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A SIMPLE MARKET EQUILIBRIUM MODEL OF A RANDOM WALK

by

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ABSTRACT

In a dynamic, logarithmic utility model of securities market equilibrium, theoretical justification is supplied for the random walk hypothesis. A random walk is also shown to be related to the existence of an unbiased term structure.
A SIMPLE MARKET EQUILIBRIUM MODEL OF A RANDOM WALK

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In another paper [1], I formulated a dynamic and stochastic model of a securities market composed of rational heterogeneous individuals. This model imposed no exogenous restrictions on the structure of security rates of return, either at the same date or over time. Since this structure was endogenously determined in equilibrium, an opportunity was presented to provide theoretical support for popular statistical characterizations of this structure. However, neither an unbiased term structure nor a random walk emerged as a necessary property of equilibrium.

Under more stringent, but by contemporary standards not unduly unrealistic assumptions, a more specific characterization of the structure of security rates of return will be derived. As an endogenous property of the equilibrium conditions, the term structure is unbiased if and only if the rate of return of the market portfolio of all securities is uncorrelated over time. It is also shown that even if the term structure is biased, then given appropriate deflation by risk-free rates, the rate of return of the market portfolio is uncorrelated over time. In other words, if the bias to the term structure is appropriately

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1 However, the complete markets assumption of [1] has been dropped.
washed out, the most significant feature\(^2\) of the random walk hypothesis is validated. Therefore, there appears to be a close relationship, not hitherto discussed to my knowledge, between these two statistical hypotheses.

I. THE ECONOMY

Consider a three-date \((t=0,1,2)\) economy consisting of identical individuals. At date \(t=0\) each individual allocates his wealth \(W_0\) among consumption \(C_0\) and securities \(j = 1,2,\ldots,J\) where \(\beta_j\) is the proportion of \(W_0-C_0\) allocated to security \(j\) so that \(\Sigma_j \beta_j = 1\). At date \(t=1\), state \(e\) occurs determining the rate of return \(r_{je}\) on each security \(j\) so that \(\Sigma_j \beta_j r_{je}\) is his portfolio rate of return. In turn, this determines his date \(t=1\) wealth \(W_e = (W_0-C_0)(1+\Sigma_j \beta_j r_{je})\). At date \(t=1\), given his past choices and the occurrence of state \(e\), each individual allocates his wealth \(W_e\) among consumption \(C_e\) and securities \(j = 1,2,\ldots,J\) where \(\beta_{je}\) is the proportion of \(W_e-C_e\) allocated to security \(j\) so that \(\Sigma_j \beta_{je} = 1\). At date \(t=2\), state \(s\) occurs determining the rate of return \(r_{js,e}\) on each security \(j\) so that \(\Sigma_j \beta_{je} r_{js,e}\)

\(^2A\) random walk is defined by statistical independence of security rates of return over time. In this paper, it is only demonstrated that the rate of return of the market portfolio is serially uncorrelated—a necessary but not sufficient condition for a random walk.

\(^3E\) xcept for logarithmic utility (see below), this assumption can be discarded without affecting the results. The theorem in Section II holds even if individuals have different wealth, different rates of patience, and nonconstant rates of patience. This should be evident from the separation property of this model. If individuals have heterogeneous beliefs, the theorem holds with respect to "consensus beliefs" as defined in [1].
is his portfolio rate of return. In turn, this determines his date
t=2 consumption \( C_{s,e} = (W_e - C_e) \left( 1 + \sum_{j} \beta_{j} r_{j,e} \right) \).

Each individual is assumed to obey the Savage axioms of
rational choice: he has beliefs and tastes representable by proba-
bilities and a utility function, so that he makes choices which
maximize expected utility. \( \pi_e \) denotes the probability that state
\( e \) will occur; given the occurrence of state \( e \), \( \pi_{s,e} \) denotes the
conditional probability that state \( s \) will occur. His utility
function is assumed to be additive logarithmic in consumption with a
constant rate of patience \( \rho \). In brief, the individual's programming
problem\(^4\) is

\[
\max_{C_0, \{\beta_j\}, \{\pi_e\}} \quad \ln C_0 + \rho \sum_e \pi_e \ln C_e + \rho^2 \sum_e \pi_e \pi_s \ln C_{s,e}
\]

s.t. \( W_e = (W_0 - C_0) \left( 1 + \sum_j \beta_j r_{j,e} \right) \) and \( \sum_j \beta_j = 1 \) \quad (all \( e \))

\[
C_{s,e} = (W_e - C_e) \left( 1 + \sum_j \beta_j r_{j,s} \right) \quad \text{and} \quad \sum_j \beta_j = 1 \quad \text{(all} \ e \ \text{and} \ s).\]

\(^4\) Nonnegatively constraints on consumption \( \{C_0, \{C_e\}, \{C_{s,e}\} \} \) are
automatically satisfied, since the marginal utility of consumption is
infinite at zero consumption.
Reformulating this in the format of dynamic programming, the date $t=1$ subproblem determines the derived utility of date $t=1$ wealth:

$$V_e(W_e) = \max_{C_e, \{\beta_j\}} \ln C_e + \rho \sum_j \pi_j e^{\ln[(W_e - C_e)(1 + \sum_j \beta_j r_j e)]} - \lambda e^{\sum_j \beta_j - 1}.$$ 

where the constraint has been substituted into the maximand and $\lambda$ is a stochastic Lagrangian multiplier. Similarly, using the derived function $V_e(W_e)$, the date $t=0$ problem becomes:

$$\max_{C_0, \{\beta_j\}} \ln C_0 + \rho \sum_j \pi_j e^{\ln[(W_0 - C_0)(1 + \sum_j \beta_j r_j e)]} - \lambda_0 e^{\sum_j \beta_j - 1}.$$ 

Solving the date $t=1$ subproblem, the consumption and portfolio problems conveniently separate so that:

$$V_e(W_e) = \max_{\{\beta_j\}} \ln C_e + \rho \ln(W_e - C_e) + \rho \max_{\{\beta_j\}} \sum_j \pi_j e^{\ln(1 + \sum_j \beta_j r_j e)} - \lambda e^{\sum_j \beta_j - 1}.$$ 

Since the second maximand is additive and does not depend on $W_e$, it can be ignored in the determination of the derived utility of wealth function (i.e., $V_e(W_e)$ is unique up to an additive constant). Solving the first maximand, $C_e^* = W_e (1 + \rho)^{-1}$ and $V_e(W_e) \sim (1 + \rho) \ln W_e$. Substituting this
into the date $t=0$ problem, the consumption and portfolio problems again separate so that we have

$$ \left( \max e \ln C_0 + \rho (1+\rho) \ln (W_0-C_0) \right) + \rho (1+\rho) \left( \max \left\{ \beta_j \right\} \ln (1+\sum_j \beta_j r_{je}) - \lambda_0 [\sum_j \beta_j - 1] \right). $$

In summary, the dates $t=0$ and $t=1$ portfolio problems reduce respectively to

$$ \max \left\{ \beta_j \right\} \ln (1+\sum_j \beta_j r_{je}) - \lambda_0 [\sum_j \beta_j - 1] $$

and

$$ \max \left\{ \beta_{je} \right\} \ln (1+\sum_j \beta_j r_{je}) - \lambda_e [\sum_j \beta_j - 1] \quad \text{(all e).} $$

Differentiating partially by the portfolio proportions, the necessary and sufficient conditions for an optimum can be demonstrated equivalent to

$$ e \hat{\pi} = \frac{1+r_{je}}{1+r_{Me}} = 1 \quad (\text{all } j) \quad \text{and} \quad s \hat{\pi} = \frac{1+r_{je}}{1+r_{Me}} = 1 \quad (\text{all } s \text{ and } j) $$

where $r_{Me} = \sum_j \beta_j r_{je}$ and $r_{Ms} = \sum_j \beta_j r_{je}$. That is, at each date, portfolio proportions are so chosen that the expected ratio of (one plus) the rate of return of any security to (one plus) the
portfolio rate of return equals 1. To transform this immediately into a market equilibrium condition, since all individuals are identical and all securities must be held, \( M \) must be the market portfolio of all securities.

II. RATES OF RETURN

Assume among securities \( j = 1, 2, \ldots, J \) at date \( t=0 \) are short-term and long-term bonds. The short-term bonds guarantee rate of return \( r_F \) realized at date \( t=1 \) and the long-term bonds guarantee a compound rate of return \( R_F \) at date \( t=2 \). While individuals are certain about this compound rate, they are uncertain about the intermediate period by period rates which underlie it. In particular, if \( r_{F1e} \) and \( r_{F2e} \) are the rates of return realized at dates \( t=1 \) and \( t=2 \), respectively if state \( e \) occurs, then for all states \( e \)

\[
(1+R_F)^2 \gamma(1+r_{F1e})(1+r_{F2e})
\]

If the term structure (in inverse rates\(^5\)) is unbiased, then

\[
(1) \quad (1+R_F)^{-2} = (1+r_F)^{-1} \sum_e \pi_e (1+r_{F2e})^{-1}.
\]

\(^5\) [1] suggests that inverse rates of return (i.e., the present price of a future dollar) are more fundamental than the rates of return themselves.
In words, (the inverse of one plus) today's forward rate equals today's expectation of (the inverse of one plus) tomorrow's spot rate.

Similarly, define the compound rate of return of the market portfolio by

\[(1 + \frac{R_{\text{MSe}}}{1 + r_{\text{Me}}})(1 + r_{\text{Me}}).\]

If the rate of return of the market portfolio (in inverse rates) is serially uncorrelated, then

\[(2) \quad \Sigma \pi \Sigma e \pi s e (1 + \frac{R_{\text{MSe}}}{1 + r_{\text{Me}}})^{-2} = [\Sigma \pi e (1 + r_{\text{Me}})^{-1}][\Sigma \pi s e (1 + r_{\text{MSe}})^{-1}].\]

In words, the expected (inverse of one plus) compound rate of return of the market portfolio equals the product of the expectations of the (inverse of one plus) period by period rates of return.\(^6\) Note that these expectations have all been assessed with respect to beliefs held at date \(t=0\).

Define:

\[1 + \frac{\hat{R}_{\text{MSe}}}{1 + \hat{r}_{\text{Me}}} = (1 + \frac{R_{\text{MSe}}}{1 + r_{\text{Me}}}) \]

\[1 + \hat{r}_{\text{Me}} = (1 + r_{\text{Me}})/(1 + r_{\text{F}})\]

\[1 + \hat{r}_{\text{MSe}} = (1 + r_{\text{MSe}})/(1 + r_{\text{F}})\]

\[1 + \hat{r}_{\text{MSe}} = (1 + r_{\text{MSe}})/(1 + r_{\text{F}})\]

\(^6\)This makes use of the statistical property that two random variables are uncorrelated if and only if the expectation of their product equals the product of their expectations.
These rates of return, the result of standardization by the appropriate risk-free rates, will be said to be deflated. If the deflated rate of return of the market portfolio (in inverse rates) is serially uncorrelated, then

\[(3) \sum_{e} \pi_{e} \pi_{s} (1 + r_{Mse})^{-1} = [\sum_{e} \pi_{e} (1 + r_{Me})^{-1}] [\sum_{e} \pi_{s} (1 + r_{Ms} e)^{-1}].\]

We are now prepared to prove the key theorem of this paper:

**Theorem:** The term structure (in inverse rates) is unbiased if and only if the rate of return of the market portfolio (in inverse rates) is serially uncorrelated. Irrespective of the term structure, the deflated rate of return of the market portfolio (in inverse rates) is serially uncorrelated.

**Proof:** Using the final results of Section I, for the short-term bond

\[(4) (1 + r_{F})^{-1} = \sum_{e} \pi_{e} (1 + r_{Me})^{-1}\]

For the long-term,

\[(5) (1 + r_{F2e})^{-1} = \sum_{e} \pi_{s} (1 + r_{Ms} e)^{-1} \quad \text{(all e)}\]

and since \(1 + r_{F1e} = \frac{(1 + r_{F})^{2}}{(1 + r_{F2e})},\)

\[(6) (1 + r_{F})^{-2} = \sum_{e} \pi_{e} (1 + r_{Me})^{-1} (1 + r_{F2e})^{-1}.\]

Substituting (5) into (6),
\begin{equation}
(1 + R_F)^{-2} = \Sigma_e \pi_e \Sigma_s \pi_s \star e (1 + R_{Mse})^{-2}
\end{equation}

and summing (5) over \( \pi_e \),

\begin{equation}
\Sigma_e \pi_e (1 + R_{F2e})^{-1} = \Sigma_e \pi_e \Sigma_s \pi_s \star e (1 + R_{Mse})^{-1}.
\end{equation}

As an immediate consequence of (4), (7) and (8), (1) if and only if (2)
which proves the first part of the theorem. To prove the second part,
from (4), (5) and (7),

\begin{equation}
\Sigma_e \pi_e (1 + R_{Me})^{-1} = \Sigma_s \pi_s \star e (1 + R_{Mse})^{-1} = \Sigma_e \pi_e \Sigma_s \pi_s \star e (1 + R_{Mse})^{-2} = 1 \quad \text{(all } e \text{)}.
\end{equation}

Moreover, since \( \Sigma_s \pi_s \star e (1 + R_{Mse})^{-1} = 1 \), then \( \Sigma_e \pi_e \Sigma_s \pi_s \star e (1 + R_{Mse}) = 1 \).

(3) follows as an immediate consequence. Q.E.D.

An intuitive explanation for at least part of this theorem can be given. The portfolio problem at date \( t = 1 \) will generally depend on state \( e \) through the dependence of wealth \( W_e \) on state \( e \) and through the serial dependence of rates of return on securities. However, if as in the logarithmic case, the date \( t=1 \) portfolio problem is independent of wealth \( W_e \) and if we also assume serial independence of rates of return on risky securities, then the date \( t=1 \) portfolio problem can depend on state \( e \) only through serial dependence of the rate of return on the long-term bond. However, if this is the only source through which the date \( t=1 \) portfolio problem is dependent on state \( e \), in equilibrium this dependence must disappear. Consequently, in equilibrium \( r_{F2e} \) must be independent of state \( e \) and the term structure will be trivially unbiased. Evidently, with logarithmic utility, these sufficient conditions
for an unbiased term structure may be weakened to a serially uncorrelated rate of return of the market portfolio of all securities.\(^7\)

\(^7\)Although the term structure will remain unbiased, \(r_Z\) will not generally be independent of state \(e\).
REFERENCES