Sequential Investment-Consumption Strategies for Individuals and Endowment Funds with Lexicographic Preferences

By

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SEQUENTIAL INVESTMENT-CONSUMPTION STRATEGIES FOR INDIVIDUALS
AND ENDOWMENT FUNDS WITH LEXICOGRAPHIC PREFERENCES

by

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ABSTRACT

This paper derives and explores the implications of certain lexicographic preference structures in the investment area. The idea that preferences are hierarchical in nature was first put forth by Maslow in a now classic paper [27]. While the problems relating to the numerical representation of such preferences have received considerable attention, lexicographic models of decision problems are practically non-existent in the financial and economic literature--this despite the fact that hierarchical preference structures break the total problem into smaller sub-problems.

Four fairly simple (although sequential and stochastic) investment models are examined. Some of these explicitly recognize the existence of a subsistence level for consumption or spending; the typical effect of this recognition is to inhibit risk-taking. In the university model, the optimal investment policy is not only "consistent" with the recommendation of a recent Ford Foundation advisory committee but provides the specificity (presently lacking) needed for its implementation.
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I. INTRODUCTION

As far as I have been able to discover, there was not a single formal model of the consumer-investor in the economics literature ten years ago which incorporated two of the more fundamental aspects of life: the fact that decisions are taken sequentially and the fact that they are made under uncertainty. In contrast, the last ten years, and in particular the last five, have seen a virtual explosion in the construction of sequential stochastic models of the consumer and quasi-consumer. Before examining the methodological foundations which made this progress possible, a very brief summary of the main papers to date dealing with the consumer-investor's recurring decision problem under risk may be appropriate.

The earliest papers appear to be due to Phelps [31] and to Yaari [40]. Phelps studied an individual with a certain income stream and a single risky investment opportunity, while Yaari examined a model of a consumer with an uncertain lifetime but facing riskless investment returns.

The Phelps' investment-consumption model (with income stream) was generalized to the case of one riskless and several risky assets by Hakansson [14, 17] and by Leland [25], and, without income stream, by Levhari and Srinivasan [26], Samuelson [34], Merton [28], Hahn [13],

-1-
and Pye [33]. Hakansson later extended his model to reflect an uncertain lifetime and the possibility of life insurance [15] and finally (without income stream), to incorporate a Markovian investment environment and state-dependent time preferences [19]. The properties of the induced utility of wealth of stochastic sequential models of the consumer-investor have been examined by Fama [11], Hakansson [16], and Neave [30]; their implications with respect to the demand for liquid assets have been studied by Dixit and Goldman [9]. Finally, Yaari [41] has tackled a sequential model in which the consumer faces an uncertain income stream.

In Section II, I shall review briefly the methodological foundations which have made possible the recent advances in the consumption-investment area. Section III then introduces certain lexicographic preference structures for the purpose of representing various behavioral phenomena. The implications of such preferences are examined in Sections V-VIII in the context of some examples. In particular, the effects of lexicographic orderings concerning total consumption, subsistence level consumption, bequests, moonlighting, the maintenance of educational quality on the part of universities, and perpetual giving by foundations, especially on optimal risk taking, are examined and related to the properties of "conventional" models.

II. THE METHODOLOGICAL FOUNDATIONS

Since the focus of this conference is on methodology, I shall review, very briefly, the tools used in the development of the models referred to in the preceding section. The methodology itself warrants
few comments since it relies almost exclusively on postulation, deduction, and induction and (formal) empirical investigation has so far not accompanied any of the studies.

Whether one is concerned with prescriptive models or with descriptive representations of the consumer-investor, there seem to me to be five ingredients which are crucial to success. Four of these fall in the analytical area; the fifth I shall loosely refer to as empirical relevance. Since empirical relevance must ultimately determine the applicable analytical tools in any of the empirical sciences, I shall begin with this component.

While mathematicians and logicians may be interested in deductive systems per se, the empirical scientist generally does not build deductive systems for their own sake. His systems are usually intended to be representations of real world phenomena about which he wishes to inquire. The interface between model and reality, or between conception and perception, is now readily seen to be crucial. This is where the concept of empirical relevance becomes important.

Since no model can be a perfect replica of reality, a natural measure of the empirical scientist's skill in his choice of model is the extent to which the essence of (the) reality (modeled) is captured in the postulates. No objective measure of this skill is possible, since the phenomena which give rise to this skill, perception and the translation of perceived ideas into concepts, are highly subjective. However, the importance of this skill is brought out by the fact that the theorems of a model have empirical content only if the postulates do. No theory can be more secure than its foundations.
It was noted earlier that elementary consideration of empirical relevance demand that models of the consumer-investor recognize that economic decisions are made sequentially and under uncertainty. The first of these facts means that once a sequential model is constructed, the derivation of theorems requires skill in what is known as sequential decision theory, dynamic programming, and control theory. A certain familiarity with the analytical tool known as sequential decision theory is therefore indispensable to the researcher interested in models of the consumer-investor.

The recognition of uncertainty in a model is generally accomplished via probabilities. The presence of this ingredient in a sequential model clearly demands considerable knowledge of probability theory and stochastic processes on the part of the scientist who is searching for (new) theorems.

A third crucial element of a model of the consumer-investor is the preference structure. Its representation as well as the subsequent search for model implications demands a thorough knowledge of utility theory. Since empirically relevant cardinal utility functions are almost always nonlinear, skills in nonlinear programming are also necessary in extracting conclusions from a sequential, stochastic model of the investor-consumer.

In sum, the primary analytical tools which form the methodological foundation for studies of the consumer-investor are sequential decision theory, probability theory, utility theory, and nonlinear programming. In addition, success in this area of study also requires an ability to build empirical relevance into the model used. At the present time, most
researchers in the consumption-investment field possess strength in three or four of the above areas. Those with pronounced skills in all five areas are few indeed.

As we look to the future, the most likely addition to the model building kit will be Bayesian processing of empirical observations in the form of econometric information (see Drèze [10]). The effect of this will be to add econometrics to the bag of skills which will be needed by the researcher in the consumption-investment area.

III. LEXICOGRAPHIC PREFERENCES

Existing models of the consumer-investor are based on the Fisherian concept that consumption is the spring of all ultimate utility although some also couple this with a Marshallian bequest motive. The point I want to stress is that the assumed preference structures obey Archimedes' axiom that everything has its price except, in some models, zero consumption, which is then avoided at all cost. But when this exception occurs, it does so by default since it appears only if the utility function is unbounded and a utility function which is consistent with the von Neumann-Morgenstern postulates [39] is always bounded [2]. In other words, present utility theory is founded on the notion that no part of an economic position is immune to an economic bribe. As long as my consumption level in some period is positive, the theory claims, I am always willing to give up some of this consumption for a chance at a higher consumption level in the same (or a different) period. Specifically, if consumption level \( x \) is preferred to level \( y \) which
in turn is preferred to level \( z \) \( (x \succ y \succ z) \), the von Neumann-Morgenstern theory asserts that there exist a number \( p \), \( 0 < p < 1 \), such that I'm indifferent between receiving \( y \) for sure and receiving \( x \) with probability \( p \) or \( z \) with probability \( 1 - p \). Thus, if \( x > y > z \geq 0 \) and \( y \) is the subsistence level, that theory suggests that, for a price \((x - y\) with probability \( p < 1\)), we are willing, upon consulting our preferences, to bring our consumption level in some years to zero, or at least arbitrarily close to zero, with positive probability. But is that really the case? If there is such a thing as a subsistence level, and the empirical evidence is pretty persuasive on this point, there is probably no amount of additional consumption in the present period or in future periods that would induce one to risk going below the subsistence level (presumably survival at less than the subsistence level is impossible in any period; remember also that we are speaking about preferences, or indifference curves, apart from opportunities).

The preceding suggests that preferences for consumption above the subsistence level are of a lower order (infinity) than for subsistence level consumption. Perhaps trade-offs between periods exist only if the subsistence level is equalled or exceeded in each period. In other words, preferences may be lexicographic: every (stochastic) consumption program which insures subsistence in every period is preferred to every program that risks providing less than the subsistence level in some period(s). This, of course, does not rule out preferences among subsubsistence consumption programs. They, in turn, may be lexicographic: possibly every program which insures subsistence in the first \( n \) periods
(n < N, the total (maximum) life-time in periods) is preferred to every program which does not insure subsistence in one or more of the first n periods.

Other examples of lexicographic orderings suggest themselves. A person may so prefer a way of living (consumption program) that he will gladly work hard enough to afford it; furthermore, given this pattern, he may gladly bequeath any fortune he might have, no matter how large, rather than spend anything more on himself or his family. Private universities seem unwilling to reduce internal endowment growth in the short run in order to get by with less fund raising efforts for a given expenditure level. And charitable endowment funds appear to regard a perpetual life of substantial giving as ranking higher than anything else. We shall examine each of these situations in some detail in the latter part of the paper.

Before proceeding to the analysis proper, it is of interest to note that a lexicographic preference structure is implicitly assumed in a class of investment-consumption models that have been studied by mathematicians. These studies have focused attention on finding the investment strategies which minimize the probability of ruin when the individual must pay a cost of living charge each period (Truelove [37, 38], Ferguson [12]). The notion that human preferences are hierarchial in nature also has wide acceptance in the behavioral sciences. In a classic paper, Maslow, for example, wrote [27: 394-395]:

There are at least five sets of goals which we may call basic needs. . . . These basic goals are related to one another, being arranged in a hierarchy of prepotency. This means that the most prepotent goal will monopolize consciousness and will tend of itself to organize the recruitment of the various
capacities of the organism. The less prepotent needs are minimized, even forgotten or denied. But when a need is fairly well satisfied, the next prepotent ('higher') need emerges, in turn to dominate the conscious life and to serve as the center of organization of behavior, since gratified needs are not active motivators.

The problem of representing lexicographic preferences under uncertainty by means of utilities has been intensively studied by Thrall and Dalkey [36], Hausner [23], Thrall [35], and Chipman [7]. Their central result, which also applies to the certainty case (Debreu [8]), is that an ordered vector space is required for representing non-Archimedean preferences.

As noted, ordinary utility theory rules out lexicographic preferences; hierarchial preference structures are specifically excluded by the continuity postulate. This suggests a possible methodological shortcoming facing the theory of the consumer-investor if we accept the existence of lexicographic preferences. However, the present framework is still of use in that it enables the determination of conditionally optimal investment and spending strategies "within layers"; the derivation of such policies will be the focal point of the following sections. In other words, even if von Neumann-Morgenstern utility theory is inappropriate for preference representation at the global level, it still serves well locally, that is, within each hierarchial layer. Clearly, it now becomes necessary to first determine which hierarchial "preference layer" applies in a given environmental situation.

IV. PRELIMINARIES

In each case we assume that opportunities for decision occur at equally spaced points in time. The period of time between decision points \( j \) and \( j + 1 \) will be denoted period \( j \).
The following basic notation will be adopted:

\( x_j \) = amount of equity capital at decision point \( j \) (the beginning of the \( j \)th period)

\( M_j \) = the number of investment opportunities available in period \( j \)

\( r_{j-1} \) = rate of interest in period \( j \), where \( r_j > 1 \)

\( \beta_{ij} \) = proceeds per unit of capital invested in opportunity \( i \), where \( i = 2, \ldots, M_j \), in the \( j \)th period (random variable)

\( F_j(y_2, y_3, \ldots, y_{M_j}) \equiv \Pr\{\beta_{2j} \leq y_2, \beta_{3j} \leq y_3, \ldots, \beta_{M_j} \leq y_{M_j}\}. \)

\( z_{ij} \) = amount lent in period \( j \) (negative \( z_{ij} \) indicate borrowing) (decision variable)

\( z^*_{ij}(x_j) \) = an optimal lending (borrowing) policy in period \( j \)

\( z_{ij} \) = amount invested in opportunity \( i \), \( i = 2, \ldots, M_j \) at the beginning of the \( j \)th period (decision variable)

\( z_j \equiv (z_{2j}, \ldots, z_{M_j}) \)

\( z^*_j(x_j) \) = an optimal investment policy in period \( j \)

\( c_j \) = total amount consumed or expended on operations in period \( j \) (allocated at beginning of period)

\( c^*_j(x_j) \) = an optimal consumption or spending policy in period \( j \)

\( y_j \) = non-capital income or gifts received at end of period \( j \)

\( Y_j \equiv \frac{y_j}{r_j} + \frac{y_j}{r_j \cdots r_N} \)

\( f_j(x_j) \) = the (induced) utility of wealth at decision point \( j \) under an optimal policy
We assume that the distribution functions $F_j$ satisfy the boundedness conditions

\begin{align*}
(1) & \quad \Pr\{0 \leq \beta_{ij} \leq B_i\} = 1 \quad i = 2, \ldots, M_j, \text{ all } j, \\
(2) & \quad E[\beta_{ij}] \geq r_j + \eta \quad \eta > 0, \text{ some } i, \text{ all } j,
\end{align*}

and the "no-easy-money condition"

\begin{align*}
(3) & \quad \Pr\left\{ \frac{1}{M_j} \sum_{i=2}^{M_j} (\beta_{ij} - r_j) \theta_i < \delta \right\} > 0 \quad \text{for all } j \text{ and all } \theta_i \text{ such that } \\
& \quad \sum_{i=2}^{M_j} \theta_i = 1 \quad \text{and } \theta_i \geq 0,
\end{align*}

where $\delta < 0$.

Conditions (1) reflect the fact that one can at most lose one's investment in a long position and that a finite investment will always bring a finite return over a finite time period. The "no-easy-money condition" (3) states that no combination of investments is available which guarantees a return at least equal to the rate of interest. (3) appears to be the weakest condition that the distribution functions $F_j$ must satisfy when the prices of the various capital assets are in equilibrium in the absence of artificial margin requirements.

To keep the technical details at a minimum, we also assume in this paper that there exists a number $p > 0$ such that for all values $y_2, y_3, \ldots, y_{M_j}$ jointly assumed by $\beta_{2j}, \beta_{3j}, \ldots, \beta_{M_j}$

\begin{align*}
(4) & \quad \Pr\{\beta_{2j} = y_2, \beta_{3j} = y_3, \ldots, \beta_{M_j} = y_{M_j}\} \geq p \quad j = 1, 2, \ldots
\end{align*}
We also assume that there are no limits on borrowing but that all debts must be fully secured. The latter requirement is necessary to achieve consistency with the standard assumption of risk-free lending (see [22]) and to cope with the requirements of a multiperiod model of the reinvestment type.

V. THE UNWORLDLY PROFESSOR

We first consider an individual, A (the unworldly professor) whose preferences exhibit the following properties:

A1. A very strong liking of a particular life-style. This life-style requires consumption expenditures of $c_j (>0)$ at the beginning of each of the remaining periods $j = 1, \ldots, N$ and brings an income stream of $y_j \geq 0$ at the end of each period $j = 1, \ldots, N$.

A2. If initial wealth $x_1$ and the income stream $y_1, \ldots, y_N$ should be insufficient to support the expenditure pattern $c_1, \ldots, c_N$ with probability 1, the individual would prefer 1) bringing his income up to the level necessary to afford $c_1, \ldots, c_N$ for sure by extra work (moonlighting) to 2) compromising in any way on $c_1, \ldots, c_N$.

A3. If initial wealth $x_1$ and the income stream $y_1, \ldots, y_N$ should be sufficient to support the expenditure pattern $c_1, \ldots, c_N$ with probability 1, the individual would prefer 1) bequeathing any accumulated wealth at the end of period $N$, $x_{N+1}$, to 2) increasing his consumption above $c_1, \ldots, c_N$.
The utility of a bequest \( x_{N+1} \geq 0 \) is \( b(x_{N+1}) \), where \( b \) is monotone increasing and strictly concave.

A4. The avoidance of any extra work, no matter how small the probability of its necessity, is always preferred to any bequest \( x_{N+1} \).

Let us first find the conditions under which the resources \( x_1 \) and \( y_1, \ldots, y_N \) are sufficient for implementing \( c_1, \ldots, c_N \) without extra work with probability 1. As noted, we have assumed that each of these elements are known with certainty at decision point 1.

At decision point \( N \), A can borrow a maximum of \( y_N/r_N \) against income \( y_N \) to be received at the end of period \( N \) without risk of default. Thus, \( x_N \) and \( y_N \) will be sufficient to support \( c_N \) (with probability 1) if and only if

\[
x_N + \frac{y_N}{r_N} \geq c_N
\]

or, equivalently,

\[
x_N \geq c_N - \frac{y_N}{r_N}
\]

since if \( x_N < c_N - \frac{y_N}{r_N} \), additional borrowed funds cannot be fully secured (by the "no-easy-money condition" (3)) and would therefore be unavailable. Moving backwards in time, we obtain by the same reasoning

\[
x_j \geq c_j - y_j \quad j = 1, \ldots, N
\]

and specifically,

\[
x_1 \geq c_1 - y_1
\]
where
\[ c_j = c_j + \frac{c_{j+1}}{r_j} + \cdots + \frac{c_N}{r_j \cdots r_{N-1}}, \]
as a necessary and sufficient condition for A to be able to avoid extra work with probability 1. The sufficiency of (6) follows from the observation that when (6) holds, the strategy
\[ z_{ij}(x_j) = x_j - c_j \]
(7)
\[ z_{ij}(x_j) = 0 \quad i = 2, \ldots, M_j \]
for \( j = 1, \ldots, N \) always yields (5). The necessity of (6) can be seen from the observation that (7) and
\[ x_1 < C_1 - Y_1 \]
(8)
\[ x_j < C_j - Y_j \quad j = 2, \ldots, N ; \]
furthermore, (8) and any policy differing from (7) in some period implies, in view of (3),
\[ \Pr\{x_j < C_j - Y_j\} > 0 \quad j = 2, \ldots, N \]
(10)
In other words, (10) states that A may have to do extra work if the present value of his income stream plus his wealth is less than the present value of his idealized consumption pattern. We shall not consider here the nontrivial problem of how he might distribute the extra
work between the periods and/or how he might attempt to alleviate it altogether by risky investments when (8) holds. Instead, we shall focus our attention on the case when

(6) \[ x_1 \geq C_1 - Y_1. \]

However, it is clear that (6) is the dividing line which in effect determines where in his preference hierarchy A finds himself and which "prepotent goals will monopolize consciousness" at the beginning of period 1. If (8) holds, the desire to achieve \( c_1, \ldots, c_N \) "will tend of itself to organize the recruitment of the various capacities of the organism" (see A2). If (6) holds or (5) results in some period \( j > 1 \) (at least with strict inequality), "the next prepotent need" (the bequest motive) emerges "to serve as the center of organization of behavior" (see A3) and the need to moonlight can be "forgotten" (just how it can be "forgotten," in view of A4, will be shown later).

Suppose first that equality holds in (6) and consider any decision point \( j \). If \( x_j - c_j \leq 0 \), there is nothing to invest in period \( j \) and no lender in sight to borrow beyond \( c_j - x_j \) from since 100% collateral cannot be offered by (3). If \( x_j - c_j > 0 \), \( x_j - c_j \) would be invested at the safe return \( r_j - 1 \); this avoids any need for extra work, as shown earlier, whereas if some portion of \( x_j - c_j \) were placed in risky assets, (3) gives

(11) \[ \Pr\{x_n < C_n - Y_n\} > 0 \quad n = j + 1, \ldots, N \]

and

(12) \[ \Pr\{x_{N+1} > 0\} > 0 \]
Thus, there would be some chance of extra work and some chance for a
bequest. But by A4, no bequest, no matter how large, combined with an
amount of extra work, no matter how small the probability of its
necessity, is preferred to no work plus no bequest. Consequently, when
(6) holds with equality, A would engage only in lending and borrowing
(financial opportunities). Thus, holdings of risky assets are non-
optimal whenever

\[(13) \quad x_1 = c_1 - y_1.\]

In this case, extra work can and will be avoided, and no bequest will
be made. The policy (7) is optimal.

Consider now the case when (6) holds with strict inequality.
This means that A will consume amounts \(c_1, \ldots, c_N\) and avoid all extra
work; furthermore, he can begin to think about the possibility of
making a positive bequest \(x_{N+1}\). The necessity of 100% collateral
implies that the constraint

\[(14) \quad \Pr\{x_{N+1} \geq 0\} = 1\]

must be observed. (14), in turn, induces the constraints

\[(15) \quad \Pr\{x_j \geq c_j - y_j\} = 1 \quad j = 2, \ldots, N\]

in view of the "no-easy-money condition" (3). Note that the constraints
(15) are also implied by A4 since they are necessary to avoid extra work
with probability 1. By the principle of optimality [4: 83], we now
obtain
\[(16)\quad f_j(x_j) = \max \mathbb{E}\left[ \frac{f_{j+1}(x_{j+1})}{z_j} \right] \quad j = 1, \ldots, N,\]

where

\[(17)\quad f_{N+1}(x_{N+1}) = b(x_{N+1}),\]

subject to

\[(18)\quad \Pr\{x_{j+1} \geq c_{j+1} - y_{j+1}\} = 1 \quad j = 1, \ldots, N\]

(where clearly \(c_{N+1} - y_{N+1} = 0\)). Since expenditures must equal available funds at each decision point \(j\) and available funds at each following decision point are made up of the investment proceeds at that point, we have

\[(19)\quad \sum_{i=1}^{M_j} z_{ij} + c_j = x_j \quad j = 1, \ldots, N\]

and

\[(20)\quad x_{j+1} = r_j z_{1j} + \sum_{i=2}^{M_j} \beta_{ij} z_{ij} + y_j \quad j = 1, \ldots, N\]

Combining (19) and (20), we obtain

\[(21)\quad x_{j+1} = \sum_{i=2}^{M_j} (\beta_{ij} - r_j) z_{ij} + r_j (x_j - c_j) + y_j \quad j = 1, \ldots, N,\]

(16) and (17) now give

\[(22)\quad f_N(x_N) = \max z_N \mathbb{E}\left[ b\left( \sum_{i=2}^{M_j} (\beta_{iN} - r_N) z_{iN} + r_N (x_N - c_N) + y_N \right) \right] \]

subject to

\[(23)\quad \Pr\{ \sum_{i=2}^{M_j} (\beta_{iN} - r_N) z_{iN} + r_N (x_N - c_N) + y_N \geq 0 \} = 1\]
But in view of (3), (23) can only be satisfied if

\[(24) \quad x_N - C_N + Y_N \geq 0\]

Thus, \(f_N(x_N)\) exists for \(x_N \geq C_N - Y_N\); furthermore, it is monotone increasing and strictly concave (see Fama [11]). Since (24) coincides with constraint (18) for \(j = N - 1\), we obtain by induction that \(f_j(x_j)\) exists for \(x_j \geq C_j - Y_j\), \(j = 1, \ldots, N\), and is monotone increasing and strictly concave since we get, for any \(j < N\),

\[(25) \quad f_j(x_j) = \max \mathbb{E} \left[ f_{j+1}(\sum_{i=2}^{M_j} \beta_{ij} x_j - r_j) z_{ij} + r_j (x_j - c_j) + y_j \right] \]

subject to

\[\text{Pr}(x_{j+1} \geq C_{j+1} - Y_{j+1}) = 1\]

which gives the indicated result.

When (2) holds, the monotonicity and strict concavity of \(f_1(x_1)\) implies that \(z_{i1}^* > 0\) for some \(i \geq 2\) whenever

\[(26) \quad x_1 > C_1 - Y_1\]

(see Hakansson [21]). Thus, when (6) holds with strict inequality, risky investments are always optimal. This in turn, in view of the "no-easy-money" condition (3), implies that a (positive) bequest will be left with positive probability when (26) holds.
If, and only if,

\[
 b(x_{N+1}) = \begin{cases} 
 -x_{N+1} & \gamma < 0 \\
 \log x_{N+1} & \gamma = 0 \\
 x_{N+1}^\gamma & 0 < \gamma < 1 
\end{cases}
\]

(27) \( f_j(x_j) \) has the form

\[
 f_j(x_j) = a_j b(x_j - c_j + y_j) + k_j 
\]

(28) \( j = 1, \ldots, N \),

where \( a_j \) and \( k_j \) are constants, and the optimal investment strategy satisfies the separation property [29]. Thus, the optimal investment policy in period \( j \) has the form

\[
 z_{ij}^* (x_j) = v_{ij}^* (x_j - c_j + y_j) 
\]

(29) \( i = 2, \ldots, M_j \)

\[
 z_{ij}^* (x_j) = x_j - c_j - \frac{1}{2} \sum_{i=2}^{M_j} z_{ij}^* (x_j) 
\]

(30)

For each \( j \), the \( v_{ij}^* \) are constants which depend only on the return distribution \( F_j \), on the interest rate \( r_j - 1 \), and on (27); that is, the optimal policy is myopic (see [29, 20]). Thus, the important property of myopia remains conditionally optimal in the particular case of lexicographic preferences we have considered.

The larger the ideal consumption stream \( c_1, \ldots, c_N \), or, more precisely, the larger its present value \( C_j \), the more conservative the optimal investment policy \( z_j^* \) would be (see (29)). That is, the
optimal amount invested in opportunity $i$ is smaller the larger the consumption "demand" $C_j$ is, or the smaller income $Y_j$ is, ceteris paribus. However, this statement cannot be made in general; when $b(x_{N+1})$ is increasingly risk-averse in the Pratt–Arrow sense [32, 3], as it would be if $b$ were quadratic, for example, the opposite is true, at least in some periods. But one thing that can safely be said is that the "investment base exposable to risk," $x_j - C_j + Y_j$, is always decreasing in $C_j$ and increasing in $Y_j$.

Significant complications arise if one or more of the consumption stream $c_1, ..., c_N$, the interest rates $r_1, ..., r_N$, and the noncapital income stream $y_1, ..., y_N$ are viewed as stochastic. The same is true if the individual's life-time is considered to be random, with one exception. If A believes or acts as if his horizon is $N$ but any lender attaches a positive probability to his death in each period $j$, the solvency constraints

$$Pr\{x_{j+1} \geq 0\} = 1 \quad j = 1, ..., N-1$$

must be added to (18). Note that when

$$-C_j + Y_j \leq 0 \quad j = 2, ..., N$$

holds,

$$x_{j+1} - C_{j+1} + Y_{j+1} \geq 0 \quad \text{implies} \quad x_{j+1} \geq 0, \quad j = 1, ..., N-1$$

Thus, (31) is never effective whenever (32) holds and can be ignored in that case; A would leave an estate (possibly zero) even if he dies "pre-maturely."
VI. THE BURDENED FAMILY MAN

We now turn our attention to individual B, who, unlike A, does not mind spending immodest sums on himself and on his family should the opportunity arise. However, while less charitable than A, B will not permit his consumption level to drop below a certain floor, which we shall call the subsistence level. We will not concern ourselves here with whether the "subsistence level" is physiologically or psychologically based, only with its existence. The following additional notation will be needed:

\[ s_j = \text{amount of subsistence level consumption in period } j \]
\[ S_j = s_j + \ldots + \frac{s_N}{r_j \ldots r_{N-1}} \]
\[ e_j = c_j - s_j \]

*Given* that the subsistence level can be assured, we assume that B's preferences for excess consumption (consumption above the subsistence level) and his bequest \( x_{N+1} \) are representable by the functions

\[
U_j(e_j, e_{j+1}, \ldots, e_N, x_{N+1}) = u(e_j) + \alpha_j u(e_{j+1}) + \ldots + \alpha_j \ldots \alpha_{N-1} u(e_N) + \\
+ \alpha_j \ldots \alpha_n b(x_{N+1}) \quad j = 1, \ldots, N
\]

Again, we must first find the conditions under which the preferences (33) will be "pre-dominant" (applicable).
The difference equations (21) now become

\[ x_{j+1} = \sum_{i=2}^{M_j} (1_j - r_j)z_{ij} + r_j(x_j - e_j - s_j) + y_j, \quad j = 1, \ldots, N \]  

By the reasoning in Section V, three situations must be distinguished:

\[ x_1 < S_1 - Y_1 \]  
\[ x_1 = S_1 - Y_1 \]  
\[ x_1 > S_1 - Y_1 \]

If (35) holds, B cannot achieve the subsistence level with probability 1. If his preferences in this situation are similar to A's (see A2), then he may choose to moonlight in order to reach it. The particular strategy to be used would depend on his disutility over time for extra work and the chances of gains from investments. The applicable model would be rather complex and will not be developed here (see footnote 1). However, if B's preferences are indeed lexicographic, then the utility functions (33) are only latent and therefore not applicable as long as

\[ x_j < S_j - Y_j \]

for the decision point in question, \( j \).

In analogy with (13), (36) rules out risky investment for all time, in view of (3), if B does not wish to risk having his resources fall below the subsistence level with positive probability in order to be able to exceed it with positive probability (see A4). In fact, this aspect of B's preferences induces the constraints
(39) \[ \Pr \{ \mathcal{S}_j - Y_j \geq X_j \} = 1 \quad j = 1, \ldots, N, \]

while the solvency requirement implies the constraint

(14) \[ \Pr \{ X_{N+1} \geq 0 \} = 1 \]

It should be noted, in analogy with Section V, that (14) also implies

(39). B's optimal strategy then becomes when (36) holds, for \( j = 1, \ldots, N \):

(40) \[ c_j^*(x_j) = s_j \]

(41) \[ z_{ij}^*(x_j) = 0 \quad i = 2, \ldots, N_j, \]

(42) \[ z_{1j}^*(x_j) = x_j - s_j \]

Finally, when (37) holds, the utility function (33) is truly
dominant and will remain so in view of (14) and (39) (unless \( x_j \)
decreases to \( S_j - Y_j \) in some period, in which case (40)-(42) apply
from that point on).

When the conditional utility functions (33) apply at some
decision point \( n \), B's decision problem may be stated, by the principle
of optimality, as

Problem B:

(43) \[ f_j(x_j) = \max_{e_j, z_j} \left\{ u(e_j) + \alpha_j E[f_{j+1}(x_{j+1})] \right\}, \quad j = n, \ldots, N, \]

where

(17) \[ f_{N+1}(x_{N+1}) = b(x_{N+1}), \]

subject to
(14) \[ \Pr\{x_{N+1} \geq 0\} = 1 \]

and (hence)

(39) \[ \Pr\{x_j \geq S_j - Y_j\} = 1 \quad j = n, \ldots, N \]

As noted, a solution to Problem 3 exists whenever

(44) \[ x_n \geq S_n - Y_n \]

When \( b(\cdot) = u(\cdot) \) and

\[
u(e_j) = \begin{cases} -e_j^\gamma & \gamma < 0 \\ \log e_j & \gamma = 0 \\ e_j^\gamma & 0 < \gamma < 1, \end{cases}
\]

the optimal strategies are analogous to those in [17] and it is readily verified that the solution has the form, for \( j = n, \ldots, N \),

(46) \[ f_j(x_j) = A_j u(x_j - S_j + Y_j) + D_j \]

(47) \[ c^*_j(x_j) = B_j (x_j - S_j + Y_j) + s_j \]

(48) \[ z^*_{ij}(x_j) = (1 - B_j) \psi_{ij}^* (x_j - S_j + Y_j) \quad i = 2, \ldots, M_j \]

(49) \[ z_{1j}^*(x_j) = x_j - c_j^*(x_j) - \sum_{i=2}^{M_j} z_{ij}^*(x_j) \]

where \( A_j, B_j \) and \( D_j \) are constants. As in [17] and [15], \( 0 < B_j < 1 \) for all \( j \) and the constants \( \psi_{ij}^* \) depend only on \( u, F_j \), and \( r_j \). On the whole, the properties of the optimal strategies (47)-(49) differ from
those in [17] and [15] in only one essential respect (the presence of $s_j$ and $S_j$). This difference will now be examined more closely.

From (47), we see that $c_j^*(x_j) \geq s_j$ in each period (not surprising in view of our model). But when a subsistence level is not present (as in previous models), the optimal consumption level may drop down at least arbitrarily close to zero if the investments should turn out badly under the optimal policy. This appears intrinsically unreasonable even though for such a drop to be possible under the optimal policy, of course, the chances for a lavish consumption level in the future would have to be substantial.

The net impact of the subsistence level is to inhibit risk-taking. In the model we have examined, B would use the same investment mix (mutual fund) of risky assets as his neighbor for whom $S_1 = 0$. But B would expose to risk only fraction

$$\frac{x_j - s_j + y_j}{x_j + y_j}$$

of what his neighbor would if he had resources $x_j$ and $y_j$. As a result, B will experience a smoother consumption pattern.

In reviewing the papers which have combined the additive utility function form (33) with the iso-elastic functions (45), Pye [33] noted that the resulting utility of wealth functions $f_j(x_j)$ display stationary risk aversion over time (at least in the absence of a non-capital income stream). This led him to investigate the multiplicative form in conjunction with the class (45). His results showed that $f_j(x_j)$ now either displays increasing or decreasing risk aversion over time. Empirical observation and folklore both suggest that investment behavior becomes
more conservative as the typical individual advances from middle to old age. For individuals whose earning power is greatest in their middle years and who retire at the normal time with or without a pension, $Y_j - S_j$ will typically decrease in $j$. Whenever this is the case, it is easily verified (see (45) and (46)) that $f_{j+1}(x)$ is more risk averse than $f_j(x)$ for all $x$ and $j$ in the present model. However, if wealth is defined as $x_j + Y_j$, a more reasonable definition in the present model since $y_j, \ldots, y_N$ can be exchanged for $Y_j$, then risk aversion at any given wealth level decreases with age whenever $S_j$ decreases in $j$.

VII. THE EMBATTLED UNIVERSITY

Consider a (private) university which has "arrived" in the sense that it already has the physical plant, the faculty, and the students required to offer a "quality" education. The "building" period having been completed, one might expect that the goals which were predominant during the development phase would fade into the background and be replaced by a new predominant set of goals, that of maintaining the quality of the university for an indefinite future "at all cost." Let us consider the higher-order specific preferences which might reasonably be expected to emerge in relation to this goal.

Assume that the current spending level is $c_1$ and that an average compound long-run growth rate, $g$, of that level is visualized as necessary if the university is to be able to continue to offer a quality education. While the growth rate in total spending may depart from $g$ in any given year, wide fluctuations cannot be tolerated if quality is to be maintained--assume therefore that proportion $p$ of $c_1$, which we denote $s_1$, is required to grow at rate $g$ without
fluctuation. Denoting the endowment and other spendable funds at the beginning of period \( j \) by \( x_j \) and "income" from outside sources in period \( j \) by \( y_j \) (tuition, grants, gifts, etc.), we obtain the following difference equation in analogy with (34) (assuming for simplicity stationary investment returns and opportunities, we can drop subscript \( j \) on \( \beta_{ij}, M_j, \) and \( r_j \)):

\[
x_{j+1} = \sum_{i=2}^{M} (\beta_{i-r}z_{ij} + r(x_{j-e}-s_j) + y_j) \quad j = 1, 2, \ldots
\]

Again, it is clear that for the university to be able to maintain quality under rising costs, its resources \( x_1 \) and its "income" \( y_1, y_2, \ldots \) must in some sense be sufficient. Typically, of course, these resources are not adequate without considerable activity in the fund-raising area (one of the "components" of \( y_j \)). But it is also clear that a shrewd investment policy can help to offset the need for gifts. Since investment decisions generally require less sweat and are more pleasurable to conduct than fund-raising activities (at least from the point of view of university people), a basic preference for "minimizing" the necessary fund-raising by "maximizing" investment results would appear to be held quite naturally among university administrators. Lexicography in this area, and at this level, seems self-evident: would anyone prefer more fund-raising, with some positive probability, to an avoidable "sub-optimal return" on invested funds (no matter how small the probability)?

Operationally, of course, there is no way the preferences just mentioned can be implemented in a single period since investment returns
are random (how do you "maximize" a random variable?). In a multi-
period context, however, the average return generally begins to obey the
law of large numbers, at least in a compound sense. But if we consider
several periods, the disutilities of fund-raising in different periods
must be related to each other; this relationship is usually referred to
as the time-preference aspect. Suppose, for purposes of exposition,
that the university administration wishes to keep the differences $d_j$
given by 3

$$d_j = y_j - rs_j \quad j = 1, 2, \ldots$$

constant. Since $s_j$ grows (exponentially) at rate $g$, the assumption
that

$$d_j = d \quad j = 1, 2, \ldots$$

implies that the level of fund-raising activities in various years, or
at least total "income" $y_j$, be closely related to the spending pattern
in those years. While assumption (52) simplifies our analysis (and
does not seem unreasonable), it is not critical for the main result to
be derived.

Let

$$D = \frac{d}{r - 1},$$

and

$$p_j = \frac{e_j}{x_j + D} \quad j = 1, 2, \ldots.$$
(55) \[ R_j(z_j) = \sum_{i=2}^M \frac{\beta_i^r}{(\pi_j + D)(1-p_j)} + r \quad j = 1, 2, \ldots \]

Clearly, \( D \) is the present value of the stream \( d_1, d_2, \ldots = d, d, \ldots \) discounted at \( r - 1 \). It corresponds to \( Y_j - C_j \) in Section V and to \( Y_j - S_j \) in Section VI; in the present model this difference is the same in each period. Thus, the university can achieve the "bare-bones" spending levels \( s_1, s_2, \ldots \) with probability 1 if and only if

(56) \[ x_j \geq -D \quad j = 1, 2, \ldots \]

so that, in analogy with our previous models,

(57) \[ \Pr\{x_{j+1} \geq -D\} = 1 \quad j = 1, 2, \ldots \]

becomes an investment constraint in each period \( j \). It is also clear that minimizing \( D \) also minimizes the necessary fund-raising in each period: the smaller \( D \) is the smaller \( d \) is (see (53) and the smaller \( d \) is the smaller the stream \( y_1, y_2, \ldots \), which includes gifts, is (see (51)). \( p_j \), a decision variable (except for \( j = 1 \) since \( e_1 = c_1 - s_1 \) is fixed), is simply the proportion of the "free" funds \( x_j + D (\geq 0) \) spent on operations in period \( j \) in excess of the "subsistence" level \( s_j \), while \( R_j(z_j) \), a random variable, is 1 + the return on the "free" funds actually invested in period \( j \), \((1-p_j)(x_j + D)\). Note that the constraint (57) is equivalent to the constraint

(58) \[ \Pr\{R_j(z_j) \geq 0\} = 1 \quad j = 1, 2, \ldots \]

The difference equation (50) now becomes, using (51), (54), and (55),

\[ \text{...} \]
(59) \[ x_{j+1} = (1-p_j)(x_j+D)(R_j-r) + r(x_j-p_j(x_j+D) + \frac{d}{r}) \]

\[ = (1-p_j)(x_j+D)R_j - D \]

Thus, we obtain, in view of (58),

(60) \[ x_{j+1} + D = (x_j+D)(1-p_j)R_j \]

\[ = (x_j+D)(1-p_j)\exp(\log R_j(z_j)) \]

which in turn gives

(61) \[ x_{j+1} + D = (x_1+D)K_j^j \]

where \( K_j \), a random variable, is given by

(62) \[ K_j \equiv \prod_{n=1}^{j} (1-p_n)^{-\frac{1}{r}} \exp\left( \frac{1}{r} \sum_{n=1}^{j} \log R_n(z_n) \right) \]

But for investment policies such that \( \log R_1, \log R_2, \ldots \) obey the law of large numbers, we obtain for all \( \varepsilon > 0 \)

(63) \[ Pr(\left| K_j - E[K_j] \right| > \varepsilon) \to 0 \quad \text{as} \quad j \to \infty \]

But \( c_j \) can only grow at an average compound rate \( g \) in the long run if its two components \( s_j \) and \( e_j \) do. Since \( s_j \) grows at rate \( g \) in each period by assumption, it follows that \( e_j \) must grow at rate \( g \) also but only in a long run sense. Since \( p_j \) is constrained by \( p_j < 1 \) (otherwise \( e_n \) cannot exceed \( 0 \) for \( n = j+1, j+2, \ldots \); see (57) and (60) and recall the implications of (36)), this is only possible if \( x_1 + D \) grows at an average compound long-run growth rate of at least \( g \) (see (54) and (61)). Thus, we must have, in view of (61) and (63),
\[(64) \quad E[K_j] \geq 1 + g \quad j \text{ large}\]

for the university to be able to satisfy its educational goals. (61), (62) and (63) now imply that the larger \(p_1, p_2, \ldots\), are, the larger \(E[\log R_1], E[\log R_2], \ldots\) must be for \(g\) to be achieved. But the larger \(p_1, p_2, \ldots\) are, the smaller \(D\) can be (see (54)). Thus, to minimize \(D\), and hence the necessary fund-raising, the university should invest in such a way as to

\[(65) \quad \text{Maximize } E[\log R_j(\tilde{z}_j)] \quad \text{each } j\]

with respect to \(\tilde{z}_j\). The investment policy which maximizes (65), known as the growth-optimal policy, is unique, myopic, does not risk ruin, is consistent with the von Neumann-Morgenstern postulates, and calls for substantial, but not excessive, risk aversion which is decreasing in the level of assets \(x_j\) (see Hakansson [21] for a more complete discussion; also Breiman [5] and Latané [24]). All of these properties are highly desirable and/or valuable; the only drawback of the growth-optimal policy resides in its computational complexity.

The preceding result is of some interest in relation to the central conclusion of the Ford Foundation's Advisory Committee on Endowment Management [1: 45]:

In our opinion, the most important present responsibility of the trustees of these institutions with respect to endowment is to shift their objective to maximizing the long-term total return. We believe the total return can be increased sufficiently to permit both a larger annual contribution to operations and greater long-term growth.
Strictly speaking, of course, this statement is, as we have noted, non-operational since long-run return is a random variable and random variables cannot be maximized. But, as (63) shows, the average compound return in the long run tends to its expected value\(^6\) for most policies by the law of large numbers and this expectation certainly can be maximized. Since this maximization implies the investment rule (65) and the resulting policy yields compliance with the law of large numbers [21], this rule is not only "consistent" with the Advisory Committee's conclusion but provides the specificity needed for its implementation.

The logarithmic utility function of wealth implied by (65) [21] automatically finds "the best reward-to-risk potentials" (see [1: 34]). Note that (65) does not imply "maximum uncertainty with respect to short-term fluctuations in the value of portfolios" [1: 53]. This is because (65) does not choose the riskiest portfolio in each period: the consequences of such choices are often disastrous in the long run (see Hakansson [18] [22]). In other words, maximization of expected average compound return over the long run does not imply maximization of expected return in each period. The first lesson to be learned from (65) is that to be successful in the long run, one must be risk averse. But this is not sufficient. Poor performance in the long run can be due to too little risk taking as well as to too much exposure of one's capital to potential gains and losses. The beauty of investment rule (65), besides its simplicity, is that it subjects the endowment to just the right amount of risk in each period in terms of maximal long-run investment results.
An Example

As an illustration of the rather dramatic savings in fund-raising and tuition which are made possible by even modest departures from the optimal management of the endowment, let us consider a simple example:

\( g \) (the required long-run growth rate in operating expenditures to maintain educational quality) \( 6.5\% \)

\( c_1 \) (current operating budget) \( $88 \) million

\( s_1 \) ("subsistence" level which must grow at rate \( g \) (75% of \( c_1 \)) \( $66 \) million

\( x_1 \) (initial (present) endowment) \( $600 \) million

\[ r \text{ (1+ return on bonds)} = 1.05 \quad (5\%) \]

\[ \beta_2 \text{ (1+ return on stocks)} \begin{cases} .70 \text{ with probability .5} & (-30\%) \\ 1.75 \text{ with probability .5} & (75\%) \end{cases} \]

For simplicity, we assume there is only one stock \((M=2)\) available for investment and that stationary \( p_j \) are desired.\(^8\) Maximizing (65), we obtain

\[ z_{2j}^*(x_j) = .75(x_j + D)(1 - p) \quad j = 1,2,\ldots \]

i.e., that three quarters of the funds available for investment should be put into stocks each period. This results in a long-run compound growth rate of investable ("free") capital almost certainly equal to 11.4% since

\[ \exp\{E[\log \: \tilde{z}_j^*(\tilde{z}_j)]\} = 1.114 \]
(see footnote 6). We now obtain from (62)-(64)

$$ (1-p)1.114 = 1 + g = 1.065 $$

or \( p = 0.044 \). But by (54),

$$ p_1 = \frac{88.66}{600+D} = 0.044 = p $$

which gives \( D = -100 \) and \( d = -5 \) (see (53)). Thus, (51) gives

$$ y_1 = d + rs_1 = -5 + 1.05 \times 66 = 64.3 $$

which means that tuition, grants, and gifts must total $64.3 million in the first year. In the \( j \)th year, they would have to be

$$ y_j^* = -5 + 1.05 \times 66 \times 1.065^{j-1} \quad j = 1, 2, \ldots $$

Suppose now that a sub-optimal investment policy \( z_2' \) is used which yields a long-run average compound growth rate on investible ("free") capital of only 8.9%\(^9\) rather than the optimal 11.4%. The preceding calculations now yield \( p' = 0.022 \) and \( d' = 20 \) so that we obtain

$$ y_j' = 20 + 1.05 \times 66 \times 1.065^{j-1} \quad j = 1, 2, \ldots $$

Since \( y_j' - y_j^* = 25 \) million for each \( j \), the sub-optimal investment policy implies that the university must raise $25 million more every single year from tuition, grants and gifts than would be necessary under an optimal investment policy.

Under the optimal policy, the university would allocate its $600 million in initial resources as follows for the first year:
To operations $ 88 \text{ million}
To investment in stocks (see (67)) 358.5 \text{ million}
To investment in bonds 153.5 \text{ million}

Total $600 \text{ million}

Furthermore, total operating expenditures would almost certainly grow from $88 \text{ million}$ for $c_1$ at an average compound rate, over many periods, of 6.5% and would satisfy

\[ c_j \geq 66 \cdot 1.065^{j-1} \quad j = 1, 2, \ldots \]

i.e., they would be supported by a substantial growing floor even in the worst circumstances of "bad luck" in the investment area.

It should be emphasized that the investment policy (65) is optimal with respect to the minimization of fund-raising activities under much more general conditions than those assumed here. Stationarity in any form or aspect is not crucial. In particular, the time-pattern for fund-raising given by (52) can be relaxed without effecting the optimal investment policy.

VIII. THE PERPETUAL FOUNDATION

As the reader might infer, the investment policy which is optimal for the university described in Section VII is also virtuous from the point of view of charitable endowment funds. Assume that the first-order goal of such funds is to be able to continue a program of substantial giving eternally. Clearly, a sufficient condition for this to be possible is that
\[ r_j \geq \varepsilon \quad j = 1, 2, \ldots \]

for some \( \varepsilon > 1 \), i.e., that the interest rate does not converge to zero.

Either (2) or (70) turns out to be a necessary condition for the first-order goal to be feasible if no additional infusion of capital is made, i.e., \( y_j = 0, j = 1, 2, \ldots \). In practice, the fund's management may prefer, above all else, that the assets of the fund actually grow at some average rate to compensate for the effects of inflation. Given this primary goal, a sensible secondary goal would appear to be the giving of as "much as possible each year."

Let \( x_1 \) denote the present net worth of the fund or the bequest establishing it and assume for simplicity that no further capital infusions are expected \( (y_1 = y_2 = \ldots = 0) \). Under stationary investment returns, we then obtain (see (50))

\[
x_{j+1} = \sum_{i=2}^{M} (\beta_i - r)z_{ij} + r(x_j - c_j) \quad j = 1, 2, \ldots
\]

\[= x_1 L_j^j,\]

where

\[
L_j = \prod_{n=1}^{j} (1-q_n)^{\frac{1}{n}} \exp \left( \frac{1}{\log R_n(z_n)} \right)
\]

\[
q_j = \frac{c_j}{x_j}
\]

i.e., \( q_j \) is the annual donation in period \( j \) as a proportion of \( x_j \) and, as before, \( R_j(\tilde{v}_j) = x_{j+1}/(x_j - c_j) \), is \( 1+ \) plus the return on investable capital. If the time-preference for giving is "stationary"
with respect to assets, i.e., if it is desired that

\[ q_j = q \quad \text{all } j, \]

then we can indeed "maximize the donation in each period" (q) while achieving an average compound long-run growth rate of the fund's net assets of \( \gamma \) since we can, as before, obtain for all \( \varepsilon > 0 \)

\[
\begin{align*}
\Pr[|L_j - 1 - \gamma| > \varepsilon] & \to 0 & \text{as } j \to \infty
\end{align*}
\]

with appropriate choices of \( q \) and investment policies \( z_1, z_2, \ldots \).

In analogy with Section VII, we obtain from (72) and (74) that

\[ q^* = 1 - \frac{1 + \gamma}{\exp\{E[\log R_j(z^*)]\}} \quad \text{any } j \]

where \( z^*_j \) is the solution to (65). Thus, if the available investment opportunities are given by (66) and the fund's average compound long-run growth rate \( \gamma \) is desired to be 1/4% per annum, then the fund could give away 10% of its net worth each year by employing a growth-optimal investment policy. Under any other (substantially different) policy, it would have to be content with giving away less than 10% of net assets.

IX. CONCLUDING NOTE

The notion of lexicographic preferences has close ties to Maslow's theory of human behavior. While the von Neumann-Morgenstern postulates are not consistent with such preferences at the global
level, no conflict arises within the various levels of the preference hierarchy, i.e., the postulates are valid for "local" outcomes. Operationally, the effect of this is that the expected utility function to be maximized at each level (except the lowest) becomes coupled to one or more constraints which must hold with probability 1.

The employment of lexicographic preference structures in economic models generally enables one to obtain more realistic implications than in the absence of such preferences under uncertainty. For example, they are apparently necessary to prevent choices which may reduce the "optimal" consumption level below the subsistence line. The assumption of lexicographic preferences also uncomplicates the decision analysis in that the total decision problem is broken down into sub-problems, only one of which is applicable at any point in time, in line with Maslow's theory.
FOOTNOTES

1. The applicable preferences appear difficult to capture. At one extreme, optimal behavior may call for going to work immediately so as to minimize the value of \( j \) for which (5) can be satisfied with probability 1. The optimal strategy for other individuals may be to postpone moonlighting as long as possible, investing in the meantime so as to minimize the probability of having to moonlight at all.

2. Growth in operating costs, of course, does not imply that the student body and/or the faculty must grow. These may conceivably be reduced although there would appear to be a minimum physical size associated with the depth and breadth that is necessary to provide a "quality" education.

3. The factor \( r \) (1 plus the interest rate) appears because operating funds were assumed to be set aside at the beginning of the year while inflows were assumed to be available at the end of the year.

4. When (1) and (3) hold, a bounded optimal policy always exists; when (4) holds, the solvency constraint (58) is never binding.

5. Except that the logarithmic function is unbounded.

6. Under stationary returns and policies, \( E[K_j - 1] \) decreases in \( j \) and becomes in the limit, whenever (63) holds,

\[
(1 - p_j) \exp(E[\log R_j(z_j)]) - 1 \quad \text{any } j
\]

where \( R_j(z_j) - 1 \) is the periodic distribution of return on capital available for investment, \( (1 - p_j)(x_j + D) \) \([22]\).
7 This figure is the estimate arrived at by Cheit [6: 138-139].

8 This would essentially agree with the conclusion of the Ford study group [1: 46].

9 In our example, this rate would be achieved by investing either

\[ z_{2j}(x_j) = 1.22(x_j + D)(1-p) \]

or

\[ z_{2j}(x_j) + .28(x_j + D)(1-p) \]

in each period, i.e., by investing 122% (this would require margin purchases) or 28% of "free" capital in stocks.

10 In relation to present assets.
REFERENCES


