RESEARCH PROGRAM IN FINANCE

WORKING PAPER NO. 21

A DISCRETE-TIME SYNTHESIS OF FINANCIAL THEORY:
PART II. VALUATION AND EFFICIENCY

by

Mark Rubinstein

June 1974

Mark Rubinstein is Assistant Professor, Graduate School of Business Administration, University of California, Berkeley. Research for this paper was supported in part by a grant from the Dean Witter Foundation. The author wishes to thank Barr Rosenberg and David Ng for useful discussions.
A DISCRETE-TIME SYNTHESIS OF FINANCIAL THEORY:
PART II. VALUATION AND EFFICIENCY

This paper is the second of a series of three that provides a synthetic treatment of financial theory, working from the context of a complete securities market. Multiperiod two-parameter and multiparameter valuation relationships are derived in an economic environment permitting any arbitrary discrete-time intertemporal stochastic process of security prices. The present value of a security is shown to depend not only on the correlation of its rate of return with the market portfolio but also with an annuity yielding equal certain payments at all future dates. In the two-parameter case, the limit of the discrete-time solution as the trading interval approaches zero is equivalent to the continuous-time solution. With logarithmic utility, the underlying real stochastic process of the rate of growth of aggregate consumption is shown to have its properties almost precisely mirrored in the equilibrium intertemporal financial process of security rates of return. In particular, positive liquidity premiums in the term structure are explained by an inverse growth rate of aggregate consumption, which is positively correlated over time. The remainder of the paper identifies necessary and sufficient conditions for economic efficiency and their implications for corporate capital structure and capital budgeting decisions and for the dissemination of information. In contradiction to the existing literature, even in an economy with exogenous production decisions, it is shown that full dissemination of all available information is Pareto-efficient under realistic conditions.

This paper is the second of a series of three\(^1\) that provides a synthetic treatment of financial theory. The first paper provides a description of the economy analyzed here, as well as closed-form solutions

\(^1\)See Rubinstein (1974b, 1974c).
for optimal decision and sharing rules under various special cases.

This paper continues with a derivation of several potentially empirically
testable security valuation equations and analyzes the efficiency prop-
erties of the economies and their implications for corporate financial
decisions and the dissemination of information. Familiarity with the
results and notation of the first paper are presumed.

In section I, the aggregation problem of each economy is easily
solved by the construction of an individual with composite resources,
beliefs, and tastes. With the aggregation problem solved, section II
develops closed-form multiperiod valuation equations in terms of aggre-
gate consumption variables, and section III extends these results to the
more complex valuation equations stated in terms of aggregate wealth
variables. These multiperiod equations are derived in two-parameter and
multiparameter forms. Since these results permit an arbitrary intertemp-
oral stochastic process of security prices, they will not (except for
singular cases) reduce to a successive application of the valuation
equations derived in a single-period context. For example, in the two-
parameter case, the present value of a security or any cash flow is de-
determined not only by the correlation of its rate of return with the mar-
ket portfolio, but also with an annuity yielding equal certain payments
at all future dates. Given the generalized separation results of the
previous paper, this result is not unexpected. Moreover, in the two-
parameter case, the limit of the discrete-time solution as the trading
interval approaches zero is shown to be equivalent to Merton's (1973)
continuous-time solution, if his state variable reflecting intertemporal shifts in opportunities is suitably reinterpreted.

In section IV, the equilibrium intertemporal structure of rates of return is examined for the existence of an unbiased term structure and a random walk. With logarithmic utility, if properly defined, the existence of these classic patterns of rates of return depends on the underlying real intertemporal stochastic process of aggregate consumption. This real process is uncorrelated if and only if financial processes are uncorrelated. Positive liquidity premiums in the term structure are explained by an inverse growth rate of aggregate consumption, which is positively correlated over time.

The remainder of the paper moves to a more general context of any economy satisfying properties P1-P3 and investigates necessary and sufficient conditions for economic efficiency. To focus more sharply on separate aspects of efficiency, the following taxonomy is used:

exchange-efficiency: exchange (of claims to consumption) is Pareto-efficient.

production-efficiency: production (of commodities) is Pareto-efficient.

information-efficiency: the production and exchange of information is Pareto-efficient.

An economic process is said to be Pareto-efficient if by changing it no individual can be made better off without some individual made worse off. If better or worse off is measured in terms of expected utility assessed at the beginning of the economy, then Pareto-efficiency is said to be ex ante; if better or worse off is measured in terms of the utility of
realized consumption over time, then Pareto-efficiency is said to be *ex post*. Of course, under certainty *ex ante* and *ex post* Pareto-efficiency are equivalent. However, under uncertainty with choices over time, an economy which is *ex ante* efficient in some respect may not be *ex post* efficient.\(^2\) Unless otherwise indicated, all efficiency concepts are regarded *ex ante*.

Using these definitions, sections V and VI isolate conditions under which a perfect and competitive securities market is exchange- and production-efficient. Under certainty these conditions are well known, but under uncertainty there has recently been some controversy. Rather than offer a formal analysis, since it is already available in scattered sources throughout the literature, I merely bring together in one place these scattered conclusions with verbal supportive arguments. Seen in this light, the confusion surrounding these efficiency issues is easily dispelled. In particular, necessary and sufficient conditions for corporate capital budgeting decisions by competitive value-maximizing firms to be unanimously preferred by all shareholders are explained. Exchange-efficiency is not only closely related to these conditions, but also closely related to production-efficiency.

\(^2\)See Starr (1972). It is well known that an Arrow-Debreu economy is *ex ante* exchange-efficient. Starr shows that if individuals are concerned with consumption only at one date or they have homogeneous beliefs, then an Arrow-Debreu economy is also *ex post* exchange-efficient. In general, with consumption at two or more dates, when uncertainty is resolved at the second date there will be at least two individuals who will regret their consumption at the first date; thus, one would realize more utility with less initial consumption and the other would realize more utility with more initial consumption. Consequently, if only they could have forecast the future with certainty, they would have made a mutually beneficial exchange at the first date. Therefore, an Arrow-Debreu economy is not generally *ex post* exchange-efficient.
The final section of the paper considers the comparatively neglected issue of information-efficiency in an economy with exogenous production (of commodities) decisions where there are no explicit costs associated with the exchange of information about supply conditions. 3 This issue has been addressed recently in the seminal contribution of Hirshleifer (1971). However, it is shown here that parts of his analysis are misleading. Indeed, in a securities market in which prices fully reflect all available information, full exchange of information is Pareto-efficient. Since existing empirical work indicates that most available information is fully reflected in security prices, it behooves us to measure the extent to which the exchange of information has been carried in the economy. The volume of trading is shown to be a useful barometer, and for some specific economies the speculative and nonspeculative demands for securities are compared. The analysis suggests that most trading volume in actual well-developed economies is likely to be speculative and therefore symptomatic of information-inefficiency.

I. AGGREGATION

As discussed in Rubinstein (1974a), the chief difficulty beclouding the analysis of securities market equilibrium is the aggregation

3 The efficiency taxonomy highlights the separate roles of the four major categories of economic information: (1) personal information: a consumer's knowledge of his own resources, beliefs, tastes, and past decisions; (2) technological information: a producer's knowledge of his own production opportunities, objectives and past decisions; (3) information about supply: knowledge of the aggregate supply conditions of commodities and securities; and (4) information about demand: knowledge of the aggregate demand conditions of commodities and securities.

Hirshleifer (1971) makes a similar distinction referring roughly to (2) as discovery, (3) as foreknowledge, and (4) as market
problem. In particular, when individual decisions are aggregated to form valuation relationships, individual-specific choice information must be eliminated and replaced by exogenous parameters of the economy. In a single-period economy, it was shown that for economies (i), (ii), and (iii), where in (i) individuals have the same patience, security prices are set as if there exist only identical individuals who are composites of the actual individuals.⁴ This theorem carries over trivially to its multiperiod extension and will be stated for reference.

Theorem (aggregation): Consider the following sets of homogeneity conditions:

0. (Identical individuals) All individuals have the same resources \( W_0 \), beliefs \( \{ \pi_e \}, \{ \pi_{s-e} \} \), and tastes \( (\rho_1, \rho_2, \psi) \).

1. (Generalized power and logarithmic utility) All individuals have the same beliefs \( \{ \pi_e \}, \{ \pi_{s-e} \} \), patience \( (\rho_1, \rho_2) \), and taste parameter \( B \) where \( B \neq 0 \).

ii. (Exponential utility) All individuals have the same taste parameter \( B \) where \( B = 0 \).

iii. (Generalized logarithmic utility) All individuals have the same resources \( W_0 \) and tastes \( (\rho_1, \rho_2, A, B) \) where \( B = 1 \).

⁴Actually, Rubinstein restricted economy (iii) to logarithmic utility \( (A = 0, B = 1) \); however, his results are easily extended to generalized logarithmic utility \( (B = 1) \).
Equilibrium prices are determined by

\[ U'(C_0) = (\rho_1 \pi_e / p_e) U'(W_e) \quad (\text{all } e) \]

\[ U'(C_e) = (\rho_2 \pi_{s\cdot e} / p_{s\cdot e}) U'(W_{s\cdot e}) \quad (\text{all } e \text{ and } s) \]

where \( C_0 \equiv C_{0\cdot I} \), \( C_e \equiv C_{e\cdot I} \) and \( W_{s\cdot e} \equiv W_{s\cdot e\cdot I} \).

For economy (i) \( A = A_{M\cdot I} \); for economy (ii) \( A = A_{M\cdot I} \),

\[ \rho_e \equiv \Pi_i \rho_i^{A_{1\cdot M}} \quad \pi_e \equiv \Pi_i \pi_i^{A_{1\cdot M}} \quad \text{and} \quad \pi_{s\cdot e} \equiv \Pi_i \pi_{s\cdot e}^{A_{1\cdot M}} ; \]

for economy (iii) \( \pi_e \equiv \Sigma_e \pi_i^{A_{1\cdot I}} \) and \( \pi_{s\cdot e} \equiv \Sigma_i \pi_{s\cdot e}^{A_{1\cdot I}} / \pi_i \).

Since these economies permit aggregation, they may prove useful approximations of actual economies for understanding equilibrium in securities markets.

To translate \((1')\) and \((2')\) in terms of "actual" securities, note that an actual security can be interpreted as a portfolio of state-contingent securities. Suppose actual security \( j \) pays off \( N_{je} \) units of wealth if state \( e \) occurs. Since all economies under consideration are exchange-efficient, by the market equivalence theorem of part I, their price systems in terms of actual securities are compatible with their price systems under complete markets. Therefore, \( P_j = \Sigma_e N_{je} P_e \).

From this it follows that
\[
\frac{N_{je}^e}{P_j} = \frac{N_{je}^e}{\sum_e N_{je}^e} = \beta_{je} \quad \text{(all } j \text{ and } e)
\]

where \( \Sigma_e \beta_{je} = 1 \). Substituting into (1'),

\[
U'(C_0)\beta_{je} = \rho_1 \pi \sum_e V'(w_e)(N_{je}^e/P_j) \quad \text{(all } e)
\]

Since by definition, one plus the rate of return on security \( j \) if state \( e \) occurs, \( 1 + r_{je} \equiv N_{je}^e/P_j \) and summing over all \( e \)

(1)

\[
U'(C_0) = \rho_1 \sum_e \pi \sum_e V'(w_e)(1 + r_{je}) \quad \text{(all } e)
\]

and similarly in place of (2')

(2)

\[
U'(C_e) = \rho_2 \sum_s \pi_s \sum_e U'(w_{s,e})(1 + r_{js,e}) \quad \text{(all } e \text{ and } s).
\]

Observe that like conditions (1') and (2'), these revised conditions are sufficient to determine rates of return in terms of exogenous parameters only, i.e., the aggregation problem has been solved. To see this, recall that tastes \( \{\rho_{11}, \rho_{12}\}, \{U_1\} \), beliefs \( \{\pi_e^{i}, \pi_{s,e}^{i}\} \), population \( I \), and the aggregate supply of consumption at each date and state \( \{C_0^M, \{C_e^M\}, \{W_{s,e}^M\}\} \) are taken as exogenous, and the composite taste and belief parameters are merely functions of these exogenous parameters.

5The distribution of endowed resources among individuals is also exogenous; however, when a composite individual can be constructed, the price system is insensitive to the distribution of endowed resources. See section V.
A final relationship will also prove useful. In the description of the multiperiod economy in section IV of part I, each individual solved a two-stage programming problem. An individual can be alternatively and equivalently viewed as solving the following single-stage programming problem

$$\max_{C_0, \{c_{e}\}, \{w_{s\cdot e}\}} U(C_0) + \rho_1 \sum_{e \in e^t} U(c_{e}) + \rho_2 \sum_{s \in s^t} \sum_{e \in e^s} U(w_{s \cdot e})$$

$$- \lambda \left[ C_0 + \sum_{e \in e^t} p_{e \cdot c_{e}} + \sum_{e \in e^s} p_{s \cdot e \cdot w_{s \cdot e}} - W_0 \right]$$

with first-order conditions

$$U'(C_0) = (\rho_1 p_{e \cdot c_{e}}) U'(c_{e}) \quad \text{(all } e)$$

$$U'(c_{e}) = (\rho_2 p_{s \cdot e \cdot w_{s \cdot e}}) U'(w_{s \cdot e}) \quad \text{(all } e \text{ and } s)$$

$$W_0 = C_0 + \sum_{e \in e^t} p_{e \cdot c_{e}} + \sum_{e \in e^s} p_{s \cdot e \cdot w_{s \cdot e}}$$

Since this is merely a reformulation of the two-stage problem, these conditions must be consistent with conditions (6)-(9) of the previous paper. This consistency requires that

$$U'(c_{e}) = V'(w_{e}) \quad \text{(all } e)$$

at the optimum. In other words, at his optimal consumption plan, an individual must be indifferent among all alternative expenditures of a marginal increment of wealth at each date and state. Equation (3) is just
another example of the delicate balancing act performed by individuals at their optimum choices.

Sections II-IV of this paper will presume that the aggregation problem has been solved and will therefore work directly with the aggregated conditions for equilibrium.

II. VALUATION IN TERMS OF CONSUMPTION

Applying the aggregated conditions (1)-(3), a multiperiod extension of the single-period valuation theorem in Rubinstein (1974a) follows.

Theorem (valuation with consumption\(^6\)): Whenever a composite individual can be constructed, in equilibrium

\[
(4) \quad 1 + r_F = \frac{U'(C_0)}{E[U'(C_1)]} \quad \text{and} \quad 1 + r_{F2:e} = \frac{U'(C_e)}{E[U'(W_{2:e})]} \quad (\text{all } e)
\]

\[
(5) \quad E(r_j) = r_F + \lambda \kappa(r_j, - U'(C_1)) \text{Std } r_j \quad (\text{all } j)
\]

where \( \lambda \equiv \text{Std}[U'(C_1)]/E[U'(C_1)] > 0 \)

\[
(6) \quad E_e(r_{j2}) = r_{F2:e} + \lambda^e \kappa_e(r_{j2}, - U'(W_{2:e})) \text{Std } r_{j2}
\]

where \( \lambda^e \equiv \text{Std}_e[U'(W_{2:e})]/E_e[U'(W_{2:e})] > 0 \).

Proof is omitted since it follows trivially from conditions (1)-(3) and is analogous to the single-period proof in Rubinstein (1974a). Equations

\(^6\) \( E \) is an expectation operator, \( \text{Std} \) a standard deviation operator, \( \kappa \) represents the correlation coefficient of the two variables following it in parentheses, and \( \lambda \) is a constant as defined and is not to be confused with a Lagrangian multiplier. The subscript \( e \) in equation (6) indicates that expectations are assessed with respect to beliefs held if state \( e \) occurs.
(4)-(6) can be translated into a simple formula for the present value of an uncertain stream of income. For example, denote the uncertain cash flows from security \( j \) by \( \{D_{je}, \{D_{js, e}\}\} \). The present value of these flows may be written as

\[
\frac{E\left[ U'(C_1) \right]}{1 + r_F} + \frac{E\left[ U'(W_2) \right]}{(1 + r_F)^2}
\]

where expectations are assessed with respect to beliefs \( \pi_e \pi_{s, e} \) and \( (1 + r_f)^2 \equiv \frac{E[U'(C_0)]}{\rho_1 \rho_2 E[U'(W_2)]} \) represents the compound rate of return on a riskfree bond issued at date \( t = 0 \) and maturing at \( t = 2 \).

Although the aggregation problem has been solved, the valuation equations given in (4)-(6) are not readily empirically testable, because they contain unobservable information about tastes. However, for certain specifications of tastes and/or beliefs, information about tastes can be buried in nonrandom parameters. For example, consider quadratic utility \( (\beta = -1) \). In this case, \( U'(C_0) = A - C_e = A - C_0(1 + r_{Ce}) \) (all \( e \)) where \( r_{Ce} \) is defined as the rate of growth of the aggregate supply of consumption in the first period, i.e., \( 1 + r_{Ce} \equiv C_e / C_0 \). Consequently (5) can be rewritten as

\[
E(r_j) = r_F + \lambda\mathbb{K}(r_j, r_{Ce})\text{Std } r_j \quad (\text{all } j)
\]

Since this holds as well for all possible portfolios of securities, it holds in particular for the market portfolio \( M \) so that
\[ \lambda = \frac{E(r_M) - r_F}{\kappa(r_M, r_C) \text{Std } r_M}. \]

As a result, if ex post data may be used to infer ex ante expectations, these specialized valuation relationships are potentially empirically testable. Correspondingly, with quadratic utility, the present value of cash flows \( \{D_{je}\}, \{D_{je*e}\} \) simplifies to

\[ \frac{E(D_{j1}) - \lambda_1 \kappa(D_{j1}, C_{j1}) \text{Std } D_{j1}}{1 + r_F} + \frac{E(D_{j2}) - \lambda_2 \kappa(D_{j2}, W_2) \text{Std } D_{j2}}{(1 + r_F)^2} \]

where \( \lambda_1 = \frac{E(r_M) - r_F}{\kappa(r_M, r_C) \text{Std } r_M} \), \( \lambda_2 = \frac{E(1 + R_{M2})^2 - (1 + R_{F2})^2}{\kappa[(1 + R_{M2})^2, (1 + R_{C2})^2] \text{Std}(1 + R_{M2})^2} \)

\( (1 + R_{M2*e})^2 = \frac{W_{s*e}}{W_0} \) and \( (1 + R_{C2*e})^2 = \frac{W_{s*e}}{C_0} \).

III. VALUATION IN TERMS OF WEALTH

Although the specialized results of the previous section are empirically testable, they ask us to think in unaccustomed terms. Rather than project the correlation of the rate of return on a security with the market rate of return, \( \kappa(r_j, r_M) \), as in the popular single-period two-parameter model, we instead must predict \( \kappa(r_j, r_C) \) and \( \kappa(r_M, r_C) \).

However, using the derived utility of wealth function \( V_e(W_e) \) and its multivariate Taylor series expansion, the previous multiperiod results can be recast in a form more directly comparable to single-period models.
Despite the absence of any restrictions on the intertemporal
stochastic process of security prices, for the economies of part I the
derived utility of wealth function \( V_e(W_e) \) depended on only three state
variables: \( W_e, \phi_e \), and \( k_e \). That is, despite the arbitrary transition
from state to state over time, these three are the sufficient decision-
relevan statistics describing shifts in opportunities and resources.
Moreover, when \( B = 1 \) or \( A = 0 \), \( V_e(W_e) \) depends on only two state vari-
ables, \( W_e \) and \( \phi_e \) or \( W_e \) and \( k_e \), respectively; when both \( B = 1 \) and
\( A = 0 \), then \( V_e(W_e) \) depends only on \( W_e \). This derived simplification in
the decision-relevant state description opens the door to the construction
of useful valuation equations stated in terms of the rate of growth of
wealth \( (r^u) \) rather than of consumption \( (r^c) \).

As a preliminary step, since \( k_e \) enters the derived utility of
wealth function \( V_e(W_e) \) multiplicatively, it can be eliminated by redefin-
ing beliefs and tastes. Recall that for \( B \neq 0,1 \)

\[
V_e(W_e) \propto k_e^b \frac{b}{1-b} (A\phi_e + BW_e)^{1-b} \text{ where } k_e \equiv 1 + \sum_s p_s e (\rho_s \pi_s e / P_s e)^B.
\]

Following Savage that beliefs and tastes are not objective but rather
inferred from behavior, beliefs can be redefined so that \( \pi_e \equiv \pi_e^b / (\sum_e \pi_e^b) \)
and patience redefined as \( \rho^b_1 \equiv \rho^b_1 e \). Consequently, the expected util-
ity of the state-dependent derived utility function

\[7\text{Similar results hold for } B = 0, \text{ and when } B = 1 \text{ no transforma-
tion of beliefs and tastes is necessary.}\]
\[ \rho_1 \sum_e \pi_e V_e(W_e) = \bar{\rho}_1 \sum_e \bar{\pi}_e V(W_e, \phi_e) \]

where \( V(W_e, \phi_e) = \frac{b}{1-b} (A\phi_e + BW_e)^{1-b} \). That is, the behavior implied by beliefs \( \{\pi_e\} \) and tastes \( (\rho_1, V(W_e, \phi_e)) \) is indistinguishable from the behavior implied by beliefs \( \{\bar{\pi}_e\} \) and tastes \( (\bar{\rho}_1, V_e(W_e)) \).

We are now left with a derived utility function \( V(W_e, \phi_e) \) defined on two state variables. Expanding this in a multivariate exact Taylor series expansion around \( E(W_1) \) and \( E(\phi_1) \)

\[
V(W_e, \phi_e) = V + \frac{V_W}{2!} (W_e - E_W)^2 + \frac{V_{W\phi}}{1!1!} (W_e - E_W)(\phi_e - E_{\phi_1})

+ \frac{V_{\phi\phi}}{2!} (\phi_e - E_{\phi_1})^2 + \ldots
\]

where \( V \) is evaluated at \( E_W \) and \( E_{\phi_1} \) and \( V_W = \frac{\partial^2 V}{\partial W^2} \), \( V_{W\phi} = \frac{\partial^2 V}{\partial W \partial \phi} \), and \( V_{\phi\phi} = \frac{\partial^2 V}{\partial \phi^2} \). Suppose that \( S_j \) is the amount of initial wealth invested in actual security \( j \) so that \( W_e = \Sigma_j S_j(1+r_{je}) \).

Taking expectations of \( V(W_e, \phi_e) \) (with respect to beliefs \( \{\pi_e\} \)) and differentiating partially with respect to \( S_j \)

\[
\frac{\partial E[V(W, \phi)]}{\partial S_j} = E(1+r_j)E[V'(W_1, \phi_1)] + V_{W'}^V\text{Cov}(r_j, W_1) + V_{\phi}\text{Cov}(r_j, \phi_1)

+ \ldots
\]

where \( V'(W_e, \phi_e) \equiv \frac{\partial V(W_e, \phi_e)}{\partial W_e} \). Examining only the first three terms, this expression decomposes the marginal contribution of security \( j \) to expected utility into its separate marginal contributions to expected
future wealth (first term) and the variance of future wealth (second term), and its marginal contribution to expected utility by providing a hedge against unfavorable shifts in future opportunities (third term).

The first period programming problem in terms of aggregated parameters and choices may be written

$$\max_{C_0, \{S_j\}} U(C_0) + \tilde{\rho}_1 E[V(W_1, \phi_1)] - \lambda[C_0 + \sum_j S_j - W_0].$$

Differentiating partially with respect to $S_j$,

$$\frac{\partial E[V(W_1, \phi_1)]}{\partial S_j} = \frac{\lambda}{\tilde{\rho}_1} \quad \text{(all } j).$$

Since this also holds for riskfree security $F$, applying the above decomposition of expected utility, in equilibrium

$$E(1+r_j)E[V'(W_1, \phi_1)] + V_{W_1} \operatorname{Cov}(r_j, W_1) + V_{W\phi} \operatorname{Cov}(r_j, \phi_1) + \ldots$$

$$= (1+r_F)E[V'(W_1, \phi_1)].$$

From this very general equilibrium condition, it is a short step to the following theorem:

**Theorem** (two-parameter valuation): Whenever a composite individual can be constructed with quadratic utility, then in equilibrium at date $t = 0$

$$E(r_j) = r_F + \lambda_1 \kappa(r_j, r_M) \operatorname{Std} r_j + \lambda_2 \kappa(r_j, r_N) \operatorname{Std} r_j \quad \text{(all } j)$$

where $M$ is the market portfolio, $N$ is an annuity yielding equal certain payments at all future dates, and
\[ \lambda_1 \equiv \frac{-V_{WW}\text{Std } W_1}{E[V'(W_1, \phi_1)]} > 0 \quad \text{and} \quad \lambda_2 \equiv \frac{-V_{W\phi}\text{Std } \phi_1}{E[V'(W_1, \phi_1)]} < 0. \]

**Proof:** The theorem follows immediately from equilibrium condition (7) noting that with quadratic utility all terms in the continuation are zero, \( W_e = (W_0 - C_0)(1 + r_{Me}) \), \( \phi_e = (\phi - 1)(1 + r_{Ne}) \), \( -V_{WW} = 1 \), and \( -V_{W\phi} = -A \). Q.E.D.

The standard single-period valuation model is amended by the addition of risk premium \( \lambda_2 \kappa(r_j, r_N)\text{Std } r_j \), a term reflecting the value of security \( j \) as a hedge against unfavorable future shifts in opportunities. \( \lambda_2 = 0 \) if and only if \( \text{Std } \phi_1 = 0 \) so that future riskless spot rates are certain at date \( t = 0 \). Only in this instance, or if \( \kappa(r_j, r_N) = 0 \), will the multiperiod valuation model reduce in form to the single-period model. Consequently, it is generally incorrect to assume that the single-period model may be applied successively over time in a multiperiod context.

Valuation equation (8) also holds for any portfolio of securities. To see this, consider arbitrary portfolio \( P \) with rate of return \( r_{Pe} = \sum_j \beta_{Pj} r_{je} \) where \( \{\beta_{Pj}\} \) measures the proportionate value-weighted composition of the portfolio. Multiplying (8) by \( \beta_{Pj} \) and summing over \( j \) proves that

\[ E(r_P) = r_F + \lambda_1 \kappa(r_P, r_N)\text{Std } r_P + \lambda_2 \kappa(r_P, r_N)\text{Std } r_P. \]

Since this holds for any portfolio, in particular it holds for \( M \) and \( N \). Consequently,
\begin{align*}
(9) \quad E(r_M) - r_F &= [\lambda_1 + \lambda_2 \kappa(r_N, r_M)] \text{Std } r_M \\
(10) \quad E(r_N) - r_F &= [\lambda_1 \kappa(r_N, r_M) + \lambda_2] \text{Std } r_N.
\end{align*}

Equations (9) and (10) may be regarded as two equations in two unknowns, \( \lambda_1 \) and \( \lambda_2 \). Solving for \( \lambda_1 \) and \( \lambda_2 \):

\[
\lambda_1 = \frac{E(r_M) - r_F}{\text{Std } r_M[1-\kappa(r_N, r_M)^2]} - \frac{[E(r_N) - r_F]\kappa(r_N, r_M)}{\text{Std } r_N[1-\kappa(r_N, r_M)^2]},
\]

\[
\lambda_2 = \frac{E(r_N) - r_F}{\text{Std } r_N[1-\kappa(r_M, r_N)^2]} - \frac{[E(r_M) - r_F]\kappa(r_M, r_N)}{\text{Std } r_M[1-\kappa(r_M, r_N)^2]}.
\]

At future dates, a valuation equation similar to (8) holds at each date except where expectations are assessed with respect to (possibly transformed) state-conditional probabilities appropriate to that date. Only at the last date before the horizon will the standard single-period two-parameter model appear, because at this date the annuity \( N \) is equivalent to a riskless short-term bond. Without recourse to aggregate consumption variables, attempting to write the present value of an uncertain stream of income as the summation over time of discounted risk-adjusted payments is extremely messy. The problem of valuing an uncertain stream is best visualized in terms of dynamic programming. For example, denote again the uncertain cash flows from security \( j \) by \( \{D_{je}\}, \{D_{je*}\} \). Starting at date \( t = 1 \), calculate the date \( t = 1 \) state \( e \) conditional
present value of \( D_{je} \); add this to the corresponding value of \( D_{je} \); finally calculate the date \( t = 0 \) present value of this sum.

Barr Rosenberg has suggested to me that Merton's (1973, p. 882, equation 32) quite similar continuous-time solution, properly reinterpreted, may be derived as the limit of the discrete-time quadratic utility solution as the trading interval approaches zero. Let the trading interval (elapsed time between dates) be denoted by \( \Delta \). If \( W_{\Delta}^e \) and \( \phi^e \) represent state-conditional date \( t = 1 \) wealth and annuity price with trading interval \( \Delta \) between dates \( t = 0 \) and \( t = 1 \), then

\[
\lim_{\Delta \to 0} V_{WW}^\Delta = \v^2 V(W_0, \phi)/\v^2 W_0^2 \equiv V_{WW}^o, \quad \lim_{\Delta \to 0} V_{W}^\Delta = \v^2 V(W_0, \phi)/\v W_0 \v \phi \equiv V_{W}^o
\]

and

\[
\lim_{\Delta \to 0} E[V'(W_{\Delta}^\Delta, \phi_{\Delta}^\Delta)] = V'(W_0, \phi)
\]

provided the limits exist. Moreover, if \( C_0^\Delta \) represents date \( t = 0 \) consumption with trading interval \( \Delta \), then

\[
\lim_{\Delta \to 0} Std W_1^\Delta = \lim_{\Delta \to 0}(W_0 - C_0^\Delta)Std \quad r_M = W_0 Std \quad r_M
\]

provided the limit exists, where \( r_M \) is the instantaneous rate of change, i.e., rate of return, in the value of the market portfolio. Rewriting equation (8) in parallel with Merton's equation (32)

\[
E(r_j) - r_F = \left[ \frac{-W_0 V_{WW}^o}{V'(W_0, \phi)} \right] \text{Cov}(r_j, r_M) + \left[ \frac{-\v^2 W_0^2}{V'(W_0, \phi)} \right] \text{Cov}(r_j, r_N)
\]
where rates of return are instantaneous. With state-variable \( r_{Ne} \) suitably reinterpreted, Merton's equation (32) for continuous-time is identical to equation (12) for the limit of the discrete-time solution as the trading interval approaches zero.

Merton somewhat arbitrarily assumes the shift in opportunities over time is fully described by one state variable, the change in the instantaneous riskfree rate. In discrete time, this state variable between dates \( t = 0 \) and \( t = 1 \) becomes \( r_{F2}e^{-r_F} \) where \( r_{F2}e \) is the state \( e \) conditional riskfree rate in the next period. Knowing \( r_{F2}e^{-r_F} \), Merton assumes each individual also knows \( E(e^{r_j}) \) for all \( j \) and \( Cov(e^{r_j}, e^{r_k}) \) for all pairs of securities \( j \) and \( k \). In contrast, no restriction has been placed here on the intertemporal stochastic process of security rates of return. Rather \( r_{Ne} \), the rate of return on an annuity maturing at the horizon, was derived as the sufficient decision-relevant statistic describing shifts in opportunities. Therefore, substituting \( N \) for Merton's asset "n" (which is perfectly negatively correlated with \( r_{F2}e^{-r_F} \)), converts Merton's equation (32) into equation (12).

The convenient quadratic simplicities of working in continuous-time, which also stop his expansion just before the continuation notation in condition (7), permit Merton to impose few assumptions on tastes. By contrast, this paper not only works in the more difficult context of discrete-time where quadratic simplifications are not generally available, but also examines explicit derived utility of wealth functions without
restricting the intertemporal stochastic process of security prices, completely solves the aggregation problem, and has developed continuous-time results quite simply as the limit to the discrete-time solution as the trading interval approaches zero. As payment for working within the HARA class of utility functions, the discrete-time results of this paper dodge certain paradoxes that arise in continuous-time,\(^8\) and perhaps most significantly do not presume features of the stochastic process of security prices that should be determined endogenously in equilibrium and not imposed from without. Indeed, in a multiperiod setting, a model such as Merton's, which takes significant features\(^9\) of this stochastic process as given, should not be regarded as a complete model of equilibrium.

Unfortunately, relaxing the quadratic utility assumption creates considerable complication in the discrete-time valuation results stated in terms of wealth.\(^10\)

\(^8\) For example, as/Barc Rosenberg have mentioned to me, the limit to the discrete-time decision rules as the trading interval approaches zero is not generally the continuous-time solution. With constant proportional risk aversion in discrete-time an individual never borrows or lends at the riskfree rate since to do so chances bankruptcy forcing expected utility to be negative infinite; however, some borrowing is frequently optimal in continuous-time. Rosenberg has also noted that in general the volume of trading required to continuously maintain the optimal portfolio in continuous-time is infinite for any finite length of time. Not only is trading volume finite in discrete-time, but it is also the subject of analysis in section VII.

\(^9\) Merton effectively assumes a lognormal process of prices and severely restricts its stochastic dependence over time. A more basic approach would be to assume a stochastic process for aggregate consumption and infer from it the equilibrium stochastic behavior of prices. This occurs automatically in a single-period context, and section IV investigates some special cases in a multiperiod context.

\(^10\) The temptation to derive simplified valuation relationships in the text, assuming a composite individual who makes normal probability
Theorem (multiparameter valuation): Whenever a composite individual can be constructed with utility in the HARA\(^{11}\) class and \(B \neq 0\), in equilibrium at date \(t = 0\)

\[
(13) \quad E(r_j) = r_F + \sum_{n=2}^{\infty} \theta_n \left[ \sum_{m=0}^{n-1} \sigma_{n,m}(r_j, r_N, r_M) \right] \quad (\text{all } j)
\]

where \(M\) is the market portfolio, \(N\) is an annuity yielding equal certain payments at all future dates,

\[
\theta_n \equiv -\frac{\nu^{(n)}}{E[V'(W_1, \phi_1)]} \quad (\text{all } n \geq 2) \quad \text{where } \nu^{(n)} \text{ is the nth}
\]

assessments at each date, has been resisted. Not only can the same objection be lodged against this as against Merton's lognormal assumption, but normality contradicts the nonnegativity property P5 and requires that utility be defined, strictly increasing and concave for negative values of consumption and arbitrarily large positive values of consumption. Ignoring these difficulties, using a method of proof similar to Rubinstein (1973c), it is possible to show for \(B \neq 0\) that in equilibrium at date \(t = 0\) equation (8) holds with \(\lambda_1\) and \(\lambda_2\) redefined as

\[
\lambda_1 \equiv -\frac{E[V''(W_1, \phi_1)]}{E[V'(W_1, \phi_1)]} \text{Std } W_1 > 0 \quad \text{and} \quad \lambda_2 \equiv -\frac{(A/B)E[V''(W_1, \phi_1)]}{E[V'(W_1, \phi_1)]} \text{Std } \phi_1
\]

where \(V''(W_e, \phi_e) \equiv \partial^2 V(W_e, \phi_e) / \partial W_e^2\). The proof utilizes the fact that the argument of the HARA-derived utility of wealth function, \(A\phi_e + BW_e\), is itself normally distributed if all securities have normally distributed rates of return. For the \(B = 0\) case, the corresponding argument \(W_e(A\phi_e)^{-1}\) does not have this property. It may be worth noting that again the limit of the above discrete-time solution, as the trading interval approaches zero, is the continuous-time solution of Merton with state-variable \(r_N\) properly reinterpreted.

\(^{11}\)Again the similar \(B = 0\) case is omitted.
derivative of $V(W_e, \phi_e)$ evaluated at $EW_1$ and $E\phi_1$, and

$$G_{n,m}(r_j, r_N, r_M) = \frac{[A(\phi-1)/B]^n-1-m(W_0-C_0)^m}{m!(n-l-m)!} E[(r_j-Er_j)(r_N-Er_N)^{n-1-m}(r_M-Er_M)^m].$$

(all $j$, $n$ and $m$).

Proof: For $B \neq 0$, either $V(W_e, \phi_e) \propto \frac{b}{1-b} (A\phi_e + BW_e)^{1-b}$ or $V(W_e, \phi_e) \propto \ln(A\phi_e + BW_e)$. Define function $G(A\phi_e + BW_e) = V(W_e, \phi_e)$. If the required central moments exist, by the expectation of the exact Taylor series expansion of $G(A\phi_e + BW_e)$ around $E(A\phi_1 + BW_1)$,

$$E[V(W_1, \phi_1)] = E[G(A\phi_1 + BW_1)] = \sum_{n=2}^{\infty} \frac{G^{(n)}}{n!} \mu_n$$

where $G^{(n)}$ is the $n$th derivative of $G$ evaluated at $E(A\phi_1 + BW_1)$ and

$$\mu_n = E\left\{[(A\phi_1 + BW_1) - E(A\phi_1 + BW_1)]^n\right\}$$

is the $n$th central moment of $A\phi_e + BW_e$. Substituting this into the derived first period programming problem

$$\max_{C_0, \{S_j\}} \left[ U(C_0) + \bar{p}_1 \sum_{n=0}^{\infty} \frac{G^{(n)}}{n!} \mu_n - \lambda[C_0 + \sum_j S_j - W_0] \right]$$

with first order conditions

\footnote{This proof is quite similar to the analogous single-period proof in Rubinstein (1973a).}
The first bracketed expression is simply \( E[G'(A\phi_1 + BW_1)] \). Moreover, since \( G'(A\phi_e + BW_e) = bV'(W_e, \phi_e) \), then \( E[G'(A\phi_1 + BW_1)] = bE[V'(W_1, \phi_1)] \) and \( G^{(n)} = bV^{(n)} \). Finally,

\[
\frac{\partial \mu_n}{\partial S_j} = nE\left\{(r_j - Er_j)[A(\phi_1 - E\phi_1) + B(W_1 - EW_1)]^{n-1}\right\} \quad (\text{all } n \geq 2)
\]

and \( \mu_0 = 1 \) and \( \mu_1 = 0 \). Substituting these results into the previous equation

\[
E(1 + r_j + r_j)bE[V'(W_1, \phi_1)] + \sum_{n=2}^{\infty} \frac{bV^{(n)}(n)}{(n-1)!} E\left\{(r_j - Er_j)[A(\phi_1 - E\phi_1) + B(W_1 - EW_1)]^{n-1}\right\} = \lambda / \bar{\rho}_1.
\]

The term \( [A(\phi_e - E\phi_1) + B(W_e - EW_1)]^{n-1} \) is a binomial of the form \( (x + y)^{n-1} \). By the formula for the binomial expansion

\[
(x + y)^{n-1} = \sum_{m=0}^{n-1} \frac{(n-1)!}{m!(n-1-m)!} x^{n-1-m} y^m.
\]

Applying this formula to the previous equation, rearranging, and noting that \( (1 + r_p)bE[V'(W_1, \phi_1)] = \lambda / \bar{\rho}_1 \) yields of equation (13). Q.E.D.

To interpret this result, writing the valuation equation term by term

\[
E(r_j) = r_p + \theta_2 [\text{Cov}(r_j, W_1) + (A/B)\text{Cov}(r_j, \phi_1)] + \theta_3 \left[ \frac{1}{2!} \text{Cos}(r_j, W_1, W_1) + \frac{(A/B)^2}{2!} \text{Cos}(r_j, \phi_1, \phi_1) \right] + \ldots
\]
where if \( X_1, X_2 \) and \( X_3 \) are any three random variables,

\[
\text{Cov}(X_1, X_2) = \mathbb{E}[(X_1 - \mathbb{E}X_1)(X_2 - \mathbb{E}X_2)] \quad \text{and} \quad \text{Cov}(X_1, X_2, X_3) = \mathbb{E}[(X_1 - \mathbb{E}X_1)(X_2 - \mathbb{E}X_2)(X_3 - \mathbb{E}X_3)].
\]

The equilibrium expected rate of return of any security equals the risk-free rate plus a series of risk premiums measuring the collective effect of each joint central moment of \( r_j, r_N, \) and \( r_M \). These moments reflect the contribution of a marginal increase in the holdings of a security to the simple central moments of \( r_M \) and the joint central moments of \( r_N \) and \( r_N \). Of course, with HARA cubic utility, the above equation cuts off just before the continuation notation. With constant proportional risk aversion, \( A = 0 \) and the multiperiod valuation equation reduces to the same equation that would have been derived in a single-period context except that the probabilities used to assess expectations are suitably transformed. With logarithmic utility, \( A = 0 \) and \( B = 1 \), no such transformation is required. A similar reduction to a single-period context is also achieved if \( \text{Std } r_N = 0 \).

In the sequel paper in this series, these valuation results will be extended to incorporate uncertain relative and absolute inflation where the impact of inflation appears explicitly in the derived utility of wealth functions and again the aggregation problem is solved.
IV. INTERTEMPORAL STRUCTURE OF RATES OF RETURN

Since the multiperiod economies analyzed in sections I-III place no exogenous restrictions on the intertemporal stochastic process of rates of return, they present an ideal laboratory to examine the intertemporal structure of rates of return generated in equilibrium. While there has been considerable empirical testing of this structure, very few theoretical results have been forthcoming. Most empirical work has tested the two classic hypotheses of

1) unbiased term structure,
2) random walk.

Rubinstein (1973a) has shown that properly defined these two hypotheses are intimately related. This section will extend his analysis by considering the effect of the assumed exogenous stochastic process of the rate of growth of aggregate consumption on the equilibrium intertemporal structure of rates of return, when a composite individual can be constructed with logarithmic utility.

We will first digress to establish an economically interesting definition of an unbiased term structure. The essence of an unbiased term structure is that an investment in a sequence of riskfree short-term bonds (with proceeds at each date reinvested) up to any date \( t \) is expected to yield the same inverse compound rate of return as a single investment in a riskfree long-term bond maturing at date \( t \). However, the usual statement of an unbiased term structure--forward rates equal the corresponding expected future spot rates--errs on two counts. First,
Nelson (1972) has shown that this more popular definition is generally inconsistent with the italicized definition (omitting the word "inverse").

Second, Nelson himself errs by failing to use inverse rates.

To be precise, for an economy with horizon $T = 3$, define

$$(1 + r_F)^{-1} = \Sigma e P_{e} e, \quad (1 + R_{F2})^{-2} = \Sigma e P_{e} \Sigma s P_{s} s e$$

$$\Sigma e P_{e} \Sigma s P_{s} s e a a s e,$$

where if both states $e$ and $s$ occur at dates $t = 1$ and $t = 2$, $P_{a s e}$ is the price paid at date $t = 2$ for a state-contingent claim to one unit of wealth at date $t = 3$ received if and only if state $a$ occurs at that date. $r_F$, $R_{F2}$, and $R_{F3}$ are then the compound rates of return on riskfree bonds of successive maturities. Also define

$$(1 + r_{2F s e})^{-1} = \Sigma s P_{s e} e$$

and

$$(1 + r_{3F s e})^{-1} = \Sigma a P_{a s e}$$
as the successive future spot rates. The usual statement of an unbiased term structure is then

$$(1 + R_{F2})^2/(1 + r_F) = E(1 + r_{F2}) \quad \text{and} \quad (1 + R_{F3})^3/(1 + R_{F2})^2 = E(1 + r_{F3}).$$

By contrast, Nelson's more fundamental statement of the hypothesis is that\footnote{13}{Here expectations are assessed with respect to beliefs held at date $t = 0$.}

$$(1 + R_{F2})^2 = (1 + r_F)E(1 + r_{F2}) \quad \text{and} \quad (1 + R_{F3})^3 = (1 + r_F)E[(1 + r_{F2})$$

$$(1 + r_{F3})].$$

If horizon $T = 2$, there is no conflict; however for $T > 2$, Nelson shows that the two definitions are inconsistent if the future spot rates are
correlated. To see this, by the popular definition

\[(1 + R_{F3})^3 = (1 + r_F)E(1 + r_{F2})E(1 + r_{F3})\]

and by Nelson's definition

\[(1+R_{F3})^3 = (1+r_F)E(1+r_{F2})E(1+r_{F3}) + (1+r_F)\kappa(r_{F2}, r_{F3})\text{Std } r_{F2}\text{Std } r_{F3}.\]

Therefore, the two definitions are inconsistent if and only if

\[\kappa(r_{F2}, r_{F3}) \neq 0.\]

For example, if \[\kappa(r_{F2}, r_{F3}) > 0,\] then an unbiased term structure in Nelson's sense implies a positive liquidity premium, i.e.

\[
(1 + R_{F3})^3 / (1 + R_{F2})^2 > E(1 + r_{F3}),
\]

using the popular definition. Moreover, Nelson's own empirical test confirms positive serial correlation of short-term rates.

Rubinstein (1973a, 1973b) has shown that inverse (one plus) rates of return, i.e., the present price of a future dollar, are theoretically more fundamental than the (one plus) rates of return themselves. Inverting the rates of return, the economically interesting statement of an unbiased term structure is then

\[
(1+R_{F2})^{-2} = (1+r_F)^{-1}E[(1+r_{F2})^{-1}] \quad \text{and} \quad (1+R_{F3})^{-3} = (1+r_F)^{-1}E[(1+r_{F2})^{-1} \cdot (1+r_{F3})^{-1}].
\]

As long as future spot rates are uncertain, this statement of an unbiased term structure conflicts with Nelson's. To see this for the two-period long-term bond, observe that for any positive random variable \(X_e\) which
is not a constant, the inverse of the expected inverse of \(X\) is less
than the expected value of \(X\); that is \([E(X^{-1})]^{-1} < E(X)\).\(^{14}\) Now compar-
ing the two definitions, for unbiasedness, Nelson requires

\[(1 + R_{F2})^2 = (1 + r_F)E(1 + r_{F2})\]

and we require instead

\[(1 + r_{F2})^2 = (1 + r_F)^{-1}E[(1 + r_{F2})^{-1}]^{-1} \cdot\]

Consequently, since \(E[(1 + r_{F2})^{-1}]^{-1} < E(1 + r_{F2})\) as long as \(r_{F2} \neq 0\)
is positive and not a constant, if the term structure is unbiased in in-
verse rates, then there is a negative liquidity premium in terms of the
rates of return themselves (i.e. \((1 + r_{F2})^2 / (1 + r_F) < E(1 + r_{F2})\)). A similar
result also applies to the three-period long-term bond.

Despite the suggested use of inverse rates, using the relation-
ship between \(r_F\) and the first period rates of return \(r_{N2}\) and \(r_N\) on
annuities maturing at dates \(t = 2\) and \(t = 3\), respectively, an unbiased
term structure in our sense may be equivalently defined in terms of rates
of return. By definition,

\(^{14}\)To see this, note that \(\text{Cov}(X, X^{-1}) = 1 - E(X)E(X^{-1}) < 0\).
\[ 1 + r_{N2:e} = \frac{1 + (1 + r_{F2:e}^{-1})^{-1}}{(1 + r_F^{-1} - 1) + (1 + r_{F2}^{-2})^{-2}} \]

and

\[ 1 + r_{Ne} = \frac{1 + (1 + r_{F2:e}^{-1})^{-1} + (1 + r_{F2:e}^{-1})^{-1}(1 + r_{F3:e}^{-1})^{-1}}{(1 + r_F^{-1} - 1) + (1 + r_{F2}^{-2} + (1 + r_{F3}^{-3})^{-3}}. \]

From this, it is easy to show the term structure is unbiased (in our sense) if and only if \( r_F = E(r_{N2}) = E(r_N) \). That is, the term structure is unbiased if and only if the expected first period rates of return on riskfree annuities of successive maturities through the horizon are the same. By contrast, the so-called liquidity preference hypothesis postulates a positive premium on longer-term investments, e.g.,

\[ (1 + r_{F2}^{-1})/(1 + r_F^{-1}) > E(1 + r_{F2}). \]

This implies \( r_F < E(r_{N2}) < E(r_N) \), but the converse is false. In a nutshell:

- \( r_F > E(r_{N2}) > E(r_N) \) → negative liquidity premiums
- \( r_F = E(r_{N2}) = E(r_N) \) → negative liquidity premiums
- \( r_F < E(r_{N2}) < E(r_N) \) → negative, zero, or positive liquidity premiums.

Let \( 1 + r_{C_e} = C_e^t / C_0^0 \) and \( 1 + r_{C_s:e} = C_{s:e}^t / C_e^t \) denote the rate of growth of aggregate consumption in periods one and two, respectively.\(^{15}\)

Let \( 1 + r_{Me} = W_e^t / (W_0 - C_0) \) and \( 1 + r_{Ms:e} = W_{s:e}^t / (W_e - C_e) \) denote the (one

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\(^{15}\)With only three dates, wealth \( W_{s:e} \) and consumption \( C_{s:e} \) at date \( t = 2 \) are the same.
plus) rate of return on the market portfolio in periods one and two, respectively. With these definitions, we are prepared for the following theorem:

**Theorem (intertemporal structure):** Whenever a composite individual can be constructed with logarithmic utility, any one of the following statements is true if and only if the other statements are true:

1. The term structure is unbiased;\(^{16}\) that is

\[
\frac{1}{(1+r_{F2})^2} = \frac{1}{1+r_{F}} \mathbb{E} \left[ \frac{1}{1+r_{F2}} \right],
\]

2. The (inverse one plus) rate of return of the market portfolio is uncorrelated; that is

\[
\mathbb{E} \left[ \frac{1}{(1+r_{ML}) (1+r_{M2})} \right] = \mathbb{E} \left[ \frac{1}{1+r_{ML}} \right] \mathbb{E} \left[ \frac{1}{1+r_{M2}} \right].
\]

3. The (inverse one plus) rate of growth of consumption is uncorrelated; that is

\[
\mathbb{E} \left[ \frac{1}{(1+r_{C1}) (1+r_{C2})} \right] = \mathbb{E} \left[ \frac{1}{1+r_{C1}} \right] \mathbb{E} \left[ \frac{1}{1+r_{C2}} \right].
\]

**Proof:** With logarithmic utility, conditions (1'), (2') and (3) reduce to

\[(a) \quad 1 + r_{Ce} = \rho_1 \pi_e / p_e \quad \text{(all e)}\]

\[(b) \quad 1 + r_{Cs} e = \rho_2 \pi_s / p_s e \quad \text{(all e and s)}\]

\[(c) \quad C_0 (1+r_{Ce}) = (1+r_{M})^{-1} (W_0 - C_0) (1+r_{Me}) \quad \text{(all e)}\]

\(^{16}\)Applying the model in footnote 10 to logarithmic utility, it is easy to show the term structure is unbiased, \(\mathbb{E}(r_e) = r_F\), if and only if \(\kappa(r_{N1}, r_{M1}) = 0\).
respectively. Dividing (a) by \(1 + r_{Ce}\), multiplying through by \(P_e\) and summing over \(e\)

\[
\frac{1}{1 + r_p} = \rho_1 \mathbb{E} \left[ \frac{1}{1 + r_{C1}} \right].
\]

Similarly, from (b)

\[
\frac{1}{1 + r_{F2,e}} = \rho_2 \mathbb{E} \left[ \frac{1}{1 + r_{C2,e}} \right].
\]

Therefore,

\[
(d) \quad \frac{1}{1 + r_p} \mathbb{E} \left[ \frac{1}{1 + r_{F2}} \right] = \rho_1 \rho_2 \mathbb{E} \left[ \frac{1}{1 + r_{C1}} \right] \mathbb{E} \left[ \frac{1}{1 + r_{C2}} \right].
\]

Combining (a) and (b)

\[
P_e P_{s,e} = \rho_1 \rho_2 \mathbb{E} \left[ \frac{1}{(1 + r_{Ce})(1 + r_{Cs,e})} \right].
\]

Summing this over all \(e\) and \(s\)

\[
(e) \quad \frac{1}{(1 + r_{F2})^2} = \rho_1 \rho_2 \mathbb{E} \left[ \frac{1}{(1 + r_{C1})(1 + r_{C2})} \right].
\]

By the consumption decision rule\(^{17}\) for logarithmic utility,

\[C_0 = \frac{W_0}{(1 + \rho_1 + \rho_1 \rho_2)}\]

so that from (c)

\[^{17}\text{See section V of Rubinstein (1974).}\]
(f) \[ 1 + r_C e = \rho_1 (1 + r_{M e}) \quad (\text{all } e) \]
and similarly using the definition of \( r_{M s, e} \)

(g) \[ 1 + r_{C s, e} = \rho_2 (1 + r_{M s, e}) \quad (\text{all } e \text{ and } s). \]

Putting this together, from (d) and (e), \((1) \iff (3)\) and from (f) and (g), \((2) \iff (3)\). Consequently, \((1) \iff (2) \iff (3)\). Q.E.D.

Whether or not the term structure is unbiased \((1)\) and the market portfolio follows a random walk\(^\text{18}\) \((2)\) depends crucially on the stochastic process governing aggregate consumption over time. Indeed, with logarithmic utility only if this process is uncorrelated will the classical hypotheses be verified. The underlying real stochastic process of aggregate consumption carries directly over to the equilibrium financial stochastic process governing security prices. For example, it follows trivially from the theorem that if the (inverse one plus) growth rate of aggregate consumption is positively correlated over time, then the term structure will be biased with positive liquidity premiums. To my knowledge, this explanation of the term structure has never been given, and yet it can explain the chief observed tendency of the term structure: normal yield curves predominate during the recovery phase of the business cycle, and inverted yield curves predominate near the peak. During recovery, individuals anticipate an increasing rate of growth in consumption with

\[\text{---18---}\]

\(^{18}\) Strictly speaking, a random walk requires that the rate of return of the market portfolio be stochastically independent, not merely uncorrelated, over time.
more optimistic future expectations as their present expectations are verified. As the peak of the cycle nears, individuals remain optimistic about the near-term (hence the "hump" in the yield curve) but are pessimistic relative to the present about long-term prospects of aggregate consumption.

The role of annuity $N$ as providing protection against future unfavorable shifts in opportunities is highlighted by the logarithmic utility case. Recall that by definition,

$$1 + r_{Ne} = \left[ 1 + (1+r_{F2,e})^{-1} \right]/\left[ (1+r_{F})^{-1} + (1+r_{F2})^{-2} \right].$$

Since from the previous proof $(1+r_{F2,e})^{-1} = E_e \left[ \left( 1+r_{M2,e} \right)^{-1} \right]$, $r_{Ne}$ is perfectly positively correlated with $E_e \left[ \left( 1+r_{M2,e} \right)^{-1} \right]$. As a result, the rate of return on annuity $N$ in the first period is generally negatively correlated with the rate of return of the market portfolio expected in period two. Consequently, purchase of the annuity provides an excellent hedge against downturns in the business cycle.

As a final observation, the logarithmic utility case provides a simple illustration of the inference of the probability law governing financial processes from the probability law governing real processes. For example, since $1 + r_{Ce} = \rho_1 (1+r_{Me})$, if $1 + r_{Ce}$ were normally or lognormally distributed, then $1 + r_{Me}$ would also be normally or lognormally distributed with greater mean and variance if $\rho_1 < 1$ (impatience). Such an inference is the chief missing element in Merton's (1973) analysis of equilibrium. Whatever the probability law, $r_{Ce}$ and $r_{Me}$ are perfectly positively correlated. Consequently, in the logarithmic economy,
the securities market does not lead but is coincident with the business cycle. The observed tendency for the stock market to lead the business cycle in the United States may be attributed to ratchet effects in consumption, which have not been modeled.\footnote{19\textsuperscript{19}Unfortunately, these simple logarithmic utility results do not carry over to other members of the HARA class of utility functions. For example, whenever a composite individual can be constructed with power utility, the term structure is unbiased if and only if (1+r_Ce)^b and (1+r_{Ce}, e)^b are uncorrelated. Consequently, the term structure can be biased even though (1+r_Ce)^{-1} and (1+r_{Ce^}, e)^{-1} are uncorrelated. For power utility, it can also be shown that \kappa \left[ \frac{r_{ML}}{\kappa}, \frac{r_{CL}}{\kappa}, (1+r_{E} (1+r_{CZ, e})^{-b+1}) \right] = 1 so that r_{ME} and r_{CE} are no longer perfectly positively correlated. In this case zero correlation in the real stochastic process of aggregate consumption is not generally mirrored in the financial stochastic process of the market portfolio.}

V. EXCHANGE-EFFICIENCY

A perfect and competitive securities market is exchange-efficient if individuals are not motivated to create exchange arrangements not already provided by the market. That the complete market multiperiod economy described in section IV of the previous paper is exchange-efficient follows from Arrow's (1953) seminal work on complete markets. In Arrow's single-period economy, individuals allocate their initial wealth among (claims to) state-contingent commodities. With $C(c = 1, 2, \ldots, C)$ commodities deliverable under each of $E(e = 1, 2, \ldots, E)$ states, a total of CE state-contingent commodities are available at date $t = 0$. At date $t = 1$, each individual learns the true state and cashes in his valuable contingent commodities. Arrow proves that the existence of a complete set of
state-contingent commodities tradable only at date $t = 0$ in a perfect
and competitive market by individuals with concave utility functions is
sufficient for exchange-efficiency. The essence of his argument is that
if individuals can trade all the arguments of their utility functions,
then a perfect and competitive economy is exchange-efficient. Let $x_{ce}$
represent the quantity of commodity $c$ realized by an individual if and
only if state $e$ occurs; and let $\bar{u}(x_{11}, \ldots, x_{Cl}, \ldots; x_{1e}, \ldots, x_{Ce})$ be his
(total) utility function. This utility function may or may not be consis-
tent with expected utility maximization. That is, it is not necessary
that there exist beliefs $\{\pi_e\}$ and a function $u$ such that

$$\bar{u}(\cdot) = \sum_e \pi_e u(x_{1e}, \ldots, x_{Ce}, \ldots, x_{Ce}).$$

As long as all the arguments $\{x_{ce}\}$ of $\bar{u}$ are initially tradable, a per-
fектив and competitive economy achieves exchange-efficiency.

It is very easy to see why this must be true. Suppose an equi-
librium exists in a perfect and competitive economy (implied by the con-
cavity of utility functions). If the market is complete, no buyer and
seller of a state-contingent commodity (which is all they care about)
will be able to both benefit by trading outside the market: if they set
a price (i.e., exchange ratio) at the same level as in the market, they
would be equally as well off if they had just used the market; if a higher
price is set, the buyer is worse off and so will not agree to the exchange;
if a lower price is set, the seller is worse off and so will not agree to
the exchange. However, can the two create a new state-contingent commodity or portfolio of state-contingent commodities not already provided by the market? If so, they may find it beneficial to trade in this new opportunity. However, because all possible state-contingent commodities are already tradable in the market, i.e., it is complete, and any portfolio may be manufactured out of them, no new opportunities can be created. Consequently, the complete market must be exchange-efficient.

The complete market single-period economy described in section I of the previous paper is both a special case and a generalization of Arrow's economy: only one commodity is traded and enters utility functions, but this commodity is desired at both dates \( t = 0 \) and \( t = 1 \). That is, (total) utility is \( \bar{U}(C_0; W_1, \ldots, W_E) \). Nevertheless, since both current consumption as well as claims to future wealth for every state are tradable at date \( t = 0 \), the resulting final allocation must be exchange-efficient.

In the multiperiod economy described in section IV, (total) utility is

\[
\bar{U}(C_0; C_1, \ldots, C_E; W_{1.1}, \ldots, W_{S.1}; \ldots; W_{1.E}, \ldots, W_{S.E}).
\]

where any one of \( S(s = 1,2,\ldots,S) \) states can occur at date \( t = 2 \).

While initial consumption and a full set of state-contingent claims to consumption at date \( t = 1 \) is tradable at date \( t = 0 \), a full set of state-contingent claims to date \( t = 2 \) wealth is not initially available. Instead, the occurrence of state \( e \) at date \( t = 1 \) conditions the opening of new markets to those claims. Will this delayed trading opportunity
make up for the incomplete initial markets so that the economy will be exchange-efficient?

Arrow's second result shows this will be true if individuals maximize expected utility. He considers a modified economy which at date \( t = 0 \) permits trading only in dollars of future consumption, i.e., state-contingent claims; the occurrence of state \( e \) at date \( t = 1 \) conditions the opening of spot markets in each of the \( C \) commodities. He proves that the opportunity to make a delayed decision with the opening of these spot markets yields the same final allocation as in his original economy with state-contingent commodities. Since the original economy is exchange-efficient, then his modified economy is also exchange-efficient. The general principle is that individuals are indifferent between relatively rich forward markets with few revision opportunities and relatively poorly developed forward markets with many future trading dates.

To see this intuitively, a price system may be fundamentally interpreted as a means of diffusing the decision-relevant personal information about the distribution of resources, beliefs, and tastes to all individuals in the economy so that they all can make efficient choices.\(^\text{20}\)

That is, if each individual utilizes only his own personal information (his own resources, beliefs, and tastes) as well as the opportunities (prices) made available to him by the economy, his choices will be exchange-efficient, even though he is ignorant of the personal information

\(^{20}\) For an excellent description of this view of a competitive price system, see Hayek (1945).
of all other individuals and they are ignorant of his personal information. An exchange-efficient price system therefore is a distilled summary of decision-relevant personal information. Consequently, any two price systems which permit a cross-inference of each other contain the same decision-relevant information and will lead to the same final allocations.

As an example, consider the multiperiod economy described in section IV of the previous paper. In that case, individuals can purchase state-contingent claims to date \( t = 1 \) wealth at date \( t = 0 \) prices \( \{P_e\} \); and, depending on which state \( e \) occurs, at date \( t = 1 \) individuals can purchase state-contingent claims to date \( t = 2 \) wealth at date \( t = 1 \) prices \( \{P_{s \cdot e}\} \). Call this a modified economy. Now imagine the original economy, identical in all respects, except at date \( t = 0 \) individuals can purchase state-contingent claims to date \( t = 1 \) wealth at prices \( \{P_e\} \) and a full set of state-contingent claims to date \( t = 2 \) wealth at prices \( \{P_{se}\} \) that pay off if and only if both states \( e \) and \( s \) occur. From the previous discussion, clearly this original economy must be exchange-efficient, because all arguments which enter the (total) utility function are initially tradable. Will it yield the same final allocation as the second economy? Observe that, if arbitrage were permitted across the two economies, in equilibrium \( P_{se} = P_e P_{s \cdot e} \) for all \( e \) and \( s \) since purchasing a claim to state \( e \) and, if state \( e \) occurs, exchanging it for claims to state \( s \) must be equivalent to initially purchasing a claim that pays off if and only if both states \( e \) and \( s \) occur. Using this arbitrage condition, it is possible to cross-infer the prices
of the two economies: from the original economy, knowing \( \{p_e\} \) and \( \{p_{se}\} \), the prices of the modified economy can be inferred; from the modified economy, knowing \( \{p_e\} \) and \( \{p_{se}\} \), the prices of the original economy can be inferred. The two price systems therefore contain equivalent information and will lead to the same final allocations. Since the original economy is exchange-efficient, then the modified economy is also exchange-efficient.

As Stiglitz (1969) initially demonstrated, while a complete market (possibly completed by revision opportunities) is sufficient for exchange-efficiency, it is not necessary. As an extreme example, in the single-period economy described in section I of the previous paper, if all individuals in equilibrium choose to purchase state-contingent claims in the same proportions, i.e., universal complete portfolio separation, exchange-efficiency can be achieved with just one actual security, i.e., the market portfolio, available. In this case, for all individuals the desired arguments of their utility functions are implicitly tradable; and this is all that counts. More generally, if there exists a subset of the full set of state-contingent securities that all individuals desire to hold in the same proportions within the subset, then the subset may be replaced by an actual security (representing an appropriate combination of the state-contingent securities in the subset) and the final exchange-efficient allocation will remain unchanged. The special case economies of the previous paper with generalized portfolio separation are illustrations of this principle.
Indeed, even stronger results can be obtained. In Arrow's economy as well as its multiperiod extension presented in this series, every Pareto-efficient final allocation can be achieved by an appropriate redistribution of resources. This is the chief economic justification of a competitive economy. If final allocations violate consensus values of social equity, then a decentralized competitive allocation system need not be debunked and centralized; rather equity can be restored by the imposition of taxes correcting the distribution of initial wealth, yet leaving individuals free to make their own choices and preserve their private access to their personal information. Generally, each distribution of resources will require a different price system to be exchange-efficient. However, there are interesting special cases in which all Pareto-efficient allocations of resources are associated with the same price system, that is, when the price system is insensitive to all possible redistributions of resources. Rubinstein (1974a) shows that for economy (i) where all individuals have the same patience, economy (ii) and economy (iii), the price system is indeed insensitive to the allocation of resources. Consequently, in these cases, any interference by a central planner that alters prices must ipso facto lead to exchange-inefficiency.

Whenever an incomplete market is exchange-efficient, as has been argued, there must exist combinations of implicit state-contingent claims that all individuals desire to hold in common. As one would expect, whenever an economy is exchange-efficient the packaging of securities to
the market by financial intermediaries or productive firms makes no difference as long as these desired combinations can be constructed. For example, with universal portfolio separation, i.e., all individuals choose to divide their wealth (after consumption) between the market portfolio and a default-free annuity irrespective of the variety of available securities, corporate capital structure (even if bankruptcy is possible) and nonsynergistic mergers (even if bankruptcy is possible) have no effect on final allocations as long as the default-free annuity is not destroyed in the process.\footnote{21} Moreover, the price systems of the economies before and after the repackaging of securities are compatible.\footnote{22}

The irrelevancy of certain repackaging decisions however may extend even into economies that are not exchange-efficient. For example, \footnote{21}{For this to be true, capital structure must also be "nonsynergistic," an implication of perfect and competitive markets. That is, the total cash flows to all security holders of a firm must be independent of the capital structure decision. This rules out corporate taxes (if the government is not viewed as an implicit security holder), direct costs associated with bankruptcy, and the potential for the probability of bankruptcy to influence either the cost of productive factors and the prices of outputs, or the operating decisions of management. For example, other things equal, the government may prefer one aerospace contractor to another the less the probability of bankruptcy if there are direct costs associated with switching contractors in the event of bankruptcy. As a second example, increased debt may subject management to protective covenants restricting the range of their operating decisions, or in the absence of protective covenants, equity-oriented management may be more willing to make long-shot operating decisions.}

\footnote{22}{Rubinstein (1974a) shows that with universal portfolio separation, the present value of a portfolio of securities when its constituent securities cannot be separately traded equals the sum of the present values of the constituent securities when they can be separately traded.}
consider an all-equity firm in an economy for which a riskfree opportunity exists but which is nonetheless exchange-inefficient, e.g., an incomplete markets version of economy (ii). If the firm now alters its capital structure by issuing debt to repurchase its shares, as long as it does not issue enough debt to create a positive probability of bankruptcy in the view of any individual, then final allocations will remain unchanged: those individuals wishing to hold the old equity will simply repackage it themselves and those wishing to hold a riskfree investment could have done so anyway. More generally, even if the firm were to issue enough debt to create a positive probability of bankruptcy as long as the pattern of returns across both dates and states of the two risky securities it was creating could be duplicated by forming portfolios of already existing securities, then the final allocations would remain unchanged. This property of existing securities is known as spanning. 23

There is a third and final hybrid case for which capital structure decisions will be irrelevant. Although the economy is exchange-inefficient and existing securities do not span the securities created by certain capital structure decisions, the new patterns of returns created by the securities may still not be desired since all individuals might choose to repackage them as they were before by their own portfolio decisions. Although other new securities might have patched up the exchange-inefficiencies, these new securities do not. This property of existing

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23 This concept as applied to security markets is developed by Ekern and Wilson (1974).
securities and the desires of individuals in the economy will be called partial spanning.

These remarks can be summarized as follows:

**Theorem (exchange-efficiency):** Given an economy satisfying properties P1-P3:24 corporate capital structure decisions will not affect final allocations if and only if

1) the economy is exchange-efficient before and after the decision, or

2) existing securities span the securities created (or destroyed) by the decision, or

3) existing securities partially span the securities created (or destroyed) by the decision.

Both (1) and (2) may be satisfied simultaneously as in the case of a complete market; however, in general they will not overlap. In the absence of a complete market, (1) may be viewed as a restriction primarily on beliefs and tastes of all individuals in the economy, (2) as a restriction on opportunities, i.e., the set of existing securities and firm capital structure decisions, and (3) as a joint restriction on beliefs, tastes, and opportunities.25

One further issue might generate confusion. If conditions (1), (2), or (3) hold, then the price of the firm’s equity (as well as the total value of the firm’s securities) is unaffected by its capital

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24 Property P1 may be relaxed to require only an ordinal ordering over consumption.

25 These same three conditions apply to the effect of nonsynergistic mergers on final allocations. However, as Fischer Black has reminded me, if debt contracts are not renegotiated, since a merger will generally reduce the likelihood of bankruptcy, it may create a windfall gain to debt holders and a windfall loss to equity holders.
structure decisions. However, suppose the economy violates these conditions, would an all equity firm by issuing enough debt to create a new desired security become more valuable? To see why it might not, consider a homely apples-and-oranges analogy. Suppose there are only two groups of individuals in the economy, those who only like apples and those who only like oranges. However, supermarkets trying to enhance sales only offer apples and oranges in a basket containing an equal number of each (1:1 ratio) and, moreover black markets (side arrangements between the two groups) are prohibited by law. What would happen to the prices of apples and oranges if the supermarkets were to sell them separately (or side arrangements became legal)? On the one hand, prices would tend to rise, because each group now has more money to spend on their favorite fruit; they no longer have to waste money on the fruit they dislike just to purchase the one they like. On the other hand, prices would tend to fall, because the demand for each fruit from the disaffected group would disappear; no individual would have to buy a fruit he disliked. Which of these offsetting tendencies would prove strongest would depend on other factors not specified (the resources, beliefs, tastes, and other trading opportunities of all individuals). There is no general conclusion.

VI. PRODUCTION-EFFICIENCY

In a perfect and competitive securities market under certainty, it is well known that an economy of competitive value-maximizing firms achieves production-efficiency. Recently Ekern and Wilson (1974),
Leland (1973), and Rubinstein (1974a) have isolated sufficient conditions for production-efficiency under uncertainty. Taking their results together provides necessary and sufficient conditions that parallel those for exchange-efficiency.

**Theorem (production-efficiency):** Given an economy satisfying properties P1-P3: corporate capital budgeting decisions in an economy of competitive value-maximizing firms will be unanimously preferred by all shareholders if and only if

1) the economy is exchange-efficient before and after the decisions, or
2) existing securities span the pattern of returns of all feasible production plans, or
3) existing securities partially span the pattern of returns of all feasible production plans.

By definition, a competitive firm creates no production externalities (i.e., its production decisions do not affect the production functions of other firms), has decreasing or constant returns to scale, and acts as if it cannot affect the final allocation across individuals, dates, and states. If a composite individual can be constructed, Rubinstein (1974a) shows this last condition of competition simplifies to the firm acting as if it cannot affect the aggregate amount of production across

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26 Note the paradox implied by this attitude: while a firm is exhorted to act in the best interests of its shareholders, it is advised to pretend it cannot affect those interests. When is it reasonable to expect a firm to ignore its effect on final allocations? In the usual scenario of atomistic competition, although a firm has some influence it is negligible and may be ignored as a good first-order approximation. As another possibility, there may exist as few as two firms where each firm believes the output of the other firm will exactly compensate for its output, leaving total output unaffected. See Fama and Laffer (1972) for an analysis of competition with perfectly reacting firms.
both dates and states, since given this the allocation across individuals is determined: By definition, a *value-maximizing* firm makes production decisions that maximize the price of its shares set as if it were indeed true that it could not affect the final allocation across individuals, dates, and states.

To see why this theorem is true, observe that under a complete market (sufficient for (1)) or more generally with spanning (2), a competitive firm can infer the present value of its production plan from the prices of existing securities; there is always a definite standard against which its plan may be assessed. A competitive firm can measure the present value of its production plan merely by constructing a portfolio of existing securities with the same pattern of returns as its plan and measuring the present value of this portfolio by using existing market prices. The securities market is rich enough to convey the necessary information to firms to permit them to evaluate their alternative production plans.

However, suppose a firm could choose a production plan with a pattern of returns that cannot be duplicated by a portfolio of existing securities. The present value of this plan cannot be inferred solely from prices of existing securities; the firm will also require information about the resources, beliefs, and tastes of individuals. For example, in the simple two-parameter case described in Rubinstein (1973b) to make efficient production decisions (without trial and error), a firm must know the homogeneous beliefs held by all individuals to estimate means, variances, and covariances, the "market price of risk," which itself is
an aggregation of individual attitudes toward risk, as well as the current price of a riskfree security. However, if the economy is exchange-efficient (1) or with partial spanning (3), no individual wishes to hold the pattern of returns created by the production plan separately in his portfolio. As an alternative interpretation, shareholders can be viewed as already holding the pattern of returns implicitly as a part of a larger portfolio of existing securities. Consequently, as in the exchange-efficiency analysis, production decisions by value-maximizing competitive firms are Pareto-efficient.

VII. INFORMATION-EFFICIENCY

It is useful to divide the question of information-efficiency into two parts: the exchange of information and the production of information. Pure exchange issues are comparative statics questions arising from the comparison of an economy with heterogeneous beliefs to an otherwise similar economy with homogeneous beliefs. With full exchange of information so that all individuals have homogeneous beliefs, pure production issues are comparative statics questions arising from the comparison of two otherwise similar economies but with different homogeneous beliefs. The analysis of the social value of information in Hirshleifer (1971) suffers from a failure to draw this distinction sharply. His chief conclusion is that "the community as a whole obtains no benefit under pure exchange, from either the acquisition or the dissemination of private foreknowledge." While this statement is true for the acquisition, i.e.,
production, of information, it is not generally true with regard to the dissemination; i.e., exchange, of information. Indeed, in interesting cases, the full exchange of information is Pareto-efficient.

Although Hirshleifer couches his analysis of information production in an economy consisting of identical individuals, as Ng (1974) shows, his results are quite general, carrying over to perfect and competitive economies with arbitrary heterogeneity among individuals (but with the same beliefs). In particular, Ng isolates which classes of individuals are benefited and which classes of individuals are hurt by the production of good or bad news.

Although the production of information cannot be Pareto-efficient in a perfect and competitive economy with exogenous production, the exchange of information can be. A securities market consisting of individuals with homogeneous beliefs, i.e., all information is shared, will be called strongly information-efficient. When individuals have heterogeneous beliefs, the concept of consensus as developed in Rubinstein (1973b) proves useful. Consensus beliefs are those beliefs, which if held by all individuals in an otherwise similar economy, would generate the same equilibrium security prices as in the actual heterogeneous economy. An individual with consensus beliefs is said to perceive the securities market as information-efficient; that is, he perceives all his information as "fully reflected" in security prices since if everyone had his beliefs prices would remain unchanged.  

27 In a multiperiod context, Rubinstein (1973b) makes a further distinction between "new" and "old" information and develops corresponding efficiency concepts. New information is received after the first trading
For example, consider single-period economy (iii) as described in the previous paper (generalized logarithmic utility with heterogeneous beliefs). The composite beliefs, \( \pi_e = \sum_i \pi_e^i / I \) for all \( e \), are the consensus beliefs since if \( \pi_e^i = \pi_e \) for all \( e \) and \( i \), security prices would remain unchanged.\(^28\) If each individual were to publicize all his private information, all individuals would then have homogeneous beliefs.

If these revised but now homogeneous beliefs coincide with the consensus date. Surprisingly, the necessary and sufficient condition for an individual to perceive his new information as fully reflected in security prices is independent of his resources and tastes, depending only on the relationship of his beliefs to security prices. As David Ng has suggested to me, this condition permits a precise and meaningful definition of an individual's subjective valuation of a security. For example, in the multiperiod economy described in section IV of part I, for individual \( i \), \( P_e^{i} \) may be defined as his subjective valuation given state \( e \) of a claim to one unit of wealth if state \( s \) occurs where

\[
P_{s \cdot e}^{i} = \frac{\pi_{s \cdot e}^{i} \sum_{e} P_{e}^{i} C_{e}^{i}}{P_{e}^{i} \sum_{e} \pi_{s \cdot e}^{i}}.
\]

If the actual price \( P_{s \cdot e}^{i} \) \((= <)\) \( P_{s \cdot e}^{i} \), then the individual perceives the claim as over- (properly, under-) valued.

\(^28\) To see this, from the first-order condition for this economy \((A = 0)\)

\[
U'(C_{0}^{i}) = (\pi_{e}^{i} / P_{e}) U'(W_{e}^{i}),
\]

so that \( P_{e} W_{e}^{i} = \rho \pi_{e}^{i} C_{0}^{i} \). But since \( C_{0}^{i} = W_{e}^{i} / (1 + \rho) \), initial consumption is the same for all individuals, so that \( P_{e} W_{e}^{i} = \rho \pi_{e}^{i} C_{0}^{i} \). Summing this over all individuals and dividing by \( I \),

\[
P_{e} = \rho \pi_{e}^{i} (C_{0}^{i} / W_{e}^{i}), \text{ where } W_{e}^{i} = W_{e}^{i} / I.
\]

Consequently, price \( P_{e} \) is determined by the "average belief" \( \pi_{e}^{i} = \sum_{i} \pi_{e}^{i} / I \) and is otherwise independent of the distribution of beliefs among individuals.
beliefs of the heterogeneous economy, the actual securities market will be called weakly information-efficient. That is, the consensus beliefs of a weakly information-efficient securities market "fully reflect" the pool of all available (public and private) information.

To show that in a weakly information-efficient securities market the full exchange of information is Pareto-efficient, consider again single-period economy (iii). For purposes of exposition, suppose the economy consists of three individuals \( i = 1, 2, 3 \) facing an uncertain two-state \( e = 1, 2 \) future. Recall the investment-sharing rule for \( A = 0 \)

\[
W_e^{i} = \left( \frac{\pi_e^i - \pi_e^0}{\pi_e^0} \right) W_e + W_e, \quad \text{where} \quad \pi_e^0 = \sum \pi_e^i / I (\text{all } e).
\]

Assume that for state \( e = 1 \), \( \pi_1^1 > \pi_1^2 > \pi_1^3 \) and \( \pi_1^2 = (\pi_1^1 + \pi_1^2 + \pi_1^3) / 3 \). Consequently, individual two makes no side bets and has consensus beliefs. That is, if individuals one and three were to revise their beliefs so that they agreed with individual two, security prices would remain unchanged. Consequently, security prices fully reflect the beliefs of individual two; security prices do not fully reflect the beliefs of the other individuals who, as a result, make side bets with each other. Assume also that the securities market is weakly information-efficient so that the consensus beliefs fully reflect the pool of all available information. Consequently, if the three individuals were to share all their private information, then individual one (the optimist) and individual three (the pessimist) would be converted to the beliefs of individual two (the realist). While the securities market is weakly information-efficient, it is not however strongly information-efficient since individuals have different beliefs.
As a result, the optimist and pessimist are making nonoptimal side bets, since if all private information were publicized they would not make them. Moreover, since security prices would remain unchanged, the realist would face the same opportunities as before. The exchange of private information therefore has social value since by converting the beliefs of the optimist and pessimist to the beliefs of the realist, everyone will be at least as well off as before and some better off. Even the realist can be better off if he charges for his private information. Consequently, if the securities market is approximately weakly information-efficient as most empirical work indicates, a strong argument on Pareto-efficient grounds (not merely on equity grounds) can be made for the exchange of private information. Even if some private information is not fully reflected in consensus beliefs (and hence in security prices), it generally may still be Pareto-efficient for insiders to disseminate their information before taking side bets, provided outsiders are willing to pay direct compensation for the information. If this were the case, informational inefficiencies would quickly be removed by the profitable dissemination of information.²⁹

While actual security markets may be weakly information-efficient, they unquestionably are not strongly information-efficient. How serious a social problem does this create? What price should the society be

²⁹In their book, Beat the Market, Thorp and Kassouf ask why they are disclosing their profitable trading methods since disclosure will quickly eliminate their effectiveness. They conclude that the expected profit from disclosure, i.e., sale of their book, exceeds the expected profit from future side bets.
willing to pay to eliminate this inefficiency? To begin to answer these questions, we need a barometer of the degree of inefficiency. As this analysis suggests, the volume of speculative trading provides a natural measure. Trading volume on major stock exchanges are quite visible numbers; the chief problem in an analysis of volume is to separate speculative from nonspeculative trading. Again economies (ii) and (iii), which permit strongly information-inefficient markets, provide a useful context to make this distinction.

First, consider single-period economy (iii) and assume individuals have identical endowments \( \bar{\bar{w}}_e^i = \bar{w}_e \) across all states \( e \). In this extreme case, the only motive for exchange is differences in beliefs among individuals so that all trading is speculative. From the investment-sharing rule, the volume of trading of individual \( i \) in claims to state \( e \) for \( A = 0 \) may be represented by

\[
\nu_e^i = |\bar{w}_e^i - \bar{\bar{w}}_e^i| = \bar{w}_e \left| \frac{\pi_e^i - \pi_e}{\pi_e} \right|
\]

The volume of trading for the market as a whole for claims to state \( e \) may be represented by

\[
\nu_e = \frac{\Sigma_i \nu_e^i}{2} = \frac{\bar{w}_e}{2} \Sigma_i \left| \frac{\pi_e^i - \pi_e}{\pi_e} \right|
\]

where the divisor 2 considers that a minimum of two individuals are required for each trade. More are required if some individuals function
as financial intermediaries. The total volume of trading (in all state-contingent claims\textsuperscript{30}) is represented by $v = \sum e v_e$. The total number of outstanding claims is $W = \sum e \sum i W_e^i$. Therefore, trading volume as a percentage of the total outstanding claims may be represented by

$$\frac{v}{W} = \frac{\sum e \sum i \left| \frac{\pi_i - \pi_e}{\pi_e} \right|}{\sum e W_e^i}.$$

As an example of the magnitude of this ratio, assume the economy consists as before of three individuals ($i = 1, 2, 3$) facing an uncertain two-state ($e = 1, 2$) future. Let $\pi^1_1 = 2/3$, $\pi^2_2 = 1/2$, and $\pi^3_1 = 1/3$; and $W_1 = 200$ (prosperity) and $W_2 = 100$ (recession). In this simple case, $v/W = 1/9$, all speculative trading. If the securities market is weakly information-efficient, then all trading is undesirable. Out of a total of 900 claims, individuals one and three have unfortunately exchanged 100 claims.

Second, consider single-period economy (ii), where individuals are heterogeneous in all respects except for taste parameter $B = 0$

\textsuperscript{30}Since actual securities represent standardized packages of state-contingent claims, the trading volume measured in terms of actual securities may be considerably less than the trading volume in terms of state-contingent claims. However, in the absence of exchange costs, information-inefficiency is monitored with greater accuracy in terms of implicit state-contingent claims. With exchange costs, the issue becomes more complex. Now the damage to society by the resources wasted in informationally inefficient trading also needs to be measured to assess the impact of information-inefficiency. In this case, the standardized packaging of state-contingent claims in actual securities may reduce the waste in resources.
(exponential utility). With this economy, we can compare trading volume generated by speculative and nonspeculative motives. Recall the investment-sharing rule

\[ \bar{w}_{ie} = A_i \ln(\pi_i^e/\pi_e) - (A_i/\phi) \sum_{e} P_e \ln(\pi_i^e/\pi_e) + (A_i/\phi) \ln(\rho_i/\rho) \]

\[ + (A_i/\phi)^{-1} [A_i \bar{w}_{ie}^i - A_i \bar{w}_{ie}^N] + (A_i/A_M) \bar{w}_{ie}^M \]

where \( A_i \equiv \sum A_i \), \( \rho \equiv \pi_i \rho_i \), and \( \pi_i \equiv \Pi_i \pi_i^e \). To focus attention on the trading volume created by heterogeneous beliefs (speculative) and heterogeneous patience (nonspeculative), assume that the ratio \( A_i/\bar{w}_{ie}^i \) is the same for all individuals and that endowments differ by a multiplicative constant across all states so that \( \bar{w}_{ie}^i / \bar{w}_{ie}^N = \bar{w}_{ie}^M / \bar{w}_{ie}^M \) for all \( i \) and \( e \). These two simplifying assumptions reduce the volume of trading of any individual to the absolute value of the first three terms of the sharing rule,

\[ v_{ie} = (A_i/\phi) \phi \ln(\pi_i^e/\pi_e) - \sum_{e} P_e \ln(\pi_i^e/\pi_e) + \ln(\rho_i/\rho) \] (all \( e \) and \( i \)).

To obtain a fair comparison, assume beliefs and patience have the same differential magnitude across individuals. In particular, assume the economy consists of two individuals \((i = 1, 2)\) facing an uncertain two-state \((e = 1, 2)\) future where \( \rho_1 = \pi_1 = \pi_2 \) and \( \rho_2 = \pi_1 = \pi_2 \). With these specifications, nonspeculative demand for individual \( i \) equals
\[(A_i/\phi) \ln(p_i/p)\]

and speculative demand for individual \(i\) for claims to state \(e\) equals \(^{31}\)

\[(A_i/\phi)[\phi \ln(p^1_e/p^1_e) - \sum_e \pi_e \ln(p^1_e/p^1_e)] = (A_i/\phi) \ln(p_i/p) (1 + 2F^F - 2F^e).\]

Consequently, speculative demand is almost \textit{double} nonspeculative demand. The total volume of trading (in all state-contingent claims)

\[v = 2A_i |\ln(p_i/p)|.\]

These specific results mildly suggest that speculative volume is apt to be high relative to nonspeculative volume. This tendency will be compounded in a multiperiod economy with trading in actual rather than state-contingent securities. \(^{32}\) The sharing-rule analysis of the previous paper indicated that it was relatively easy for securities to be packaged to diminish trading volume (measured now in units of actual securities) for nonspeculative reasons. Indeed in a strongly information-efficient securities market (where all individuals have the same taste parameter \(\varphi\)), individuals simply buy and hold the market portfolio and buy and hold a portfolio of short- and long-term default-free bonds, managing quite nicely by living off the income and never revising their portfolios. The

\(^{31}\text{Recall that } F^F = \sum_e \pi_e.\)

\(^{32}\text{Indeed, Rubinstein (1973b) shows that under quite general conditions if at the first trading date the securities market provides a rich enough variety of futures contracts, then }\forall\text{ trading at later dates is speculative.}\)
variety of securities required hardly taxes the market. However, in a securities market that is only weakly information-efficient or inefficient, the profusion of securities possibly required for exchange-efficiency without revision strains that imagination.\textsuperscript{33} If these securities are not initially available, they must be created later, and revision will be required to maintain exchange-efficiency. The joint effect of these considerations will be increased speculative trading spread out over time; nonspeculative motives will be satisfied by relatively stable portions of each individual's portfolio. Although the analysis here is highly simplified, it argues with some force for careful public scrutiny of the speed of information dissemination and the pervasiveness of its chief symptom, speculative trading.

\textsuperscript{33}Rose (1974) demonstrates that options issued on common stock have a surprising capability of repairing exchange-inefficiencies.
REFERENCES


