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COMPOUND-RETURN MEAN-VARIANCE EFFICIENT PORTFOLIOS
NEVER RISK RUIN

by

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I. INTRODUCTION

In an earlier paper [4], one of the authors examined the implications of concentrating on the lowest moment(s) of average compound return over $N$ periods (abbreviated ACRN) in making investment decisions. In particular, maximization of expected average compound return was shown to imply the existence of a utility of wealth function in each period with the "right" properties for all finite $N \geq 2$ as well as in the limit. More importantly, mean-variance efficient portfolios with respect to average compound return were also shown to be closely related to the class of utility of wealth functions $(1/\gamma)x^\gamma$, $\gamma \leq 1/N$, for large $N$ as well as in the limit.

The properties of this class renders it both empirically plausible and highly attractive analytically: among them are monotonicity, strict concavity, and decreasing risk aversion. Moreover, the optimal mix of risky assets is independent of initial wealth (providing a basis for the formation of mutual funds) and the optimal investment policy is myopic. The purpose of this paper is to extend the class of return distributions for which the preceding results hold and to demonstrate that portfolios which are efficient with respect to average compound return do not risk ruin either in a short-run or a long-run sense.

II. PRELIMINARIES

The following notation is continued from [4]:

-1-
\[ x_j = \text{amount of investment capital at decision point } j \text{ (the beginning of the } j\text{-th period)} \ (x_j > 0) \]
\[ M_j = \text{the number of investment opportunities available in period } j, \text{ where } M_j \leq M \]
\[ S_j = \text{the subset of investment opportunities which it is possible to sell short in period } j \]
\[ r_j - 1 = \text{rate of interest in period } j \]
\[ \beta_{1j} = \text{proceeds per unit of capital invested in opportunity } i, \text{ where } i = 2, \ldots, M_j, \text{ in the } j\text{-th period (random variable). That is, if we invest an amount } \theta \text{ in } i \text{ at the beginning of the period, we will obtain } \beta_{1j} \theta \text{ at the end of that period} \]
\[ z_{1j} = \text{amount lent in period } j \text{ (negative } z_{1j} \text{ indicate borrowing) (decision variable)} \]
\[ z_{ij} = \text{amount invested in opportunity } i, i = 2, \ldots, M_j, \text{ at the beginning of the } j\text{-th period (decision variable)} \]
\[ F_j(y_2, y_3, \ldots, y_{M_j}) \equiv \Pr \{ \beta_{2j} \leq y_2, \beta_{3j} \leq y_3, \ldots, \beta_{M_j} \leq y_{M_j} \} \]
\[ z_j \equiv (z_{2j}, \ldots, z_{M_j}) \]
\[ v_{ij} \equiv \frac{z_{ij}}{x_j} \quad i = 1, \ldots, M_j \]
\[ v_j \equiv (v_{2j}, \ldots, v_{M_j}) \]
\[ \langle v_N \rangle \equiv v_1, v_2, \ldots, v_N \]
\[ E_j \equiv \{ v_{ij}; v_{ij} < 0, \ i \in S_j, \ \sum_{i=2}^{M_j} |v_{ij}| = 1 \} \]
\( v_{ij} \) clearly denotes the proportion of capital \( x_j \) invested in opportunity \( i \) at the beginning of period \( j \). Thus, \( \langle v_{ij} \rangle \) uniquely identifies the investment policy over the first \( N \) periods.

We assume that the return distributions \( F_j \) are independent and satisfy the boundedness conditions

1. \( \beta_{ij} \geq 0 \) all \( i,j \)
2. \( E[\beta_{ij}] \leq K \) all \( i,j \)
3a. \( E[\beta_{ij}] \geq r_j + \eta_2 \), where \( r_j \geq \eta_1 > 1, \eta_2 > 0 \), and \( \beta_{ij} \leq L \), some \( i \), all \( j \).
3b. \( \beta_{ij} \leq L \) \( i \in S_j \), all \( j \)

and the "no-easy-money condition"

\[
\text{Pr} \left\{ \sum_{i=2}^{M_j} (\beta_{ij} - r_j) \theta_i < \delta_1 \right\} > \delta_2 \text{ for all } j \text{ and all } \theta_1 \text{ such that } \sum_{i=2}^{M_j} |\theta_i| = 1 \text{ and } \theta_1 \geq 0 \text{ for all } i \notin S_j, \text{ where } \delta_1 < 0, \delta_2 > 0
\]

(1)-(4) are weaker than the corresponding conditions in [4], where the \( \beta_{ij} \)'s themselves (as opposed to their first moments) were assumed to be uniformly bounded and to take on their values with a (joint) probability at least equal to some number \( p > 0 \).

We also assume, as before, that the investor must remain solvent in each period, i.e., that he must satisfy the solvency constraints

\[
\text{Pr}(x_{j+1} \geq 0) = 1 \quad j = 1, 2, \ldots
\]
This constraint is necessary to achieve consistency with the standard assumption of risk-free lending and to cope with the requirements of a multiperiod model of the reinvestment type.

Since the end-of-period capital position is given by the proceeds from current savings, or the negative of the repayment of current debt plus interest, plus the proceeds from current risky investments, we have

\[ x_{j+1} = r_j z_{1j} + \sum_{i=2}^{M_j} \beta_{ij} z_{ij} \quad j = 1, 2, \ldots, \]

where

\[ \sum_{i=1}^{M_j} z_{ij} = x_j \quad j = 1, 2, \ldots. \]

Combining the preceding we obtain

\[ (6) \quad x_{j+1} = \sum_{i=2}^{M_j} (\beta_{ij} - r_j) z_{ij} + r_j x_j \quad j = 1, 2, \ldots \]

\[ = x_j R_j(\bar{v}_j) \quad j = 1, 2, \ldots, \]

where

\[ (7) \quad R_j(\bar{v}_j) \equiv \sum_{i=2}^{M_j} (\beta_{ij} - r_j) v_{ij} + r_j. \]

Note that \( R_j(\bar{v}_j) \) is 1 plus the return on the whole portfolio \( \bar{v}_j \) in period \( j \). The portfolio problem at decision point \( j \) is now seen to be one of choosing the vector of risky investments \( \bar{z}_j \equiv (z_{2j}, \ldots, z_{M_j j}) \) (proportions \( \bar{v}_j \)) so as to produce the most "favorable" distribution of end-of-period capital \( x_{j+1} \) (or \( R_j(\bar{v}_j) \)). Clearly,
\[ v_{1j} = 1 - \sum_{i=2}^{M} v_{ij}. \]

Note that when \( x_j > 0 \), the solvency constraint (5) is equivalent to the constraint

\[ \Pr(R_j(v_j) \geq 0) = 1 \quad j = 1,2,\ldots. \]  

Solving (6) recursively, we obtain

\[ x_{N+1} = x_1 \prod_{j=1}^{N} R_j(v_j) \quad N = 1,2,\ldots. \]  

Denoting the average compound return over \( N \) periods, ACRN, by

\[ C_N(\langle v_N \rangle) = 1, \text{ where} \]

\[ C_N(\langle v_N \rangle) = \left( \prod_{j=1}^{N} R_j(v_j) \right)^{1/N} \quad N = 1,2,\ldots, \]  

(9) may now be written

\[ x_{N+1} = x_1 C_N(\langle v_N \rangle)^N \quad N = 1,2,\ldots. \]  

In the following, it will be convenient to define the variables

\[ S_N(\langle v_N \rangle) = \sum_{j=1}^{N} \log R_j(v_j) \quad N = 1,2,\ldots, \]

and

\[ C_N(\langle v_N \rangle) = \frac{S_N(\langle v_N \rangle)}{N} \quad N = 1,2,\ldots \]

which gives
(14) \[ C_N(\langle \overline{v}_N \rangle) = \exp \{ C_N(\langle \overline{v}_N \rangle) \} \quad N = 1, 2, \ldots \]

By (9), we may now write

\[ x_{N+1} = x_1 \exp \{ s_N(\langle \overline{v}_N \rangle) \} \quad N = 1, 2, \ldots \]

\[ = x_1 \exp \{ NC_N(\langle \overline{v}_N \rangle) \} \quad N = 1, 2, \ldots \]

Lemma 1 in [3] now generalizes trivially to

**Lemma 1:** Let (1), (2), (3), and (4) hold. Then the sets \( V_j \) of portfolios \( \overline{v}_j \) which satisfy the short-sale constraints

\[ v_{ij} \geq 0 \quad i \notin S_j \]

and the solvency constraints (8) contain risky portfolios in addition to the riskless ones and are closed, uniformly bounded and convex. When \( x_j = 0 \), only the portfolios \( \overline{z}_j = (0, \ldots, 0) \) satisfy the short-sale constraints

\[ z_{ij} \geq 0 \quad i \notin S_j \]

and the solvency constraints (5).

The following corollaries are immediate:

**Corollary 1:** For every \( \overline{v}_j \in V_j \), there is a \( \overline{v} \in E_j \) and a number \( \lambda(\overline{v}_j) \geq 0 \) such that \( \overline{v}_j = \lambda(\overline{v}_j)\overline{v} \); moreover, the \( \lambda(\overline{v}_j) \) are uniformly bounded, i.e., there exist a \( \overline{\lambda} \) such that

\[ \lambda(\overline{v}_j) \leq \overline{\lambda} \quad \overline{v}_j \in V_j, \text{ all } j. \]
Corollary 2: For every feasible portfolio $\overline{v}_j$ containing risky assets, the variance of return is positive, i.e.,

$$\text{Var}[R_j(\overline{v}_j)] > 0 \quad \overline{v}_j \in V_j, \overline{v}_j \neq (0,\ldots,0), \quad \text{all } j.$$ 

III. EFFICIENCY AND THE CENTRAL LIMIT PROBLEM

The portfolio sequence $\langle \overline{v}_N \rangle$ is said to be efficient with respect to ACRN if there exists no other (feasible) sequence $\langle \overline{v}_N' \rangle$ such that

$$E[C_N(\langle \overline{v}_N' \rangle)] \geq E[C_N(\langle \overline{v}_N \rangle)] \quad \text{and}$$

$$\text{Var}[C_N(\langle \overline{v}_N' \rangle)] \leq \text{Var}[C_N(\langle \overline{v}_N \rangle)]$$

where one of the inequalities holds strictly.

For any finite $N$, the efficient sequences at the two "extremes" are readily identified. By Corollary 2 and the independence of the distribution functions $F_j$, there is only one sequence for which $\text{Var}[C_N(\langle \overline{v}_N \rangle) - 1] = 0$, namely $\overline{v}_1, \overline{v}_2, \ldots, \overline{v}_N = (0,0,\ldots,0)$; its expected ACRN is, using (3a),

$$E[C_N(0,0,\ldots,0)] - 1 = (r_1 r_2 \ldots r_N)^{1/N} - 1$$

$$\geq n_1 - 1 > 0.$$

There is also only one sequence which achieves the highest expected ACRN; the $j$-th component of that sequence is that portfolio $\overline{v}_j$ which maximizes

$$E[R_j(\overline{v}_j)^{1/N}]$$
subject to (8) and (16). This result was derived in [4:IV] and is not dependent on the special assumptions of that paper.\footnote{Maximization of (19) is equivalent to maximization of}

\[ E[u(x_{j+1})] \]

at each decision point \( j = 1,2,\ldots,N \), where

\[ u(x) = x^{1/N}. \]

Thus, the efficient sequence at the "upper extreme" is obtained by myopic application of the single-period utility of wealth function (20). Among the properties of (22) are monotonicity, strict concavity, and decreasing risk aversion, \footnote{i.e. \( -u''(x)/u'(x) \) is decreasing in \( x \), and the optimal investment policy is myopic and satisfies the separation property [4], i.e. the optimal mix of assets is independent of wealth.} the optimal mix of assets is independent of wealth.

The remaining efficient sequences are clearly dependent on the distributions \( F_1,\ldots,F_N \). Simple decision rules for their (exact) generation do not appear to be available. But when the central limit law operates, we found in [4] that close approximations to the "nonextreme" efficient sequences could be obtained by the myopic application of the members of the set of single-period utility of wealth functions

\[ u(x_{j+1}) = \frac{1}{\gamma} x_{j+1}^{\gamma} \quad \gamma < 1/N. \]

Presumably, the approximation can be expected to be quite close for large \( N \), although numerical results in [4] for \( N=2 \) were surprisingly accurate.

The basis for the preceding result is the following. Using the central limit theorem approximation, when applicable for a portfolio
sequence $\{\overline{v}_N\}$, we obtain, denoting the mean and variance of $\overline{v}_N$ by $\mu_N$ and $s_N^2$,

$$E[C_N(\overline{v}_N)] = \exp\left\{\frac{1}{N} \mu_N + \frac{1}{2N^2} s_N^2\right\}$$

$$\text{Var}[C_N(\overline{v}_N)] = \exp\left\{\frac{2}{N} \mu_N + \frac{1}{N^2} s_N^2\right\}(\exp\left\{\frac{1}{N^2} s_N^2\right\} - 1).$$

Provided 1) that the central limit theorem holds for the efficient sequences, and 2) that the efficient frontier (in mean-variance space) is strictly concave to the origin, an approximation to the efficient sequences was shown in [4] to be obtained by maximizing

$$\mu_N + \frac{a}{2N} s_N^2$$

subject to (8) and (16) for all values of $a \leq 1$; These sequences, in turn, were found to be generated by repeated application of the class of single-period utility of wealth functions

$$u(x) = \frac{1}{\gamma} x^\gamma \quad \gamma \leq 1/N.$$ 

Concerning 1), we will show that, under the general conditions of this paper, the utility functions (23) generate sequences which indeed satisfy the conditions of the central limit theorem; we will also demonstrate that sequences which do not are not efficient, at least in the limit. Furthermore, the resulting objective functions (see (24)) will be found to have unique maxima, which, as in [4], insures that the efficient frontier satisfies 2) when 1) holds. In contrast to [4], however, constraints
(8) may be binding in the present paper which means that each $\gamma$ in (23) need not generate a different (efficient) portfolio sequence.

While for many sequences both $x_{N+1}$ and $C_N(\overline{v}_N)$ are approximately lognormally distributed for large $N$, it should be noted that $x_{N+1}$ and $C_N(\overline{v}_N)$ can never be exactly lognormal unless $R_j(\overline{v}_j)$ is, $j = 1, \ldots, N$. But $R_j(\overline{v}_j)$ is never lognormal even if the $\beta_{ij}$ are since the lognormal distribution does not reproduce itself under addition (see (7)).

In the remainder of this section we will establish that the sequences $\log R_1(\overline{v}_1), \log R_2(\overline{v}_2), \ldots$ generated by each of the functions (23) have uniformly bounded means and variances and that they satisfy both the central limit law and the law of large numbers. To do this, we require some lemmas.

**Lemma 2:** Let $\overline{v}_{1\gamma}, \overline{v}_{2\gamma}, \ldots$ be a portfolio sequence which maximizes

$$h_j(\overline{v}_j) = \frac{1}{\gamma} E[R_j(\overline{v}_j)^\gamma]$$

subject to (8) and (16) for some fixed $\gamma$, $-\infty < \gamma < 1$. Then the maximizing sequence is unique and there exist numbers $K_1$ and $K_2$ independent of $\gamma$ such that

$$E[R_j(\overline{v}_j)^\gamma] \leq K_1$$

for all $j$.

$$E[R_j(\overline{v}_j)^{\gamma-1}] \leq K_2$$

for all $j$.

**Proof:** The feasible sets $V_j$ are convex, uniformly bounded, closed, and nonempty by Lemma 1. The function $h_j$ is continuous on the interior of $V_j$, and since it is also upper semicontinuous on $V_j$, the maximum is achieved. Uniqueness follows from the strict concavity of $h_j$. 

The upper bound $K_1$ follows from the fact that the means of the
$\beta_{ij}$ are uniformly bounded and that the sets $V_j$ are uniformly bounded.
Using Corollary 1 and (2), we set $K_1 = \frac{1}{2}K$.

The inequality (26) is more difficult to prove since $y^{-1}$ gets
large as $y$ gets close to zero. For each $\bar{v}_j$ there is a $v \in E_j$ (the
set of feasible $v$ with a norm of 1) and a $\lambda_{jY}$ such that $\lambda_{jY}v = \bar{v}_j$.
Upon differentiating (24) or

$$\frac{1}{y} \mathbb{E}[R_j(\lambda v)^y]$$

with respect to $\lambda$, we obtain, for $\lambda = \lambda_{jY}$,

$$\text{E}[R_j(\lambda_{jY}v)^{-1} \left( \sum_{i=2}^{M_j} (\beta_{ij} - r_j)v_{1Y} \right) \geq 0,$$

where equality holds if $\lambda_{jY} < \lambda_{Y}$, where $\lambda_{Y}$ is the largest $\lambda$ for
which $\lambda v$ satisfies (8). (27) may be written

$$\int R_j(\lambda_{jY}v, \omega) dP$$

$$\omega \sum_{i=2}^{M_j} (\beta_{ij} - r_j)v_{1Y} < 0$$

$$\int R_j(\lambda_{jY}v, \omega) dP$$

$$\omega \sum_{i=2}^{M_j} (\beta_{ij} - r_j)v_{1Y} \geq 0,$$

where $P$ is the relevant probability measure. Denoting the two integrals
by $A_j$ and $B_j$, we have (remembering (6))

$$A_j < 0, \quad B_j > 0, \quad A_j + B_j \geq 0.$$  \hspace{1cm} (29)

In the case of $B_j$, $R_j (\lambda_j, v_j, \omega) \geq 1$ everywhere so that, in view of (2) and the fact that $v_j \in E_j$,

$$B_j \leq r_j^{-1} \left[ \sum_{i=2}^{M_j} (\beta_{i j} (\omega) - r_j) v_{i j} \right] dP \leq r_j^{-1} K < K$$

$$\omega: \sum_{i=2}^{M_j} (\beta_{i j} (\omega) - r_j) v_{i j} \geq 0 \quad \text{all } j.$$

Thus, (29) gives

$$|A_j| \leq K \quad \text{all } j. \hspace{1cm} (30)$$

Define

$$A_j' = \int R_j (\lambda_j, v_j, \omega)^{Y-1} dP$$

$$\omega: R_j (\lambda_j, v_j) \leq 1$$

When

$$R_j (\lambda_j, v_j) \leq 1, \quad \sum_{i=2}^{M_j} (\beta_{i j} (\omega) - r_j) v_{i j} \leq \frac{1 - r_j}{\lambda_j} < 0,$$

so that
\[
A_j \leq \frac{\lambda_Y}{1-\tau_j} \int \frac{M_j}{\omega} \left( \sum_{i=2}^{M_j} (\beta_{ij} (\omega) \tau_j) \gamma_\omega \right) d\omega \\
\omega: R_j (\lambda_j \gamma) \leq 1
\]

\[
A_j \leq \frac{\lambda_Y}{1-\tau_j} \leq \frac{\gamma}{n_1} \quad x \in k_2
\]

by (3a), (30), and Corollary 1, which completes the proof.

**Lemma 3:** Let \( X \) be a random variable such that

(a) \( X \geq c > 0 \)

and

(b) \( \text{Var}[X^{-1}] \geq \epsilon_1 > 0 \) for \( -\infty < \gamma < 1 \).

Then \( \text{Var}[\log X] \geq \epsilon_2 (\epsilon_1, \gamma) > 0 \).

**Proof:** \( \text{Var}[\log X] = E[(\log X - \log \mu_1)^2] \), where \( \mu_1 \) satisfies

\( E[\log X] = \log \mu_1 \). We obtain

\( \text{Var}[\log X] \geq E[(\log X - \log \mu_2)^2] \),

where \( \mu_2 \) satisfies \( E[X^{-1}] = \mu_2 ^{-1} \), by the well-known fact that \( E[(X-\alpha)^2] \) is minimized by \( \alpha = E[X] \).

Since \( E[(X^{-1} - \mu_2^{-1})^2] = \text{Var}[X^{-1}] \geq \epsilon_1 \), the lemma is established if there is a \( c(\gamma)^2 \leq \infty \) such that

(31) \( (x^{-1} - \mu_2^{-1})^2 \leq (\log x - \log \mu_2)^2 c(\gamma)^2 \)

for all points \( x, \mu_2 \geq c \).

We may write

\[
x^{-1} - \mu_2^{-1} = \int_{\mu_2}^{x} (\gamma - 1) y^{-2} dy
\]

and
\[ \log x - \log \mu_2 = \int_{\mu_2}^{x} y^{-1} dy. \]

Now set \( c(\gamma) = e^{Y-1-\gamma}. \) Then \( |\log x - \log \mu_2| |c(\gamma)| \geq |x^{Y-1} - \mu_2^{Y-1}| \) for all \( x, \mu_2 \geq c, \) which gives (31).

**Lemma 4:** Let \( \tilde{v}_{1\hat{\gamma}}, \tilde{v}_{2\hat{\gamma}}, \ldots \) be defined as in Lemma 2. Then

\[ \text{Var}[\log R_{\hat{j}}(\tilde{v}_{\hat{j}Y})] \geq K_3(\gamma) > 0 \quad \text{all } j. \]

**Proof:** Let \( i \) be the investment of assumption (3a). If \( \sum_{k=2}^{M_j} |v_{k\hat{\gamma}}| \geq 1/2L, \) then by applying the bounds \( \delta_1 \) and \( \delta_2 \) in (4) we can find a \( K_3(\gamma) > 0 \) such that \( \text{Var}[\log R_{\hat{j}}(\tilde{v}_{\hat{j}Y})] > K_3(\gamma). \) Hence the remainder of the proof is concerned with the case \( \sum_{k=2}^{M_j} |v_{k\hat{\gamma}}| < 1/2L. \)

Since \( \beta_{k\hat{j}}(\omega) \leq L \) if \( k \in S_j \), we have \( R_{\hat{j}}(\tilde{v}_{\hat{j}Y}) \geq 1/2 \) and condition (a) of Lemma 3 is fulfilled for \( X = R_{\hat{j}}(\tilde{v}_{\hat{j}Y}). \) Also, \( \tilde{v}_{\hat{j}Y} \) is in the interior of \( V_j \) so that

\[ \frac{\partial h_{\hat{j}}(\tilde{v}_{\hat{j}Y})}{\partial v_{\hat{j}Y}} = E[R_{\hat{j}}(\tilde{v}_{\hat{j}Y})^{-1}(\beta_{\hat{j}j} - r_{\hat{j}})] = 0. \]

Let \( \overline{R}_{\hat{\gamma}} \) be the mean of \( R_{\hat{j}}(\tilde{v}_{\hat{j}Y})^{-1}. \) \( \overline{R}_{\hat{\gamma}} \) has a strictly positive lower bound \( \varepsilon(\gamma) \) since, for example, the values of \( R_{\hat{j}}(\tilde{v}_{\hat{j}Y}) \) cannot go beyond \( K + r_{\hat{j}} \) with probability greater than one half. Thus, using (3a),

\[ E[\overline{R}_{\hat{\gamma}}(\beta_{\hat{j}j} - \overline{R}_{\hat{\gamma}})] \geq \varepsilon(\gamma)n_2 = \varepsilon > 0. \]

If we subtract the above from (32) and use the fact that \( (E[R_{\hat{j}}(\tilde{v}_{\hat{j}Y})^{-1}] - \overline{R}_{\hat{\gamma}})r_{\hat{j}} = 0, \) we get

\[ E[(R_{\hat{j}}(\tilde{v}_{\hat{j}Y})^{-1} - \overline{R}_{\hat{\gamma}})\beta_{\hat{j}j}] \geq - \varepsilon. \]
For $t = -\varepsilon/2K$ we obtain

$$E[(R_j(\bar{v}_{j\gamma})^{-1} - \bar{R}_\gamma)\beta_{i,j}] =$$

$$\int (R_j(\bar{v}_{j\gamma})^{-1} - \bar{R}_\gamma)\beta_{i,j} dP + \int (R_j(\bar{v}_{j\gamma})^{-1} - \bar{R}_\gamma)\beta_{i,j} dP$$

$\omega: R_j(\bar{v}_{j\gamma})^{-1} - \bar{R}_\gamma \geq t \quad \omega: R_j(\bar{v}_{j\gamma})^{-1} - \bar{R}_\gamma < t.$

The value of the first integral is clearly greater than $Kt = -\varepsilon/2$.

By (33) the second integral must be less than $-\varepsilon/2$. This gives

$$-\varepsilon/2 > \int (R_j(\bar{v}_{j\gamma})^{-1} - \bar{R}_\gamma)\beta_{i,j} dP \geq \frac{1}{t} \int (R_j(\bar{v}_{j\gamma})^{-1} - \bar{R}_\gamma)^2 dP$$

$\omega: R_j(\bar{v}_{j\gamma})^{-1} - \bar{R}_\gamma < t \quad \omega: R_j(\bar{v}_{j\gamma})^{-1} - \bar{R}_\gamma < t$

$$\geq \frac{1}{t} \text{Var}[R_j(\bar{v}_{j\gamma})^{-1}].$$

Therefore

$$\text{Var}[R_j(\bar{v}_{j\gamma})^{-1}] > (-\varepsilon/2)(t/L) = \frac{\varepsilon^2}{4LK} > 0.$$  

Now apply Lemma 3 and the proof is complete.

**Theorem 3:** Let $\bar{v}_{1\gamma}, \bar{v}_{2\gamma}, \ldots$, be a portfolio sequence which maximizes

$$(24) \quad h_j(\bar{v}_j) = \frac{1}{\gamma} E[R_j(\bar{v}_j)^\gamma]$$

subject to (8) and (16) for some fixed $\gamma$, $-\infty < \gamma < 1$. Then the variables $\log R_1(\bar{v}_{1\gamma}), \log R_2(\bar{v}_{2\gamma}), \ldots$, where $\mu_{j\gamma} = E[\log (R_j(\bar{v}_{j\gamma}))]$ and, as before, $\sigma_{\gamma}^2 = \text{Var}[S_n(\bar{v}_{\gamma})]$, satisfy
\begin{align}
\frac{1}{N} \sum_{j=1}^{N} \mathbb{E}[|\log R_j(\nu_{jY}) - \mu_{jY}|^2] & \to 0 \\
\text{and} \\
\frac{1}{3 \sigma_{NY}^2} \sum_{j=1}^{N} \mathbb{E}[|\log R_j(\nu_{jY}) - \mu_{jY}|^3] & \to 0.
\end{align}

i.e. they obey the law of large numbers and the central limit theorem (Loève [6, p. 275]).

Proof: For large \( y \), \( y \geq \log^3 y \) and for small \( y \), \( |y^{-1}| \geq |\log y|^3 \).

Therefore Lemma 2 implies that the second and third moments of

\( |\log R_j(\nu_{jY}) - \mu_{jY}| \) are uniformly bounded (call this bound \( B \)). By Lemma 4,

\( \sigma_{NY} \geq \sqrt{\frac{N}{K(\gamma)}} \); upon insertion into the left side of (35) we obtain

\[ \frac{NB}{N^{3/2}K(\gamma)^{3/2}} \to 0. \]

The proof of (34) is even simpler and is omitted.

If we strengthen assumption (2) to read

\begin{equation}
\mathbb{E}[\beta_{ij}]^{2+\delta} \leq K \quad \text{for some } \delta > 0, \quad \text{all } i,j,
\end{equation}

then we can also establish

**Theorem 2:** Let \( \nu_{1Y}, \nu_{2Y}, \ldots \) be a sequence of Theorem 1 and assume that (36) holds. Then \( \text{Var} [C_N \langle \nu_{NY} \rangle] \to 0. \)

**Proof:** Letting

\[ \mu_N = \mathbb{E}[S_N(\nu_{NY})] \]

and
\[ y_N = S_N \langle \tilde{y}_{N,Y} \rangle - \mu_N, \]

we obtain

\[
\text{Var} [C_N \langle \tilde{y}_{N,Y} \rangle] = \text{Var} [\exp(G_N \langle \tilde{y}_{N,Y} \rangle)] \\
\leq \mathbb{E}[(\exp((y_N + \mu_N)/N) - \exp(\mu_N/N))^2] \\
= \mathbb{E}[\exp(2(\mu_N/N))(\exp(2(y_N/N)) + 1 - 2\exp(y_N/N))].
\]

The factor \( \exp(2(\mu_N/N)) \) is uniformly bounded by \( (1+N)^2 \), \( (y_N/N) \), and hence \( 2(y_N/N) \), both converge in probability to the point 0 by Theorem 1. Since \( e^Y \) is a continuous function \( \exp(y_N/N) \) and \( \exp(2y_N/N) \) both converge in probability to the point 1 (Billingsley [2, Corollary 2, p. 31]). The proof will be complete when we establish that \( \exp(y_N/N) \) and \( \exp(2y_N/N) \) are uniformly integrable and hence \( \mathbb{E}[\exp(y_N/N)] \) and \( \mathbb{E}[\exp(2y_N/N)] + \mathbb{E}[1] = 1 \) (Billingsley [2, Theorem 5.4, p. 32]). In particular we will establish that

\[
\sup_N \mathbb{E}[(\exp(2y_N/N))^{1+\epsilon}] < \infty \quad \text{for some positive } \epsilon, \text{ which is a sufficient condition for } \exp(2y_N/N) \text{ to be uniformly integrable (Billingsley [2, p. 32]). Write}
\]

\[
\mathbb{E}[(\exp(2y_N/N))^{1+\epsilon}] = \exp(-2+\delta)\mu_N/N \mathbb{E}[\exp((2+\delta)(y_N + \mu_N)/N)].
\]

The first factor is bounded since \( \mu_N \) is bounded below \( (26) \) and the proof of Theorem 1). The second factor equals

\[
\mathbb{E}[\left(\sum_{j=1}^N \tilde{V}_j^{1/N}\right)^{2+\delta}] \leq \mathbb{E}[\left(\sum_{j=1}^N \tilde{V}_j^{1/N}\right)^{2+\delta}] \]

using the arithmetic-geometric mean inequality. The latter is less than or equal to \( \mathbb{E}[\sum_{j=1}^N (\tilde{V}_j^{1/N})^{2+\delta}] \) by the convexity of \( y^{2+\delta} \). But
$E[(R_j\overline{v}_{jy})^{2+\delta}] = E[(\sum_{i=2}^{M_j} (\beta_{i j} - r_j)v_{i j y} + r_j)^{2+\delta}]$ is uniformly bounded since $E[\beta_{i j}^{2+\delta}] \leq K$ and $\overline{v}_{jy}$ is uniformly bounded. This proves the $\exp(2y_N/N)$ case; the $\exp(y_N/N)$ case is similar.

Thus, while condition (2) (as Theorem 1 shows) is sufficient for the distribution of ACRN to tend to its mean, the stronger condition (36) is necessary for the variance of ACRN to vanish with respect to the distributions generated by the class (23).

IV. EFFICIENCY IN THE LIMIT

A portfolio sequence $\langle \overline{v}_N \rangle$ is said to be efficient in the limit with respect to ACRN if there exists no other (feasible) sequence $\langle \overline{v}_N' \rangle$ such that

$$\lim \sup (E[C_N(\overline{v}_N')] - E[C_N(\overline{v}_N)]) \geq 0$$

(37)

and

$$\lim \sup (\text{Var}[C_N(\overline{v}_N')] - \text{Var}[C_N(\overline{v}_N)]) \geq 0,$$

with one of the inequalities holding strictly.

**Theorem 3:** Let $\overline{v}_1, \overline{v}_2, \ldots$, be the portfolio sequence which maximises

$$E[\log R_j(\overline{v}_j)], \quad j = 1, 2, \ldots$$

subject to (8) and (16) (the $\gamma = 0$ case), and strengthen assumption (2) to

$$E[\beta_{i j}]^{1+\delta} \leq K$$

for some $\delta > 0$. 
Then for any feasible sequence $\bar{v}_1', \bar{v}_2', \ldots, \bar{v}_N'$ we have

$$\lim \sup \mathbb{E}\left[ C_N(\langle \bar{v}_N' \rangle) \right] - \mathbb{E}\left[ C_N(\langle \bar{v}_N'' \rangle) \right] \leq 0.$$  

**Proof:** We begin by considering the portfolio sequence $\bar{v}_1^\varepsilon, \bar{v}_2^\varepsilon, \ldots, \bar{v}_N^\varepsilon$ defined by $\bar{v}_j^\varepsilon = \bar{v}_j'(1-\varepsilon)$, $j = 1, 2, \ldots, N$, where $\bar{v}_j'$ is any feasible sequence and $\varepsilon > 0$ and small. The point of introducing $\langle \bar{v}_N^\varepsilon \rangle$ is that $\text{Var}[\log R_j(\bar{v}_j^\varepsilon)]$ is uniformly bounded since the fact that (8) holds for $\langle \bar{v}_N' \rangle$ implies $\Pr\{R_j(\bar{v}_j') > c_{r_j} \} \geq 1$, $j = 1, \ldots, N$.

For $j = 1, \ldots, N$,

$$\frac{1}{1/(1-\varepsilon))}(\beta_{i1}(\omega) - r_j)\bar{v}_j'((1-\varepsilon) + r_j) > \frac{1}{\sum_{i=2}^{M_j} (\beta_{i1}(\omega) - r_j)\bar{v}_j' + r_j}$$

which is equivalent to

$$(1/(1-\varepsilon))R_j(\bar{v}_j^\varepsilon, \omega) > R_j(\bar{v}_j', \omega) \quad \text{for every } \omega.$$  

Hence

$$\log(1/(1-\varepsilon)) + \log(R_j(\bar{v}_j^\varepsilon, \omega)) > \log R_j(\bar{v}_j', \omega)$$

for every $\omega$. Since $e^y$ is monotone,

$$(39) \quad \mathbb{E}\left[ \exp\left( \frac{1}{j=1} \log(1/(1-\varepsilon)) + \log R_j(\bar{v}_j^\varepsilon)/(N) \right) \right]$$

and

$$(1/(1-\varepsilon))\mathbb{E}[C_N(\langle \bar{v}_N^\varepsilon \rangle)] > \mathbb{E}[C_N(\langle \bar{v}_N' \rangle)].$$
Returning to the two original policies,

\[ \lim \sup (E[C_N^c(\mathbf{v}_N^c)] - E[C_N(\mathbf{v}_N^*)]) \]

\[ < \lim \sup ((1+\epsilon/(1-\epsilon))E[C_N^c(\mathbf{v}_N^c)] - E[C_N(\mathbf{v}_N^*))] \]

\[ = \lim \sup E[(1+\epsilon/(1-\epsilon))\exp\{(\mu_N^c + y_N^c)/N\} - \exp\{(\mu_N^* + y_N^*)/N\}] \]

where

\[ \mu_N^c = E[S_N(\mathbf{v}_N^c)] \]

and

\[ y_N^c = S_N(\mathbf{v}_N^c) - \mu_N^c, \]

with analogous definitions for \( \mu_N^* \) and \( y_N^* \). By the definition of \( \mathbf{v}_1^*, \mathbf{v}_2^*, \ldots, \mathbf{v}_N^* \geq \mu_N^c \), so that (40) is less than or equal to

\[ \lim \sup E[\exp(\mu_N^*/N)(\exp(y_N^c/N) - \exp(y_N^*/N))] \]

\[ + \epsilon/(1-\epsilon)\exp\{(\mu_N^* + y_N^c)/N\}] \]

The \( \lim \sup \) of the expression in front of the plus sign is zero since \( \exp(\mu_N^*/N) \) is uniformly bounded by \( \lambda K \) and \( E[\exp(y_N^c/N) - \exp(y_N^*/N)] \to 0 \) by the arguments used in Theorem 2. The second term is bounded by \( \epsilon \lambda K \) and since \( \epsilon \) is arbitrary the proof is complete.

Under the assumption (36), i.e.

\[ E[\beta_{ij}^{2+\delta}] \leq K \] some \( \delta > 0 \), all \( i,j \)

we have the following two corollaries.
Corollary 3: The portfolio sequences $v^*_1, v^*_2, ...$ of Theorem 3 is efficient in the limit when (36) holds.

Proof: Apply Theorems 3 and 2.

Corollary 4: No portfolio sequence $<v^*_j>$ such that $\lim \sup \text{Var}[C_j(<v^*_j>)] > 0$ is efficient in the limit when (36) holds.

Proof: Let $<v^*_j>$ be the portfolio sequence of Theorem 3. We have

$$\lim \inf (E[C_j(<v^*_j>)] - E[C_j(<v^i_j>))] \geq 0$$

and a fortiori

$$\lim \sup (E[C_j(<v^*_j>)] - E[C_j(<v^i_j>))] \geq 0.$$

Since $\text{Var}[C_j(<v^*_j>)] \to 0$ by Theorem 2,

$$\lim \sup (\text{Var}[C_j(<v^*_j>)] - \text{Var}[C_j(<v^i_j>))] \geq 0.$$

Thus, the results of this section confirm that the crucial quantity in the limit is $\lim E[C_j(<v^*_j>)].$ As a result, any portfolio sequence which is not asymptotically close to the unique sequence $v^*_1, v^*_2, ...$ generated by (38) cannot be efficient in the limit.

V. EFFICIENCY AND LOGNORMALITY

In Section III, we noted that when 1) the central limit theorem holds for the efficient sequences and 2) the efficient frontier (in mean-variance space) is strictly concave to the origin, an approximation to the efficient sequences is obtained by maximizing (22) or, in each period, (23), subject to (8) and (16) when $N$ is large. We have already shown
that the portfolio sequences generated by (23) obey the central limit theorem and satisfy the concavity requirement (since they give unique solutions) (Theorem 1). We have also established that in the limit, portfolio sequences which are "substantially different" from the sequence \( \vec{v}_1, \vec{v}_2, \ldots \) generated by (23) when \( \gamma = 0 \) are not efficient. What remains to be ascertained is that no portfolio sequence (other than the riskless one) can be efficient for large \( N \) (based on our approximation) if it does not obey the central limit law. Only when this is done can we be sure that the class (23) generates the efficient sequences or a close approximation. There are two kinds of portfolio sequences \( \{ \vec{v}_N \} \) which do not satisfy the central limit theorem: those for which \( \log R_1(\vec{v}_1), \log R_2(\vec{v}_2), \ldots \) have "fat" tails and those for which these variables have progressively thinner tails. Of these two, the first category is more important and we will limit our attention to it.

The problem we face is that we do not know the rate at which the first two actual moments of \( C_N(\{ \vec{v}_N \}) \) converge to the corresponding moments of \( C_N(\{ \vec{v}_N \}) \) based on the central limit theorem approximation. This is not only a difficult but important unresolved question since \( \text{Var} [C_N(\{ \vec{v}_N \})] \) itself vanishes, at least when (36) holds (Theorem 2). Based on limited computational evidence, we conjecture that the moment approximation error is small compared to \( \text{Var} [C_N(\{ \vec{v}_N \})] \).

Conjecture: Let \( \mathcal{V}^* \) denote the set of sequences \( \{ \vec{v}_N \} \) which satisfy

\[
\sum_{i=2}^{M_j} |v_{ij}| \geq \alpha \quad \alpha > 0 \quad j = 1, 2, \ldots
\]
and

\[ E[|\log R_j(\bar{\nu}_j) - E[\log R_j(\bar{\nu}_j)]|^3] \leq L' \quad j = 1, 2, \ldots \]

for some number \( L' \). Then there exist numbers \( \delta(N) \to 0 \) as \( N \to \infty \) such that

\[ |E[C_N(\langle \bar{\nu}_N \rangle)] - \exp\left(\frac{\lambda}{N} \langle \bar{\nu}_N \rangle + \frac{1}{2N^2} s^2_N(\langle \bar{\nu}_N \rangle)\right)| \leq \delta(N) \]

and

\[ |\text{Var}[C_N(\langle \bar{\nu}_N \rangle)] - \exp\left(\frac{\lambda}{N} \langle \bar{\nu}_N \rangle + \frac{1}{2N^2} s^2_N(\langle \bar{\nu}_N \rangle)\right)| \leq \delta(N) \]

for all \( \langle \bar{\nu}_N \rangle \in \mathbb{R}^k \), where \( \delta(N) \) is small compared to \( \text{Var}[C_N(\langle \bar{\nu}_N \rangle)] \).

Since (41) and (42) imply (35), the variables \( \log R_1(\bar{\nu}_1), \log R_2(\bar{\nu}_2), \ldots \) obey the central limit law. Furthermore, the distributions \( s_N(\langle \bar{\nu}_N \rangle) \)

converge at a uniform rate to the normal distribution with means \( \mu_N(\langle \bar{\nu}_N \rangle) \), and variances \( s_N^2(\langle \bar{\nu}_N \rangle) \) since the bounds in (41) and (42) are fixed (Loève [6:288]).

Assuming that the conjecture holds, we proceed as follows. Let \( \Delta > 0 \) and \( A \geq \frac{\lambda}{N} \) be given, set \( \bar{\gamma} = 1/N \), and consider the portfolio sequence \( \bar{\nu}_1, \bar{\nu}_2, \ldots, \bar{\nu}_N \), where

\[ \sum_{j=1}^{N} \text{Var}[\log R_j(\bar{\nu}_j)] > s_N^2(\bar{\gamma}) + A \]

and, as before,
\[
S_N^2 = \sum_{j=1}^{N} \text{Var}[\log R_j(\tilde{V}_N^{i})] \quad j = 1, 2, \ldots .
\]

We can then show that the portfolio sequence \(\tilde{V}_N\) is \(\Delta\)-inefficient for \(N \geq n\) with respect to the mean and variance of ACRN, \(C_N = 1\), i.e. that there is a feasible sequence \(\tilde{V}_N\) such that
\[
E[C_N(\tilde{V}_N')] \leq E[C_N(\tilde{V}_N'')] + \Delta \quad N \geq n
\]
and
\[
\text{Var}[C_N(\tilde{V}_N')] \geq \text{Var}[C_N(\tilde{V}_N'')] - \Delta \quad N \geq n
\]
where at least one inequality is strict. 3

For efficient sequences, the expected ACRN ranges from \((r_1 r_2 \ldots r_N)^{1/N} - 1\) to \(E[C_N(\tilde{V}_N'')] - 1\). Thus, by (45) and Theorem 1,
\[
(46) \quad E[C_N(\tilde{V}_N')] < E[C_N(\tilde{V}_N'')].
\]

If in addition
\[
(47) \quad E[C_N(\tilde{V}_N')] \leq (r_1 r_2 \ldots r_N)^{1/N},
\]
then \(\tilde{V}_N\) is clearly inefficient by (18). If on the other hand, (47) does not hold, then there exists an \(\varepsilon > 0\) such that by moving some density from the interval \([0, \varepsilon]\) to the interval \([\varepsilon, 2\varepsilon]\) of the variables \(R_1(\tilde{V}_1'), R_2(\tilde{V}_2'), \ldots\), we obtain new variables \(R_1(\tilde{V}_1, \varepsilon), R_2(\tilde{V}_2, \varepsilon), \ldots\) (with means \(\mu_1, \mu_2, \ldots\) and variances \(s_1^2, s_2^2, \ldots\)) that obey the central limit theorem. In addition, the transformation can be done in such a way as to also satisfy
(48) \[ E[C_N(\tilde{\mathbf{v}}_{N}^i, \varepsilon)] \leq E[C_N(\tilde{\mathbf{v}}_{NY}^i)] \]

and

(49) \[ s_{Ne}^2 \geq s_{NY}^2. \]

Clearly,

(50) \[ E[C_N(\tilde{\mathbf{v}}_{N}^i)] < E[C_N(\tilde{\mathbf{v}}_{NY}^i), \varepsilon)], \]

(51) \[ \text{Var}[C_N(\tilde{\mathbf{v}}_{N}^i)] > \text{Var}[C_N(\tilde{\mathbf{v}}_{NY}^i), \varepsilon)], \]

and, in view of (43), (48) gives

(52) \[ \exp \left( \frac{1}{N} \mu_{Ne} + \frac{1}{2N^2} s_{Ne}^2 \right) \leq \exp \left( \frac{1}{N} \mu_{NY} - \frac{1}{2N^2} s_{NY}^2 \right) + 2\delta(N). \]

Case 1: If

(53) \[ \frac{1}{N} \mu_{Ne} + \frac{1}{2N^2} s_{Ne}^2 \geq \frac{1}{N} \mu_{NY} - \frac{1}{2N^2} s_{NY}^2, \]

the central limit theorem approximation gives, using (44) and (49),

(54) \[ \text{Var}[C_N(\tilde{\mathbf{v}}_{N}^i, \varepsilon)] \]

\[ \geq \exp \left( \frac{1}{N} \mu_{Ne} + \frac{1}{2N^2} s_{Ne}^2 \right) \left( \exp \left( \frac{1}{N} s_{Ne}^2 \right) \right) - 1) - \delta(N) \]

\[ \geq \exp \left( \frac{1}{N} \mu_{NY} - \frac{1}{2N^2} s_{NY}^2 \right) \left( \exp \left( \frac{1}{N} s_{NY}^2 \right) \right) - 1) - \delta(N) \]

\[ \geq \text{Var}[C_N(\tilde{\mathbf{v}}_{NY}^i)] - 2\delta(N) \]

\[ \geq \text{Var}[C_N(\tilde{\mathbf{v}}_{NY}^i)] - \Delta. \]
for \( N \) sufficiently large, (46), (51), and (54) now give the result; if the conjecture is true, \( \text{Var}[C_N(\langle \tilde{v}_N \rangle)] \) is large compared to \( \Delta \).

**Case 2:** If (53) does not hold, we obtain by minimizing \( N \sum_{j=1}^{N} \text{Var} \log R_j(\tilde{v}_j) \) subject to

\[
\frac{1}{N} \mu_N(\langle \tilde{v}_N \rangle) + \frac{1}{2N} s^2(\langle \tilde{v}_N \rangle) = \frac{1}{N} \mu_N \pi + \frac{1}{2N} s^2_N \pi
\]

a feasible sequence \( \langle \tilde{v}_N'' \rangle \) such that

\[
s''_N = \sum_{j=1}^{N} \text{Var}[\log R_j(\tilde{v}_j)] < s''_{N\gamma}
\]

since \( \mu_N(\langle \tilde{v}_N \rangle) \) and \( s^2_N(\langle \tilde{v}_N \rangle) \) are continuous in \( \langle \tilde{v}_N \rangle \) and \( \lambda \tilde{v}_{N\gamma} \) satisfies (55) for some \( \lambda < 1 \). Thus, using (43) and (44),

\[
E[C_N(\langle \tilde{v}_N'' \rangle, \varepsilon)] \leq \exp\left[\frac{1}{N} \mu_N \pi + \frac{1}{2N^2} s^2_N \pi \right] + \delta(N) = \exp\left[\frac{1}{N} \mu''_N + \frac{1}{2N^2} s''_N \pi \right] + \delta(N) \\
\leq E[C_N(\langle \tilde{v}_N'' \rangle)] + 2\delta(N)
\]

and

\[
\text{Var}[C_N(\langle \tilde{v}_N'' \rangle, \varepsilon)] \geq \exp\left[\frac{2}{N} \mu_N \pi + \frac{1}{N^2} s^2_N \pi \right] \left(\exp\left[\frac{1}{N^2} s^2_N \pi \right] - 1\right) - \delta(N) = \exp\left[\frac{2}{N} \mu''_N + \frac{1}{N^2} s''_N \pi \right] \left(\exp\left[\frac{1}{N^2} s''_N \pi \right] - 1\right) - \delta(N) \\
\geq \text{Var}[C_N(\langle \tilde{v}_N'' \rangle)] - 2\delta(N).
\]
For $N$ sufficiently large (say $N \geq n$), $2\Delta(N) \leq \Delta$, so that (50), (57), (51) and (58) give

$$E[C_N(\bar{\nu}_N)] < E[C_N(\bar{\nu}_N^*)] + \Delta, \quad N \geq n$$

and

$$\text{Var}[C_N(\bar{\nu}_N)] > \text{Var}[C_N(\bar{\nu}_N^*)] - \Delta, \quad N \geq n$$

which concludes the demonstration.

VI. EFFICIENCY AND THE RISK OF RUIN

We have found that ACRN for efficient portfolio sequences $\langle \bar{\nu}_{NY} \rangle$, when $N$ is large, is approximately lognormally distributed with mean

$$E[C_N(\bar{\nu}_{NY})] \geq \eta_1 > 1. \quad (59)$$

This means that, on the basis of the lognormal approximation,

$$\Pr[C_N(\bar{\nu}_{NY}) = 0] = 0. \quad (60)$$

Furthermore, by Theorem 2,

$$\Pr(|C_N(\bar{\nu}_{NY}) - E[C_N(\bar{\nu}_{NY})]| > \varepsilon) \rightarrow 0$$

for every $\varepsilon > 0$ as $N \rightarrow \infty$. This implies

$$\Pr[x_{N+1} > x_1] \rightarrow 1 \quad (61)$$

so that a portfolio policy which is efficient with respect to ACRN does
not risk ruin. Both (59) and (60), and hence (61), are properties not shared by portfolio sequences which are efficient under the traditional (single-period) mean-variance criterion [5]. For such policies we may have, for any $\alpha > 0$,

$$\Pr(x_{N+1} < \alpha) = 1.$$ 

Since (60) is based on an approximation, it may be reassuring to confirm (60) on the basis of individual periods for the portfolio sequence $< \tilde{v}_N >$. In view of (28), (30) can only be true if

$$\Pr(R_j(\tilde{v}_j) = 0) = 0 \quad j = 1, \ldots, N.$$ 

In fact, when, as in [4], the possible values assumed by $R_j(\tilde{v}_j)$ have a probability of at least $p > 0$, there is a lower bound $b_\gamma > 0$ such that $\Pr(R_j(\tilde{v}_j) \geq b_\gamma) = 1$ for all $j$; by (26)

$$b_\gamma \gamma^{-1} p \leq K_2,$$

which gives

$$\frac{1}{b_\gamma} \geq (K_2/p)^{-1} > 0.$$
FOOTNOTES

1. The possibility that (16) may be binding under the present assumptions is of no import.

2. The empirical significance of this property is described in Arrow [1, pp. 35-43].

3. This result follows from Corollary 3 when (36) holds. The following argument is included to indicate how the normal approximation applies in the case where $N$ is large enough for normality to be a realistic assumption, but not so large that the variance is negligible.

REFERENCES


