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SECURITIES MARKET EFFICIENCY IN AN ARROW-DEBREU ECONOMY

by

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ABSTRACT

In the context of a dynamic, uncertain and heterogeneous Arrow-Debreu economy, precision is given to the concept that present prices "fully reflect information" about future prices ("efficiency"). After taking an initial portfolio position, an individual is said to perceive new information that becomes available to him as fully reflected in revised security prices if and only if he has "nonspeculative beliefs": beliefs for which no portfolio revision is an optimal strategy. Before taking an initial portfolio position, an individual is said to perceive all information that becomes available to him as fully reflected in security prices if and only if he has "consensus beliefs": beliefs, which if held by all individuals in an otherwise similar economy, would generate the same equilibrium prices as in the actual heterogeneous economy. Personal characteristics (resources, beliefs and tastes) of individuals with consensus beliefs are identified. Although they imply restrictions on the intertemporal structure of security rates of return, these are not strong enough to create an unbiased term structure or random walk. Alternative definitions of efficiency provided by Fama and Samuelson are found to be inadequate.
The word "efficiency" as applied to securities markets has unfortunately been used to represent a variety of logically distinct concepts. In particular, it may mean:

a. Exchange-efficiency: participants are not motivated to create exchange arrangements not already provided by the market.


c. Information-efficiency: present security prices are costlessly known to all participants and information about future security prices is "fully reflected" in these present prices.

To focus undiluted attention on information-efficiency, efficiency concepts a and b will be automatically satisfied by the Arrow-Debreu model used in this paper.\(^1\) Moreover, since transactions costs are assumed nonexistent (i.e., perfect market), the remaining question and chief focus of this paper is: when will present security prices "fully reflect information" about future prices?

This question should first serve to raise the logically prior

\(^1\)In the context of an incomplete market for securities, exchange- and production-efficiency hold whenever there is universal portfolio separation so that at each date all individuals divide their wealth among consumption and the same two mutual funds, one of which is risk-free. Refer to Rubinstein [9], [10].
question, what do we mean by the phrase "fully reflect information?" In a perfect and competitive economy composed of rational individuals with homogeneous beliefs about future prices, by any meaningful definition, present security prices must fully reflect all available information about future prices. According to Fama [6, p. 387]:

First, it is easy to determine sufficient conditions for capital market efficiency. For example, consider a market in which (i) there are no transactions costs in trading securities, (ii) all available information is costlessly available to all market participants, and (iii) all agree on the implications of current information for the current price and distributions of future prices of each security. In such a market, the current price of a security obviously "fully reflects" all available information.

Moreover, the model of the securities market developed here will be informationally-efficient in this trivial case. The conditions for efficiency only become interesting when participants have heterogeneous beliefs (or face positive transactions costs, not considered here). In this case, it seems obvious that some may perceive their information as "fully reflected" in present prices, while others may not. The challenge is to identify who is who, or to put it somewhat more abstractly, to identify what information is and is not reflected in present prices. As I shall later argue, after a precise analytical definition of information-efficiency is developed, despite some attempts—Fama [6] and Samuelson [11]—its definition under nontrivial circumstances has remained ambiguous.

Section I details a model constructed to answer these questions. Section II develops a precise definition of information-efficiency and
identifies the characteristics of participants who believe present security prices fully reflect their information. Section III examines implications for the intertemporal structure of security rates of return (random walk, unbiased term structure). Section IV contrasts the definition of information-efficiency with those of Fama and Samuelson. Section V mentions the chief limitations of the model in view of the issues addressed.

I. THE ECONOMY

Appropriate answers to the questions posed require a model of a dynamic, uncertain, heterogeneous economy. Assume each individual in the economy makes decisions over a three-date \((t=0,1,2)\) horizon. At date \(t=0\), he chooses present consumption \(C_0\) and makes a provisional choice of future consumption by selecting a portfolio of contingent claims \(\hat{C}_a\) to date \(t=1\) consumption which pay off only if state \(e\) occurs at date \(t=1\), and by selecting a portfolio of contingent claims \(\hat{C}_s\) to date \(t=2\) consumption which pay off only if state \(s\) occurs at date \(t=2\). Consequently, if \(\{P_e\}\) and \(\{P_s\}\) denote, respectively, the date \(t=0\) prices to these claims, the individual will divide his date \(t=0\) wealth, \(W_0\) so that \(C_0 + \sum e P_e \hat{C}_e + \sum s P_s \hat{C}_s = W_0\).

\(^2\)Specifically, resources, beliefs and tastes are generally different for different individuals, while they all share the same opportunities (i.e., face the same prices of securities—an implication of competitive markets).

\(^3\)A type of complete markets has been assumed. For complete markets in the Debreu \([3, \text{pp. 98-102}]\) sense, a complete set of
At date \( t=1 \), the true state \( e \) occurs: contingent claims to that state pay off, contingent claims to other states at that date become worthless, and new information becomes available and is dispersed perhaps unevenly throughout the economy. This information revises beliefs held by individuals concerning prices to rule in the market at date \( t=2 \). As a consequence, and also due to the mere passage of time, the equilibrium prices of contingent claims to date \( t=2 \) consumption are revised from their prior levels set at date \( t=0 \). Since these revised prices will in general depend on state \( e \), they are denoted consumption claims to date \( t=2 \) consumption would need to be available at date \( t=0 \) with payoffs if and only if both state \( s \) and state \( e \) occur. With these claims available, the exchange opportunities at date \( t=0 \) would be sufficiently rich that no individual would need to revise his portfolio at date \( t=1 \) and a market at that date would be redundant.

It is well known that if such claims exist, then the equilibrium would be Pareto-optimal; in particular, no individual would have an incentive to create additional securities or engage in transactions outside the market. However, these claims do not exist in the economy described in this paper; nonetheless, the equilibrium will still be Pareto-optimal since it can be shown to yield exactly the same allocations as if these claims did exist. The general principle is that individuals are indifferent between relatively rich forward markets with few future revision opportunities and relatively poorly developed forward markets with many future trading dates. In reality, market set-up costs and economies of scale in exchange will help determine this trade-off. The objections raised against the assumption of rich forward markets lose a great deal of force if one recognizes that individuals may be able to make the same final choices even in the absence of some forward markets, provided sufficient opportunities for revision exist. One need only recall the virtually continuous trading provided by stock exchanges to lend strength to this position. See Arrow [1] and commentary by Dreze [5; pp. 144-145] for the genesis of this argument.
The individual's wealth $W_e$ at date $t=1$ will, therefore, depend both on state $e$ and on his prior choices so that $W_e = \hat{C}_e + \sum_s P_{s,e} \hat{C}_s$. In view of the revised prices, each individual with his own revised beliefs will in general desire to revise his provisional choices $\hat{C}_e$ and $\{\hat{C}_s\}$ made at date $t=0$. To accommodate this, denote $C_e$ and $\{C_{s,e}\}$ as his revised (and final) choice of date $t=1$ consumption and portfolio of contingent claims to date $t=2$ consumption (in general, $C_e \neq \hat{C}_e$ and $\{C_{s,e} \neq \hat{C}_s\}$). The subscript $e$ reflects that his revision at date $t=1$ will in general depend on the true state at that date. Consequently, after revision, to satisfy his wealth constraint $C_e + \sum_s P_{s,e} C_{s,e} = W_e$, Date $t=1$ is the last time choices are made. At date $t=2$ the true state $s$ occurs: contingent claims to that state pay off, and contingent claims to other states become worthless. Consequently, the individual simply consumes $C_{s,e}$.

All individuals are presumed to obey the Savage axioms of rational choice: they have heterogeneous beliefs and tastes, representable by probabilities and utility functions, so that they make choices which maximize expected utility. $\pi_e$ denotes the probability held by an individual that state $e$ will occur; given the occurrence of state $e$, $\pi_{s,e}$ denotes the conditional probability that state $s$ will occur; by the laws of probability, the unconditional probability that state $s$ will occur $\pi_s = \sum_e \pi_e \pi_{s,e}$. Just as prices of contingent claims to date $t=2$ consumption may be revised at date $t=1$, beliefs may be similarly revised (in general, $\{\pi_{s,e} \neq \pi_s\}$). Each individual is assumed to maximize an
additive, state-independent utility function over his final consumption plan \((C_0, \{C_e\}, \{C_s\})\). That is, the utility of consumption \(U_t(\cdot)\) at date \(t\) is evaluated independently of consumption at all other dates (but may depend on the date itself) and independent of states.\(^4\)

The following programming problem summarizes the above description:\(^5\)

\[
\max \quad U_0(C_0) + \sum_{e \in \hat{C}} U_e(C_e) + \sum_{s \in \hat{C}} \sum_{e \in \hat{C}} U_2(C_s, e)
\]

\[
C_0, \{C_e\}, \{C_s\}, \{e\}, \{e, s\}
\]

\[
s.t. \quad C_0 + \sum_{e \in \hat{C}} C_e + \sum_{s \in \hat{C}} C_s = W
\]

\[
C_e + \sum_{s \in \hat{C}} C_s e = W_e \quad (all \ e).
\]

Factoring and using Lagrangian multipliers,\(^6\) the problem may be restated

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\(^4\)For a more general portfolio model, excepting its assumption of constant proportional risk aversion, see Hakansson [7], who allows for incomplete markets, uncertain lifetime, and state-dependent utility. The salient feature of his model, as well as the one presented in this paper, is the freedom of prices to follow the most generalized (discrete time) stochastic process.

\(^5\)Assume that \(U^1_t > 0\) and \(U^2_t < 0\). Nonnegativity constraints on actual consumption \((C_0, \{C_e\}, \{C_s\})\) are presumed to be satisfied although they are not explicitly incorporated into the analysis. For at least one class of utility functions, \(U_t\), those with constant proportional risk aversion, the nonnegativity constraints are automatically satisfied since for this class the marginal utility of consumption is infinite at zero consumption.

\(^6\)We have made use here of stochastic Lagrangian multipliers, which are a natural generalization of standard Lagrangian techniques. For
\[
\max \quad U_0(C_0) + \sum_e \pi_e U_1(C_e) + \sum_s s^e U_2(C_s^e) - \lambda_e [C^e + \sum_s s^e] C^s_e
\]

\[
\frac{d}{dc} \left[ U_0(C_0) + \sum_e \pi_e U_1(C_e) + \sum_s s^e U_2(C_s^e) - \lambda_e [C^e + \sum_s s^e] C^s_e \right] - \lambda_0 [C_0 + \sum_e e^e C_e + \sum_s s^s C_s] - W_0.
\]

The necessary and sufficient conditions for an optimum are determined by differentiating the maximand partially with respect to each choice variable and repeating the constraints. Differentiating successively by \(C_0, \{C_e\}, \{C_s\}, \{C_s^e\}\) and \(\{C_s^e\},\)

\[
U'_0(C_0) = \lambda_0 \quad \pi_s \lambda_s = \lambda_0^p \quad \text{(all e)} \quad \sum_e \pi_e^e \lambda_s^e s^e = \lambda_0^s \quad \text{(all s)}
\]

\[
U'_1(C_e) = \lambda_e \quad \text{(all e)} \quad \pi_s^e U'_2(C_s^e) = \lambda_s^e \quad \text{(all s and e)}
\]

example, see Dreze [5]. Formulating the programming problem in this way permits it to be solved without explicitly working backwards via a derived utility function as in dynamic programming. Had we done so, for the date \(t=1\) subproblem contained in the large brackets, the derived utility of wealth function

\[
V_e(W_e) \equiv \max \quad U_1(C_e) + \sum_s s^e U_2(C_s^e) - \lambda_e [C^e + \sum_s s^e] C^s_e - W_e
\]

where \(W_e \equiv C^e + \sum_s s^e C_s^e\). In general, even though utility over consumption is state-independent, the derived utility function \(V_e(W_e)\) of date \(t=1\) wealth will not be state-independent. It will depend on state e not merely indirectly through \(W\) but also through the revised state-dependent prices and beliefs. With this subproblem solved, his date \(t=0\) problem reduces to

\[
\max \quad U_0(C_0) + \sum_e \pi_e V_e(W) - \lambda_0 [C_0^e + \sum_s s^e C_e + \sum_s s^s C_s - W_0].
\]
The stochastic Lagrangian multipliers \( \{ \lambda_e \} \) play the role of linking the decision problems at dates \( t = 0 \) and \( t = 1 \). Eliminating the Lagrangian multipliers and rearranging, the necessary and sufficient conditions for an optimum can be symmetrically represented by

1. \[ P_s = \mathbb{E}_e P_e S_e \quad \text{(all } s \text{)} \]

2. \[ \pi_e^1 U_e^1 (C_e^e) = U_0^1 (C_0^e) P_e \quad \text{(all } e \text{)} \]

3. \[ \pi_{s_e}^1 U_{s_e}^1 (C_{s_e}^e) = U_{s_e}^1 (C_{s_e}^e) P_{s_e} \quad \text{(all } e \text{ and } s \text{)} \]

4. \[ \mathbb{E}_s P_s \hat{C}_s + \mathbb{E}_e P_e \hat{C}_e + C_0 = w_0 \]

5. \[ \mathbb{E}_s P_s \hat{C}_e + C_e = \mathbb{E}_s P_s \hat{C}_s + \hat{C}_e \quad \text{(all } e \text{)}. \]

Condition (1) is an arbitrage relationship among prices, paralleling \( \pi_s = \mathbb{E}_e \pi_e S_e \), which must hold in equilibrium. Conditions (2) and (3) are the standard single-period optimality conditions linking marginal utility to prices, and conditions (4) and (5) repeat the constraints.

Let \( i \) superscript individuals and \( \{ \hat{C}_i^0, \{ \hat{C}_i^e \}, \{ \hat{C}_i^s \} \} \) represent the endowment of individual \( i \) so that \( w_i^0 = \hat{C}_i^0 + \mathbb{E}_e P_e \hat{C}_i^e + \mathbb{E}_s P_s \hat{C}_i^s \). The closure conditions then needed to endogenously determine equilibrium
prices would be

\[ \Sigma_i C_0^i = \Sigma_i \overline{C}_0^i \]

\[ \Sigma_i C_e^i = \Sigma_i \hat{C}_e^i = \Sigma_i \overline{C}_e^i \quad \text{(all } e) \]

\[ \Sigma_i C_{s,e}^i = \Sigma_i \hat{C}_{s,e}^i = \Sigma_i \overline{C}_{s,e}^i \quad \text{(all } e \text{ and } s). \]

The \( i \) superscript will continue to be omitted, unless there is danger of ambiguity.
II. NONSPECULATION AND CONSENSUS

We will now use conditions (1) - (5) to determine what information about future prices is "fully reflected" in present prices. We shall first provide a precise definition of prices that fully reflect the new information that becomes available to an individual at date \( t=1 \); then this will be extended to a definition of prices that fully reflect all information available to an individual at both dates \( t=0 \) and \( t=1 \).

An individual will be said to perceive the new information that becomes available to him as fully reflected in revised security prices if and only if he has "nonspeculative beliefs." By definition, nonspeculative beliefs are those beliefs for which no portfolio revision is an optimal strategy.\(^7\) The model described in Section I has been constructed to motivate this definition.

Observe that the variety of consumption claims available at date \( t=0 \) is just sufficient for an individual to move directly to his optimal

\(^7\) Stiglitz [12] also identifies speculation with portfolio revision. Using a three-date model, he examines the relationship between the term structure and an individual's demand for short- versus long-term bonds. At date \( t=0 \), individuals choose between short- and long-term bonds. At date \( t=1 \), the previous short-term bonds mature and the previous long-term bonds now become the only existing short-term bonds. Individuals can revise their portfolios, choosing between consumption and the short-term bonds. At date \( t=2 \), they consume their remaining wealth. Stiglitz introduces uncertainty by assuming the date \( t=1 \) price of the long-term bonds is uncertain at date \( t=0 \). Unfortunately, since Stiglitz admits no purely risky securities into his model, in an equilibrium context his economy is riskless and the injected uncertainty disappears together with the purpose of his model.
final consumption plan by choosing \( \{C^{*}_{e} = \hat{C}^{*}_{e}\} \) and \( \{C^{*}_{s,e} = \hat{C}^{*}_{s,e}\} \). In this case, he opts not to revise his portfolio at date \( t=1 \), but stays with his date \( t=0 \) provisional choices. This might well be optimal since one of the chief reasons for portfolio revision—availability of new consumption claims at future dates—is absent. At date \( t=0 \), an individual can purchase claims to consumption at every state that might occur at dates \( t=1 \) and \( t=2 \). Other possible causes of portfolio revision—barriers to marketability occasioned by nonmarketable resources, short selling restrictions, and finite transactions costs, as well as state-dependent tastes—are omitted from the model.

In fact, the economy has been so constructed as to leave only one reason for portfolio revision: a change in prices to claims to date \( t=2 \) consumption that is not fully offset by revised beliefs. In general, for claims under some state \( s \), price changes from \( P_{s} \) at date \( t=0 \) to \( P_{s,e} \) at date \( t=1 \) if state \( e \) occurs. Similarly, on the basis of newly received information about the future state \( s \), an individual revises his belief from \( \pi_{s} \) at date \( t=0 \) to \( \pi_{s,e} \) at date \( t=1 \) if state \( e \) occurs. Since his portfolio chosen at date \( t=0 \) was based on price \( P_{s} \) and belief \( \pi_{s} \), with revised price and belief at date \( t=1 \), the individual will generally desire to revise his portfolio. For example, even without formal analysis, it should be intuitively obvious that if \( P_{s,e} < P_{s} \) and \( \pi_{s,e} > \pi_{s} \), then the individual will shift more of his holdings into consumption claims to state \( s \) (i.e., \( C^{*}_{s,e} > \hat{C}^{*}_{s} \)).
However, if the change in price from $P_s$ to $P_{s,e}$ were in some sense precisely offset by the change in belief from $\pi_s$ to $\pi_{s,e}$ and this occurred for all states $\omega$ and $e$, then the individual would plan not to revise his portfolio at date $t=1$ regardless of the state $e$ that occurs at that date. If this offsetting effect occurs, then we shall say the individual has nonspeculative beliefs and revised prices fully reflect the new information received by the individual at date $t=1$. Consequently, to identify nonspeculative beliefs and prices that fully reflect new information, we need only examine the circumstances under which no portfolio revision is optimal.

**Theorem (nonspeculation condition):** No portfolio revision under all states $e$ is optimal ($\{C^*_e = \hat{C}^*_e\}$ and $\{C^*_{s,e} = \hat{C}^*_{s,e}\}$) if and only if

$$\frac{\pi_s}{P_s} = \frac{\pi_{e,s,e}}{P_{e,s,e}} \text{ (all } s \text{ and } e).$$

**Proof:** Applying conditions (1)-(5), we will (i) develop necessary and sufficient conditions for $C^*_{s,e}$ to be independent of $e$ and (ii) show that, in this case, given $C^*_{s,e}$ and $C^*_e$, it is feasible to move immediately to these choices at date $t=0$.

(i) Substituting condition (2) into (3): $\frac{U_0'(C^*_e)}{U_1(C^*_{s,e})} = \frac{\pi_s \pi_{s,e}}{P_s P_{s,e}}$.

If $C^*_{s,e}$ is independent of $e$, then there exists a number $k_s$ independent of $e$ so that $k_s \equiv (\pi_s \pi_{s,e})/(P_s P_{s,e})$. Summing over states $e$,
\[ k\left(\sum_{s} \pi_{s} P_{s,e} e\right) = \sum_{e} \pi_{e} P_{s,e} e, \] which by condition (1) and the laws of probability implies \( k P_{s} = \pi_{s} \). Consequently, \( \pi_{s}/P_{s} = (\pi_{e} P_{s,e})/(P_{e} P_{s,e}) \). Moreover, if this last equation holds, then \( U_{0}^{1}(C_{s}^{*})/U_{0}^{2}(C_{s}^{*}) = \pi_{s}/P_{s} \) so that \( C_{s}^{*} \) must be independent of \( e \). In summary, we can define \( C_{s}^{*} \) so that \( C_{s}^{*} e = C_{s}^{*} \) if and only if \( \pi_{s}/P_{s} = (\pi_{e} P_{s,e})/(P_{e} P_{s,e}) \). This holds for all \( s \) and \( e \).

(ii) It remains to show \( \{C_{s}^{*} = \hat{C}_{s}^{*}\} \) and \( \{C_{e}^{*} = \hat{C}_{e}^{*}\} \). To accomplish this we need only verify that these choices are feasible at date \( t=0 \), since in this case the individual would be willing to move immediately to \( \{C_{s}^{*}\} \) and \( \{C_{e}^{*}\} \) at date \( t=0 \) and not revise his portfolio at date \( t=1 \). Multiplying condition (5) by \( P_{e} \)

\[ \sum_{e} (\sum_{s} P_{s,e} e) C^{*} + \sum_{e} P_{e} C^{*} = \sum_{s} (\sum_{e} P_{s,e} e) \hat{C}^{*} + \sum_{e} P_{e} \hat{C}^{*} \]

and summing over \( e \)

\[ \sum_{s} (\sum_{e} P_{s,e} e) C^{*} + \sum_{e} P_{e} C^{*} = \sum_{s} (\sum_{e} P_{s,e} e) \hat{C}^{*} + \sum_{e} P_{e} \hat{C}^{*}. \]

By conditions (1) and (4)

\[ \sum_{s} P_{s} C^{*} + \sum_{e} P_{e} C^{*} = \sum_{s} P_{s} \hat{C}^{*} + \sum_{e} P_{e} \hat{C}^{*} = W_{0} - C_{0} \]

so that

\[ \sum_{s} P_{s} C^{*} + \sum_{e} P_{e} C^{*} + C_{0} = W_{0}. \]
Consequently, \( \{C^*_s\} \) and \( \{C^*_e\} \) could have been chosen immediately at
date \( t=0 \). Indeed, since the equilibrium must be unique, \( \{C^*_s = \hat{C}^*_s\} \) and
\( \{C^*_e = \hat{C}^*_e\} \), Q.E.D.

A similar result has been derived by Hirshleifer and Rubinstein
[8] in a somewhat less general context. In that paper, it was emphasized
that, surprisingly, the nonspeculation condition depends only on the
relationship between beliefs and prices, not on resources and tastes.\(^8\)

Apparently, the variety of consumption claims available in the market at
date \( t=0 \) is sufficiently rich that desired exchange generated solely
by differences among individuals in resources and tastes is satisfied by
exchange permitted at date \( t=0 \). In this paper, another implication
will be emphasized. Rewriting the nonspeculation condition,

\[
\frac{P_{s,e}/P_s}{\pi_{s,e}/\pi_s} = \frac{\pi_e}{P_e}.
\]

The right-hand side is a number fully determined before state \( e \) becomes
known; it is a constant of proportionality between revised prices and be-
liefs, predetermined at date \( t=0 \), which must hold for no revision to be
optimal. Imagine the following vignette:

At date \( t=0 \), prices \( P_e = 1/2 \) and \( P_s = 1/20 \) are set in the
market. From the point of view of one individual \( \pi_e = 1/2 \) and
\( \pi_s = 1/10 \). Consequently, \( \pi_e/P_e = 1 \). On the basis of new information

\(^8\) However, while the decision to revise does not depend on re-
sources and tastes, the extent of revision does.
received at date $t=1$ under the occurrence of state $e$, the individual doubles the probability he now attaches to state $s$ ($\pi_{s,e} = 1/5$). Before consulting the market, the individual decides to purchase more contingent claims to consumption under state $s$ if its price has not risen sufficiently. However, new information (but not necessarily the same information) may also have become available to other participants in the market, causing them also to revise their beliefs. As a result, a new equilibrium price will be set on the contingent claims. In fact, if the price exactly doubles ($P_{s,e} = 1/10$), then the individual will stand pat and not alter his holdings. Why? Because from his point of view, the new information he has received is already fully reflected in the revised price $P_{s,e}$.

If all individuals have homogeneous beliefs, do they all regard the new information they receive at date $t = 1$ as fully reflected in revised prices? If the above definition of "fully reflected" is meaningful, the answer should be positive. Fortunately,

**Corollary:** If all individuals have homogeneous beliefs, then all individuals have nonspeculative beliefs.

**Proof:** It follows trivially from the previous theorem that

$$C_{s,e}^* > (\langle C_{s,e}^* \rangle_{s,e}$$

if and only if

$$\frac{\pi_{s,e}}{P_{s,e}} < (\langle \frac{\pi_{s,e}}{P_{s,e}} \rangle_{s,e}$$

(all $s$ and $e$).

If one individual revises his portfolio so that $C_{s,e}^* > \hat{C}_{s,e}$, by closure there must be some other individual for whom $C_{s,e}^* < \hat{C}_{s,e}$; but in this
case, by the above, the beliefs of both cannot be the same. Hence, the theorem is proved by contradiction. Q.E.D.

Although homogeneous beliefs are sufficient for each individual to see his new information fully reflected in the revised prices, they are not necessary. From the nonspeculation condition, a necessary as well as sufficient condition is that all individuals must have nonspeculative beliefs, that is, agree on the ratio \( \pi_{s,e}/\pi_s \), but with possible disagreement over its components.

For all information, not merely new information, to be fully reflected in prices, a stronger condition than nonspeculative beliefs is required. We will say an individual perceives all the information he has and will receive as fully reflected in security prices if and only if he has "consensus beliefs." By definition, consensus beliefs are those beliefs which, if held by all individuals in an otherwise similar economy, would generate the same equilibrium prices as in the actual heterogeneous economy. In other words, consensus beliefs are the homogeneous beliefs that would sustain the same equilibrium prices. To motivate this definition, imagine the following vignette:

Jack, curious to learn if all his information is fully reflected in prices, visits the Great Economist in the Sky. Knowing the resources and tastes of every participant in the economy, as well as the actual equilibrium prices, the Great Economist calculates whether or not those same prices would rule if every individual had the same beliefs as Jack. Discovering that different equilibrium prices would emerge, the Great Economist concludes that either Jack has valuable
information (i.e., it would alter prices if publicly announced) not fully reflected in prices or that other individuals have information affecting prices not known to Jack.

Unfortunately, unlike nonspeculative beliefs, consensus beliefs admit of no simple and general mathematical formulation; they will depend directly on the resources, tastes, and beliefs of all individuals in the economy.\footnote{This raises the prior question of conditions for the existence of consensus beliefs. In the usual existence problem, we input heterogeneous probabilities and solve for nonnegative homogeneous prices. In this case, we input homogeneous prices (which hold in the heterogeneous economy) and solve for homogeneous probabilities (i.e., nonnegative real numbers which sum to one). Unlike prices, these homogeneous probabilities may not exist. While the derivation of necessary and sufficient conditions for their existence is not contemplated here, some sufficient conditions are given as an illustration.}

In particular, suppose all individuals maximize weighted logarithmic utility functions with the same constant rate of patience; that is, \( U_t^i(*) = \rho_t^x \ln(*) \) where \( \rho > 0 \). In this case, it is easily shown that at the optimum \( C_t^i = W_t^i(1+p+p^2)^{-1} \). Substituting this into conditions (2)

\[
\frac{i}{e} C_0^i = \frac{e}{e} P
\]

\[
\rho(1+p+p^2)^{-1} \pi_e^i W_0^i = \frac{e}{e} P
\]

and summing over \( i \) and by closure,

\[
\rho(1+p+p^2)^{-1} \Sigma_i \pi_e^i W_0^i = P \Sigma_i \bar{C}_e^i = P \Sigma_i \bar{C}_e^i.
\]

If \( \pi_e^m \) were the consensus belief for state \( e \), then this condition in the homogeneous economy would simplify to

\[
\rho(1+p+p^2)^{-1} \pi_e^m W_0^i = P \Sigma_i \bar{C}_e^i.
\]

Comparing these two results, the consensus belief
If this interpretation of consensus beliefs is to be meaningful, then surely an individual with consensus beliefs must see his new information at date $t=1$ as fully reflected in revised prices. This is indeed the case since

**Corollary:** If an individual has consensus beliefs, then he has nonspeculative beliefs.

**Proof:** By definition of consensus beliefs, an individual who has them can be viewed as a member of an otherwise similar economy, except in which these beliefs are held by everyone. Moreover, by the previous corollary, everyone in a homogeneous economy has nonspeculative beliefs. But since both economies have the same equilibrium prices, nonspeculative beliefs are the same in both economies. Therefore, the individual with consensus beliefs must also have nonspeculative beliefs in the actual heterogeneous economy. Q.E.D.

It should be similarly clear that the converse of this corollary is false.

\[
\pi_e^m = \frac{\sum_i w_i^1 \pi_i^e}{\sum_i w_i^0} \quad \text{(all } e) \tag{all e}
\]

which is indeed a nonnegative real number such that $\sum_e \pi_e^m = 1$. Since similar results hold for conditions (3), and (1), (4) and (5) are trivially satisfied, consensus beliefs exist. In the logarithmic case, they are simply weighted arithmetic averages of actual beliefs. See Wilson [13] for illustrations of consensus beliefs if all individuals have exponential utility, in which case they are weighted geometric averages of actual beliefs.
In sum, we have a triplet of increasingly stronger conditions that capture what is meant by prices fully reflect information:

(1) nonspeculative beliefs,

(2) consensus beliefs,

(3) homogeneous beliefs.

(3) implies (2) and (2) implies (1), but the converses of these implications are false. (3) is necessary and sufficient for all individuals to perceive all information fully reflected in prices. (2) is necessary and sufficient for an individual to perceive all his information fully reflected in prices. (1), if true for all individuals, is necessary and sufficient for all individuals to perceive new information fully reflected in prices; and (1) is necessary and sufficient for an individual to perceive his new information fully reflected in prices.

Of course, all effects of uncertainty (without assuming it away) on the allocation of resources can be completely exorcised from the economy if individuals have "clairvoyant homogeneous beliefs." By definition, clairvoyant homogeneous beliefs are beliefs that generate the same allocation of resources as would have emerged in otherwise similar but certain economies, one for each possible sequence of states. An economy composed of identical individuals is a trivial instance of clairvoyance.
III. RATES OF RETURN

The past decade has witnessed the development and testing of theoretical models of the contemporaneous structure of security rates of return under uncertainty. These models, as typified by variants of the popular mean-variance model, describe the relationship among rates of return of different securities at the same date. Concurrently, there has been considerable empirical research on the intertemporal structure of rates of return of the same security at different dates. This research has been directed at verification of a variety of statistical hypotheses such as the existence of an unbiased term structure and characteristics of a random walk. However, excepting the incipient work of a very few authors—Jerry Green and John Long—no one has attempted to provide a theoretical basis for these hypotheses in an equilibrium context.

The nonspeculation condition implies restrictions on the intertemporal structure of security rates of return as viewed by individuals who perceive the securities market as informationally-efficient. To see this, translating prices into rates of return, let

\[
1 + r_{1s}e = \frac{P_{s}e}{P_{e}} \quad 1 + r_{2s}e = \frac{P_{e}}{P_{s}}
\]

\[
1 + r_{1F} = (\Sigma P_{e})^{-1} \quad 1 + r_{2Fe} = (\Sigma P_{s}e)^{-1} \quad (1 + R_{2F})^2 = (\Sigma P_{s})^{-1}
\]
Using these definitions, we state the following theorem:

**Theorem (nonspeculative intertemporal structure):** In terms of nonspeculative beliefs,

\[ E[(1+r_{1s})(1+r_{2s})] = (1+r_{1P})E[1+r_{2s}], \]

where the expectations are assessed with respect to beliefs held at date \( t=0 \).

**Proof:** By the nonspeculation condition, \( \frac{\pi}{P_s}^{e} \cdot P_e = \frac{\pi_s^e}{P_s} \cdot P_e \).

Summing over states \( e \) and rearranging

\[ \pi_{s}^{-1} = (\sum_{e} \pi_{s}^{-1} e\cdot P_{s}^{-1} e). \]

Since \( \pi_{s} = \sum_{e} \pi_{e} \cdot s_{e} \cdot e \),

\[ \sum_{e} \pi_{e} \cdot s_{e} \cdot e \left( \frac{P_{s}^{-1} e \cdot P_{s}^{-1}}{P_{s}} \right) = (\sum_{e} \pi_{e}^{-1} e \cdot s_{e} \cdot e \cdot P_{s}^{-1}). \]

The conclusion follows immediately upon substituting rate of return notation. Q.E.D.

In words, the intertemporal structure of rates of return implied by nonspeculative beliefs (and hence by consensus or homogeneous beliefs) must have the following characterization: *the expected one plus compound rate of return on any security discounted by one plus the first-period risk-free rate equals its expected one plus second-period rate of return, where*
the expectations are assessed with respect to beliefs at the beginning of the first period.\textsuperscript{10}

If we assume this minimal characterization of the intertemporal structure, can we confirm or reject any popular statistical hypotheses? Some trial hypotheses are listed below:

\begin{align*}
(A) & \quad 1 + r_{1F} = E[1+r_{1s}] & \text{(no risk adjustment)} \\
(B) & \quad E[(1+r_{1s})(1+r_{2s})] = E[1+r_{1s}]E[1+r_{2s}] & \text{(uncorrelated rates)} \\
(C) & \quad E[(1+r_{1s})^{-1}(1+r_{2s})^{-1}] = E[(1+r_{1s})^{-1}]E[(1+r_{2s})^{-1}] & \text{(uncorrelated inverse rates)} \\
(D) & \quad (1+r_{2F})^{-2} = (1+r_{1F})^{-1}E[(1+r_{2F})^{-1}] & \text{(unbiased inverse term structure)}
\end{align*}

It follows immediately from the previous theorem that, in terms of non-speculative beliefs, (A) and (B) are equivalent. That is, the expected first-period rate of return on any security equals the first-period risk-free rate if and only if its intertemporal sequence of one plus rates of return are serially uncorrelated. Consequently, we cannot simultaneously believe in a risky economy composed of risk-averse individuals and observe

\textsuperscript{10}Caution: This characterization will in general hold only for rates of return of each contingent claim, not for rates of return on portfolios of contingent claims.
ex ante serially uncorrelated (and, hence, serially independent) one plus rates of return. However, although hypotheses (A) and (B) may be rejected, neither hypotheses (C) nor (D) can be confirmed or rejected in the presence of nonspeculative beliefs. Apparently, to obtain more powerful results, we must derive a more specific characterization of the intertemporal structure from assumptions stronger than mere nonspeculation.
IV. CONCEPTS OF INFORMATION-EFFICIENCY

It is instructive to contrast the concept of information-efficiency developed in this paper with other discussions in the literature. To paraphrase Fama [6, p. 384], efficiency is defined as follows:

Using information set $\phi$, forecast the probability distribution of the prices of securities that will be realized at date $t=1$. Input this data into a model of market equilibrium determining expected rates of return. From these and the expected date $t=1$ prices, the date $t=0$ prices, given information set $\phi$ and the market equilibrium model, can be computed. Compare these computed prices with the prices actually observed in the marketplace. If they are the same, then actual security prices are said to "fully reflect" $\phi$.

Although less precise, the implied beliefs of this definition appear similar to consensus beliefs. However, Fama's definition and subsequent use of it suffer from several deficiencies. First, his definition is unnecessarily couched in terms of expected returns. Second, suppose $\phi$ is fully reflected by his definition and an individual knows only some proper subset of $\phi$. He may then assess different date $t=0$ prices and, although he will be unable to make superior returns relative to $\phi$, his mistaken estimate of date $t=0$ prices will lead him to perceive some securities as over- and under-valued. As a consequence, he may neglect to diversify efficiently. By Fama's definition, an efficient market may
coexist with this nonoptimal behavior. Third, unable to draw nontrivial, testable implications from his definition, Fama is forced to supplement it with highly specialized and unrealistic models of equilibrium.

By contrast, the definition developed in this paper arises from within an explicit but quite general equilibrium context. Despite its generality, precise characteristics are derived describing individuals who view the market as efficient (i.e., nonspeculation condition). An important distinction is made between new information and all information. The definition is also linked with the volume of trading. For example, if every individual viewed the market as efficient, there would be no trading at date \( t=1 \). In a more general model, with the continuous emergence of new securities and new participants, as well as barriers to marketability, the model holds out the promise of distinguishing trading volume for these reasons from trading for speculative purposes. The volume of speculative trading could then be used as a barometer of information-efficiency.\(^{11}\)

Samuelson [11] attempts to justify martingale characterizations of efficiency. He proves the following exceedingly simple theorem:\(^{12}\)

**Theorem (Fair Game):** If \( P_s = \pi_s \) and \( P_{s\cdot e} = \pi_{s\cdot e} \) (all \( e \)), then \( P_s = \sum e \pi e P_{s\cdot e} \).

\(^{11}\)Downes and Dyckman [4, pp. 314–316] offer a less specific but similar suggestion.

\(^{12}\)Samuelson actually proves a more general version that allows for continuous probability distributions, any finite number of dates, and incomplete markets. However, nothing of significance is lost in the more straightforward theorem given here.
Proof: By the laws of probability, $\pi_s = \sum_{e} \pi_{s,e}$ and by assumption $P_s = \pi_s$ and $P_{s,e} = \pi_{s,e}$ (all $e$). Q.E.D.

$P_s$, it will be recalled, is the date $t=0$ price of one unit of consumption received at date $t=2$ if and only if state $s$ occurs. Since the date $t=2$ price of one unit of consumption received at date $t=2$ is just 1, $\pi_s$ may be interpreted as the expected date $t=2$ price of one unit of consumption received at date $t=2$ if and only if state $s$ occurs, where the expectation is assessed with respect to beliefs held at date $t=0$. Similarly, given that state $e$ has occurred, $P_{s,e}$ is the date $t=1$ price of one unit of consumption received at date $t=2$ if and only if state $s$ occurs. Given that state $e$ has occurred, $\pi_{s,e}$ may be interpreted as the expected date $t=2$ price of one unit of consumption received at date $t=2$ if and only if state $s$ occurs, where the expectation is assessed with respect to beliefs held at date $t=1$. Loosely speaking, then, the fair game theorem says: If the price of a security at every date always equals its expected price at some terminal date, then its price at every date equals its expected price at any future date prior to the terminal date. This stochastic process of price behavior is called a martingale.

The theorem is obvious. Since its assumption ignores time preference and risk aversion, it is not surprising to learn that the date $t=0$ price of a consumption claim equals its expected date $t=1$ price.

13 Alternatively, it ignores time preference and presumes a riskless economy in which risky securities may exist as long as, through offsetting effects, aggregate consumption is certain.
As it stands, the theorem is a trivial characterization of efficiency, or in Samuelson's words, "properly anticipated prices." Possibly realizing this, Samuelson generalizes the result to allow for positive risk-free interest and risk aversion:

**Corollary:** If \( P_s = \lambda_s^{-1} \lambda_{2s}^{-1} \pi_s \) and \( P_{s,e} = \lambda_{2s}^{-1} \pi_{s,e} \) (all \( e \)),

then \( P_s = \lambda_s^{-1} \sum_e \pi_e P_{s,e} \).

\( \lambda_s \) and \( \lambda_{2s} \) are time and risk adjusted discount rates that may be different for different consumption claims (i.e., depend on state \( s \)), but that are independent of state \( e \). Indeed, in this lies the content of the theorem, since if \( \lambda_s^{-1} \) in the second equality were replaced by \( \lambda_{2s}^{-1} \), the assumption of the corollary would be vacuously true. Examination of the proof of the fair game theorem provides an immediate proof of this generalization.

What is the actual content of these results and how are they related to the definition of efficiency provided in this paper? In view of the restrictive equilibrium implications of the fair game assumptions, we will turn more hopefully to Samuelson's generalization. However, in the context of nonspeculative beliefs (and, hence, consensus or homogeneous beliefs), a significant inconsistency emerges. By the assumption of his corollary, \( \pi_s/P_s = \lambda_s \lambda_{2s} \) and \( \pi_{s,e}/P_{s,e} = \lambda_{2s} \) (all \( e \)). Substituting these relationships into the nonspeculation condition \( \pi_s/P_s = (\pi_e \pi_{s,e})/(P_e P_{s,e}) \) (all \( e \)), we derive \( \pi_e/P_e = \lambda_s \) (all \( e \)). This not only means \( \pi_e/P_e \) must be independent of \( e \) but also that \( \lambda_s \) must be independent of \( s \). Indeed, by summing over \( e \),
\(\lambda_{is} = 1/(\sum e^e)\). Interpreting this verbally, all claims to consumption at date \(t=2\) have the same risk-adjusted discount rate between dates \(t=0\) and \(t=1\) equal to one plus the risk-free rate. In effect, despite Samuelson's attempt to generalize to risk-adjusted discount rates, an individual with nonspeculative (or consensus, or homogeneous) beliefs will use the risk-free rate, not adjusting for risk.

As Samuelson asks at the end of his paper:

I have not here discussed where the basic probability distributions are supposed to come from. In whose minds are they ex ante? ... Are they supposed to belong to the market as a whole? And what does that mean? Are they supposed to belong to the "representative individual," and who is he? ... This paper has not attempted to pronounce on these interesting questions.

While the present paper has not unveiled this mystery man, if securities are not all to be discounted at the risk-free rate, he cannot have homogeneous, consensus or even nonspeculative beliefs. In short, he may not be an economically interesting individual.

Another tempting definition is to associate with information-efficiency the holding of (possibly levered) value-weighted balanced portfolios. However, we know from Cass and Stiglitz [2] that even in a static homogeneous economy, individuals will not generally desire to hold (possibly levered) miniatures of the market portfolio. Yet, by any meaningful definition, homogeneous beliefs is surely a sufficient condition for information-efficiency.

"Intrinsic value" is another recurrent phrase associated with efficiency. Clearly, nothing has "intrinsic value" apart from the resources, tastes, and beliefs of all individuals in the economy.
However, taking a more charitable view toward intrinsic value, perhaps it refers to valuations based on the pool of all available information. In other words, intrinsic values are those equilibrium prices set as if each individual publicized all his information. In this "as if" economy, all individuals would have homogeneous beliefs. If these "as if" homogeneous beliefs happen to coincide with the actual consensus beliefs, only then, by this definition, will the actual market be efficient.

Although this definition is perhaps closest in spirit to the implicit definitions used in most of the empirical literature, it still suffers from the coexistence of nonoptimal speculative trading. That is, just because equilibrium prices are set as if each individual publicized all his information, does not mean that any individual bases his choices on the pooled set of available information.
V. LIMITATIONS

The chief difficulty with the concept of information-efficiency developed in this paper has been the absence of transactions costs. Without these costs, in a heterogeneous economy, complete markets are a natural evolution of the structure of exchange opportunities. At a deeper level, the choice, cost and communication of information which creates transactions cost has not received even implicit attention.

It is useful to distinguish between two types of information about security prices—information about the supply of aggregate consumption and information about the demand for it. The dynamic optimization problem envisaged here requires a considerable degree of forethought: an individual must not only choose \((C_0, \{\hat{C}_e\}, \{\hat{C}_{s,e}\})\), but also decide in advance at date \(t=0\) how he will revise these provisional choices at date \(t=1\) (choose \((C_e, \{C_{s,e}\})\) for all states \(e\)). In this planning process, at date \(t=0\) the individual utilizes date \(t=1\) equilibrium state conditional prices even before the date \(t=1\) market convenes! In effect, we have assumed that given the resolution of uncertainty surrounding the aggregate supply of consumption, demand conditions and hence prices are known with certainty.\(^{14}\) In a more general setting, even

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\(^{14}\) Actually, this statement only holds for results, such as the identification of consensus beliefs, which require a market equilibrium setting. The identification of non-speculative beliefs requires only a model of individual behavior in which uncertainty about future prices can derive from both supply and demand conditions.
if supply uncertainty were resolved, lacking information about the resources, beliefs and tastes of other individuals, an individual could not forecast demand and hence prices with certainty. Unfortunately, for issues examined in this paper, absence of demand uncertainty is particularly objectionable since it cavalierly ignores the Keynesian or psychological view of the securities market, in which speculative bubbles are a possibility.  

While the reduction of demand uncertainty motivates transactions costs, supply uncertainty also motivates the acquisition of information. The model developed in this paper contained no explicit analysis of its choice—beliefs were taken as data. Moreover, although different individuals had different beliefs, this information was implicitly assumed to be costless. With explicit consideration of productive choice and delegation of authority to managers of firms, supply uncertainty would become more complex. Individuals would not only be uncertain about the outcomes of "natural processes," but also about the behavior of managers. In turn, this additional cause of uncertainty would derive from two sources—uncertainty about their reported production choices and uncertainty about discrepancies between their actual and reported choices.

A theoretical exploration of the meaning of the phrase "security prices fully reflect information," in the presence of the choice, cost and communication of information potentially capable of reducing demand and supply uncertainty, must await another opportunity.

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15 This distinction between supply and demand uncertainty would seem to be at the root of the separation of so-called "fundamental" and "technical" approaches to security analysis.
REFERENCES


