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Gordon Pye is Professor, Graduate School of Business Administration, University of California, Berkeley. The research for this paper was supported in part by a grant from the Dean Witter Foundation.
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It is widely assumed in portfolio theory that investors are risk-averse, expected-utility maximizers. There is a good theoretical reason for assuming expected-utility maximization. Such behavior is well known to be consistent with several quite plausible postulates of rationality [5]. On the other hand, the main empirical foundation for such behavior in portfolio selection appears to be the observation of diversification. Risk-averse, expected-utility maximization implies diversification in portfolio selection, and investors are observed to diversify.

In recent papers it has been argued that investors in some situations appear to display risk aversion to opportunity loss rather than to outcome. Thus I have argued [3] that investors are frequently jittery with very large cash positions as well as with very small ones. A natural explanation is that they fear being caught with a large cash position, if the market should rise, just like they fear being caught with a large stock position, if the market should fall. Being subject to a large opportunity loss of either type is cause for concern. The risk-averse expected-utility maximizer, on the other hand, presumably feels quite secure when he has little doubt about the outcome as he does with a large cash position.
This is not the only evidence consistent with risk aversion to opportunity loss: I have argued [2] that the familiar strategy of dollar averaging is designed to hedge against large opportunity losses rather than against outcome variability. In particular, under conditions which may be taken to approximate those under which dollar averaging is advocated, it has been shown that dollar averaging is a nonsequential, minimax strategy and cannot be a nonsequential, risk-averse, expected-utility maximizing strategy. Under similar conditions, Bawa [1] has recently pointed out that a sequential rather than nonsequential minimax policy can also be interpreted in terms of a familiar prescription for stock market trading. That prescription is to let your profits run but to cut your losses.

Since the main empirical support for risk-averse, expected-utility maximization is diversification, the question arises as to whether diversification may be used to discriminate between these two alternatives under uncertainty. Several years ago in this journal, Paul Samuelson [4] proved a strong diversification theorem. His theorem says that, if the joint distribution of returns for the available securities is symmetric and has finite first and second moments, a risk-averse, expected-utility maximizer will invest an equal amount in each security. The same diversification property also holds for an investor who is risk-averse to opportunity loss. Thus, merely the general observation of diversification does not allow one to discriminate between these two hypotheses. In fact, risk aversion to opportunity loss implies diversification in some cases in which risk-averse, expected-utility maximization does not. For instance,
an investor will hold some stock and some cash to hedge against the opportunity loss of a market rise even if the expected return on stock is the same as that on cash. A risk-averse, expected-utility maximizer in such a case will not hold any stock. Thus, some may feel risk aversion to opportunity loss implies too much diversification. The point of this note is that more delicate data than simply the observation of diversification are required to establish risk-averse, expected-utility maximization as the empirically relevant alternative.

**MINIMAX DIVERSIFICATION**

Let \( y_i \) denote the future value of a dollar's worth of the \( i^{th} \) security and let \( y = (y_1, y_2, \ldots, y_n) \). The possible values of \( y \) are taken to be contained in the set \( \mathcal{Y} \). The elements of \( \mathcal{Y} \) are taken to be nonnegative and bounded. Following Samuelson, the elements of \( \mathcal{Y} \) are taken to be symmetric in the sense that if \( y \) is contained in \( \mathcal{Y} \), then any renumbering of the elements of \( y \) is also contained in \( \mathcal{Y} \). Also following Samuelson, the trivial case will be ruled out in which \( \mathcal{Y} \) consists only of those \( y \) which have all their elements equal. The proportion of the portfolio invested in the \( i^{th} \) security will be denoted by \( z_i \) with \( z = (z_1, z_2, \ldots, z_n) \). The feasible values of \( z \) are taken to be contained in the set \( \mathcal{Z} \). Following Samuelson, \( \mathcal{Z} \) will be taken to require \( z \geq 0 \) and \( \sum_{i=1}^{n} z_i = 1 \). Without loss of generality, the wealth to be invested in the portfolio will be set equal to 1.
Let the regret or opportunity loss of a portfolio \( z \), given the value of \( y \), be denoted by \( R \). \( R \) is equal to the amount obtained from the best portfolio for that \( y \) less the amount obtained from the given portfolio for that \( y \):

\[
R = \max_{z \in \mathcal{Y}} \left( \sum_{i=1}^{n} y_i z_i \right) - \frac{\sum_{i=1}^{n} y_i}{n} z_i
\]

\[
= \max\{y_1, y_2, \ldots, y_n\} - \frac{\sum_{i=1}^{n} y_i}{n} z_i.
\]

It will next be shown that making all values of the \( z_i \) equal (\( z_i = 1/n; i = 1, 2, \ldots, n \)) satisfies \( \min_{z \in \mathcal{Y}} \max_{y \in \mathcal{Y}} R \). Thus, investing equal amounts in all securities is a minimax strategy, if the distribution of \( y \) is symmetric.

For each \( z_i \) in any portfolio \( z \), define \( e_i = z_i - 1/n \). Note that \( \sum_{i=1}^{n} e_i = 0 \), since \( \sum_{i=1}^{n} z_i = 1 \). Substituting \( z_i = e_i + 1/n \) in (1) and using \( e_i = -\frac{\sum_{i=1}^{n} y_i}{n} \) gives:

\[
R = \max_{y \in \mathcal{Y}} \left( \sum_{i=1}^{n} y_i - \frac{\sum_{i=1}^{n} y_i}{n} e_i \right)
\]

\[
= \max\{y_1, y_2, \ldots, y_n\} - \frac{\sum_{i=1}^{n} y_i}{n} e_i.
\]

Let \( y^* \) satisfy \( \max_{y \in \mathcal{Y}} \left( \max\{y_1, y_2, \ldots, y_n\} - \frac{\sum_{i=1}^{n} y_i}{n} \right) \). Note that \( y^* \) gives the maximum possible value of \( R \) when \( z_i = 1/n \) as in this case \( e_i = 0 \).

The portfolio with \( z_i = 1/n \) for all \( i \) must be a minimax portfolio if there exists a \( y \in \mathcal{Y} \) for any \( z \in \mathcal{Y} \) which makes \( R \) at least as large as

\[
\max\{y_1^*, y_2^*, \ldots, y_n^*\} - \frac{\sum_{i=1}^{n} y_i^*}{n}.
\]

To see that this is the case, let \( k \) be the number of nonnegative values of the \( e_i \) for any alternative \( z \in \mathcal{Y} \). Let the elements of \( y^* \) be ranked according to their value from smallest to
largest. Select term $k+1$ in this rank order and renumber it $y_1'$. 

Renumber the $k$ smaller elements of $y^*$ to correspond with the indexes of those $k$ of the $e_i$ with nonnegative values. Also, renumber the $n-k-1$ remaining elements of $y^*$ to correspond with the indexes of the remaining $e_i$ with negative values. As constructed, $y'$ is feasible from the symmetry of $Y$ since $y'$ is merely a reordering of the elements of $y^*$. Also, as constructed $y'$ must be such that $(y_i' - y_1')e_i \leq 0$ for all $i$. It follows, therefore, that $R$ must satisfy

$$
R = \text{Max}(y_1', y_2', \ldots, y_n') = \frac{n}{1}y_1'/n - \frac{n}{2}(y_1'-y_1)e_i 
$$

$$
\geq \text{Max}(y_1', y_2', \ldots, y_n') = \frac{n}{1}y_1'/n
$$

$$
= \text{Max}(y_1^*, y_2^*, \ldots, y_n^*) = \frac{n}{1}y_1^*/n .
$$

The argument holds for any set of $e_i$, and therefore the maximum possible value of $R$ for any possible set of $e_i$ or $z_i$ must be at least as large as the maximum possible value of $R$ when $z_1 = 1/n$ for all $i$. Therefore, equal investment in all securities must be a minimax policy as was to be proved.

With a more judicious choice of $y'$ (and a more laborious argument), it can be shown that $R$ can be made strictly larger for any portfolio policy other than equal investment in all securities. Thus, equal investment in all securities can be shown to be the only minimax policy under the assumed conditions.

Suppose the investor has a disutility function for regret and that his objective is to minimize its expected value. Let this disutility function be given by $-U$. It is assumed that disutility increases with
regret and at an increasing rate reflecting risk aversion. Therefore, the first two derivatives of $-U$ are taken to be strictly positive: $-U' > 0$, $-U'' > 0$. Minimizing the expected value of a disutility function can alternatively be formulated as maximizing the expected value of a utility function:

$$\min_{Z \in W} E[-U(R)] = -\max_{Z \in W} E[U(W)].$$

In this form the problem is formally much the same as that studied by Samuelson. Here the argument of the utility function is regret ($R = \max(y_1, y_2, \ldots, y_n) - \sum_1^n y_i z_i$) while in Samuelson's it is final wealth ($W = \sum_1^n y_i z_i$). The crucial question, however, is the relation between $V(z) = E[U(R)]$ and $V(z) = E[U(W)]$. It is easily seen in both cases that $V(z)$ is increasing in each of its arguments, strictly concave, and symmetric. These are sufficient conditions on $V(z)$ to have $\max_{Z \in W} V(z)$ satisfied by having all the $z_i$ equal in value. Thus, suppose that $z'$ satisfies $\sum_1^n z_i = 1$ but has at least two of its elements unequal. Interchange these values giving $z''$. From symmetry it follows that $V(z') = V(z'')$. From this and concavity it follows that

$$V(\lambda z' + (1-\lambda)z'') > \lambda V(z') + (1-\lambda) V(z'') = V(z')$$

where $\lambda$ satisfies $0 < \lambda < 1$. Thus, $\lambda z' + (1-\lambda)z''$ is feasible and gives a higher value to $V(z)$ so that $z'$ cannot maximize $V(z)$ if any of its elements are unequal. For a symmetric distribution of returns, therefore, either risk aversion to opportunity loss or risk-averse,
expected-utility maximization implies equal investment in all securities. This holds whether the risk aversion to opportunity loss takes the form of the simple minimax rule or risk-averse, expected-disutility of regret minimization of the firm just considered.

A NONSYMMETRIC CASE

In his paper, Samuelson also demonstrated another weaker form of diversification for returns that are not symmetrically distributed. In particular, he showed that a security with an independently distributed return would always be included in an optimal portfolio for a risk-averse, expected-utility maximizer, if the securities had a common mean return and positive and finite variances. Consider an investor who is a risk-averse, expected-disutility of regret minimizer of the form just considered. Assume his portfolio choice is between two securities with common mean return. Using the budget constraint to eliminate \( z_2 \) in the expression for \( \mathbb{R} \) gives \( V(z_1) \) as the function to be maximized:

\[
V(z_1) = E\left\{ U(\text{Max}(y_1 - y_2, 0) - (y_1 - y_2)z_1) \right\}.
\]

Differentiating with respect to \( z_1 \) and setting \( z_1 = 0 \) gives:

\[
V'(0) = E\left\{ -(y_1 - y_2)U'(\text{Max}(y_1 - y_2, 0)) \right\}.
\]

\( V'(0) \) is equal to the covariance of \( y_1 - y_2 \) and \(-U'(\text{Max}(y_1 - y_2, 0))\) since \( E[y_1 - y_2] = 0 \). Moreover, \(-U'(\text{Max}(y_1 - y_2, 0))\) is an increasing (but not strictly increasing) function of \( y_1 - y_2 \). Therefore, \( V'(0) \) must be positive because an increasing function is positively correlated.
with its argument, as long as the function is strictly increasing over a set of possible values of the argument. It must be true that

\[ y_1 - y_2 > 0 \]

for some set of possible \( y_1 \) and \( y_2 \), as otherwise \( y_1 \leq y_2 \) for all \( y_1 \) and \( y_2 \). The latter is impossible if \( E(y_1) = E(y_2) \), as has been assumed, unless \( y_1 = y_2 \) for all possible \( y_1 \) and \( y_2 \). This last possibility has been ruled out. Similarly, it can be shown that \( V'(1) \) is necessarily negative. Having \( V'(0) > 0 \) and \( V'(1) < 0 \) means that \( V'(z_1) = 0 \) for some \( z_1 \) which satisfies \( 0 < z_1 < 1 \). Moreover, this \( z_1 \) is unique as \( V''(z_1) < 0 \). The optimal portfolio must contain a mixture of both securities.

The interesting point is that no assumption has been required about the correlation of \( y_1 \) and \( y_2 \). Nor is it necessary to assume that the returns on both securities are stochastic. For his risk-averse, expected-utility maximizer, Samuelson found it necessary to assume independence or negative correlation between the securities. It was also necessary to assume that they were both stochastic. Neither of these assumptions is necessary for an investor who is risk-averse to opportunity loss to diversify between two securities. Such an investor will buy some of a security with a stochastic return even though it offers no higher an expected return than a riskless security. More generally, he will buy some of a much more volatile security even if its return is highly, positively correlated with another security with the same mean return. The observation of diversification or absence thereof under such conditions provides a means of discriminating between the two modes of behavior under uncertainty. No empirical test of this type appears to have been made as yet.
REFERENCES


