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CONVERGENCE TO ISOELASTIC UTILITY IN MULTIPERIOD

PORTFOLIO CHOICE: FURTHER RESULTS*

by

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I. INTRODUCTION

This paper considers the "reinvestment problem," i.e., the problem of the investor who has numerous opportunities for revising his portfolio and whose choices are governed by a utility function defined on "terminal" wealth, \( U_0(x_0) \). Attention is focused on the behavior of the induced utility functions of intermediate wealth with \( n \) periods to go, \( U_n(x_n) \), and the associated investment policies.

Mossin [10] also studied this problem and found that the functions \( U_n(x_n) \) will tend to isoelasticity if the terminal function has linear risk tolerance. These results were extended by Leland [9] to cover all functions for which the (Arrow-Pratt [1], [11]) relative risk aversion of \( U_0(x_0) \) converges as \( x_0 \to \infty \). In this paper, the conditions for convergence are weakened further, to the point where they appear sufficiently broad to encompass perhaps most utility functions of practical interest. In any case, only the behavior of \( U_0(x_0) \) for very large \( x_0 \) is of consequence; the shape of \( U_0(x_0) \) for small \( x_0 \) is of no import. The significance of the preceding is that "reinvestors" with a sufficiently distant horizon can in fact behave optimally by behaving myopically; this is almost never true if the horizon is not far away (Mossin [10], Hakansson [7]). In addition, due to the separation property of the isoelastic class, one mutual fund for each level of relative risk aversion "associated" with the set of terminal utility functions is both necessary and, apart from differing probability beliefs, sufficient to serve all such investors in the economy (see Cass and Stiglitz [2], Hakansson [5]).

The paper proceeds as follows. The underlying model is developed in Section II, along with some preliminary results. Section III contains further
background results plus a simple but revealing set of sufficient conditions for convergence; these conditions are further illuminated by a pair of corollaries. The main result can be found in Section IV. Section V considers the kind of conditions which must hold for convergence in general (they appear rather weak but are somewhat lacking in operational value), and Section VI contains further discussion.

II. PRELIMINARIES

Aside from the assumptions concerning preferences, the premises employed in portfolio theory are highly standardized. A positive risk-free interest rate at which funds can be both borrowed and lent and the existence of one or more risky investment opportunities with stochastically constant returns to scale is usually postulated. Furthermore, perfect liquidity and divisibility of the assets at each (fixed) decision point and absence of transaction costs and taxes are also implicitly assumed in most instances, along with the opportunity to make short sales. Reinvestment of all proceeds at each decision point is generally also postulated, i.e., withdrawals and capital additions are ruled out. Finally, stochastic independence of returns over time are almost universally presumed. These assumptions will again be employed in this paper; some relaxations will be considered in Section VI. The following basic notation will be adopted.

\[ x_j = \text{amount of investment capital at decision point } j \text{ (the beginning of the } j\text{-th period)} \]

\[ M_j = \text{the number of investment opportunities available in period } j \text{, where } M_j \leq M \]
\( S_j \) = the subset of investment opportunities which it is possible to sell short in period \( j \)

\( r_j - 1 \) = rate of interest in period \( j \), where \( r_j > 1 \)

\( \beta_{ij} \) = proceeds per unit of capital invested in opportunity \( i \), where \( i = 2, \ldots, M_j \), in the \( j \)-th period (random variable). That is, if we invest an amount \( \theta \) in \( i \) at the beginning of the period, we will obtain \( \beta_{ij} \theta \) at the end of that period.

\( z_{1j} \) = amount lent in period \( j \) (negative \( z_{1j} \) indicate borrowing) (decision variable)

\( z_{ij} \) = amount invested in opportunity \( i \), \( i = 2, \ldots, M_j \), at the beginning of the \( j \)-th period (decision variable)

\( F_j(y_2, y_3, \ldots, y_{M_j}) \equiv \Pr(\beta_{2j} \leq y_2, \beta_{3j} \leq y_3, \ldots, \beta_{M_j} \leq y_{M_j}) \)

\( z_j \equiv (z_{2j}, \ldots, z_{M_j}) \)

\( v_{ij} \equiv \frac{z_{ij}}{x_j} \quad i = 1, \ldots, M_j \)

\( v_j \equiv (v_{2j}, \ldots, v_{M_j}) \)

\( v_{ij} \) clearly denotes the proportion of capital \( x_j \) invested in opportunity \( i \) at the beginning of period \( j \).

We assume that the (nonstationary) return distributions \( F_j \) are independent and satisfy the boundedness conditions.
(1) \[ 0 \leq \beta_{ij} \leq K \quad \text{all } i,j \]

(2) \[ E[\beta_{ij}] = r_j + \eta_2, \text{ where } r_j \geq \eta_1 > 1, \eta_2 > 0, \]
\[ \text{some } i, \text{ all } j, \]

and the "no-easy-money condition"

\[ \Pr\left\{ \sum_{i=2}^{M_j} (\beta_{ij} - r_j)\theta_i < \delta_1 \right\} > \delta_2 \quad \text{for all } j \text{ and all } \theta_i \]

such that \[ \sum_{i=2}^{M_j} |\theta_i| = 1 \]

and \( \theta_i \geq 0 \) for all \( i \notin S_j \), where \( \delta_1 < 0, \delta_2 > 0 \).

We also assume that the investor must remain solvent in each period, i.e.,
that he must satisfy the solvency constraints

(4) \[ \Pr\{x_{j+1} \geq 0\} = 1 \quad j = 1,2,\ldots. \]

This constraint is necessary to achieve consistency with the standard
assumption of risk-free lending and to cope with the logical requirements of
a multiperiod model of the reinvestment type.

Since the end-of-period capital position is given by the proceeds from
current savings, or the negative of the repayment of current debt plus
interest, plus the proceeds from current risky investments, we have
\[ x_{j+1} = r_j z_{1j} + \sum_{i=2}^{M_j} \beta_{ij} z_{ij} \quad j = 1, 2, \ldots, \]

where

\[ \sum_{i=1}^{M_j} z_{ij} = x_j \quad j = 1, 2, \ldots. \]

Combining the preceding we obtain

\[ x_{j+1} = \sum_{i=2}^{M_j} (\beta_{ij} - r_j) z_{ij} + r_j x_j \quad j = 1, 2, \ldots, \]

\[ = x_j R_j (v_j) \quad j = 1, 2, \ldots, \]

where

\[ R_j (v_j) = \sum_{i=2}^{M_j} (\beta_{ij} - r_j) v_{ij} + r_j. \]

Note that \( R_j (v_j) \) is 1 plus the return on the whole portfolio \( v_j \) in period \( j \). The portfolio problem at decision point \( j \) is now seen to be one of choosing the vector of risky investments \( z_j \equiv (z_{2j}, \ldots, z_{M_j}) \) (proportions \( v_j \)) so as to produce the most "favorable" distribution of end-of-period capital \( x_{j+1} \) (or \( R_j (v_j) \)). Clearly,

\[ v_{1j} = 1 - \sum_{i=2}^{M_j} v_{ij}. \]

Note that when \( x_j > 0 \), the solvency constraint (4) is equivalent to the
constraint

(8) \( \Pr\{R_j(v_j) \geq 0\} = 1 \quad j = 1, 2, \ldots \)

Solving (5) recursively, we obtain

(9) \( x_{N+1} = x_1 \prod_{j=1}^{N} R_j(v_j) \quad N = 1, 2, \ldots \)

We now give some preliminary results: The following lemmas are trivial generalizations of Lemma 1 in [5].

**Lemma 1.** Let (1), (2), and (3) hold. Then the sets \( V_j \) of portfolios \( v_j \) which satisfy the short-sale constraints

(10) \( v_{ij} \geq 0 \quad i \notin S_j \)

and the solvency constraints (8) contain risky portfolios in addition to the riskless ones and are closed, uniformly bounded, and convex.

**Corollary 1.** For every feasible portfolio \( v_j \) containing risky assets, the variance of return is positive, i.e.,

(11) \( \text{Var} [R_j(v_j)] > 0 \quad v_j \in V_j, v_j \neq (0, \ldots, 0), \text{ all } j \).

**Lemma 2.** Let (1), (2), (3), and (5) hold and let \( d \) be a real number. Then the sets \( Z_j(x_j, d) \) of portfolios \( z_j \) which satisfy the short-sale constraints
\[(12) \quad z_{ij} \geq 0 \quad i \notin S_j \]

and the constraints

\[(13) \quad \Pr\{x_{j+1} \geq d\} = 1 \]

are non-empty for \( x_j \geq d/r_j \). Moreover, they contain risky portfolios in addition to the riskless ones and are closed, bounded, and convex; when \( x_j = d/r_j \), only the riskless portfolio \( z_j = (0, \ldots, 0) \) satisfies (12) and (13).

The following corollaries are immediate:

**Corollary 2.** \( z_j(x, d) \subset z_j(y, d) \) for \( x < y \), whenever \( y \geq d/r_j \), all \( j \).

**Corollary 3.** \( z_j(x_j, d_1) \subset z_j(x_j, d_2) \) for \( d_1 > d_2 \) whenever \( d_2 \leq x_j r_j \), all \( j \).

**Lemma 3.** Let \( f \) be a continuous function on \([d, \infty)\) and let (5) hold. Then \( g(x_j) \) given by

\[
g(x_j) \equiv \max_{z_j \in z_j(x_j, d)} \mathbb{E}[f(x_{j+1})] \]

exists and is continuous for \( x_j \geq d/r_j \). If \( f \) is monotone increasing, so is \( g \). If \( f \) is strictly concave, so is \( g \), and the solution \( z_j^*(x_j, d) \) is then unique.
The existence of \( g(x_j) \) is insured by the compactness of \( Z_j(x_j, d) \) (see Lemma 2), (1), and the continuity of \( E[f(x_{j+1})] \) as a function of \( x_j \). The continuity of \( g(x_j) \) follows from the additional fact that the boundary of \( Z_j(x_j, d) \) is continuous in \( x_j \). Monotonicity derives from the observation that for \( \delta > 0 \)

\[
g(x_j + \delta) - g(x_j) = E[f(x_{j+1} + r_j \delta) - f(x_{j+1})] > 0 \quad \text{if } f'(x) > 0
\]

where

\[
g(x_j) = E[f(x_{j+1})] .
\]

For a proof of the concavity property, see Fama [3].

**Lemma 4.** Let \( f \) and \( g \) be continuous and defined for \( x \geq d_f \) and \( x \geq d_g \), respectively, where \( d_f < d_g \). Moreover, let \( f \geq g \) for \( x \geq d_g \) and let (5) hold. Then \( f_1(x) \geq g_1(x) \) for \( x \geq d_g / r_j \), where

\[
f_1(x_j) = \max_{z_j \in Z_j(x_j, d_f)} E[f(x_{j+1}(x_j, z_j))]
\]

and

\[
g_1(x_j) = \max_{z_j \in Z_j(x_j, d_g)} E[g(x_{j+1}(x_j, z_j))]
\]

**Proof:** By Lemma 3, \( f_1(x) \) and \( g_1(x) \) exist and are continuous for \( x \geq d_g / r_j \). Let \( z_j(x_j) \) be the solution to (15). Since \( Z_j(x_j, d_g) \subset Z_j(x_j, d_f) \) by Corollary 3, \( z_j(x_j) \in Z_j(x_j, d_f) \); \( f \leq g \) now gives
\[
\max_{z_j \in Z_j(x_j, d_j)} \mathbb{E}[g(x_{j+1}(x_j, z_j))] = \mathbb{E}[g(x_{j+1}(x_j, z_j))]
\]

\[
\leq \mathbb{E}[f(x_{j+1}(x_j, z_j))] \leq \max_{z_j \in Z_j(x_j, d_f)} \mathbb{E}[f(x_{j+1}(x_j, z_j))],
\]

which completes the proof.

III. FIRST RESULTS

We consider the situation in which the investor, while having the opportunity to make numerous reinvestment decision at discrete points in time, is concerned only with his terminal wealth position, \(x_0\). We assume that his preferences for terminal wealth are representable by utility function, \(U_0(x_0)\), which is continuous, \(J/\) monotone increasing, and defined for \(x_0 \geq c \geq 0\). Letting \(U_n(x_n)\) be the induced utility function, with \(n\) periods to go, of wealth \(x_n\) we obtain, using the principle of optimality, the recursive relation

\[(16) \quad U_n(x_n) = \max_{z_n \in Z_n(x_n, r_0 \cdots r_{n-1})} \mathbb{E}[U_{n-1}(x_{n-1}(x_n, z_n, F_n, r_n))]\]

\[n = 1, 2, \ldots\]

where \(r_0 = 1\). By repeated application of Lemma 3, each function \(U_n(x_n), n = 1, 2, \ldots\), exists for \(x_n \geq c/(r_1 \cdots r_n)\) and is continuous and monotone. Furthermore, should \(U_0(x_0)\) be strictly concave, so is \(U_n(x_n), n = 1, 2, \ldots\).

It is clear that induced utility functions \(U_1(x_1), \ldots, U_n(x_n)\) depend on \(F_1, \ldots, F_n, r_1, \ldots, r_n\) as well as \(U_0(x_0)\), at least under assumptions (1)-(3). But it is possible that the influence of the return structure is
merely to make \( U_1(x_1), \ldots, U_n(x_n) \) (positive) linear transformations of \( U_0(x) \). This is the case if and only if \( U_0(x) \) is isoelastic, i.e., it may be written (Mossin [10])

\[
U_0(x) = \frac{1}{\gamma} x^\gamma ,
\]

where \( \gamma = 0 \) represents \( U_0(x) = \log x \).

In what follows, we shall make use of the following more general result.

**Theorem 1.** (Mossin [10]): Let \( U_0(x_0) \) have the form

\[
U_0(x_0) = \frac{1}{\gamma} (x_0 + m)^\gamma ,
\]

Then the induced functions \( \bar{U}_1(x_1), \bar{U}_2(x_2), \ldots \) given by the recursive relations

\[
\bar{U}_n(x_n) = \max_{x_n \in Z_n} \left( \frac{-m}{r_0 \cdots r_{n-1}} \right) E[\bar{U}_{n-1}(x_{n-1})] , \quad n = 1, 2, \ldots
\]

where \( \bar{U}_0(x_0) = U_0(x_0) \), have the form

\[
\bar{U}_n(x_n) = \frac{k_1(\gamma) \cdots k_n(\gamma)}{\gamma} \left( x_n + \frac{m}{r_1 \cdots r_n} \right)^\gamma , \quad n = 1, 2, \ldots
\]

\[
\sim \frac{1}{\gamma} \left( x_n + \frac{m}{r_1 \cdots r_n} \right)^\gamma
\]

where the constants \( k_1(\gamma), \ldots, k_n(\gamma) \) are given by

\[
k_j(\gamma) \equiv \gamma \max_{v_j \in V_j} E[\frac{1}{\gamma} R_j(v_j)^\gamma] .
\]
(The equivalence, written \( \sim \), follows from the fact that \( k_1(\gamma), \ldots, k_n(\gamma) \) are positive constants.)

Theorem 1 confirms that when \( m = 0 \) in \((18)\),

\[
U_n(x_n) = k_1(\gamma) \cdots k_n(\gamma) U_0(x_n) \quad n = 1, 2, \ldots
\]

so that the optimal portfolio policy in any period is myopic and based on the terminal utility function directly. Note also that \((19)\) is identical to \((16)\) when \( -m = c \). Since the solvency constraint only requires \( c \geq 0 \), and the functions \((18)\) are defined for \( x_0 \geq -m \), it follows that \((19)\) solves \((16)\) whenever \( m \leq 0 \), i.e.,

\[
U_n(x_n) = \frac{k_1(\gamma) \cdots k_n(\gamma)}{\gamma} \left( x_n + \frac{m}{r_1 \cdots r_n} \right)^\gamma \quad n = 1, 2, \ldots
\]

whenever

\[
U_0(x_0) = \frac{1}{\gamma} (x_0 + m)^\gamma \quad m \leq 0.
\]

But when \( m > 0 \) in \((18)\), \((20)\) is not a valid solution to \((16)\) (Hakansson [7]), since \((20)\) does not obtain if the solvency constraint \((4)\) (i.e., \( c \geq 0 \)) must be observed in \((19)\).

**Lemma 5.** (Hakansson and Miller [8]): Let \((1)-(3)\) hold and let \( v_{ij}^{\gamma} \) be the solution to \((21)\). Then

\[
(22) \quad \sum_{i=2}^{M_1} |v_{ij}^{\gamma}| > \varepsilon(\gamma) > 0 \quad j = 1, 2, \ldots.
\]
Thus, the optimal portfolios generated by the class of functions (17) under assumptions (1)-(3) always contain a minimal fraction of risky assets in each period.

We now give an important intermediate result.

**Lemma 6.** $k_j(\gamma)$ given by (21) is positive and increasing in $\gamma$, with $k_j(0) = 1$.

**Proof:** It is immediate from (2) and Lemma 1 that $k_j(\gamma) > 1$, $\gamma > 0$, $k_j(0) = 1$, $0 < k_j(\gamma) < 1$ for $\gamma < 0$. Thus it suffices to demonstrate that $k_j(\gamma - \delta) < k_j(\gamma)$ for $\delta > 0$ and, whenever $\gamma$ is positive, $\delta < \gamma$.

Let

$$ f(x) \equiv \frac{r^\delta}{\gamma} \frac{r_{1{\downarrow}}}{(1 - \delta/\gamma)^{\gamma-\delta}} \left[ x - \frac{\delta r_{1{\downarrow}}}{\gamma} \right] \gamma-\delta $$

$$ g(x) \equiv \frac{1}{\gamma} x^\gamma. $$

We obtain

$$ f(x_j) = g(x_j) = \frac{r^\gamma}{\gamma}, \quad f'(x_j) = g'(x_j) = r_j \gamma^{-1}. $$

We shall first show that

$$ f(x) < g(x) \quad \text{if} \quad \gamma > 0, \quad x \neq x_j $$

by examining the first derivative of

$$ A(x) \equiv \frac{f(x)}{g(x)} = \frac{r^\delta}{(1 - \delta/\gamma)^{\gamma-\delta}} \left[ x - \frac{\delta r_{1{\downarrow}}}{\gamma} \right] \frac{1}{x^\gamma} \quad \text{for} \quad x \geq \text{Max}(0, \frac{\delta r_{1{\downarrow}}}{\gamma}). $$
Differentiating with respect to $x$,

$$A'(x) = \frac{r_j^δ x^{-\gamma-1}(x - \frac{δr_j}{\gamma})^{\gamma-δ-1}}{(1 - \frac{δ}{\gamma})^\delta} (r_j - x) \quad \forall x \leq r_j,$$

which, coupled with (23), implies (24).

By Theorem 1,

$$\max_{z_j \in Z_j(x_j, r_j)} E[f(x_{j+1})] = \frac{k_j (\gamma-δ) r_j^δ}{\gamma(1-\delta/\gamma)^{\gamma-δ}} (x_j - \frac{δ}{\gamma})^{\gamma-δ} \quad x_j \geq \frac{δ}{\gamma} \quad j = 1, 2, \ldots$$

$$\max_{z_j \in Z_j(x_j, 0)} E[g(x_{j+1})] = \frac{k_j (\gamma)}{\gamma} x_j^{\gamma} \quad x_j \geq 0 \quad j = 1, 2, \ldots$$

But by (23), (24), and Lemma 4,

$$k_j (\gamma-δ) r_j^δ \frac{(x_j - \frac{δ}{\gamma})^{\gamma-δ}}{(1 - \frac{δ}{\gamma})^{\gamma-δ}} \leq k_j (\gamma) x_j^{\gamma} \quad x_j \geq \max\{0, \frac{δ}{\gamma}\}.$$

Since $\delta/\gamma < 1$, (25) must hold for $x_j = 1$; inserting $x_j = 1$, (25) gives

$$k_j (\gamma-δ) \leq k_j (\gamma) r_j^{-δ}.$$

Thus $k(\gamma-δ) < k(\gamma)$ for $δ > 0$ (provided $δ < \gamma$ should $\gamma$ be positive), which completes the proof.
Corollary 1. Let $\gamma_1 < \gamma_2$ and $k_1(\gamma_1), k_2(\gamma_1), \ldots$ and $k_1(\gamma_2), k_2(\gamma_2), \ldots$ be given by (21). Then there exists a number $k < 1$ such that

$$\frac{k_1(\gamma_1)}{k_j(\gamma_2)} \leq k \quad j = 1, 2, \ldots$$

Proof: Apply (2) to (26).

We now define

(27) $U_0(x_0) \equiv a + b u_0(x_0)$

(28) $u_n(x_n) \equiv \frac{U_n(x_n)}{k_1(\gamma) \ldots k_n(\gamma)} \quad n = 1, 2, \ldots$

where $b > 0$ and $a$ are constants and $k_1(\gamma_1), \ldots, k_n(\gamma)$ are given by (21). Since $u_1(x_1), u_2(x_2), \ldots$ are merely positive linear transformations of $U_1(x_1), U_2(x_2), \ldots$, they too are valid utility functions.

Our first result of interest is

Theorem 2. Let $u_0(x_0)$ be a continuous and monotone increasing (terminal wealth utility) function defined for $x_0 \geq c \geq 0$ satisfying

(29) $\frac{1}{\gamma} (x_0 - d)^{\gamma} - A(\gamma) \leq U_0(x_0) \leq \frac{1}{\gamma} (x_0 + d)^{\gamma} + A(\gamma)$, $x_0 \geq c$

for some numbers $d \geq c$, $\gamma$, and $A(\gamma)$, where $U_0(x_0)$ is given by (27) and $A(\gamma)$ is positive for $\gamma > 0$ and 0 otherwise. Then the induced (utility of wealth) functions $u_n(x_n)$ (given by (28), (16), and (21)) satisfy
(30) \[ \frac{1}{\gamma}(x_n - D_n) - k^nA(\gamma) \leq u_n(x_n) \leq \frac{1}{\gamma}(x_n + D_n) + k^nA(\gamma) \quad x_n \geq \frac{e}{r_1 \ldots r_n}, \]

\[ n = 1, 2, \ldots \]

where \( k < 1 \) and

(31) \[ D_n \equiv \frac{d}{r_1 \ldots r_n} \leq \frac{d}{n_1} \rightarrow 0. \]

In addition, the optimal policy \( z^*_n(x_n) \) satisfies

\[ z^*_n(x_n) \rightarrow v_{n\gamma} x_n, \]

where \( v_{n\gamma} \) is the solution to (21).

**Proof:** Let \( B_0(x_0) \equiv \frac{1}{\gamma}(x_0 - d) - A(\gamma), \ A_0(x_0) \equiv \frac{1}{\gamma}(x_0 + d) + A(\gamma) \)

and define \( B_1(x_1), B_2(x_2), \ldots \) and \( A_1(x_1), A_2(x_2), \ldots \) by the relations

(32) \[ B_n(x_n) = \max_{z_n \in Z_n(x_n, D_{n-1})} E[B_{n-1}(x_{n-1})] \quad n = 1, 2, \ldots \]

(33) \[ A_n(x_n) = \max_{z_n \in Z_n(x_n, -D_{n-1})} E[A_{n-1}(x_{n-1})] \quad n = 1, 2, \ldots \]

By Theorem 1,

(34) \[ B_n(x_n) = k_1(\gamma) \ldots k_n(\gamma) \frac{1}{\gamma}(x_n - D_n) - A(\gamma) \quad n = 1, 2, \ldots \]

(35) \[ A_n(x_n) = k_1(\gamma) \ldots k_n(\gamma) \frac{1}{\gamma}(x_n + D_n) + A(\gamma) \quad n = 1, 2, \ldots ; \]
they exist for \( x_n \geq D_n \) and \( x_n \geq -D_n \), respectively.

Since \(-d < c \leq d\), we have

\[
(36) \quad -D_n < \frac{c}{r_1 \ldots r_n} \leq D_n \quad n = 1, 2, \ldots
\]

and the functions \( U_n(x_n) \), \( n = 1, 2, \ldots \), given by (16) exist for \( x_n \geq c/(r_1 \ldots r_n) \). Thus by (29), Lemma 4, and (36)

\[
(37) \quad B_n(x_n) \leq U_n(x_n) \leq A_n(x_n) \quad n = 1, 2, \ldots
\]

By Lemma 6 and Corollary 4 there exists a \( k \) such that

\[
(38) \quad k_n(\gamma)^{-1} \leq k < 1 \quad \gamma > 0 \quad n = 1, 2, \ldots
\]

Dividing (37) by \( k_1(\gamma), \ldots, k_n(\gamma) \) and using (28), (34), (35), and (38) gives (30). (31) follows from the fact that \( \eta_1 > 1 \) (see (2)) by assumption. Since both bounds in (30) \( \rightarrow \frac{1}{\gamma} x_n \), for which the optimal policy is \( \nu_n x_n \), \( z_n^*(x_n) \rightarrow \nu_n^* x_n \). This completes the proof.

The following corollaries are immediate.

**Corollary 5.** Assume that \( u_0(x_0) \) is bounded and satisfies (29). Then \( u_n(x_n) + \frac{1}{\gamma} x_n, \gamma < 0 \), i.e., the induced utility functions will lose their lower (but not their upper) bound.

**Corollary 6.** Assume that \( u_0(x_0) \) is unbounded and satisfies (29). Then \( u_n(x_n) + \frac{1}{\gamma} x_n, \gamma \geq 0 \), i.e., if \( \gamma > 0 \), the induced utility functions will become bounded below.
Remark 1. \[ u_0(x_0) = \frac{1}{\gamma} x^\gamma - A \sum_{i=1}^{N} x_i, \] where \( \gamma, A > 0 \), \( \gamma_i < 0, \ i = 1, \ldots, N \), satisfies (29); hence \( u_n(x_n) \), \( n = 1, 2, \ldots \), satisfies (30).

Remark 2. If \( u_0(x_0) \) satisfies (29) for some \( \gamma \), then it satisfies (29) for that \( \gamma \) alone.

In interpreting Theorem 2, it is clear that the shape of the utility function for low levels of wealth is unimportant; the bounds (29) only restrict \( u_0(x_0) \) for very large wealth levels. Thus, for example, any Friedman-Savage terminal utility function [4] which satisfies (29) for large \( x_0 \) always does so for smaller \( x_0 \) and would be covered by either Corollary 5 or Corollary 6.

Let \( q^*(x_0) \equiv -x_0 u_0''(x_0)/u'(x_0) \) be the relative risk aversion function for \( u_0(x_0) \). It is readily verified that \( q^*(x_0) \) need not converge as \( x_0 \) becomes large for terminal utility functions satisfying (29) since \( q^*(x_0) \) may oscillate around \( 1-\gamma \) indefinitely and still satisfy (29). Thus, Theorem 2 goes farther than the results of Leland [9], for which \( q^*(x_0) \) is required to converge.

How broad is the class of terminal utility functions encompassed by Theorem 2? One way to throw light on this question is to identify the functions which do not satisfy (29). One class of such functions is clearly that for which, for every \( d > 0 \), and whenever \( \gamma > 0 \), every \( A(\gamma) > 0 \), there is a sequence \( \{x_i\} \), \( x_i \to \infty \), such that

\[ u_0(x_i) \leq \frac{1}{\gamma}(x_i + d)^\gamma + A(\gamma) \]

(39)
\[ u_0(x_i) \leq \frac{1}{\gamma}(x_i - d)^\gamma - A(\gamma) \]
for some \( \gamma \). (It may be noted that if (39) is true only for \( x_1 \leq x \), larger \( \alpha \) (and, if applicable, larger \( A(\gamma) \)) can be found such that (29) holds.) The remaining possibility is that there is a number \( L(\gamma) \) such that

\[
(40) \quad u_0(x) \geq \frac{1}{\gamma} x^\gamma \quad x \geq L(\gamma), \text{ every } \gamma > 0
\]

or

\[
(41) \quad 0 > u_0(x) \geq \frac{1}{\gamma} x^\gamma \quad x \geq L(\gamma), \text{ every } \gamma > 0.
\]

Consider (41). Setting \( \alpha = L(\gamma) \) we obtain

\[
\frac{1}{\gamma} (x_0 - \alpha)^\gamma < u_0(x_0) < 0 \quad \text{any } \gamma < 0, \quad x_0 \geq c.
\]

By Lemma 4 and Theorem 1, we obtain (after division by \( k_1(\gamma) \ldots k_n(\gamma) \))

\[
(42) \quad \frac{1}{\gamma} (x_n - D_n)^\gamma < u_n(x_n) < 0.
\]

But (42) holds for all \( \gamma \), no matter how negative. Since \( k_j(\gamma) \to 0 \) as \( \gamma \to -\infty \) by (26), we can choose \( \gamma \) sufficiently negative that \( u_n(x) \to -\infty \), \( x < 1 \), which proves, using analogous reasoning for (40),

**Theorem 3.** When \( u_0(x_0) \) satisfies (40), \( u_n(x_n) \to \lim_{\gamma \to \infty} \frac{1}{\gamma} x^\gamma \); when \( u_0(x_0) \) satisfies (41), \( u_n(x_n) \to \lim_{\gamma \to -\infty} \frac{1}{\gamma} x^\gamma \).

**Corollary 7.** Let \( u_0(x_0) = -e^{\gamma x} \), \( \gamma < 0 \). Then \( u_n(x_n) \to \lim_{\gamma \to -\infty} \frac{1}{\gamma} x^\gamma \).

The functions remaining to be addressed are those belonging to the class (39); they will be taken up in the following sections.
IV. THE MAIN THEOREM

We now state our central result.

Theorem 4. Let \( u_0(x_0) \) be a continuous and monotone increasing (terminal wealth utility) function defined for \( x_0 \geq c > 0 \) satisfying

\[
\frac{1}{\bar{Y}}(x_0 - d) - B(x_0 - d) \leq u_0(x_0) \leq \frac{1}{\bar{Y}}(x_0 + d) + A(x_0 + d), \quad x_0 \geq c
\]

for some numbers \( \bar{Y} \) and \( d \geq c \), where \( u_0(x_0) \) is given by (27) and \( B(x), A(x) \geq 0 \) are defined for \( x \geq 0 \). Moreover, let \( B(x) \) and \( A(x) \) be such that (i) when \( \bar{Y} > 0 \),

\[
B(x) = A(x) = Kx^{\bar{Y}},
\]

where \( K > 0, \quad 0 < \bar{Y} < \gamma \); (ii) when \( \bar{Y} \leq 0 \),

\[
\frac{1}{\bar{Y}}(x_0 + d) - A(x_0 + d) < 0
\]

and

\[
\frac{k_n(\gamma, x)}{k_n(\gamma)} \leq k < 1, \quad \frac{p_n(q, x)}{p_n(q, x)} \leq k, \quad n = 1, 2, \ldots
\]

for all positive \( q \leq 1 \) and \( x \geq 0 \), where \( k_n(\gamma) \) is given by (21),

\[
z_{n, \gamma}(x_n, r_0^{n-1}) \] is the solution to (19),

\[
k_n(\gamma, x_n^{-D_n}) \equiv \frac{E[B(x_n - D_n) - D_{n-1}]}{B(x_n - D_n)}
\]

and \( p_n(q, x) \) and \( \bar{p}_n(q, x) \) are defined by
(48) \[
\max_{z_n \in Z_n(x_{n-1}, -D_{n-1})} \mathbb{E}\left[ \frac{1}{\gamma} (x_{n-1} + D_{n-1})^\gamma + qA(x_{n-1} + D_{n-1}) \right]
\]
\[
= \mathbb{P}_n(q, x_n + D_n) \frac{1}{\gamma} (x_n + D_n)^\gamma + \mathbb{P}_n(q, x_n + D_n)qA(x_n + D_n).
\]

Then the induced (utility of wealth) functions \( u_n(x_n) \), \( n = 1, 2, \ldots \) (given by (28), (16), and (21)) satisfy

(49) \[
\frac{1}{\gamma} (x_n - D_n)^\gamma - B_n(x_n) \leq u_n(x_n) \leq \frac{1}{\gamma} (x_n + D_n)^\gamma + A_n(x_n),
\]

\[
x_n \geq \frac{c}{r_1 \cdots r_n}
\]

\( n = 1, 2, \ldots \)

where

(50) \[
A_n(x_n), B_n(x_n), D_n \to 0,
\]

and the optimal policy \( z_n^*(x_n) \) satisfies

(51) \[
z_n^*(x_n) \to v_{\gamma} x_n
\]

where \( v_{\gamma} \) is the solution to (21).

Proof: We give the proof for \( \gamma > 0 \) first. As before, let \( D_n \) be given by (31). By Lemma 6, \( k_n(\gamma) > k_n(\bar{\gamma}) \) and by Corollary 4 there exists a number \( k < 1 \) such that

(52) \[
\frac{k_n(\bar{\gamma})}{k_n(\gamma)} \leq k \quad n = 1, 2, \ldots
\]
For the feasible policy \( v_{nY} \) in (21) we obtain

\[
(53) \quad k_n(\gamma) \equiv E[R_n(v_{nY})^\gamma] \leq k_n(\overline{\gamma}) \text{ if } \overline{\gamma} > 0.
\]

Hence for \( \gamma > \overline{\gamma} > 0 \)

\[
k_1(\gamma) \frac{1}{\gamma}(x_1 - D_1)^\gamma - k_1(\gamma)K(x_1 - D_1)^{\overline{\gamma}} \leq \max_{z_1 \in Z_1(x_1, D_0)} \frac{E[\frac{1}{\gamma}(x_0 - D_0)^\gamma]}{z_1 \epsilon Z_1(x_1, -D_0)}
\]

\[-K(x_0 - D_0)^{\overline{\gamma}} \leq U_1(x_1) \leq \max_{z_1 \in Z_1(x_1, -D_0)} \frac{E[\frac{1}{\gamma}(x_0 + D_0)^\gamma]}{z_1 \epsilon Z_1(x_1, -D_0)}
\]

\[+K(x_0 + D_0)^{\overline{\gamma}} \leq k_1(\gamma) \frac{1}{\gamma}(x_1 + D_1)^\gamma + k_1(\overline{\gamma})K(x_1 + D_1)^{\overline{\gamma}},
\]

\[x_1 \geq \frac{c}{r_1},
\]

again using Lemma 4. By recursion and dividing through by \( k_1(\gamma) \ldots k_n(\gamma) \) we obtain, using (28), (52), and (53),

\[
(54) \quad \frac{1}{\gamma}(x_n - D_n)^\gamma - k_n^2K(x_n - D_n)^{\overline{\gamma}} \leq u_n(x_n)
\]

\[\leq \frac{1}{\gamma}(x_n + D_n) + k_n^2K(x_n + D_n)^{\overline{\gamma}}, \quad x_n \geq \frac{c}{r_1 \ldots r_n},
\]

where \( k_n^2K(x_n + D_n)^{\overline{\gamma}} \to 0 \) since \( k < 1 \) and \( D_n \leq d \), and \( D_n \to 0 \) because \( n_1 > 1 \) (see (31)). Since both bounds in (54) \( \to \frac{1}{\gamma} x_n^\gamma \), for which the optimal policy is \( v_{nY} x_n \), \( z_n^\#(x_n) \to v_{nY} x_n \).

When \( \gamma \leq 0 \), we obtain from (47) and Lemma 4, since \( z_n^Y(x_n, D_{n-1}) \) is a feasible policy,
\[ (55) \quad \frac{k_\gamma(y)}{y} (x_\gamma - D_\gamma)^y - k_\gamma(y, x_\gamma - D_\gamma)B(x_\gamma - D_\gamma) \leq \max_{z_\gamma \in Z_\gamma(x_\gamma, D_0)} E\left[ \frac{1}{y} (x_0 - D_0)^y - B(x_0 - D_0) \right] \leq u_1(x_\gamma) . \]

(45), (48), and Lemma 4 give

\[ (56) \quad u_1(x_\gamma) \leq \max_{z_\gamma \in Z_\gamma(x_\gamma, -D_0)} E\left[ \frac{1}{y} (x_0 + D_0)^y + A(x_0 + D_0) \right] \]

\[ = \frac{p_\gamma(1, x_\gamma + D_\gamma)}{y} (x_\gamma + D_\gamma)^y + p_\gamma(1, x_\gamma + D_\gamma)A(x_\gamma + D_\gamma) \]

\[ < 0 \quad \quad \quad \quad \quad \text{for } x_\gamma \geq c/r_\gamma \]

where, by (21),

\[ p_\gamma(1, x_\gamma + D_\gamma) \geq k_\gamma(\gamma) . \]

Defining \( c_\gamma(x_\gamma) \) by

\[ (57) \quad c_\gamma(x_\gamma) p_\gamma(1, x_\gamma + D_\gamma) = k_\gamma(\gamma) , \quad x_\gamma \geq -D_\gamma , \]

we have

\[ (58) \quad 0 < c_\gamma(x_\gamma) \leq 1 \]

and hence from (56) and (57)

\[ (59) \quad u_1(x_\gamma) \leq \frac{k_\gamma(\gamma)}{\gamma} (x_\gamma + D_\gamma)^y + c_\gamma(x_\gamma) p_\gamma(1, x_\gamma + D_\gamma) A(x_\gamma + D_\gamma) , \]

\[ < 0 \quad \quad \quad \quad \quad \text{for } x_\gamma \geq c/r_\gamma \].
By assumption (46)

\[
\frac{k_n(y, x_n - D_n)}{k_n(y)} \leq k, \quad x_n \geq D_n, \quad n = 1, 2, \ldots
\]

\[
\frac{p_n(k^{n-1}, x_n + D_n)}{p_n(k^{n-1}, x_n + D_n)} \leq k, \quad x_n \geq -D_n, \quad n = 1, 2, \ldots
\]

for some $0 < k < 1$. Dividing (55) and (59) by $k_n(y)$, we obtain, using (28), (57), (60), and (61),

\[
\frac{1}{\gamma} (x_1 - D_1)^Y - kB(x_1 - D_1) \leq u_1(x_1)
\]

\[
\leq \frac{1}{\gamma} (x_1 + D_1)^Y + kA(x_1 + D_1),
\]

\[x_1 \geq c/x_1.\]

Since $k < 1$ in (62) and $D_0 > D_1 > D_2 \ldots$, we can repeat the process for $n = 2, 3, \ldots$, which gives

\[
\frac{1}{\gamma} (x_n - D_n)^Y - k^nB(x_n - D_n) \leq u_n(x_n)
\]

\[
\leq \frac{1}{\gamma} (x_n + D_n)^Y + k^nA(x_n + D_n),
\]

\[x_n \geq \frac{c}{r_1 \ldots r_n}.
\]

Again, both bounds converge to $\frac{1}{\gamma} x^Y$, which completes the proof.
The following corollary is immediate.

**Corollary 8.** Assume that $u_0(x_0)$ satisfies (43) for some $\gamma > 0$, where $A(x) = B(x) \geq 0$ for $x \geq 0$, and

$$k_k(x_0) = \max_{z \in Z_n(x_0,0)} \frac{B[A(x_n) - z]}{A(x_n)}, \quad n = 1, 2, \ldots$$

for all $x_n \geq 0$ and some $k < 1$. Then the induced utility functions $u_n(x_n)$, $n = 1, 2, \ldots$, satisfy (63) and hence (49) and (50).

**Remark 3.** Let $U_0(x_0) = \sum_{i=1}^{I} \frac{a_i}{y_i} x_0$ where the $a_i$ are positive constants and $\max_{i} y_i > 0$. Then $U_0(x_0)$ satisfies (43), and $u_n(x_n)$, $n = 1, 2, \ldots$, satisfies (49) and (50), for $\gamma = \max_{i} y_i$.

In words, if the terminal utility of wealth function is a positive linear combination of power functions and at least one power is positive, the induced utility of wealth functions converge to that power function in the linear combination which has the largest exponent.

When $\gamma < 0$ for some $u_0(x_0)$ in (43), (49), and (50) do not appear to hold for the induced utility functions $u_n(x_n)$ when $A(x)$, $B(x)$ are of the form (44) with $\bar{\gamma} < \gamma$ except in special cases. One such case is when

$$Pr[R_n(v_n) \geq 1] = 1\,, \quad n$$

which is consistent with assumptions (2) and (3) and may well be satisfied for sufficiently risk-averse investors.

Significant extension of the boundaries $A(x)$ and $B(x)$ beyond the limits given in (44) is not possible since if we attempt to make $\bar{\gamma} = \gamma$, convergence of $u_n(x_n)$ to $\frac{1}{\gamma} x_n^\gamma$ need not occur, as the following counterexample shows.
Let \( u_0(x_0) \) be strictly concave and such that

\[
\frac{1}{\gamma} x_0^\gamma \leq u_0(x) \leq \frac{s}{\gamma} x_0^\gamma \quad s > 1
\]

where \( u_0(x) \) has contact with the lower bound at points \( \frac{1}{4} x', x', 4x', \ldots \) and the upper bound at points \( \frac{1}{4} x'', x'', 4x'', 16x'', \ldots \). Assume that \( R_j(v_{j\gamma}) \) (where \( v_{j\gamma} \), as before, is the policy which maximizes (21)) assumes the values 1/2 and 2 with equal probability in each period and let \( v_1^*(x_n) x_n \) be the optimal policy with \( n \) periods to go. Then it is readily verified (recall that \( U_0(x_0) \) is strictly concave) that \( v_1^*(x_1) = v_{1\gamma} \) for \( x_1 = \ldots \frac{1}{2} x', 2x', 8x', \ldots \frac{1}{2} x'', 2x'', 8x'', \ldots \) which gives

\[
U_1(x_1) = \frac{k(\gamma)}{\gamma} x_1^\gamma \quad x_1 = \ldots \frac{1}{2} x', 2x', 8x', \ldots
\]

\[
U_1(x_1) = \frac{sk(\gamma)}{\gamma} x_1^\gamma \quad x_1 = \ldots \frac{1}{2} x'', 2x'', 8x'', \ldots
\]

Continuing in the same fashion, each of the functions \( u_2(x_2), u_3(x_3), \ldots \) is continuous and strictly concave by Lemma 3 and satisfies

\[
\frac{1}{\gamma} x_n^\gamma \leq u_n(x_n) \leq \frac{s}{\gamma} x_n^\gamma
\]

by Lemma 4, with \( u_n(x_n) \) alternatingly touching both boundaries at points \( \ldots \frac{1}{4} x', x', 4x', \ldots \frac{1}{4} x'', x'', 4x'', \ldots \) (n even) or points \( \ldots \frac{1}{2} x', 2x', 8x', \ldots \frac{1}{2} x'', 2x'', 8x'', \ldots \) (n odd). Thus, \( u_n(x_n) \) does not converge to an isoelastic function of any power.
V. A SUFFICIENT CONDITION FOR CONVERGENCE

We have noted that (49) and (50) fail to hold only if the terminal utility function $U_0(x_0)$ satisfies (40) or (41) or oscillates rather violently around some $\frac{1}{\gamma} x^\gamma$ and does so indefinitely. However, failure of the bounds in (43) to converge does not necessarily mean that $u_n(x_n)$ fails to converge to an isoelastic function. For example, if $R_i(v_i)$ were to have a unimodal density, the functions (64) would converge in the sense that $s$, instead of remaining fixed, would $\to 1$ in (65).

The theorem below illustrates the kind of requirement that must be satisfied for a utility function not satisfying (43) to converge to an isoelastic one. The following notation and lemma will be needed. Let

\begin{equation}
(66) 
 u_n(x_n) \equiv \frac{U_n(x_n)}{k_{21} \cdots k_{2n}} \quad n = 1, 2, \ldots
\end{equation}

where $U_1(x_1), U_2(x_2), \ldots$ are given by (16) and $k_{21}, k_{22}, \ldots$ are given by

\begin{equation}
(67) 
 k_{1n} \equiv \gamma_{i, n-1} \max_{v_n \in V_n} \frac{1}{\gamma_{i, n-1}} R_n(v_n) Y_{i, n-1} 
 i = 1, 2, \quad n = 1, 2, \ldots;
\end{equation}

in (67), $\gamma_{2n}$ is the smallest value of $\gamma$ such that

\begin{equation}
(68) 
 U_n(x_n) \leq k_{21} \cdots k_{2n} \gamma_n (x_n + D_{2n})^{\gamma_n} \quad x_n \geq c_n \quad n = 1, 2, \ldots,
\end{equation}

where

\begin{equation}
(69) 
 c_n \equiv \frac{c}{r_1 \cdots r_n} \quad n = 1, 2, \ldots
\end{equation}

Also, let
Lemma 7. If \( \frac{A(x) - x}{\gamma_2 - \gamma_1} \geq \epsilon > 0 \)

where \( \gamma_1 < \gamma_2 \), \( x > 1 \), then

\[
A(x) > x + \epsilon(\gamma_2 - \gamma_1), \quad x > 1.
\]

The proof is based on the strict convexity of \( x^\gamma \) in \( \gamma \), \( x > 1 \).

As an example, for utility functions unbounded above we can state

Theorem 5. Let \( U_0(x_0) \) be a continuous and monotone increasing (terminal wealth utility) function defined for \( x_0 \geq c \geq 0 \) satisfying

\[
(x_0 - D_{10})^{\gamma_{10}} \leq U_0(x_0) \leq (x_0 + D_{20})^{\gamma_{20}}
\]

for some numbers \( \gamma_{20} > \gamma_{10} \geq 0 \) and \( D_{10} > c \), \( D_{20} > 0 \). Then the induced utility functions \( u_n(x_n) \), \( n = 1, 2, \ldots \), (given by \( 66 \), \( 16 \), and \( 67 \)) satisfy

\[
(x_n - D_{1n})^{\gamma_{1n}} \leq u_n(x_n) \leq (x_n + D_{2n})^{\gamma_{2n}},
\]

where both bounds are tight.
(72) \[ D_{in} = \frac{D_{10}}{x_1 \cdots x_n} \quad i = 1, 2, \quad n = 1, 2, \ldots, \]

\[ \gamma_{21} \geq \gamma_{22} \geq \ldots, \quad \text{and} \]

(73) \[ \gamma_{2n} - \gamma_{1n} \leq (1 - \varepsilon)^{n-L} (\gamma_{20} - \gamma_{10}), \quad n \geq L \geq N \]

for some number \( L \) whenever there exists an \( \varepsilon > 0 \) such that

\[ \frac{k_n(x_n)}{k_{2n}} u_n(x_n) - (x_n - D_{1n})^{\gamma_{1n,1}} \]

(74) \[ \frac{k_{2n}}{(x_n + D_{2n})^{\gamma_{2n,1}} - (x_n - D_{1n})^{\gamma_{1n,1}}} \geq \varepsilon \quad x_n \geq D_{1n}, \quad n \geq N. \]

Proof: By (66) and (68) we have

(75) \[ u_n(x_n) \leq \frac{1}{\gamma_{2n}} (x_n + D_{2n})^{\gamma_{2n}} \quad x_n \geq c_n \quad n = 1, 2, \ldots \]

where \( \gamma_{21} \geq \gamma_{22} \geq \ldots \). For \( n = N - 1 \) there also exists, given that \( D_{10} \) is sufficiently large, a \( \gamma_{1, N-1} \) (which may be less than \( \gamma_{10} \)) such that

(76) \[ u_{N-1}(x_{N-1}) \geq (x_{N-1} - D_{1,N-1})^{\gamma_{1,N-1}} \quad x_{N-1} \geq D_{1,N-1} \]

with equality holding for some values of \( x_{N-1} \). Using (66) and (69) we obtain

\[ u_n(x_n) = \frac{k_n(x_n)}{k_{2n}} u_{n-1}(x_n) \quad n = 1, 2, \ldots. \]

Thus, by assumption (74),
(77) \[
\frac{u_n(x_n) - (x_n - D_{1n})^{\gamma_{1,n-1}}}{(x_n + D_{2n})^{\gamma_{2,n-1}} - (x_n - D_{1n})^{\gamma_{1,n-1}}} \geq \varepsilon \quad x_n \geq D_{1n}, \quad n \geq N.
\]

By Lemma 7 (remembering that \( x_n + D_{2n} > x_n - D_{1n} \)), (77) gives

(78) \[
u_N(x_N) > (x_N - D_{1N})^{\gamma_{1,N-1}} + \varepsilon(\gamma_{2,N-1} - \gamma_{1,N-1}) \quad x_N > D_{1N} + 1.
\]

But since \( \gamma_{1,N-1} \geq 0 \),

\[
(x_N - D_{1N})^{\gamma_{1,N-1}} + \varepsilon(\gamma_{2,N-1} - \gamma_{1,N-1}) \leq (x_N - D_{1N})^{\gamma_{1,N-1}}
\]

so that

\[
u_N(x_N) \geq (x_N - D_{1N})^{\gamma_{1N}} \quad x_N \geq D_{1N}
\]

where, with equality holding for some \( x_{1N} \),

\[
\gamma_{1N} > \gamma_{1,N-1} + \varepsilon(\gamma_{2,N-1} - \gamma_{1,N-1}).
\]

Setting \( \gamma_{2,N-1} - \gamma_{1,N-1} = K(\gamma_{20} - \gamma_{10}) \), we obtain

\[
\gamma_{2N} - \gamma_{1N} \leq (1 - \varepsilon)K(\gamma_{20} - \gamma_{10}).
\]

But (77) holds for all \( n \geq N \) so that by recursion
Fig. 1
\[ \gamma_{2n} - \gamma_{1n} \leq (1 - \varepsilon)^{n-N+1} k(\gamma_{20} - \gamma_{10}) \quad n \geq N , \]
\[ \rightarrow 0 , \]

which concludes the proof.

In Fig. 1, the ratio (74) is depicted as \( \frac{A}{A+B} \). The requirement that it exceed \( \varepsilon \) seems rather innocuous. Recall that it need not be operative until \( n \geq N \). When equality holds in (70) (which is only possible over finite intervals of large \( x \) if (43) and (44) are violated since \( D_{10} > c \), \( A/(A+B) \) may in fact be zero for some \( x \) when \( n \) is small. But (3) and (22) insure that after a finite number of iterations contact between \( u_n(x_n) \) and the lower bound is limited to single points; and the same properties assure that from that point on the ratio will be nonzero.

VI. DISCUSSION

The reader will undoubtedly have noted that the preceding results do not depend on the terminal utility function being monotone everywhere. While many locally decreasing functions satisfy the necessary bounds, they were ruled out on a priori grounds. The same observation holds regarding the presence of the solvency constraint (4).

If the analysis is restricted to functions \( U_0(x_0) \) which are risk averse for large \( x_0 \), the upper bound in (1) can be relaxed to the point where only the first moments are bounded.

The preceding results can also be extended to the case in which there is no riskless asset, provided the returns on the risky assets are sufficiently favorable.
Since utility functions of type (40) are clearly of no interest, functions of type (41) seem to be of at most limited interest, and (74) would, for realistic return structures, appear satisfied by many functions not satisfying the requirements of Theorem 4, the asymptotic relevance of the class (17) extends to a substantial portion of the terminal utility functions one might expect to encounter in real situations. The significance of this is threefold. First, "qualifying" investors can without sacrifice behave myopically when their horizon is distant (even though most of them can only do so at a considerable price when the horizon is near), since (17) is the only class for which optimal behavior is always myopic. Second, since (17) exhibits the separation property but each $\gamma$ yields a different optimal mix of risky assets (which is not a linear combination of other mixes) (Cass and Stiglitz [2], Hakansson [5]), one mutual fund for each $\gamma$ associated with the set of terminal utility functions is both necessary and, in the absence of differing return assessments, sufficient to serve all long-run "reinvestors" in the economy. Finally, all long-run "reinvestors" associated with a $\gamma \leq \gamma^*$, where $\gamma^* > 0$ (and depends on $F_1, \ldots, F_n, r_1, \ldots, r_n$), never risk ruin (Hakansson and Miller [8]).
Footnotes

1. This assumption may be viewed as redundant since every von Neumann-Morgenstern utility function defined on an interval is continuous.

2. If \( a(x) \) is defined for \( x \geq x_1 \) and \( b(x) \) for \( x \geq x_2 > x_1 \), we will write \( a(x) \geq b(x), \ x \geq x_1 \), whenever the inequality holds for \( x \geq x_2 \).
References


