LIFETIME PORTFOLIO SELECTION IN CONTINUOUS TIME
FOR A MULTIPLICATIVE CLASS OF UTILITY FUNCTIONS

Gordon Pye*

In a recent paper, Meyer [5] has studied the lifetime portfolio problem in continuous time for a multiplicative class of utility functions. Though obtaining a number of general characteristics of the optimal policy, he is able to obtain an analytic solution only for a special limiting case that corresponds to the additive family. The solution for this additive family is known from the independent work of Merton [4], who for continuous time attacked this case directly. The continuous time case requires the assumption that the distribution of returns follow an infinitely divisible normal process. The discrete time case does not require this assumption. The additive family in discrete time has been studied by Phelps [6], Hakansson [2], Levhari and Srinivasan [3], and Samuelson [9].

The purpose of this note is to point out that an analytic solution can be obtained for a subclass of the multiplicative family studied by Meyer. For discrete but not continuous time, this solution was previously given by Pye [8]. This solution differs significantly from that for the additive case in either discrete or continuous time. In the additive case, for a stationary distribution of returns, the proportion of wealth invested

*The author is Professor of Business Administration at the University of California, Berkeley.
in risky securities doesn't change with age. This is not true for the solution in the multiplicative case. The proportion of wealth invested in risky securities increases or decreases with age as risk aversion is greater or less than that of the logarithmic utility function. The measure of risk aversion is the index of relative risk aversion developed by Arrow [1] and Pratt [7]. If the logarithm may be thought of as the dividing line between optimists and pessimists, this result may be interpreted as follows. Optimists tend to gamble less as they grow older as they have less to gain, whereas pessimists gamble more as they have less to lose. Unlike the additive case, risk aversion in portfolio selection also depends on impatience in the multiplicative solution.

Risk aversion in portfolio selection increases with impatience for optimists, while the reverse holds for pessimists. As the propensity to consume also depends on impatience, this means risk aversion and the propensity to consume will be correlated through their mutual dependence on impatience. For optimists, the correlation will be positive, whereas for the pessimists, it will be negative.

Consider an investor at time $t$ with a remaining lifespan of $T-t$. Following Meyer, the utility of his remaining consumption stream will be taken to be the following:

$U_t = \delta \exp \left\{ \gamma \int_t^T \alpha S u(c(s)) \, ds \right\}$

\[ \delta = \begin{cases} 
+1 & \gamma > 0 \\
-1 & \gamma < 0 
\end{cases} \]

$= \lim_{\Delta t \to 0} \delta \exp \{ \gamma \alpha u(c(t)) \Delta t \} U_{t+\Delta t}$
In (1), $u(c)$ is the utility of consuming at the rate $c$ at any instant of time, $\alpha$ is a discount factor reflecting impatience which satisfies $0 < \alpha < 1$, and $\gamma$ is a parameter which will turn out to be related to risk aversion in portfolio selection in the solution to be obtained below. It is to be noted that the additive family studied by Merton can be obtained from (1) by expanding in terms of $\gamma$, letting $\gamma$ approach zero, and disregarding terms of order $\gamma^2$ or smaller.

The investor's net worth at $t$ is denoted by $w_t$. There are two securities available for investment, one risky and the other riskless.

Let $z_{lt}$ be the proportion of net worth invested in the risky security, and let $1 - z_{lt}$ be the proportion invested in the riskless one. The risky return is taken to be infinitely divisible and normally distributed. Denoting the return over $\Delta t$ by $r_t(\Delta t)$, the mean and variance of $r_t(\Delta t)$ are taken to be $\mu \Delta t$ and $\sigma^2 \Delta t$ for any $\Delta t$. The return on the riskless security is taken to be $\rho \Delta t$. It is assumed that $\mu > \rho$. The proportionate rate at which net worth is consumed will be denoted $z_{0t}$. The consumption rate is therefore:

$$c(t) = z_{0t} w_t.$$

Let $V(w_t, t)$ denote the maximum expected value of $U_t$, given $w_t$, and that an optimal policy is followed in $t$ and subsequently. Use of (1) and application of the optimality principle gives the following relation which must be satisfied by $V$ for small $\Delta t$:

$$V(w_t, t) = \max_{z_{0t}, z_{lt}} \exp(\gamma \alpha (z_{0t} w_t) \Delta t) E[V(w_{t+\Delta t}, t+\Delta t) | w_t].$$

For small $\Delta t$, the dependence of investment income on consumption withdrawal over $\Delta t$ can be neglected so that $w_{t+\Delta t}$ is given as follows:
\[ w_{t+\Delta t} = (1 + \rho \Delta t + (r_t(\Delta t) - \rho \Delta t) z_{1t} - z_{0t}(\Delta t)) w_t. \]

Expanding the exponential in (2) with respect to \( \Delta t \); expanding \( V(w_{t+\Delta t}, t+\Delta t) \) with respect to both its arguments about \( w_t \) and \( t \), respectively; and making use of (3) and the properties of \( \mathbb{E}(r_t(\Delta t)) \) and \( \text{Var}(r_t(\Delta t)) \), one obtains for small \( \Delta t \) (neglecting terms of order \( (\Delta t)^2 \) or less):

\[
V(w_t, t) = \max_{z_{0t}, z_{1t}} \text{Max} \ V(w_t, t) + (\rho + (\mu - \rho) z_{1t} - z_{0t})(\Delta t) V_t w_t \]
\[ + \Delta t V_t(w_t, t) + \frac{1}{2} \sigma^2 \Delta t z_{1t}^2 V_{ww}(w_t, t) + \gamma \alpha^t u(z_{0t}, w) \Delta t V_t(w_t, t), \]

where \( V_w \), \( V_{ww} \), and \( V_t \) have been used to denote partial derivatives with respect to the first and second arguments of \( V \). Dropping \( V(w_t, t) \) from both sides of (4), dividing by \( \Delta t \), suppressing the \( t \) subscript on the state and decision variables, and suppressing the arguments of \( V \) and its partial derivatives gives:

\[
0 = \max_{z_{0t}, z_{1t}} (\rho + (\mu - \rho) z_{1t} - z_{0t}) w_{V_t} + V_t + \frac{1}{2} \sigma^2 z_{1t}^2 V_{ww} + \gamma \alpha^t u(z_{0t}) V. \]

The optimal policy must satisfy the partial differential equation given in (5). The equation in (5) agrees with that given by Meyer with the following exceptions: \( V_t \) is not taken to be equal to zero even when \( T = \infty \), since \( V \) and the optimal policy depend on \( t \); \( \alpha \) is not suppressed by setting \( \alpha = 1 \) since having \( 0 < \alpha < 1 \) is necessary for the proper convergence of \( V \) when \( T = \infty \); the sign of \( \gamma \) is not assumed to be negative,
as having risk aversion either increasing or decreasing with age is taken to be a logical possibility. The equation in (5) agrees with that given by Merton except that the last term containing \( u(c) \) is multiplied by \( V \), reflecting the fact that the multiplicative and not the additive family is being considered.

When \( u(c) = \ln c \), an analytic solution can be obtained to (5) for either a finite or an infinite horizon. The latter can be obtained directly or as a limiting form of the finite case, providing \( 0 < \alpha < 1 \) as is being assumed. For simplicity, only the solution to the infinite case will be derived here. Portfolio selection behavior in either case is of the same form. The solution is given in the following theorem:

**Theorem:** The solution to the lifetime portfolio selection problem in continuous time for \( u(c) = \ln c \) and \( T = \infty \) is as follows:

\[
\begin{align*}
\lambda(t) &= -\frac{\gamma \mu}{\ln \alpha} \\
\sigma^2 &= \frac{(\ln \alpha)^\lambda \exp\left(-\lambda(1+\rho/\ln \alpha)\right) (1-\lambda)(\mu-\rho)^2/2\sigma^2}{\ln \alpha} \\
\sigma^*_{\text{t}} &= -\ln \alpha \\
\sigma^*_{\text{t}} &= (\mu-\rho)/\sigma^2(1-\lambda).
\end{align*}
\]

---

1 It is to be noted that having \( \alpha = 1 \), \( T = \infty \), and \( \gamma < 0 \), as does Meyer, will mean necessarily that \( V = 0 \). This follows because a policy of investing all net worth in the riskless asset and consuming the riskless income will make \( \int_t^\infty u(c(s))ds \) unbounded with certainty and, hence:

\[
V = E\{\exp \left(-\int_t^\infty u((s))ds\right)\} = 0.
\]

Presumably, many other policies will also give \( V = 0 \), so the problem is not well formulated.
To see that (6) is the solution to (5) under the specified conditions, substitute $u(c) = \ln c$ and the $V$ given in (6) into (5). Using

$$wV = \lambda V, \quad w^2V_{ww} = \lambda(\lambda-1)V, \quad \text{and} \quad V_t = \lambda V \ln w + \dot{a}V/a$$

(where the dot notation indicates the time derivative) gives:

$$0 = \max_{z_0, z_1} V[(\rho+(\mu-\rho)z_1-z_0)\lambda + \dot{\lambda} \ln w + \dot{a}/a$$

$$+ 1/2 \lambda(\lambda-1)6^2 z_1^2 + \gamma a^\tau \ln(z_0w)].$$

Taking $\delta V$ outside the maximization operator because it is positive and independent of $z_0$ and $z_1$, and observing that $\dot{\lambda} \ln w + \gamma a^\tau \ln w = 0$, gives the following:

$$0 = \max_{z_0, z_1} \delta [\lambda(\rho+(\mu-\rho)z_1-z_0) + \dot{a}/a + 1/2 \lambda(\lambda-1)6^2 z_1^2 + \gamma a^\tau \ln z_0]$$

$$0 = \dot{a}/a + \lambda \rho + \lambda \max_{z_0} (-z_0 - \ln \alpha \ln z_0) + \lambda \max_{z_1} (\lambda \frac{\mu-\rho}{6^2 (1-\lambda)} z_1 - 1/2(\lambda-1)6^2 z_1^2).$$

Solution of these two simple maximization problems gives that $z_0^\ast = -\ln \alpha$ and $z_1^\ast = (\mu-\rho)/6^2(1-\lambda)$, as stated in (6). It is to be noted that the maximum with respect to $z_1$ has been taken as internal and independent of constraints on margin purchases or short sales. Substitution of these solutions into (8) shows that $V$ is the required solution, if $\dot{a}/a$ satisfies the following differential equation:

$$0 = \dot{a}/a + \lambda \rho + \gamma a^\tau (\ln(-\ln \alpha)-1) + \lambda(\mu-\rho)^2/26^2(1-\lambda).$$
By using the \( a(t) \) given in (6), it is easily verified that this \( a(t) \) does provide a solution to (9).

For finite \( T \), the portfolio solution will be of the same form as (6) except that \( \lambda = -\gamma \alpha^t(1 - \alpha T^{-t})/\ln \alpha \). The optimal propensity to consume is no longer independent of \( t \), being given by: \( z^*_{0t} = -\ln \alpha/(1 - \alpha T^{-t}) \).

The expression for \( a(t) \) is considerably more cumbersome.

The limiting case as \( \gamma \to 0 \) gives the additive family with a logarithmic utility function. The solution for this has been given by Merton. It corresponds to the solution in (6) for \( z^*_0 \) and \( z^*_t \) with \( \gamma = 0 \) (i.e., \( \lambda = 0 \)).

The solution in (6) for \( z^*_t \) is easily interpreted to verify the results stated in the introduction. Differentiating \( z^*_t \) with respect to \( t \) gives:

\[
\frac{dz^*_t}{dt} = -\gamma \alpha^t (\mu - \rho)/6(1 - \lambda)^2.
\]

Thus, the proportion of the portfolio invested in the risky security increases with age if \( \gamma < 0 \) and decreases with age if \( \gamma > 0 \). The additive case with \( \gamma = 0 \) forms the watershed. Referring back to (5), this is easily related to the Arrow-Pratt index of relative risk aversion in portfolio selection. The optimal portfolio at any time is that obtained by maximizing \( E\{V\} \). This may be seen as follows. For gambles, small relative to wealth, \( E\{V\} \) is equal to a linear function of mean and variance where the coefficient of variance is \( wV_{\text{var}}/2V_{\text{w}} \), or \(-1/2\) times the index of relative risk aversion. Here the gamble will be small, relative to wealth for small \( \Delta t \). From (5) the optimal portfolio is given by
\[
\max_w \frac{wV}{z_1} \left( \rho + (\mu - \rho)z_1 \right) + \frac{wV}{2\sigma} 6z_1^2
\]

where the term in brackets is the required linear function of mean and variance of return. From the \( V \) given in (6) the index of relative risk aversion in portfolio selection at any time is \( 1 - \lambda(t) \). It will be greater or less than that of the logarithm (which is one) as \( \gamma \) is negative or positive. Thus, risk aversion in portfolio selection as measured by this index is seen to rise or fall with age as risk aversion is less or greater than that of the logarithm.

Differentiating \( z_{1t}^* \) with respect to \( \alpha \) gives

\[
\frac{dz_{1t}^*}{d\alpha} = \gamma \alpha^{t-1} (1 - \ln \alpha)/(\ln \alpha)^2.
\]

(11)

Thus the proportion invested in the risky security increases or decreases with increasing patience as risk aversion is greater or less than that of the logarithm. The propensity to consume (i.e., \( z_0^* = -\ln \alpha \)) decreases with increasing patience. Therefore, risk aversion in portfolio selection and the propensity to consume will be negatively or positively correlated, as risk aversion is greater or less than that of the logarithm.
REFERENCES


