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A DISCRETE-TIME SYNTHESIS OF FINANCIAL THEORY:
PART I. OPTIMAL DECISION AND SHARING RULES

by

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A DISCRETE-TIME SYNTHESIS OF FINANCIAL THEORY:
PART I. OPTIMAL DECISION AND SHARING RULES

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This paper is the first of a series of three that provides a synthetic treatment of financial theory, working from the context of a complete securities market. After describing the economic setting for the theory developed in the series, optimal closed-form decision and sharing rules are derived for both a single-period setting and its multiperiod extension. The sharing rules are shown to provide a ready analysis of many important properties of the economy. In particular, they imply economy-wide portfolio separation properties that indicate the extent to which the complete markets context may be relaxed without interfering with the behavior of the economy. Under certain interesting conditions in the multiperiod setting, at each date all individuals in the economy divide their wealth (after consumption) among the same two risky mutual funds: The market portfolio of all securities and an annuity yielding equal certain payments at all future dates. The complete markets context is shown to be far more powerful and general than usually believed. Its scope is circumscribed by the following theorem: A complete market and an otherwise similar incomplete but exchange-efficient market are equivalent in the sense that they reach the same final allocations and have compatible price systems.

This paper is the first of a series of three that provides a synthetic treatment of financial theory: most existing major results and issues are developed and analyzed from a single integrating model.

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1See Rubinstein (1974b, 1974c).
gaps are filled, and the current state of the art is carried well beyond its present boundaries. The impetus behind the series lies in the observation that most of existing financial theory consists of cleverly selected assumptions that lead to closed-form solutions to an Arrow-Debreu (complete market) economy. Indeed, the reason financial theory has met with such considerable success in recent years is that several sets of such assumptions exist under which an incomplete securities market behaves as if it were complete. Consequently, without loss of generality relative to the current state of the art, this series of papers works directly from a complete markets context. Despite limitations of this approach revealed by increasing generalization, it remains the most powerful theoretical tool in finance capable of being shifted comfortably between individual and aggregate levels of analysis, between conditions of certainty and uncertainty, and between static and dynamic settings with the fundamental theoretical elements exhibiting a continuity and a clarity otherwise unobtainable.

All the economies analyzed will share the following three basic properties:

2 The two major exceptions to this are Lintner's (1969) exponential utility model with heterogeneous normal probability assessments and Rubinstein's (1974a) security valuation models without the availability of riskfree investments. However, it will be shown in this paper that these models are flawed by the non-Pareto-efficiency of exchange arrangements.

3 See Radner (1968).
Pl. Expected utility maximization.
P3. Perfect and competitive securities market.

That is, all individuals are rational in the sense that they have beliefs and tastes over consumption⁴ representable by subjective probabilities and utility functions; they prefer more consumption to less and avoid fair gambles; securities are perfectly divisible, there are no exchange costs, each individual acts as if he cannot influence his opportunities, and all individuals have the same opportunities. In addition, except for analysis relegated to footnotes and the concluding remarks, the economics will be consistent with

P5. Nonnegative consumption.

That is, individuals are not motivated to create exchange arrangements not already provided by the market; and no individual plans for negative consumption at any date and state. In comparison with the most important omitted work in financial theory,⁵ trading in securities takes place in


This paper describes in greater detail the economic setting for the theory developed in the series. Optimal individual consumption and investment decision rules and optimal aggregate consumption and investment sharing rules are derived for both a single-period setting and its

⁴In the third paper of the series, tastes are generalized and defined over commodity bundles rather than value of consumption.

⁵See Merton (1971, 1973). The first sequel paper does give brief consideration to continuous-time, showing that under certain conditions the limit of the discrete-time valuation solution as the trading interval approaches zero is equivalent to Merton's continuous-time solution, suitably reinterpreted.
When the burden of specializing assumptions falls upon tastes so that unlike most other work no restrictions are placed on the intertemporal stochastic process of security prices, closed-form consumption and investment decision rules are nonetheless derived for a large and important class of utility functions. The sharing rules imply economy-wide portfolio separation properties which indicate the extent to which the complete markets context may be relaxed without interfering with the behavior of the economy. In particular, under certain interesting conditions in the multiperiod setting, at each date all individuals in the economy divide their wealth (after consumption)

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Individual decision rules are functions mapping information exogenous to an individual (resources, opportunities, beliefs, and tastes) into his choice variables (consumption and portfolio choices). Sharing rules are sets of functions mapping the aggregate supply of a resource (present consumption or future wealth) into individuals' shares of it. For example, if aggregate state dependent future wealth \( W^M_e \) is allocated among \( I(i=1,2,...,I) \) individuals, then the set of functions \( f_i \) is a sharing rule if \( W^M_e = \sum f_i(W^M_e, e) \). Here, \( W^i_e = f_i(W^M_e, e) \) is individual \( i \)'s future wealth if state \( e \) occurs. Following Wilson's (1968) terminology, sharing rule \( f_i \) for individual \( i \) is said to be linear if \( \partial f_i(\cdot)/\partial W^M_e \) is independent of \( W^M_e \) and determinate if \( \partial f_i(\cdot)/\partial W^M_e \) is independent of state \( e \). All sharing rules developed in this paper are linear and all but one (economy (iii)) are determinate.

This represents an extension to the larger class of hyperbolic absolute risk-averse (HARA) functions of Hakansson's (1971) seminal work with constant proportional risk-averse functions. Foreseeing (see Rubinstein 1974b) that intertemporal equilibrium might very likely be inconsistent with a particular specification of the intertemporal stochastic process of security prices, Hakansson placed no restrictions on it. Nonetheless, despite this level of generality (as well as uncertain lifetime), he was still able to derive interesting specific results. Leland (1968) in discrete-time and Merton (1971) in continuous-time have also examined the broader HARA class of utility functions permitting withdrawals for consumption over time; however, both imposed strong arbitrary restrictions on the stochastic process of security prices. Leland, only slightly less restrictive than Merton, assumed security rates of return follow a stationary random walk.
A DISCRETE-TIME SYNTHESIS OF FINANCIAL THEORY

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among the same two risky mutual funds: the market portfolio of all securities and an annuity yielding equal certain payments at all future dates. Moreover, in these cases the sharing rules are shown to be stationary through time. Relaxation of the assumption of homogeneous beliefs is explicitly shown to lead to the creation of side bets between optimists and pessimists and a demand for a greater variety of securities. The sharing rules are also used to identify borrowers and lenders, to determine sufficient conditions for nonnegative consumption, and to develop comparative statics implications of changes in absolute and proportional risk aversion and rates of patience for future consumption.

In short, analysis of sharing rules leads to a quick and easy derivation of important properties of the economy. The paper ends with an assessment of the complete markets approach to modeling in finance. It is concluded to be far more powerful and general than is usually believed.

The first sequel paper, summarized in greater detail elsewhere, develops potentially empirically testable multiperiod valuation relationships for actual securities without restricting the intertemporal stochastic process of security prices. Special cases are examined linking the stochastic processes of real and financial variables. The remainder

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Merton (1971, 1973) also generalizes the single-period portfolio separation results of Cass and Stiglitz (1970) to a multiperiod setting. While similar to the generalization developed in this paper, he restricts the intertemporal shift in investment opportunities. Since this paper admits no such restrictions, its separation properties are both significantly different and of greater generality.
of the paper identifies necessary and sufficient conditions for economic efficiency and their implications for corporate capital structure and capital budgeting decisions and for the production and dissemination of information.

Uncertain noncapital income, commodities, and explicit productive activities by firms are the chief elements of the full-fledged Arrow-Debreu economy as described in Debreu (1959) which are missing in the economy described in this paper and its first sequel. In the second sequel paper these elements are added and again closed-form solutions for individuals and firms are derived, with both the aggregate demand and supply curves of securities and commodities simultaneously determined in equilibrium. Since the prices of commodities delivered at future dates are uncertain, this extended economy incorporates both uncertain relative and absolute inflation.

I. THE SINGLE-PERIOD ECONOMY

Consider a two-date \((t=0,1)\) Arrow-Debreu economy. The state at date \(t = 0\) is known with certainty; at date \(t = 1\) any one of \(E(1, 2, \ldots, E)\) states can occur. Each individual, at date \(t = 0\) allocates his positive present wealth \(W_0\) among present consumption \(C_0\) and \(E\) "state-contingent claims" to future wealth \(W_e\) at date \(t = 1\). Since
$P_e$ denotes the date $t = 0$ present value of a unit of wealth received at date $t = 1$ if and only if state $e$ occurs, then $W_0 = C_0 + \sum_e P_e W_e$.

Each individual is assumed to obey the Savage (1954) Axioms of Rational Choice: He has beliefs and tastes representable by probabilities and a utility function. Let $\pi_e > 0$ denote the subjective probability he attaches to state $e$ and let $\bar{U}(C_0, W_e) = U(C_0) + \rho U(W_e)$ denote his additively separable utility function over present consumption and future wealth,\(^9\) where $\rho$ is a positive constant measuring patience (i.e. at $C_0 = W_e$, $\partial U(\cdot)/\partial W_e + \partial U(\cdot)/\partial C_0 \equiv \rho$) and $U$ is a function such that $U' > 0$ and $U'' < 0$.

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\(^9\)For the single-period case, Dreze and Modigliani (1972) analyze a generalization of this utility function, $\bar{U}(C_0, W_e)$, to allow for time-complementarity of consumption. While Dreze and Modigliani provide only a comparative statics analysis, Pye (1972), using multiplicatively separable utility which permits a modest type of complementarity, is one of the few to develop closed-form solutions.

In the third paper of this series, to give recognition to the obvious complementarities among different commodities consumed at the same date, closed-form solutions will be derived from an additive-multiplicative class of utility functions which will contain Pye's analogous results as a special case.

Another apparent generalization is to permit state-dependent patience so that an individual maximizes $U(C_0) + \sum_e \pi_e \rho_e U(W_e)$. However, following the Savage notion that beliefs and tastes are not objective but rather inferred from behavior, beliefs can be redefined so that $\bar{\pi}_e \equiv \pi_e \rho_e / (\sum_e \pi_e \rho_e)$ and patience redefined as $\bar{\rho} \equiv \sum_e \pi_e \rho_e$. Consequently, the state-independent utility function $U(C_0) + \bar{\rho} \sum_e \bar{\pi}_e U(W_e)$ implies behavior indistinguishable from the behavior implied by the state-dependent function. Only if the utility function is state-dependent in a way that cannot be removed by an increasing linear transformation, will it represent a generalization over the function used in this section. The derived utility of wealth functions of section V of this section...
In brief, the individual solves the following programming problem:

\[
\max_{C_0, \{W_e\}} U(C_0) + \rho \sum_e \pi_e U(W_e) - \lambda [C_0 + \sum_e \pi_e W_e - W_0],
\]

where \(\lambda\) is a Lagrangian multiplier. Differentiating partially with respect to each choice variable, the necessary and sufficient conditions for a unique optimum are

1. \[
U'(C_0) = (\rho \pi_e / P_e) U'(W_e) \quad \text{(all}\ e)\]

2. \[
W_0 = C_0 + \sum_e \pi_e W_e.
\]

To provide for endogenous determination of security prices, we append closure conditions that effectively take aggregate production decisions across dates and states as given. If the population of the economy is \(I(i=1,2,\ldots,I)\), then,

3. \[
C_0^M = \sum_i C_0^i
\]

4. \[
W_e^M = \sum_i W_e^i \quad \text{(all}\ e)\]

Paper will be state-dependent in this nontrivial way, and the derived utility of consumption functions in the third paper in this series will be nontrivially state-dependent through uncertain commodity prices. See Koopmans (1960) for an axiomatic justification of additive separability and measuring patience independent of choice variables.

As stated, this programming problem omits the nonnegativity constraints \(C_0 \geq 0\) and \(W_e \geq 0\) for all \(e\). Rather than introduce them now, they will be assumed to be satisfied at the optimum. Sufficient conditions for them to hold are developed in section III by analyzing the corresponding sharing rules.
where \( C^M_0 \) and \( W^M_e \) denote, respectively, the exogenous aggregate supply of present consumption and future wealth if state \( e \) occurs. In equilibrium, these closure conditions (3) and (4) must hold for the choices of \( (c^1_i, \{w^1_i\}) \) for all \( i \), along with conditions (1) and (2) which must hold for each individual \( i \).

To obtain closed-form solutions to individual choice variables, assume that utility function \( U \) satisfies the following differential equation (HARA class)

\[
- U'(\cdot)/U''(\cdot) = A + B(\cdot)
\]

where \( A \) and \( B \) are fixed parameters. This equation has three solutions depending on the value of \( B \)

(a) \( U(W_e) \sim \frac{b}{1-b} (A + BW_e)^{1-b} \) \( (B \neq 0,1) \)

(b) \( U(W_e) \sim \ln(A + W_e) \) \( (B = 1) \)

(c) \( U(W_e) \sim \frac{-W_e}{A} \) \( (B = 0) \)

where \( b \equiv B^{-1} \) and \( \sim \) means "is equivalent up to an increasing linear transformation to." This class of utility functions is quite rich, containing as special cases: the constant absolute risk aversion function \( (B = 0) \), the constant proportional risk-aversion functions \( (A = 0) \), quadratic utility \( (B = -1) \), and higher-order polynomial utility \( (B = -1/2, -1/3, -1/4, \ldots) \).\(^{11}\)

\(^{11}\)The solutions listed are all those possibly consistent with \( U' > 0 \) and \( U'' < 0 \). Required for consistency and for \( U \) to be defined
II. OPTIMAL DECISION RULES

Using conditions (1) and (2) and when (5) is satisfied, the following optimal consumption and investment decision rules may be derived:12

Theorem (decision rules): The optimal consumption and investment decision rules are linear in initial wealth, and portfolio choices satisfy the portfolio separation property. In particular, for $B \neq 0$,

over positive levels of wealth near zero, in all cases $A > 0$. Moreover, for functions (a) if $A = 0$, then $B > 0$, and if $B = -1, -1/2, -1/3, \ldots$, then $-A/B > W_e$, and for function (c) $A > 0$. For function (b), if $A < 0$, then $W_e > -A$ where $-A$ may be interpreted as a subsistence level.

12 For similar decision rules, but those that do not admit initial consumption, see Cass and Stiglitz (1970, pp. 135, 136). The exponential utility case ($B = 0$) may be derived from the $B \neq 0$ decision rules by noting that $(\rho w_e/P_e)^B$ and $1 + \sum_e P_e (\rho w_e/P_e)^B$ evaluated at $B = 0$ equal 1 and $\phi$, respectively. Moreover,

$$\lim_{B \to 0} b \left[ 1 - \left( \frac{\rho w_e}{P_e} \right)^B \right] = -\ln \left( \frac{\rho w_e}{P_e} \right).$$

This investment decision rule can also be written in terms of portfolio proportions $\beta_e = P_e W_e/(W_0 - C_0)$. It is easily shown for $B \neq 0$

$$\beta_e = \frac{P_e \left( \frac{\rho w_e}{P_e} \right)^B (A \phi + B W_0) - A \left[ 1 + \sum_e P_e (\rho w_e/P_e)^B \right]}{(A + B W_0) \sum_e P_e (\rho w_e/P_e)^B - A \sum_e P_e} \quad \text{(all } e) \quad (\text{all } e).$$

Observe that for logarithmic utility ($A = 0, B = 1$), this reduces to $\beta_e = \pi_e$. 
\[ C_0 = \frac{Ab\sum e_{pe} \left[ 1 - \frac{(\rho \pi_e / P_e)^B}{1 + \sum e_p e (\rho \pi_e / P_e)^B} \right]}{1 + \sum e_p e (\rho \pi_e / P_e)^B} W_0 + \frac{1}{1 + \sum e_p e (\rho \pi_e / P_e)^B} W_0 \]

\[ W_e = \frac{Ab\left\{ \frac{\sum e_p e \left[ 1 - \frac{(\rho \pi_e / P_e)^B}{1 + \sum e_p e (\rho \pi_e / P_e)^B} \right]}{1 + \sum e_p e (\rho \pi_e / P_e)^B} \right\} - \frac{\rho \pi_e / P_e}{1 + \sum e_p e (\rho \pi_e / P_e)^B} W_0}{1 + \sum e_p e (\rho \pi_e / P_e)^B} + \frac{(\rho \pi_e / P_e)^B}{1 + \sum e_p e (\rho \pi_e / P_e)^B} W_0 \]  

(all e)

and for \( B = 0 \),

\[ C_0 = - (A/\phi)\sum e_p e \ln(\rho \pi_e / P_e) + W_0 / \phi \]

\[ W_e = - (A/\phi)\left[ \sum e_p e \ln(\rho \pi_e / P_e) - \phi \ln(\rho \pi_e / P_e) \right] + W_0 / \phi \]  

(all e)

where \( \phi = 1 + \sum e_p e \).

**Proof:** When \( B \neq 0 \), then from condition (1)

(a) \[ A + BW_e = (\rho \pi_e / P_e)^B(A + BC_0) \]  

(all e).

Multiplying both sides by \( P_e \), summing over \( e \), adding \( A + BC_0 \) to each side, and using condition (2)

(b) \[ A\phi + BW_0 = (A + BC_0)\left[ 1 + \sum e_p e (\rho \pi_e / P_e)^B \right]. \]

Solving this for \( C_0 \) yields the consumption decision rule. Substituting this into equation (a) by eliminating \( C_0 \) and solving this for \( W_e \) yields the investment decision rule. When \( B = 0 \), then from condition (1)
(a') \[ W_e = A \ln(\frac{\rho \pi_e}{P_e}) + C_0 \quad (\text{all } e). \]

Again, multiplying both sides by $P_e$, summing over $e$, adding $C_0$ to each side and using condition (2)

(b') \[ W_0 = A \sum_e P_e \ln(\frac{\rho \pi_e}{P_e}) + C_0 \phi. \]

Solving this for $C_0$ yields the consumption decision rule. Substituting this into equation (a') by eliminating $C_0$ yields the investment decision rule. To prove portfolio separation, for $B \neq 0$ it can be shown that an individual's investment decision rule can be written as

\[ W_e = (1+r_F) \left[ \frac{A(1-\phi)}{B} + \frac{(\rho \pi_e/P_e)^B}{\sum_e P_e (\rho \pi_e/P_e)^B} \left( \frac{(A\phi+BW_0) \sum_e P_e (\rho \pi_e/P_e)^B}{B \left[ 1+\sum_e P_e (\rho \pi_e/P_e)^B \right]} \right) \right] \]

and for $B = 0$

\[ W_e = (1+r_F) \left[ \frac{\phi-1)(W_0 - \sum_e P_e \ln(\frac{\rho \pi_e}{P_e}))}{\phi} + \frac{\ln(\frac{\rho \pi_e}{P_e})}{\sum_e P_e \ln(\frac{\rho \pi_e}{P_e})} \left[ \sum_e P_e \ln(\frac{\rho \pi_e}{P_e}) \right] \right] \]

where $1 + r_F \equiv (\sum_e P_e)^{-1}$ and the two bracketed expressions in each case sum to $W_0 - C_0$. Consequently, irrespective of his level of initial wealth and consumption, an individual divides his total investment among the same two mutual funds, one of which is riskfree. Q.E.D.

It will also prove useful for what follows to determine the derived utility of initial wealth functions
\[ V(W_0) = U(C_0) + \rho \Sigma_{e \in W} U(W_e) \quad \text{given} \quad W_0 \]

where \( C_0 \) and \( \{W_e\} \) are at their optimum values as given by the respective consumption and investment decision rules. Using these decision rules, it can be shown that

\[
(a') \quad V(W_0) \sim \left[ 1 + \Sigma_{e \in W} \left( \frac{\rho \pi \pi_e}{P_e} \right) \right]^{b} \frac{b}{1-b} \left( A \Phi + B W_0 \right)^{1-b} \quad (B \neq 0, 1)
\]

\[
(b') \quad V(W_0) \sim (1+p) \ln(A \Phi + W_0) \quad (B = 1)
\]

\[
(c') \quad V(W_0) \sim -e^{-1} \sum_{e \in W} \ln(\rho \pi \pi_e/P_e) - \frac{W_0}{A \pi_e} \quad (B = 0)
\]

where \( \sim \) means "is equivalent up to an additive transformation to."

Observe that the derived function is of the same form as \( U(\cdot) \) and, in particular, \(-V'(W_0)/V''(W_0) = A \Phi + B W_0\). Also, since \( U' > 0 \) and \( U'' < 0 \), then\(^{13}\) \( V' > 0 \) and \( V'' < 0 \) so that \( A \Phi + B W_0 > 0 \).

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\(^{13}\) A similar point is demonstrated by Fama (1970) for the more general case of time-complementarity of consumption. However, in the case here of additively separable utility, the strictly increasing and strictly concave properties of the derived utility of wealth function \( V \) are easy to see. With \( U \) strictly increasing and strictly concave, consider what happens to a marginal increment of initial wealth. Since \( U' > 0 \), it will not be thrown away but will be spent on present consumption or future wealth. \( U'' < 0 \) argues for moderation or balance so that both will increase. Consequently, since \( U' > 0 \) the value of \( U(C_0) + \rho \Sigma_{e \in W} U(W_e) \) at the new optimal values of \( (C_0, \{W_e\}) \) must increase so that \( V' > 0 \). Likewise, since \( U'' < 0 \), the value of \( U(C_0) + \rho \Sigma_{e \in W} U(W_e) \) at optimal values of \( (C_0, \{W_e\}) \) must increase at a decreasing rate so that \( V'' < 0 \).
III. OPTIMAL SHARING RULES

To examine the comparative behavior of different individuals in the economy as well as to allow reduction in the number of required securities, it is useful to derive optimal consumption and investment sharing rules. However, to develop interesting cases, it is necessary to assume individuals have partially similar economic characteristics, i.e., homogeneity conditions. In particular, the following special economies will be examined:

i. (Generalized power and logarithmic utility) All individuals have the same beliefs \( \{\pi_e\} \) and taste parameter \( B \) where \( B \neq 0 \).

ii. (Exponential utility) All individuals have the same taste parameter \( B \) where \( B = 0 \).

iii. (Generalized logarithmic utility) All individuals have the same resources \( W_0 \) and tastes \( \rho, A, \) and \( B \) where \( B = 1 \).

In each case, individuals may be heterogeneous in all other respects. With these homogeneity conditions, the following optimal consumption and investment sharing rules may be derived:

Theorem (sharing rules\(^{14}\)): The optimal consumption and investment sharing rules are linear and, except for economy (iii), are determinate. In particular, for economy (i) and all \( i \) and \( e \),

\(^{14}\)In Rubinstein (1974a), for economy (i) with homogeneous patience and for economies (ii) and (iii), it is shown that in each case a composite individual may be constructed with resources \( W_0 \equiv \frac{W}{W_0} \), beliefs \( \pi_e \) and tastes \( \rho, A \equiv \frac{A}{A-I} \) and \( B \) such that equilibrium security prices are
\[ C^i_0 = (b/K^e_0)[A^i_0 \bar{K}^i_0 - A^i_1 \bar{K}^i_1] + (\bar{K}^i_1/K^e_0)C^M_1 \]

\[ W^i_e = (b/K^e_0)[A^i_0 K^i_0 - A^i_1 K^i_1] + (K^i_1/K^e_0)W^M_0 \]

where \( A^i_M \equiv \sum_i A^i_i \), \( K^i_1 \equiv \left[ 1 + \sum_P \frac{\rho^P_i \pi^P_e}{P^e} \right]^{-1} (A^i_0 + BW^i_0) \),

\( \bar{K}^i_M \equiv \sum_i \bar{K}^i_i \), \( K^i_i \equiv \rho^P_i \bar{K}^i_i \) and \( K^i_M \equiv \sum_i K^i_i \); for economy (iii) for all \( i \) and \( e \),

\[ C^i_0 = -(A^i/\phi)_{\Sigma_P \sum_P \ln(\pi^P_e/\pi_e)} - (A^i_0/\phi)_{(\phi-1)_{\Sigma_P \sum_P \ln(\rho^P_i/\rho)}} + (A^i_0/\phi)_{\sum_P \sum_P \ln(\rho^P_i/\rho)}^{-1}\left[ A^i M_0 - A^i_1 W^M_0 \right] \]

\[ + (A^i_1/A^i_0)C^M_0 \]

\[ W^i_e = -A^i_0_{\Sigma_P \sum_P \ln(\pi^P_e/\pi_e)} - (A^i_0/\phi)_{\sum_P \sum_P \ln(\pi^P_e/\pi_e)} + (A^i_1/\phi)_{\sum_P \sum_P \ln(\rho^P_i/\rho)}^{-1}\left[ A^i M_0 - A^i_1 W^M_0 \right] \]

\[ + (A^i_1/A^i_0)W^M_0 \]

where \( W^M_0 \equiv \sum_i W^i_0 \), \( A^i_M \equiv \sum_i A^i_i \), \( \rho \equiv \Pi_i \rho^i_1 \) and \( \pi^i_e \equiv \Pi_i \pi^i_e \); for economy (iii) for all \( i \) and \( e \),

\[ C^i_0 = C^M_0 / I \quad \text{and} \quad W^i_e = A(\pi^i_e - \pi^e_e) / \pi^e_e + (\pi^i_e / \pi^e_e)(W^M_e / I) \]

where \( \pi^e_e \equiv \Sigma_i \pi^i_e / I \).

determined as if there exist only identical composite individuals. That is, the philosophical notion of "average man" can be given precise characterization for these economies.
Proof: For economy (i), from equation (b) in the proof of the decision rules theorem, at the individual level \( A_i + BC_i^0 = \bar{k}_i \) for all \( i \). Summing this over \( i \), at the aggregate level \( A_M + BC_M^0 = \bar{k}_M \). Combining these two equations and solving for \( C_0^i \) yields the optimal consumption sharing rule. Using both equations (a) and (b) from that same proof, at the individual level \( A_i + BW_e^i \cdot (\pi_e / p_e) K_i \) for all \( i \). Summing this over \( i \), at the aggregate level \( A_M + BW_e^M \cdot (\pi_e / p_e) K_i \). Combining these two equations and solving for \( \bar{w}_e \) yields the optimal investment sharing rule. For economy (ii), a similar proof follows from equations (a') and (b'), noting that the logarithm of a product equals the sum of the logarithms. For economy (iii), a similar proof follows from equations (a) and (b), noting that \( 1 + \sum _e p_e (\rho \sigma_e / p_e) B_e = 1 + \rho \). Q.E.D.

The sharing rules are the key to a ready analysis of many important properties of the three economies. Starting with economy (i), the consumption and investment sharing rules have the natural interpretation of providing a state-dependent dividend (last term) and a certain side payment (first term). To see this more easily, suppose all individuals in the economy have the same patience \( \rho \), then the sharing rules will simplify to

\[
C_0^i = \frac{A_M^i - A_0^i}{A_M^i + BW_0^M} + \frac{A_i^i + BW_0^i}{A_M^i + BW_0^M} C_0^M
\]

\[
W_e = \frac{A_M^i - A_0^i}{A_M^i + BW_0^M} + \frac{A_i^i + BW_0^i}{A_M^i + BW_0^M} W_e^M
\]
where \( W_0^M \equiv \sum_1 W_0^1 \). Observe that for the dividend

\[
\sum_1 \left( A_1 \phi + B_0^1 \right) \left( A_M \phi + B_0^M \right)^{-1} = 1
\]

and for the side payment

\[
\sum_1 \left( A_M W_0^1 - A_1 W_0^M \right) \left( A_M \phi + B_0^M \right)^{-1} = 0.
\]

Consequently, focusing on the investment decision, despite the variety of securities available, every individual chooses to hold the market portfolio \( M \) in proportion \( \left( A_1 \phi + B_0^1 \right) \left( A_M \phi + B_0^M \right)^{-1} \) and to borrow or lend \( \left( A_M W_0^1 - A_1 W_0^M \right) \left( A_M \phi + B_0^M \right)^{-1} \left( e \right. \left. \left. p_e \right)^{-1} \right. \) at the certain one plus rate of return \( \left( e \right. \left. \left. p_e \right)^{-1} \right. \). This can be called universal portfolio separation since all individuals invest in the same risky mutual fund and make side arrangements between each other in a risk-free investment.

Since (see end of section II) \( A_1 \phi + B_0^1 > 0 \) so that \( A_M \phi + B_0^M > 0 \), no individual sells the market portfolio short. An individual lends (borrows) if and only if \( A_M W_0^1 - A_1 W_0^M > (\leq) 0 \). Define \( W_0 \equiv W_M^1 / I \) and \( A \equiv A_M / I \) as the arithmetic average initial wealth and taste parameter, respectively. Therefore, given \( A_1 \neq 0 \), an individual lends (borrows) if and only if \( W_0^1 / A_1 > (\leq) W_0 / A \). Rich individuals tend to be lenders and individuals with high taste parameter \( A_1 \) tend to be borrowers. If these effects exactly offset so that \( W_0^1 / A_1 = W_0 / A \), then an individual only holds the market portfolio. For complete portfolio
separation to be universal, then either \( \frac{W^i_0}{A_i} = \frac{W_0}{A} \) for all \( i \) or \( A_i = 0 \) for all \( i \). In these cases, the investment sharing rule reduces to \( \frac{W^i_e}{A_i M} = \frac{W^i}{W^i_0 M} \) or \( \frac{W^i_e}{A_i M} = \frac{W^i_0}{W^i_0 M} \), respectively, for all \( i \) and \( e \), and no side payments are made.

Moreover, in these cases where differences among individuals are not sufficiently large, nonnegativity constraints \( C_0^i > 0 \) and \( W^i_0 > 0 \) are automatically satisfied. More generally, requiring both these nonnegativity constraints to be binding is equivalent to requiring\(^{15}\) for \( A_i > 0 \)

\[
\frac{W^i_0}{A_i} \geq \max \left[ \frac{W_0 - C_0^i}{A + B C_0^i}, \frac{W^i_e - W^i_0}{A + B W^i_0} \right] \quad \text{(all \( i \) and \( e \))}
\]

where \( W^i_e = \frac{W^i_e}{W^i_0} \) and \( C_0 = \frac{C_0^i}{W^i_0} \). Consequently, if \( C_0^i > W^i_0 \) and \( W^i_0 \geq W^i_0 \) for all \( i \), then the nonnegativity constraints are satisfied, irrespective of the disparity of economic characteristics among individuals. However, like Scylla and Charybdis, both conditions can only be simultaneously satisfied in a hair's-breadth scope. To see this, suppose the future were certain so that \( W^i_0 = W^i_e \) for all \( e \) and \( P_F = \sum_{e} P^e \), then \( W_0 = C_0 + P_F W^i_0 \). The first inequality \( C_0^i > W^i_0 \) is then equivalent to \( C_0(1 + P_F) > C_0 + P_F W^i_0 \), which in turn is equivalent to

\(^{15}\)To see this, from the simplified sharing rule for consumption, \( C_0^i > 0 \) if and only if \( A_i W^i_0 - A_i W^i_0 + A_i C_0^i + B W^i_0 C_0 > 0 \). Rearranging, \( A_i (C_0^i - W^i_0) + W^i_0 (A_i + B C_0) > 0 \). Now, dividing by \( i \), since \( A_i > 0 \)

\[
\frac{W^i_0}{A_i} \geq \frac{W^i_0 - C_0^i}{A_i + B C_0} \]

A similar argument holds for the sharing rule for investment. For \( A_i < 0 \), the final inequality is reversed so that \( \frac{W^i_0}{A_i} \leq \max[\ast] \) but the same conclusion holds.
\[ C_0 \geq W_1. \] The second inequality \( W_0^\phi \geq W_0 \) is then equivalent to 
\[ W_1(1 + P_F) \geq C_0 + P_F W_1, \] which in turn is equivalent to \( W_1 \geq C_0. \)
Clearly, these two inequalities can both hold if and only if \( C_0 = W_1. \)
In other words, under certainty the nonnegativity constraints will be 
automatically satisfied for all individuals in the economy irrespective 
of their heterogeneity if and only if aggregate present consumption and 
aggregate future wealth are equal. To the extent \( C_0^M > W_1^M \) or \( C_0^M < W_1^M, \)
individual differences must be circumscribed to prevent violation of the 
nonnegativity constraints. Roughly speaking, as long as the future is 
not too uncertain and as long as societal impatience approximately off- 
sets the technological advantages from roundabout production, the non-
negativity constraints will be satisfied for all individuals and all 
states.

The sharing rules are also useful for analyzing the comparative 
statics implications of taste parameter \( B. \) Since 
\[ -U'_i(W_i^e)/U''_i(W_i^e) = A_1 + BW_i^e, \] then absolute risk aversion 
\[ -U''_i(W_i^e)/U'_i(W_i^e) = (A_1 + BW_i^e)^{-1}. \]
Consequently, absolute risk aversion is increasing (constant, decreasing) 
if and only if \( B < 0 \) (\( B = 0 \), \( B > 0 \)). The implications for portfolio 
behavior can be read directly from the investment sharing rule. When 
\( B < 0, \) the proportion held of the market portfolio 
\( (A_1^\phi + BW_0^M)/(A_1^\phi + BW_0^M) \) 
is less for rich than for poor individuals. A similar interpretation of 
this proportion follows from recalling that (see end of section II) 
\[ A_1^{-1} = -V'_i(W_0^i)/V''_i(W_0^i) = A_1^\phi + BW_0^i. \] Aggregating over all individuals and 
dividing by \( I, \) \( A_1^{-1} = \Sigma_i A_1^{-1}/I = A_1^\phi + BW_0^i. \) Consequently,
\[ \frac{A_i \phi + BW_i^0}{A_i \phi + BW_i^M} \frac{W_i^M}{W_i^e} = \frac{A_i}{A_i} \frac{W_i}{W_i} \quad (\text{all } i \text{ and } e). \]

That is, the proportion of the market portfolio held by an individual varies inversely with his absolute risk aversion (evaluated at \( W_i^0 \)).

Similarly, \( \text{proportional risk aversion} = \frac{W_i^i W_i^0}{U_i^i W_i^0}/U_i^i W_i^0 = [(A_i/W_i^0) + B]^{-1} \). Consequently, proportional risk aversion is increasing (constant, decreasing) if and only if \( A_i > 0 \) (\( A_i = 0 \), \( A_i < 0 \)). The implications of proportional risk aversion can be read directly from the investment sharing rule by observing that the ratio

\[ \frac{A_i W_i^0 - A_i W_i^M}{A_i \phi + BW_i^M} = \frac{P_i - P}{A_i \phi} \quad (\text{all } i) \]

where \( P_i = -W_i^i W_i^0(U_i^i W_i^0)/U_i^i W_i^0 \) and \( P^{-1} = E_i P_i^{-1}/I \). That is, an individual lends if and only if his proportional risk aversion (evaluated at \( W_i^0 \)) is increasing.

---

16 The HARA class contains all five possible combinations of absolute and proportional risk aversion:

<table>
<thead>
<tr>
<th>Absolute Risk Aversion</th>
<th>Proportional Risk Aversion</th>
<th>B</th>
<th>A</th>
</tr>
</thead>
<tbody>
<tr>
<td>increasing</td>
<td>(increasing)</td>
<td>B &lt; 0</td>
<td>A &gt; 0</td>
</tr>
<tr>
<td>constant</td>
<td>(increasing)</td>
<td>B = 0</td>
<td>A &gt; 0</td>
</tr>
<tr>
<td>decreasing</td>
<td>increasing</td>
<td>B &gt; 0</td>
<td>A &gt; 0</td>
</tr>
<tr>
<td>(decreasing)</td>
<td>constant</td>
<td>B &gt; 0</td>
<td>A = 0</td>
</tr>
<tr>
<td>(decreasing)</td>
<td>decreasing</td>
<td>B &gt; 0</td>
<td>A &lt; 0</td>
</tr>
</tbody>
</table>

Parentheses denote a redundancy.

See Arrow (1965) for a more general comparative statics analysis of absolute and proportional risk aversion.
greater than average for the economy, and the absolute size of his position in the riskfree investment varies inversely with his absolute risk aversion (evaluated at $w_0^1$). In summary, the investment sharing rule can be written alternatively as

$$w_e^1 = \frac{p_i - p}{A_i^1 \phi} + \frac{A_i^1}{A_i^1} w_e \quad \text{(all } i \text{ and } e).$$

For economy (ii), when beliefs and patience are homogeneous, the sharing rules again collapse to

$$c_0^1 = \frac{M^0 w_0^1 - A_i^1 w_0^M}{A_i^1 \phi} + \frac{A_i^1}{A_i^1} c_0^M$$

$$w_e^1 = \frac{M^0 w_0^1 - A_i^1 w_0^M}{A_i^1 \phi} + \frac{A_i^1}{A_i^1} w_e^M$$

---

17 This paper has thus far ignored one very useful assumption in financial modeling. Rather than place the burden of assumptions on tastes, it is popular instead to require homogeneous normal probability assessments for security rates of return, while requiring tastes only to be nonsatiated and risk-averse. As a slight extension of Rubinstein (1973a), the investment sharing rule becomes

$$w_e^1 = \frac{(w_0^1 - c_0^1) \theta_i^1 - (w_0^1 - c_0) \theta_i}{\theta_i (\phi - 1)} + \frac{8}{\theta_i} w_e$$

where $\theta_i^1 \equiv -E \left[ u'(w_1^1) / u'(w_1^1) \right] / E \left[ u'(w_1^1) \right]$ and $\theta^{-1} \equiv E_i \theta_i^{-1} / \theta_i$; $E$ is an expectation operator and $w_1^1$ a random variable denoting future wealth. Rubinstein shows that $\theta_i^1$ has comparative statics implications similar to absolute risk aversion.

Again, as the sharing rule indicates, all individuals divide their wealth (after consumption) between the market portfolio and a riskfree
for all $i$ and $e$ so that the above analysis of the sharing rules for economy (i) also applies to economy (ii). However, the advantage of economy (ii) lies in the straightforward analysis of the role of heterogeneous patience and beliefs. As expected, individuals with higher than average patience ($\rho_1 > \rho$) consume less and invest more. The additional investment is accomplished only through a greater side payment (more lending). Moreover, if the side payments created by heterogeneous patience are summed over all individuals, they are self-canceling since

$$\sum_i (A_i / \phi) \ln(\rho_1 / \rho) = 0.$$  

With heterogeneous beliefs, side payments are altered by the amount $-(A_i / \phi) \sum_e P_e \ln(\pi^i_e / \pi_e)$ for each individual. An individual who tends to be optimistic (i.e. $\pi^i_e > \pi_e$ so that $\ln(\pi^i_e / \pi_e) > 0$) toward states for which contingent claims are relatively expensive to purchase, will find himself making less of a side payment. In any case, these investment. However, if tastes do not belong to the HARA class for all individuals with homogeneous taste parameter $B$, further analysis of the sharing rule is impeded. Indeed, as noted in Rubinstein (1974a), the aggregation problem arising in the derivation of equilibrium prices generally has no analytic solution.

Even more disturbing, the normality assumption violates P5 since individuals are forced to plan on a possibly small but positive probability of negative future wealth. Moreover, in the context of a multiperiod setting, to assume as Fama (1970) does that first-period rates of return are a normally distributed random walk is to assume what should be a derived endogenous property of the economy. Although Merton's (1973) lognormal random walk does not violate property P5, it still has this problem. By contrast, the results in the text of this paper and its sequels are consistent with any arbitrary discrete-time intertemporal stochastic process.
additional side payments will be self-canceling since they sum to zero over all individuals. Individuals also make side bets \( A_i \ln(\frac{\pi^i_e}{\pi_e}) \) on each state. As an unrealistic implication of constant absolute risk aversion, i.e., exponential utility, these side bets, although they depend on the taste parameter \( A_i \), are independent of initial wealth. Since for every optimist (\( \pi^i_e > \pi_e \)) there must be a pessimist (\( \pi^i_e < \pi_e \)), the side bets for every state are self-canceling; that is, \( \sum_i A_i \ln(\frac{\pi^i_e}{\pi_e}) = 0 \). When all individuals have the same beliefs for all states, then no side bets are made since \( \pi^i_e = \pi_e \) and \( \ln(\frac{\pi^i_e}{\pi_e}) = 0 \) for all \( i \) and \( e \). Consequently, even in the absence of a complete securities market, the economy can achieve the same equilibrium allocation and price system as long as a risk-free investment exists. Complete markets are only required by heterogeneous beliefs. Indeed, although individuals have heterogeneous beliefs about most states, if there were some states \( e \) which garnered agreement so that \( \pi^i_e = \pi_e \) for all \( i \), then state-contingent claims to these states need not exist.

Economy (iii) provides another opportunity to analyze the effects of heterogeneous beliefs. Rewriting the investment-sharing rule as

\[
W^i_e = \left[ \frac{\pi^i_e - \pi_e}{\pi_e} \right] (A + W_e) + W_e
\]

where \( W_e \equiv W_e^M/I_e \), with homogeneous beliefs all individuals are identical so that \( W^i_e = W_e \). With heterogeneous beliefs, individuals make side bets \((A + W_e)(\pi^i_e - \pi_e)/\pi_e\) which, unlike the constant absolute risk aversion
case, depend on wealth as well as on taste parameter $A$. Optimists $(\pi_e^1 > \pi_e^c)$ bet positively on state $e$, and pessimists negatively on state $e$, with all side bets self-canceling.

IV. THE MULTIPERIOD ECONOMY

To extend the foregoing analysis to an economy with many dates, there is no loss of generality by considering only a three-date $(t = 0, 1, 2)$ Arrow-Debreu economy. Individuals are now permitted to make consumption-investment decisions at dates $t = 0$ and $t = 1$. In brief, the individual now solves the following two-stage programming problem. At date $t = 1$, given state $e$ and $W_e$, an individual

$$V_e(W_e) = \max_{C_e, \{W_{s*e}\}} U(C_e) + \rho_2 \sum_s \pi_{s*e} U(W_{s*e}) - \lambda_e [C_e + \sum_s p_{s*e} W_{s*e} - W_e]$$

where at date $t = 2$ any one of $S$ $(s = 1, 2, \ldots, S)$ states can occur, $\pi_{s*e}$ denotes the state $e$ conditional subjective probability attached to state $s$, $p_{s*e}$ denotes the state $e$ conditional date $t = 1$ present value of a unit of wealth received at date $t = 2$ if and only if state $s$ occurs, date $t = 1$ wealth $W_e$ is allocated among date $t = 1$ consumption $C_e$ and "state-contingent claims" to date $t = 2$ wealth $W_{s*e}$, $\rho_2$ measures patience between dates $t = 1$ and $t = 2$, $\lambda_e$ is a stochastic Lagrangian multiplier, and $V_e(W_e)$ is a derived state-dependent utility function of wealth at date $t = 1$. At date $t = 0$, an individual
\[ V_0(W_0) \equiv \max_{C_0, \{\bar{W}_e\}} U(C_0) + \rho_1 \sum_{e} \bar{V}_e(W_e) - \lambda_0 [C_0 + \sum_{e} \bar{P}_e \bar{W}_e - \bar{W}_0] \]

similar to his programming problem in the single-period economy except where the utility function for future wealth has been derived assuming optimal consumption-investment decisions at date \( t = 1 \). Observe that unlike most other work no restrictions have been placed on the intertemporal stochastic process of security prices. This paper will continue to retain this level of generality with the burden of assumptions falling on specifications of tastes (equation (5)).

Differentiating these two-stage maximands partially with respect to each choice variable and rearranging the resulting necessary and sufficient conditions to eliminate the Lagrangian multipliers

\begin{align*}
(6) & \quad U'(C_e) = (\rho_2 \bar{P}_s \bar{e} / P_s \bar{e}) U'(\bar{W}_s \bar{e}) \quad (all \ e \ and \ s) \\
(7) & \quad \bar{W}_e = C_e + \sum_s P_s \bar{e} \bar{W}_s \bar{e} \quad (all \ e) \\
(8) & \quad U'(C_0) = (\rho_1 P_e / P_e) U'(W_e) \quad (all \ e) \\
(9) & \quad \bar{W}_0 = C_0 + \sum_e P_e \bar{W}_e.
\end{align*}

To provide for the endogenous determination of security prices, we require

\begin{align*}
(10) & \quad C_0^M = \sum_i C_0^i \\
(11) & \quad C_e^M = \sum_i C_e^i \quad (all \ e)
\end{align*}
(12) \[ W^M_{s*e} = \Sigma_i W^i_{s*e} \quad (\text{all } e \text{ and } s). \]

In equilibrium, conditions (10)-(12) must hold for the choices of 
\((c_0^i, \{c^i_e\}, \{\bar{W}^i_{s*e}\})\) for all \(i\), along with conditions (6)-(9) which must hold for each individual \(i\).

V. OPTIMAL DECISION RULES

The decision and sharing rules for the second period of this economy have already been derived and analyzed in sections II and III. It remains to derive these rules for the first period. Using (suitably indexed) derived utility functions \( (a')-(c') \) of section II and conditions (6) and (7)

**Theorem (decision rules):** The optimal consumption and investment decision rules for the first period are linear in initial wealth. In particular, for \(B \neq 0\)

\[
C_0 = \frac{A b \Sigma e P_e \left[ \phi - (\rho_{1e}/P_e)^{B_k} \right]}{1 + \Sigma e P_e (\rho_{1e}/P_e)^{B_k}} + \frac{1}{1 + \Sigma e P_e (\rho_{1e}/P_e)^{B_k}} W_0
\]

\[
W_e = \frac{A b \left[ \phi - (\rho_{1e}/P_e)^{B_k} \right]}{1 + \Sigma e P_e (\rho_{1e}/P_e)^{B_k}} + \frac{(\rho_{1e}/P_e)^{B_k}}{1 + \Sigma e P_e (\rho_{1e}/P_e)^{B_k}} W_0 \quad (\text{all } e)
\]
and for \( B = 0 \)

\[
C_0 = - \left( \frac{\Delta}{\phi} \right) \sum_e P_e \left[ \phi \ln(\rho_{1e}/P_e) + k_e \right] + W_0/\phi
\]

\[
\overline{W}_e^i = - \left( \frac{\Delta}{\phi} \right) \left\{ \phi \sum_e P_e \left[ \phi \ln(\rho_{1e}/P_e) + k_e \right] - \phi \left[ \phi \ln(\rho_{1e}/P_e) + k_e \right] \right\} + \phi \overline{W}_0/\phi
\]

(all \( e \))

where \( \phi_e \equiv 1 + \sum_s s_{s e} \), \( \phi \equiv 1 + \sum_e P_e \phi_e \).

for \( B \neq 0 \) \( k_e \equiv 1 + \sum_s s_{s e} \left( \rho_{2s e}/P_{s e} \right)^B \), and

for \( B = 0 \) \( k_e \equiv \sum_s s_{s e} \ln(\rho_{2s e}/P_{s e}) \).

Proof: Use the same method as for the analogous proof in section II.

Q.E.D.

To show that similar results will hold for arbitrary horizon \( T \), compare the derived utility of wealth functions for dates \( t = 1 \) and \( t = 0 \). Using the decision rules it can be shown that

\( a'' \)

\[
V_e(W_e) \propto k_e^b \frac{b}{1-b} \left( A \phi_e + BW_e \right)^{1-b}
\]

(\( B \neq 0, 1 \))

\[
V_0(W_0) \propto \frac{b}{1-b} \left( A \phi + BW_0 \right)^{1-b}
\]

\( b'' \)

\[
V_e(W_e) \propto (1+p_2) \ln(A \phi_e + W_e)
\]

(\( B = 1 \))

\[
V_0(W_0) \propto (1+p_1+p_2) \ln(A \phi + W_0)
\]
\[ V_e(W_e) \propto e^{-k_e - W_e/(A_\phi_e)} e^{\phi_{-1} k_e} \]

\[ V_0(W_0) \propto e^{-k_0 - W_0/(A_\phi)} e^{\phi_{-1} k_0} \]

for \( B = 0 \)

where for \( B \neq 0 \), \( k \equiv 1 + \sum e \ln(p_{e} p_{e}^{s} / p_{e})^{B} k e \) and for \( B = 0 \)

\[ k \equiv \sum e \ln(p_{e} p_{e}^{s} / p_{e}) + k e \].

That is, as an individual moves backwards in time further away from his horizon, his derived utility of wealth function at each date remains of the same "form," even though no restrictions have been placed on the intertemporal stochastic process of security prices.

No myopia is ever present, however, in the sense that an individual's choices are independent of the time remaining to the horizon date. To calculate the appropriate weighting factor \( k^B \) for \( B \neq 0 \), or \( e^{\phi_{-1} k} \) for \( B = 0 \), an individual needs to know the prices of all securities at every future date and state, even if these prices are intertemporally stochastically independent, i.e., \( \pi_{s,e} \) and \( p_{s,e} \) are independent of state \( e \), for all \( s \) and \( e \). However, if an individual has generalized logarithmic utility \( (B = 1) \), then partial myopia is present in the sense that an individual's choices are dependent on the time remaining to the horizon date only through dependence on his patience and the riskfree term structure through that date. To see this, define \( 1 + r_F \equiv (\sum e p_{e})^{-1} \) and \((1+R_F)^2 \equiv (\sum e p_{e} p_{s,e})^{-1} \). \( r_F \) is the riskfree rate of return on a short-term bond maturing at date \( t = 1 \), and \( R_F \) is the compound riskfree rate of return on a long-term bond maturing at date \( t = 2 \). Then, from
its definition
\[ \phi = 1 + \frac{1}{1+r_F} + \frac{1}{(1+r_F)^2} \]
so that \( \phi \) may be interpreted as the present value of an annuity yielding one dollar for sure in the present and at every future date up to and including the horizon. Clearly, with logarithmic utility since \( A = 0 \), current decisions are even independent of the term structure.

To see what happens to \( V_0(W_0) \) as \( T \to \infty \), assume for simplicity a constant riskfree rate so that in particular \( r_F = \hat{r}_F \) and that
\[ \bar{\rho} = \Sigma_{e,F} \left( \rho_{1,e} / F_{e} \right)^{B} = \Sigma_{e,S \cdot e} \left( \rho_{2,S \cdot e} / F_{S \cdot e} \right)^{B} = \ldots < 1. \]
When \( B \neq 0,1 \), then
\[
\lim_{T \to \infty} V_0(W_0) \sim (1-\bar{\rho})^{-b} \frac{b}{1-b} \left[ A \left( \frac{1+r_F}{r_F} \right) + BW_0 \right]^{1-b}
\]
and in the simpler case when \( B = 1 \), then with \( \rho \equiv \rho_t < 1 \)
\[
\lim_{T \to \infty} V_0(W_0) \sim (1-\rho)^{-1} \ln \left[ A \left( \frac{1+r_F}{r_F} \right) + W_0 \right].
\]
Observe that the familiar turnpike result of the convergence of \( V_0(W_0) \) to a constant proportional risk-averse utility function is not forthcoming, because
If, as in Mossin (1968), an individual maximizes the expected value of a terminal utility of wealth function $U(W_{s,e})$ without withdrawals for consumption, then $V_e(W_e)$ and $V_0(W_0)$ are unique up to an increasing linear transformation. As a special case of these results, it is easily shown that for $B \neq 0,1$

$$V_e(W_e) \sim \frac{b}{1-b} \left[ \frac{A}{1+r_{F,e}} + BW_e \right]^{1-b}$$

and

$$V_0(W_0) \sim \frac{b}{1-b} \left[ \frac{A}{(1+R_p)^2} + BW_0 \right]^{1-b}$$

for $B = 1$

$$V_e(W_e) \sim \ln \left[ \frac{A}{1+r_{F,e}} + W_e \right]$$

and

$$V_0(W_0) \sim \ln \left[ \frac{A}{(1+R_p)^2} + W_0 \right]$$

and for $B = 0$

$$V_e(W_e) \sim -\frac{A}{1+r_{F,e}} e^{-W_e/\left[ \frac{A}{1+r_{F,e}} \right]}$$

and

$$V_0(W_0) \sim -\frac{A}{(1+R_p)^2} e^{-W_0/\left[ \frac{A}{(1+R_p)^2} \right]}$$

In this case, myopia is present when $A = 0$ (constant proportional risk aversion); partial myopia is always present. Moreover, for $B \neq 0,1$

$$\lim_{T \to \infty} V_0(W_0) \sim \frac{W_0^{1-b}}{1-b}$$

and for $B = 1$, $\lim_{T \to \infty} V_0(W_0) \sim \ln W_0$. Unfortunately, these turnpike results, as well as their more recent generalizations by Leland, Hakansson, and Ross are destroyed if withdrawals for consumption are admitted. However, as Nils Hakansson has suggested to me, it is as yet unproven under what conditions (other than for the HARA class itself) turnpike results admitting withdrawals for consumption will hold with the larger HARA class of utility functions at the beginning of the turnpike.
VI. OPTIMAL SHARING RULES

Again to develop analytically tractable sharing rules, it is necessary to impose homogeneity conditions on the individuals in the economy. In parallel with section III, the following special economies will be examined:

i. (Generalized power and logarithmic utility) All individuals have the same beliefs \( \{ \pi_e \}, \{ \pi_e^* \} \) and taste parameter \( B \) where \( B \neq 0 \).

ii. (Exponential utility) All individuals have the same taste parameter \( B \) where \( B = 0 \).

iii. (Generalized logarithmic utility) All individuals have the same resources \( \tilde{W}_0 \) and tastes \( \rho_1, \rho_2, A \) and \( B \) where \( B = 1 \).

In each case, individuals may be heterogeneous in all other respects.

With these homogeneity conditions, the following optimal consumption and investment sharing rules may be derived:

**Theorem (sharing rules):** The optimal consumption and investment-sharing rules for the first period are linear and, except for economy (iii), are determinate. In particular, for economy (i) for all \( i \) and \( e \)

\[
C_0^i = \frac{(b/\tilde{K}_n)}{(A_i K_i - A_i \tilde{K}_n)} + \left( \tilde{K}_i/\tilde{K}_n \right) C_0^M
\]

\[
W_e^i = \left( b/\tilde{K}_n \right) \left[ \frac{K_i K_i \rho_i - B}{M_1 \rho_i 12 \tilde{K}_i K_i \rho_i - B} \right] (\pi_e/\pi_e^* B) + \left( b/K_n \right) \left[ A_i K_i - A_i K_n \right] \phi + \left( K_i/K_n \right) W_e^M
\]
where $A_M \equiv \Sigma_i A_i$, $\bar{k}_i \equiv \left[1 + \Sigma e_p e \left(\rho_{ee} / \rho_e \right)^{B e} \bar{k}_{ie}\right]^{-1} (A_i \phi + B W_i)$, $\bar{k}_M \equiv \Sigma_i \bar{k}_i$, $k_i \equiv \rho_{i}^{B} \rho_{12}^{B} \bar{k}_i$ and $k_M \equiv \Sigma_i k_i$;

for economy (ii) for all $i$ and $e$

$$C_0^i = - \left(\frac{A_i}{\phi}\right) \left(\Sigma e_p e \Sigma e_p e \ln(\rho_{12}/\rho_2) - (A_i/\phi) \Sigma e_p e \left[\phi_e \ln(\pi_e^i/\pi_e) + \Sigma e_p e \ln(\pi_{ee}/\pi_{ee})\right] + (A_i/\phi) \ln(\rho_{12}/\rho_2) \left[\phi_e (\Sigma e_p e \Sigma e_p e)\right] - (A_i/\phi) \ln(\rho_{12}/\rho_2) \left[\phi_e (\Sigma e_p e \Sigma e_p e)\right] - \phi_e (\Sigma e_p e \Sigma e_p e)\right] - (A_i/\phi) \ln(\rho_{12}/\rho_2) \left[\phi_e (\Sigma e_p e \Sigma e_p e)\right] - (A_i/\phi) \Sigma e_p e \left[\phi_e \ln(\pi_e^i/\pi_e) + \Sigma e_p e \ln(\pi_{ee}/\pi_{ee})\right] - \ln(\rho_{12}/\rho_2) + (A_i/\phi) \left[A_i W_0^i - A_i W_0^i\right] + \left(A_i/A_m\right) W_e^i$$

where $W_0^i \equiv \Sigma_1 W_0^i$, $A_M \equiv \Sigma_i A_i$, $\rho_{\pi} \equiv \Pi_i \rho_{it}$, $\pi_{ee} \equiv \Pi_i \pi_{ee}$.

and $\pi_{ee} \equiv \Pi_i \pi_{ee}$.

and $\pi_{ee} \equiv \Pi_i \pi_{ee}$.

for economy (iii) for all $i$ and $e$

$$C_0^i = C_0^M / I$$ and

$$W_e^i = A \phi_e (\pi_e^i - \pi_e) / \pi_e + (\pi_e^i / \pi_e) (W_e^i / I)$$

where $\pi_e \equiv \Sigma_i \pi_e^i / I$. 
Proof: Use the same method as for the analogous proof in section III.

Q.E.D.

For economy (i) the consumption sharing rule is similar to the single-period case. However, the investment sharing rule is notably different. In the single-period economy, all individuals divide their wealth (after consumption) between two mutual funds: the market portfolio and a riskfree portfolio. In the multiperiod economy, all individuals divide their wealth (after consumption) among three mutual funds: the market portfolio, a second risky portfolio yielding \( \phi_e \) for each state \( e \) and a third risky portfolio yielding \( (\pi_e / P_e)^B \) for each state \( e \). The second risky portfolio is simply an annuity yielding one dollar for sure at every future date up to and including the horizon. This portfolio is used to protect individuals against future uncertain shifts in investment opportunities. The third risky portfolio is more difficult to interpret, however net holdings of both risky portfolios are each self-canceling, because

\[
\sum_i [A_i K_i - A_i K_i] = 0
\]

and

\[
\sum_i [K_i K_i \rho_{12}^{-B} - K_i K_i \rho_{12}^{-B}] = 0.
\]

Moreover, the third risky portfolio is generally required by heterogeneous patience among individuals toward date \( t = 2 \) consumption, since if

\[19\] Since a similar analysis of sharing rules as in section III carries over to the multiperiod case, it is omitted.
themselves more than the average individual against uncertain shifts in their investment opportunities.  

The most interesting special case of the three economies occurs when all individuals are the same except for their resources \( \hat{w}_0^i \) and taste parameter \( A_i \).

**Corollary (universal portfolio separation):** If all individuals in the economy are the same except for their resources \( \hat{w}_0^i \) and taste parameter \( A_i \), then at each date all individuals divide their wealth (after consumption) between two mutual funds: the market portfolio of all securities and an annuity yielding equal certain payments at all future dates. Moreover, the consumption and investment sharing rules are stationary through time. In particular, for all \( i \)

\[
\begin{align*}
\hat{c}^i_0 &= \alpha_i^i + \beta_i \hat{c}_0^i \\
\hat{c}_e^i &= \alpha_i^i + \beta_i \hat{c}_e^i \\
\hat{w}^i_e &= \alpha_i^i \phi_e + \beta_i \hat{w}_e^i \\
\hat{w}^i_s &= \alpha_i^s + \beta_i \hat{w}_s^i
\end{align*}
\]

(all \( e \))

where

\[
\begin{align*}
\alpha_i^i &= \frac{A_i \hat{w}_0^i - A_i \hat{w}_0^i}{A_i \phi + \hat{w}_0^i} \\
\beta_i &= \frac{A_i \phi + \hat{w}_0^i}{A_i \phi + \hat{w}_0^i}
\end{align*}
\]

\(20\) This analysis suggests the following conjecture: whenever all individuals in the economy hold the market portfolio and a second portfolio (possibly different for different individuals) irrespective of their wealth, then the second portfolio is a weighted mixture of bonds maturing at each future date up to and including the horizon. The weights in the second portfolio will generally be different for different individuals and will depend on the time pattern of their heterogeneous patience. Other things equal, an individual will hold more bonds than another individual maturing at a date if he is more patient toward consumption at that date.
Proof: From the investment sharing rules for economy (i), setting \( \rho_{1i} = \rho_1 \) and \( \rho_{12} = \rho_2 \) for all \( i \) reduces them to

\[
W_e^1 = \frac{A_{i,0}^M - A_{i,0}^M}{A_{i,0}^M + B e_0^M} \phi_e + \frac{A_{i,0}^M}{A_{i,0}^M + B e_0^M} W_e^M \quad (\text{all } e)
\]

and

\[
W_{s,e}^1 = \frac{A_{i,0}^M - A_{i,0}^M}{A_{i,0}^M + B e_0^M} \phi_e + \frac{A_{i,0}^M}{A_{i,0}^M + B e_0^M} W_{s,e}^M \quad (\text{all } e \text{ and } s).
\]

By substituting the sharing rule for \( W_e^1 \) into

\[
(A_{i,0}^M - A_{i,0}^M)/(A_{i,0}^M + B e_0^M) \quad \text{and} \quad (A_{i,0}^M + B e_0^M)/(A_{i,0}^M + B e_0^M),
\]

these ratios can be shown equal to \( (A_{i,0}^M - A_{i,0}^M)/(A_{i,0}^M + B e_0^M) \) and

\( (A_{i,0}^M + B e_0^M)/(A_{i,0}^M + B e_0^M) \), respectively. The investment sharing rules for economy (ii) are just special cases of these where \( B = 0 \). Similar results follow for the consumption sharing rules. Q.E.D.

Under the conditions of the corollary, all individuals find it sufficient to hedge against all future shifts in investment opportunities by borrowing or lending an annuity yielding equal certain payments at all future dates. This may at first seem surprising since no restrictions have been placed on the intertemporal stochastic process of security prices. In particular, extreme forms of nonrandomness may be present. However, with the differences among individuals sufficiently curtailed, the ability to allocate their investment among two portfolios provides sufficient flexibility. This result generalizes to any horizon \( T \); if \( T = \infty \), then all individuals either borrow or lend a perpetual annuity.
in addition to a long position in the market portfolio. The investment sharing rules are stationary over time precisely because individuals are willing to take an unbiased position toward all future dates by purchasing the annuity. As a consequence, no one ever needs to revise his portfolio; at each date, the cash throw-offs from the market portfolio and the annuity are exactly enough to satisfy his optimal consumption plan. Only if the first-period rate of return of the annuity is stochastically independent of its future rates of return, i.e., in the three-date case, \( \sum_{s=1}^{3} r_s \) independent of \( e \), will the multiperiod separation property reduce to the comparatively simple single-period separation property of section III.

VII. THE SCOPE OF COMPLETE MARKETS

The principal objective of this series of papers is to demonstrate the power of Arrow's (1953) complete markets concept as a generator of financial theory. Virtually all\(^{21}\) of the theoretical work in finance (where the Savage axioms and a perfect and competitive securities market are assumed) can be (and perhaps should have been) developed in the context of complete markets without any loss of generality. Most incomplete market models are revealed, through their separation properties, as thinly disguised versions of a complete market. Consequently, since a complete

\(^{21}\)With continuous probability distributions for security rates of return, an uncountably infinite number of state-contingent securities must be assumed. Again this fiction can be relaxed after requisite separation properties are demonstrated.
market context is so much neater—technically and conceptually—to work with, it is the preferable modeling approach as long as properties P1-P6 are maintained.

It is both easier and more productive to start with the assumption of a complete market and then prove, through construction of sharing rules, that the assumption can be relaxed. As an added bonus, the resulting model is guaranteed to be exchange-efficient. Indeed, as long as markets are perfect, i.e., no indivisibilities or exchange costs, exchange-inefficiencies should be wiped out simply by individuals serving their own self-interest. The import of these ideas is best captured by the following conjecture:

Theorem (market equivalence): A complete market and an otherwise similar incomplete but exchange-efficient market are equivalent in the sense that they reach the same final allocations and have compatible price systems.

22 Note that Hakansson (1971) was unable to derive closed-form investment decision rules. However, had he only been willing to assume his investor traded in an exchange-efficient securities market, they can be derived as this paper demonstrates. It is usually easier to solve portfolio problems by first converting the prices of actual securities into implicit prices for state-contingent securities, solving for the state-contingent solution, and then converting back to the solution in terms of actual securities. As an interesting exercise, suppose an individual has logarithmic utility with \( \rho = 3/4 \). He has wealth \( W_0 = \$100 \) and can consume and invest in two securities. One is riskfree with \( r_F = 0 \) and for the other \( r_{j1} = 1 \) under state \( e = 1 \) and \( r_{j2} = -1 \) under state \( e = 2 \). What are his optimal present consumption and portfolio proportions?

23 To define compatible price systems, let \( P_j \) be the present price of any actual security \( j \) in the incomplete market yielding \( \{N_{j e}\} \) dollars in the future for each state \( e \). If \( \{P_e\} \) are the prices of state-contingent securities in the complete market, we require \( P_j = \sum_e N_{j e} P_e \).
All the economies examined in the text of this paper, as well as those based on homogeneous normal (Rubinstein, 1973a) or lognormal (Merton, 1973) probability assessments in which a riskfree investment exists, are illustrations of this theorem.

Moreover, examination of the sharing rules for economies (i)-(iii) reveals the necessary and sufficient conditions for exchange-efficiency even if a securities market were not utilized to effect exchanges. Recall that in a complete market every Pareto-efficient allocation can be reached by an appropriate redistribution of resources (initial wealth). Such a redistribution does not affect the linearity of determinateness of the sharing rules. In the economies examined, these properties therefore become the necessary and sufficient specifications for exchange-efficiency. Consequently, Lintner's (1969) exponential utility model with heterogeneous normal probability assessments is not exchange-efficient, because opportunities for sufficient side bets are not generally available. The sharing rule indicates that for exchange-efficiency these side bets are necessary irrespective of the distribution of resources. Similarly, Rubinstein's (1974a) security valuation models without the availability of riskfree investments are not generally exchange-efficient. The sharing rules indicate that for exchange-efficiency certain side payments are generally necessary irrespective of the distribution of resources.\(^{24}\)

\(^{24}\)Recall that as a special case side payments will not be utilized by any individual in the economy if and only if \(\frac{A_i}{W_i} = \frac{A}{W_0}\) for all \(i\).
Observe that in a complete market, certain side payments are always available.

It is frequently asserted that the complete markets approach to financial modeling cannot produce empirically testable security valuation equations. If no restrictions on tastes or beliefs are allowed, other than those required by P1-P6, this assertion may be true. However, if tastes or beliefs are cleverly specified, a plethora of closed-form results are derivable. In the first sequel paper, it will be shown that the generalized separation properties of this paper lead to potentially empirically testable valuation equations which subsume existing results as special cases.

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25 However, in a multiperiod setting potentially empirically testable hypotheses about the intertemporal process of security rates of return are derivable. See Rubinstein (1973b).
REFERENCES


