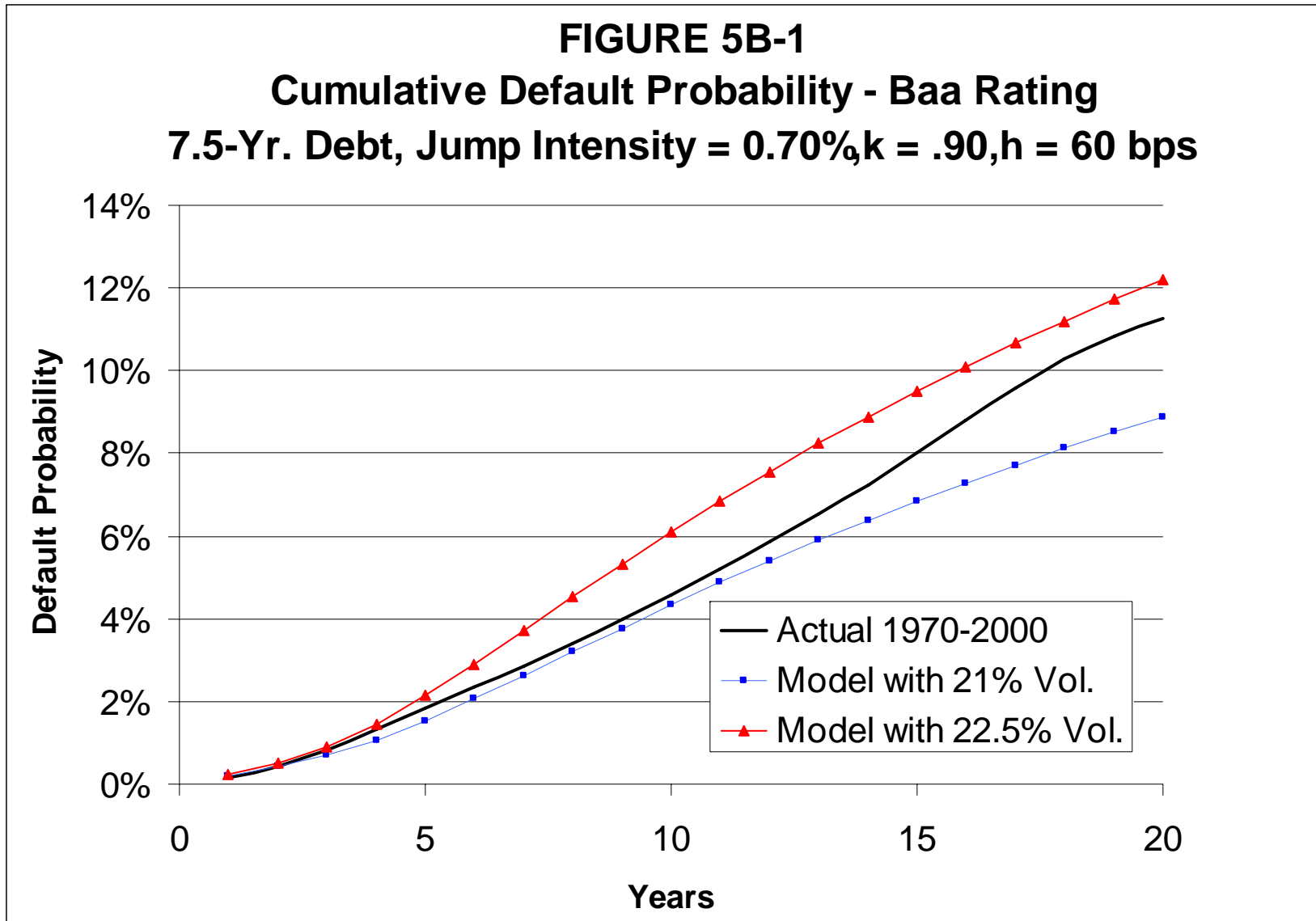
Leverage D/v	43.3%
Average Debt Maturity T	7.5 yrs.
Asset Volatility σ	22%
Payout Rate δ	6%
Tax Advantage to Debt τ	15%
Default Costs α	30%
Asset Risk Premium	4%

To match the target recovery rate and short-term default rates, the assumed (risk-neutral) jump intensity is $\lambda = 0.70\%$, with fractional value loss $k = 90\%$ if a jump occurs.

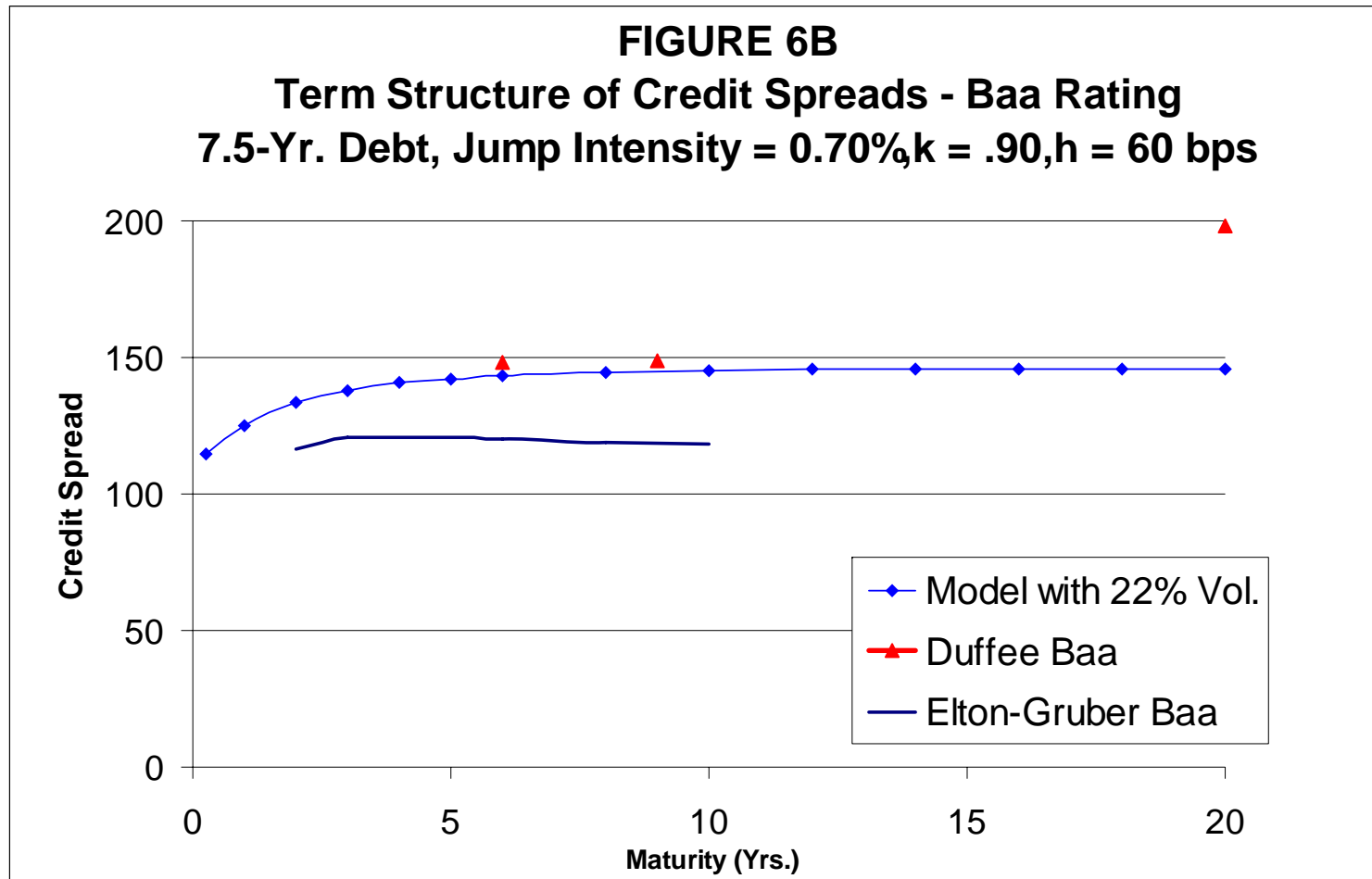
For Baa-rated debt, the model-predicted default rate at the target 22% asset volatility is reasonably close to Moody's data for Baa-rated debt, 1970-2000. The model predicts shorter-term default rates quite well. **Figure 5B-1** show that default rates over all horizons are bounded below by the model's predictions when asset volatility = 21%, and above when volatility = 22.5%. The model predicts a recovery rate of 50.8%, vs. the target of 50%.

The term structure of Baa-rated credit spreads predicted by the model with 22% volatility is given in **Figure 6B**. Spreads run from 115 bps for 3-month debt to 146 bps for 20-year debt. Spreads range from 115 to 139 (115 to 149) when volatility is 21% (22.5%).





The Term Structure of Credit Spreads: Baa-rated Debt



Predictions of Cumulative Default Risk: B-Rated Debt (Figure 5C).

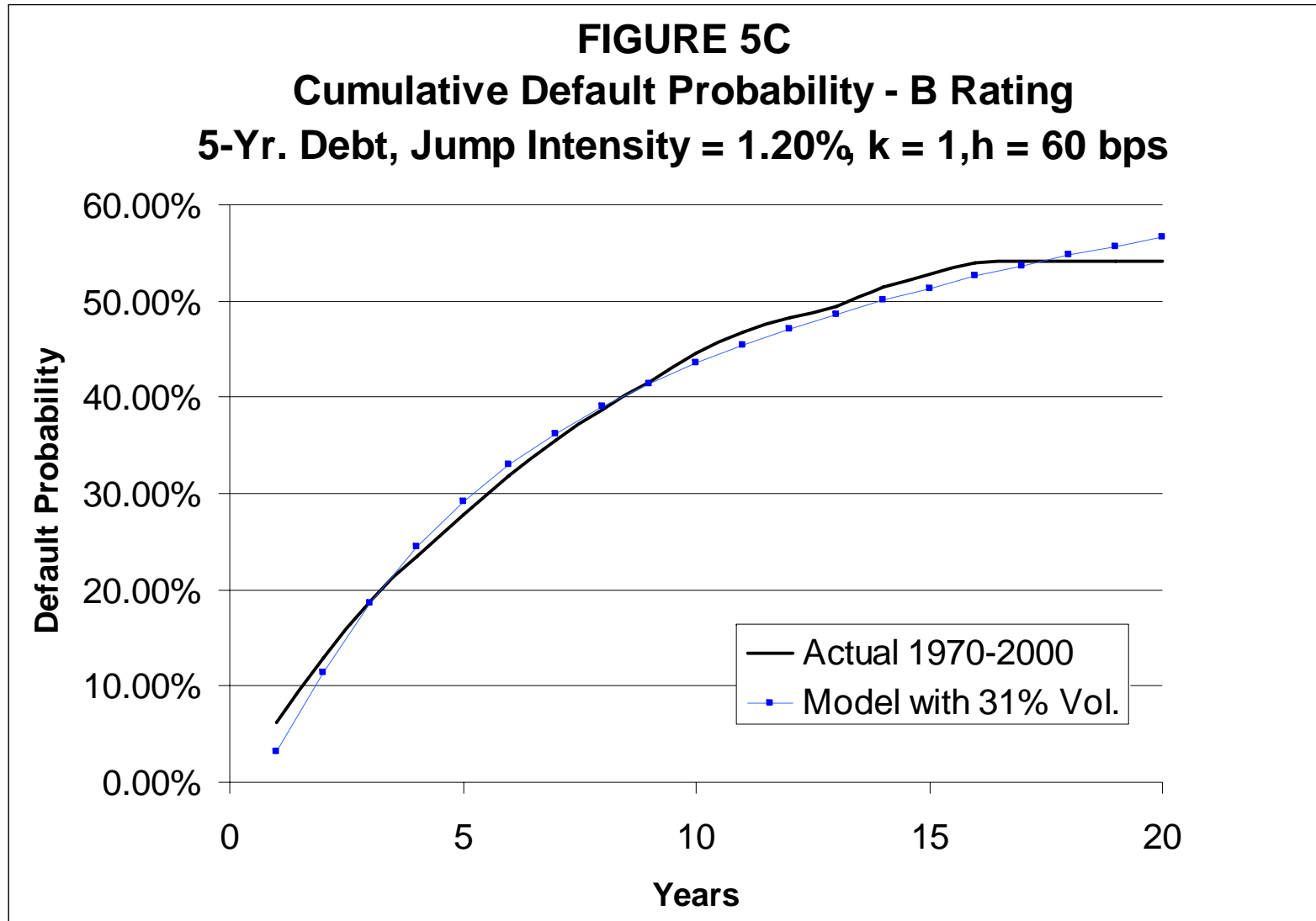
From Table 2 of Lecture 1, model parameters are

Leverage D/v	65.7%
Average Debt Maturity T	5 yrs.
Asset Volatility σ	31%
Payout Rate δ	6%
Tax Advantage to Debt τ	15%
Default Costs α	30%
Asset Risk Premium	4%

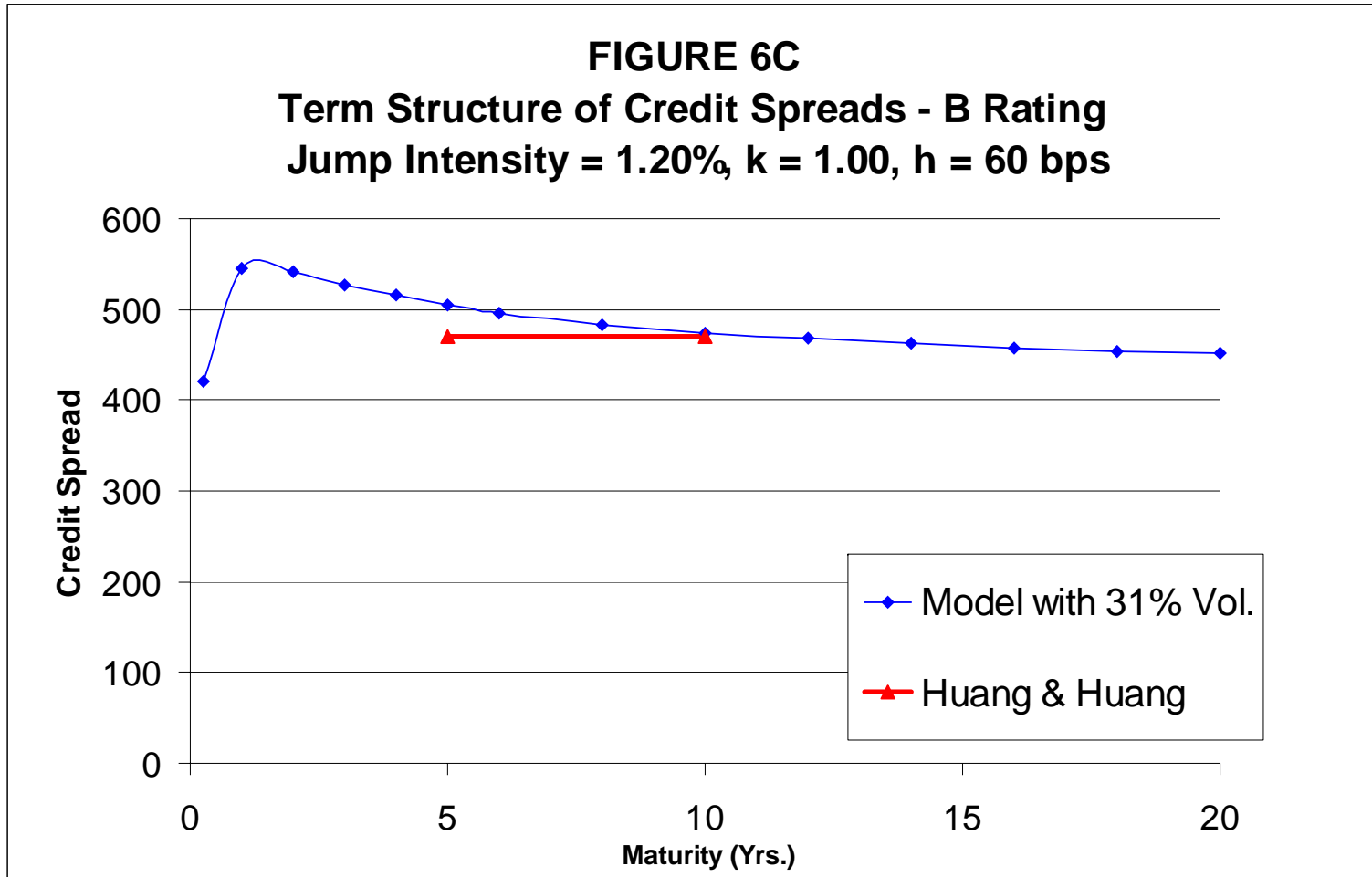
To match the target recovery rate and short-term default rates, the assumed (risk-neutral) jump intensity is $\lambda = 1.20\%$, with fractional value loss $k = 100\%$ if a jump occurs.

For B-rated debt, the model-predicted default rate at the target 31% asset volatility is quite precise relative to Moody's data for B-rated debt, 1970-2000. The model predicts shorter-term default rates well, and predicts a recovery rate of 45.1%, vs. the target of 45%.

The term structure of B-rated credit spreads predicted by the model is given in Figure 6C. Spreads decline from 545 bps for 1-year debt to 451 bps for 20-year debt, but also decline for very short-term debt (421 bps for 3-month debt).



The Term Structure of Credit Spreads: B-rated Debt



Discussion of Term Structure of Credit Spreads

- *Spreads are generally consistent with levels found in empirical research, for all bond ratings and maturities considered.**
 - Structural models that include a jump processes and a liquidity premium can explain both default probabilities and spreads.
- *The term structure of credit spreads has shapes seen in earlier research:*
 - Upward sloping (gently) for investment grade bonds
 - Humped (and mostly downward sloping) for high yield bonds
- *But an important caveat:* In using our model to predict the term structure
 - We assume that the volatility, leverage, etc. of firms offering debt of the same rating *remain constant across different debt maturities.*
 - Unclear if these parameters are constant within *actual ratings* levels...
 - Thus, comparisons with *term structure based on ratings* may be inexact.

* Our model predicts somewhat lower spreads than Duffee observes for 20-year Baa-rated debt.

APPLICATIONS OF THE MODEL

The model's success in explaining both default rates and spreads suggests that it can serve as a useful guide to firms' financial structure decisions.

1) Optimal Capital Structure

- We now drop the assumption that leverage for firms with different ratings matches the previously-specified levels (e.g. 43.3% for Baa-rated firms)
 - We consider leverage ratios that *maximize total firm value*, given the other parameters for firms in each different rating category.

- **A-rated firms:** Optimal leverage (10-yr. debt): **45.2%** (vs. actual 32.0%)
 - At optimal debt, spread 131 bps
 - >> A-rated firms appear to be somewhat *under-leveraged*
 - >> But the value loss is small from under-leveraging ($< 0.3\%$ of v)
- **Baa-rated firms:** Optimal leverage with 7.5 yr maturity debt: **46.5%**
 - This is not far from the actual *Baa average leverage* of **43.3%**
 - At optimal debt, spread = 155 bps
 - If $h = 0$, optimal leverage 49.7%.
- **B-rated firms:** Optimal leverage = **36.7%!!** (vs. actual 65.7%)
 - Less optimal leverage than Baa because volatility higher, maturity 5 yrs.
 - Spread at optimal leverage would be 240 bps, not 505 bps

—*Tentative conclusion:*

- >> Average B-rated firm in the data base is *over-leveraged*
- >> Leverage, volatility for B-rated firm likely includes *fallen angels*, whose initial leverage, and perhaps volatility, was lower

QUESTION: Should firms optimally issue “junk” bonds (spread ≥ 400 bps)?

- Preliminary analysis* suggests spreads at optimal leverage exceed 400 bps if
 - (i) *The tax advantage of debt is quite high* ($\tau > 28\%$)
 - May have been that high after 1986 Tax Reform Act—but not now
 - (ii) *Default costs are low* ($\alpha < 6\%$)
 - But this implies 55%+ recovery rates on (junk) debt

* We use the base case for B-rated debt, and change individual parameters

(iii) *Asset volatility is very high ($\sigma > 55\%$)*

—But this is much higher than average B-rated firm risk (32%)

(iv) *The equity liquidity premium exceeds the debt liquidity premium by more than five times**

--Yet debt is generally considered less liquid than equity, at least for firms with publicly-traded equity and debt.

In conclusion, it would seem that few firms would find it optimal to issue junk debt. A more favorable tax environment for debt seems necessary for substantial junk bond financing to be optimal.

*See footnote on p. 20. In the scenario of a B-rated firm with equity risk premium of 400 bps ($\Rightarrow r = 12\%$), and $h = -340$ bps (leaving a net liquidity premium for debt of 60 bps), the optimal leverage would be 56% and the credit spread would be 370 bps.

Using the Model: 2) Comparative Statics of Optimal Capital Structure

- **Optimal leverage and leverage benefits rise as** (ceteris paribus)
 - Default costs α fall
 - Volatility σ falls
 - Tax advantage τ of debt rises
 - Maturity $T = 1/m$ increases (Fig. 14)
 - Finally, optimal leverage rises as payout rate δ falls.
 - >> May seem surprising, since lower δ implies higher growth μ , and **data suggests higher growth firms have less leverage.**

—However, empirical results could be explained by *other factors that reduce leverage*:

>> Higher default costs α if growth options lost

>> Higher business risk σ , lower effective tax rate τ

>> Lenders demand shorter term debt (agency concerns?)

>> Growth firms want use short term debt for restructuring flexibility (Dangl & Zechner (2004), Ju et al. (2005))

• **Example High Tech Firm**

—Payout rate	$\delta = 0$
—Recovery rate 0% →	$\alpha = k = 1$
—Firm risk	$\sigma = 30\%$
—Tax advantage to debt	$\tau = 10\%$
—Debt maturity $T = 5$ yrs. →	$m = 0.20$

• Then optimal leverage L^* : $L^* = 4.6\%$

CONCLUSIONS

- ***Structural Models are alive and well!***

- *With the addition of a simple jump and liquidity cost, they can explain both observed credit spreads and default probabilities*
- *Closed form solutions allow easy comparative statics*
- *Valuations can be used to study optimal financial structure for firms, as well as other corporate decisions*
- *Optimal leverage is close to observed leverage for Baa-rated firms*
 - >> *A-rated firms appear to be under-leveraged relative to optimal*
 - >> *B-rated firms appear to be considerably over-leveraged*

- ***Some Desirable Extensions***

*(references have models previously addressing these issues,
without jumps or liquidity costs)*

—Dynamic Capital Structure

(e.g. Fischer, Heinkel & Zechner (1989), Collin-Dufresne & Goldstein (2001);
Goldstein, Ju, & Leland (2001); Ju, Parrino, Poteshman & Weisbach (2004))

—Optimal Investment

(e.g. Mauer & Triantis (1994); Childs, Mauer & Ott (2005); Hackbarth (2006);
Wang (2006))

—Risk Management for Firms

(e.g. Ross (1996); Leland (1998))

—Operating Decisions by Firms

(e.g. Brennan & Schwartz (1985); Mello & Parsons (1992); Fries, Miller & Perraudin (1997))

—Managerial Behavior (vs. equity value maximization)

(e.g. Hackbarth (2004))

APPENDIX A: *Notes on Hackbarth, Miao, & Morellec and on Chen & Kuo:*

HMM consider a jump process for cash flows that alternates “boom” and “recession” conditions, where recession may or may not lead to default. They use the exponential debt model and derive closed form solutions (though not for the default boundary). Their model does not include a liquidity premium.

CK have a similar agenda, although they also do not consider liquidity (or analyze default risk). CK use a more general jump process (double exponential) that requires numerical Laplace transform inversion to determine debt values, as in HR. CK cite Leland & Toft (1996), but actually use the exponential debt model (Leland (1994b, 1998)).

APPENDIX B: *Notes on the debt valuation formulas:*

The third integral term in formula (1) can be written as

$$\begin{aligned}
 Z^* &\equiv \int_0^{\infty} e^{-rt} \{e^{-mt} (1-k) E[V(t)]\} (1-F) \lambda e^{-\lambda t} dt \\
 &= \frac{\lambda(1-k)V}{r+m+\lambda-g} \left(1 - \left(\frac{V}{V_B}\right)^{-y_2}\right)
 \end{aligned} \tag{16}$$

where $E[V(t)] = e^{gt}V$ is the unconditional expected asset value at t , and V is asset value at time $t = 0$.*

Z^* is an approximation of the actual expected present value received by bondholders if there is a jump to first default at t . It is approximate because actual bondholder claims are capped (by P), and the expected asset value $V(t)$ must be conditional on not previously hitting the default boundary. Thus the actual expectation to all bondholders of payoff given jump to default at t is

$$w(t) \equiv E[\text{Min}[(1-k)V(t), P] | V_{\min}(t) \geq V_B],$$

where $V_{\min}(t)$ is the minimum asset value up to time t .

* Recall that bondholders at time $t = 0$ have claim to a fraction e^{-mt} of post-jump asset value at time t , which is the random amount $(1-k)V(t)$.

Let Z define the present value of a security paying expected value $e^{-mt}w(t)$ if a jump occurs at time t , and zero otherwise. Then (with $h = 0$) the expected present value to bondholders at $t = 0$ is

$$Z = \int_0^{\infty} e^{-rt} e^{-mt} w(t) (1 - F) \lambda e^{-\lambda t} dt$$

The value Z must satisfy the o.d.e. (see Merton (1976); HMM (2006))

$$\begin{aligned} .5\sigma^2 V^2 Z'' + gVZ' + \lambda((1-k)V - Z) &= (r+m)Z, & V_B \leq V \leq V_M \\ .5\sigma^2 V^2 Z'' + gVZ' + \lambda(P - Z) &= (r+m)Z, & V_M \leq V \end{aligned} \quad (17)$$

where $V_M = P/(1-k)$, $g = r - \delta + \lambda k$, and primes denote derivatives w.r.t. V .

Let $Z_L(V)$ denote the solution of the o.d.e. when $V_B \leq V \leq V_M$, and $Z_H(V)$ the solution when $V_M \leq V$. It is well known that

$$\begin{aligned} Z_H(V) &= C_{1H} (V/V_M)^{x_1} + C_{2H} (V/V_M)^{x_2} + \lambda P / (r+m+\lambda), & V_M \leq V \\ Z_L(V) &= C_{1L} (V/V_B)^{x_1} + C_{2L} (V/V_B)^{x_2} + \lambda(1-k)V / (r+m+\lambda-g) & V_B \leq V \leq V_M \end{aligned}$$

where $x_1 < 0$ and $x_2 > 0$ are the roots to the equation

$$0.5\sigma^2 x^2 + (g - 0.5\sigma^2)x - (r+m+\lambda) = 0.$$

Boundary conditions require that Z_H is bounded as $V \rightarrow \infty$, implying that $C_{2H} = 0$. Furthermore, $Z_L(V_B) = 0$ (since at the diffusion barrier, diffusion default will occur with probability one and the value of the contingent claim to a jump is zero). Finally, at $V = V_M$, both value-matching and smoothness conditions hold, implying (and recalling $C_{2H} = 0$) that

$$C_{1H} + \lambda P / (r + m + \lambda) = C_{1L} (V_M / V_B)^{x_1} + C_{2L} (V_M / V_B)^{x_2} + \lambda(1-k)V_M / (r + m + \lambda - g)$$

$$x_1 C_{1H} / V_M = x_1 C_{1L} (V_M / V_B)^{x_1-1} / V_B + x_2 C_{2L} (V_M / V_B)^{x_2-1} / V_B + \lambda(1-k) / (r + m + \lambda - g)$$

Jointly, the boundary conditions admit closed form solutions for $\{C_{1L}, C_{2L}, C_{1H}\}$ as follows:

$$C_{1L} = \frac{\lambda \left(P((g/K)x_1 - 1) \left(\frac{P}{(1-k)V_B} \right)^{-x_2} + (k-1)(x_1 - x_2)V_B \right)}{(K-g)(x_1 - x_2)}$$

$$C_{2L} = \frac{-\lambda \left(P((g/K)x_1 - 1) \left(\frac{P}{(1-k)V_B} \right)^{-x_2} \right)}{(K-g)(x_1 - x_2)}$$

$$C_{1H} = \frac{\lambda \left(P(1 - x_2(g/K)) - P(1 - x_1(g/K)) \left(\frac{P}{(1-k)V_B} \right)^{x_1-x_2} + (1-k)(K-g-1)(x_1 - x_2)V_B \left(\frac{P}{(1-k)V_B} \right)^{x_1} \right)}{(K-g)(x_1 - x_2)}$$

Source: Constants 4.nb

where $K = r + m + \lambda$. Equity will be valued by $E = v - D$, with the third term in equation (1), Z^* , replaced by Z . Note E will still have a closed form solution, but V_B will not, and must be determined numerically to satisfy the smooth pasting condition

$$\partial E(V) / \partial V |_{V=V_B} = 0$$

(If there is a liquidity premium $h > 0$, then r is replaced in the formulas for debt value by $r + h$).

The differences between Z and Z^* for base case parameters are very small. So too are the differences in the optimal V_B . For example, for Baa-rated debt, using Z^* rather than Z in (1) results in an optimal V_B of 35.3 rather than 35.4, an unchanged debt value (to the nearest penny, 45.12), and an unchanged yield spread (to the nearest basis point, 144). For B-rated debt, there is zero difference because $k = 1$. (With $k = .95$, the yield spread would fall from 501 bps to 500 bps).

Thus, the formula (1) (or (9)) serves as a close approximation of debt value for a wide range of parameters describing both investment grade and high yield debt.

Notes on this version of the lectures:

The current version of the Princeton Lectures 1 & 2 makes a few changes from the lectures delivered on September 20-21st, 2006 at Princeton. Most notably:

- (i) The discussion of papers that extend the basic diffusion model without jumps has been moved from the beginning of Lecture 2 to the end of Lecture 1.
- (ii) Due to correcting an error in calculating recovery ratios, the numerical examples are slightly different than originally presented. I use 5-year and 7.5-year maturities (rather than 10 years) for B-rated and Baa-rated debt.
- (iii) To make results directly comparable to Huang and Huang (HH, 2003), I have chosen to use default data for the period 1970-2000 provided in Moody's Special Comment (2001). More recent data is now available for the period 1970-2005, and was used in the original presentation. Since spreads and firm data in Tables 1 & 2 were based largely on statistics for the period 1985-1995, the earlier default period is arguably more appropriate. In any case, the results are very similar.
- (iv) I have added default and spread comparisons for A-rated debt in this version, in addition to the Baa-rated and B-rated debt examined in the original version.
- (v) Footnotes have been added for further clarification/discussion.
- (vi) The Appendices are new, and the list of references is expanded and appended to Lecture 2.
Version 3 of Lecture 2 corrects some errors in Version 2's Appendix B.

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