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ORDERING MARKETS AND THE CAPITAL STRUCTURES
OF FIRMS, WITH ILLUSTRATIONS

by

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I. INTRODUCTION AND SUMMARY

Consider a firm which is in the process of choosing one of the following capital structures:

A 100% common stock;
B 80% common stock, 20% riskless bonds;
C 75% common stock, 25% risky bonds;
D 70% common stock, 30% preferred stock;
E 80% common stock, 20% bonds and warrants;
F 70% common stock, 30% convertible bonds;
G 65% common stock, 10% preferred stock, 25% bonds.

What advice can the economist offer?

In a taxless world, given that a riskless asset is present in the market, the equivalence of A and B has been shown under increasingly general conditions [Modigliani and Miller (1958), Stiglitz (1969, 1974)]. This equivalence extends to all capital structures the firm might choose whenever the market as a whole is complete, i.e., when investors can, in effect, trade in Arrow-Debreu (1964, 1959) certificates; this would be the case when the number of linearly independent securities is the same as the number of visualized states of the world. But the benefits of a complete market can also, under specialized conditions, be attained by trade in as few as a single instrument (issued by a mutual fund holding all "regular" securities). The case when two instruments, one riskless and one risky, are sufficient to achieve the benefits of a complete market has received particular attention in the literature; essentially, it holds under any one of the following conditions: (i) homogeneous probability beliefs and utility belonging to the HARA-class, with the exponent the same for all
investors (unless utility is negative exponential); (ii) homogeneous beliefs, normally distributed returns, and utility functions (at a minimum) defined on the whole line; or (iii) homogeneous probability beliefs, continuous-time decision-making by investors, and a stochastic process of the Itô variety. The reason is that in these cases all investors mix risky assets the same way. Extensions incorporating two or more riskless assets of differing maturities have recently been made by Merton (1973) and by Rubinstein (1974).

The real world clearly lies somewhere between the highly restrictive conditions (i)-(iii) and a market which is sufficiently "rich" to be "complete" in the presence of significant diversity. In short, there is little disagreement that firms and investors find themselves making exchanges in a market in which incompleteness is a limiting factor. In such markets the capital structures of firms are not a matter of indifference, as many writers have noted [e.g., Borch (1968), Stiglitz (1972); see, also, Smith (1972) and Baron (1974)]. The purpose of this paper is to examine more fully the welfare implications of various capital structures in incomplete markets by comparisons of equilibria, under the assumption that investors are risk-averse but otherwise display essentially full heterogeneity with respect to probability beliefs and preferences. From a somewhat different vantage point, the purpose of the paper is to analyze the relationships between feasible allocations, investor welfare, and the equilibrium values of firms under investor heterogeneity.

In the following analysis, the notions of basic market equivalence and basic market dominance play a crucial role. Roughly, two markets are said to be (strongly) equivalent if they yield the same sets of (end-of-period) allocations in equilibrium for all beliefs and preferences; market $M'$ is said to dominate market $M''$ if every investor may be better off, and every
investor is never worse off, under $M'$ than he is under $M''$. The main result is that "market dominance" induces the same partial ordering of market structures as the "inclusion" relation applied to feasible (end-of-period) allocations. Thus, if some simple requirements involving linear independence and instrument dynamics are satisfied, capital structures A-C in the opening paragraph are dominated by capital structures D-G; this is true both with and without corporate income taxes. In the final analysis, then, the paper offers guidance as to (1) which capital structure changes by firms are of positive benefit to investors and (2) how overall market efficiency is most effectively improved.

The paper proceeds as follows. The basic model and its equilibrium properties are specified in Section II. Section III formalizes the relationship between market structures and feasible (end-of-period) allocations. Equivalent markets are taken up in Section IV, which also contains several illustrations and examines their relationship to the Modigliani-Miller type model. The possibilities of ordering incomplete markets are addressed in Section V, which also includes the main result (Theorem 2). It is also shown there that the equilibrium values of all firms may be the same in two markets even though each investor's expected utility in one market is higher (Corollary 5). Section VI contains a number of illustrations involving risky bonds, preferred stock, warrants, convertibles, mergers, and holding companies and comments on the welfare effects of these instruments and phenomena.

Corporate income taxes are introduced in Section VII; for riskless debt the debt-equity choice of firms is shown to either affect only their equilibrium values (and not investor welfare) or to have only redistributive effects if (to eliminate the money-pump) total corporate taxes are held constant; risky debt generally dominates riskless debt. Finally, Section VIII examines the
robustness of the partial ordering of markets obtained in this paper to alternative preference specifications, in dynamic settings, and across production decisions; it also examines the applicability of the ordering to options (puts and calls), summarizes the observed firm value/investor welfare relationships, and contains some concluding comments.

II. PRELIMINARIES

Basic Assumptions and Notation

We consider an ongoing economy in which the investors make decisions at the same (fixed) decision points. Our focus will be on an arbitrarily chosen single period. In line with the usual assumptions in this context, we postulate that all investment opportunities have (stochastically) constant returns to scale, are perfectly divisible, and can be sold short without restriction; transaction costs are ignored. Taxes are also left out initially, but corporate taxes are taken into account in Section VII. The following basic notation will be used.

\[ s \] a basic state of the world at the end of the period

\[ n \] the number of basic states, \( s \)

\[ S \] the set of basic states \( s \) (i.e., \( S = \{s_1, \ldots, s_n\} \))

\[ w_s \] total wealth available for allocation if state \( s \) occurs

\[ I \] the number of investors \( i \)

\[ y_i \] the resources of investor \( i \) available for investment (endogenous)

\[ w_i \] end-of-period wealth of investor \( i \) (random variable)

\[ d_{is} \] value assumed by \( w_i \) if state \( s \) occurs (investor \( i \)'s final allocation if state \( s \) occurs)
\( n_{is} \) the probability assessment of investor \( i \) that state \( s \) will occur

\( u_i(w_i) \) the utility function of investor \( i \)

\( J \) the number of investment vehicles \( j \) (stocks, bonds, notes, warrants, etc.) available in the market for the coming period

\( J \) the set of investment vehicles \( j \)

\( J_0 \) the number of continuing securities already present in investor portfolios \((J^0 \leq J)\) as a result of previous decisions

\( a_{js} \) proceeds per share of security \( j \) if state \( s \) occurs \((a_{js} \geq 0, \text{ all } s, a_{js} > 0, \text{ some } s, \text{ all } j)\)

\( R(J) \) rank of matrix \([a_{js}]\)

\( Z_j \) number of units of instrument \( j \) outstanding in current period

\( F \) the number of firms \( f \) at beginning of period

\( F \) the set of firms \( f \)

\( P_j \) price per share of security \( j \) at beginning of period

\( z_{ij} \) number of shares of security \( j \) demanded by investor \( i \) at beginning of period

\( z_{ij}^0 \) number of shares of security \( j \) owned by investor \( i \) at end of previous period

\( r_i \) amount of investor \( i \)'s resources not in form of regular security holdings (withdrawals, additions)

\( M(J) \) the market structure (the set of investment vehicles available in the market)
The investors' probability assessments are postulated to satisfy

\[(1) \quad \pi_{is} > 0, \quad s = 1, \ldots, n; \quad \sum_{s=1}^{n} \pi_{is} = 1, \quad \text{all } i.\]

Thus, while we permit probability assessments to differ among investors, none assigns probability 0 to any of the identified states \(1, \ldots, n\). Each investor is also assumed to be rational in the von Neumann-Morgenstern sense, with preferences that differentiate only with respect to his own end-of-period wealth; the utility functions \(u_i(w_i)\) are assumed to have the following properties:

\[(2) \quad u'_i(w_i) > 0, \quad u''_i(w_i) < 0 \quad \text{for } w_i \geq 0, \quad u'(0) = \infty, \quad i = 1, \ldots, I.\]

We assume that total wealth is positive (and finite) in each state, satisfying

\[(3) \quad 0 < W_s \leq W, \quad \text{all } s,\]

for some \(W\), and, for simplicity, that it is composed entirely of securities in positive supply, i.e.,

\[(4) \quad Z_j > 0, \quad \text{all } j,\]

some of which may be riskless.\(^5\) Thus we obtain

\[(5) \quad \sum_{j=1}^{J} a_{js} Z_j = W_s, \quad \text{all } s,\]

and

\[(6) \quad \sum_{i=1}^{I} d_{is} = W_s, \quad \text{all } s.\]

As noted, \(R(J)\) of the regular security return patterns are linearly independent; clearly, \(R(J) \leq \min(J,n)\). All securities are assumed to be issued by firms,\(^6,7\) i.e., \(F\) is a partition\(^8\) of \(J\). In the first part
of the paper we also assume that there are no "externalities" arising from capital structure decisions by firms; for any two market structures \( J' \) and \( J'' \) the net assets of all "firms" remain unchanged in each state, i.e.,

\[
b_{fs} = \sum_{j \in f} a_{js} Z_j' = \sum_{j \in f} a_{js} Z_j'', \quad f = 1, \ldots, \bar{F}, \quad s = 1, \ldots, n.
\]

This assumption will be relaxed in Section VII.

Summing the components of initial wealth to be invested we obtain

\[
y_i = \sum_{j=1}^{J_0} p_{ij} z_{ij} + r_i, \quad i = 1, \ldots, I;
\]

The numbers \( r_i \) which represent net new investment by individual \( i \) and are exogenous in the present model, are assumed to be consistent with positive prices in the budget constraint (see (12); when \( J = J_0 \), for example, this requires \( \sum_{i=1}^{I} r_i = 0 \) and with \(^9\)

\[
y_i > 0, \quad \text{all } i.
\]

The \( r_i \) thus capture consumption-induced changes in the amount to be re-invested, the proceeds from liquidated firms, "new" capital going into new firms, and the expansion of existing firms without formally modeling the underlying process. Little is lost by this short-cut since our analysis is essentially one of comparative statics and, as we shall see, capital structure decisions frequently need not cause decisions not pertaining to instrument holdings to be altered. End-of-period wealth \( w_i \) is given by

\[
w_i(s) = \sum_{j=1}^{J} a_{js} z_{ij}, \quad s = 1, \ldots, n; \quad i = 1, \ldots, I.
\]
The Market Structure

The object of this paper, as already noted, is to examine the allocation possibilities provided by, and the welfare implications of, various market structures, also referred to simply as markets, market situations, and market arrangements. For our purposes the focal point of a given market structure in these comparisons will be the set of instruments available for investment. Thus, even though in a larger sense the totality of the participants is an integral part of a complete description of a market arrangement $M$, it is generally sufficient to think of the distinctive feature of $M$ as simply the set of securities available for purchase; for these reasons we will in effect write $M(J) = J$.

Market structure $M(J)$ may be described as

\[
\max \sum_{i=1}^{n} \pi_i u_i \left( \sum_{j=1}^{J} \theta_j s_{ij} \right), \quad i = 1, \ldots, I,
\]

subject to the budget constraints

\[
\sum_{j=1}^{J} p_j z_{ij} = p_0 + r_i, \quad i = 1, \ldots, I,
\]

where the prices $p_1, \ldots, p_J$ are such that the optimal portfolios satisfy the market-clearing conditions

\[
\sum_{i=1}^{I} z_{ij} = z_j, \quad j = 1, \ldots, J.
\]

In view of assumptions (1)-(4) and (9), the equilibrium conditions for $M(J)$ consist of the following $I \times J + I + J$ equations in the $(I \times J + I + J)$ unknowns $z_{ij}, \lambda_i, p_j$ (where the $\lambda_i$ are Lagrange multipliers):
\( \sum_{s=1}^{n} \pi_{is} u_i \left( \sum_{j=1}^{J} a_{js} z_{ij} \right) a_{js} = \lambda_i \rho_j \), \( J = 1, \ldots, J \), \( i = 1, \ldots, I \);

\( \sum_{j=1}^{J} p_j z_{ij} = \sum_{j=1}^{J} p_j z_{ij}^0 + r_i \), \( i = 1, \ldots, I \);

\( \sum_{i=1}^{I} z_{ij} = z_j \), \( j = 1, \ldots, J \).

The assumptions also imply that a solution, not necessarily unique, exists \([\text{Hart (1974)}]\), and that the equilibrium holdings \( z_{ij}^* \), multipliers \( \lambda_i^* \), and prices \( p_j^* \) satisfy

\( d_{is}^* = \sum_{j=1}^{J} a_{js} z_{ij}^* > 0 \), \( s = 1, \ldots, n \), \( i = 1, \ldots, I \);

\( \lambda_i^* > 0 \), \( i = 1, \ldots, I \);

\( p_j^* > 0 \), \( j = 1, \ldots, J \).

However, in spite of (15), some of the \( z_{ij}^* \) will generally be negative, i.e., the equilibrium solution generally calls for short selling. Also, the equilibrium solution will in general be dependent on the previous holdings \( z_{ij}^0 \) (as well as the \( r_i \), the utility functions, and the probability assessments, of course). The equilibrium value of firm \( f \) is clearly

\( v_f^* = \sum_{j \in F} p_j^* z_j \), \( f = 1, \ldots, F \).

In order to meaningfully compare equilibria, we must generally make one more assumption. If we change from \( J' \) to \( J'' \), for example, some instruments already held by investors (as a part of their endowment) may cease to exist. How will these holdings be compensated? In some cases,
the solution appears straightforward. For example, suppose that a firm exchanges 10 shares of preferred stock for one (risky) bond and 10 warrants. Then, in the new market, each preferred share would presumably be entitled to receive $1/10$ of the equilibrium price of the bond plus the equilibrium price of the warrant. Other situations, such as the reverse of the previous case, leave the unit compensation to the bond and to the warrant unresolved unless the exchange ratio is further specified. The following basic assumption will be made.

**Price compensation assumption.** Suppose there is a change from instrument set $J' = J^0$ to $J'' = J^I$ at time 0; specifically, let subset $J''_{fc}$ replace subset $J'_{fc}$, $f = 1, \ldots, F$. Then instruments contained in investor endowments but ceasing to exist at time 0 ($j \in J''_{fc}$) are compensated at positive exchange ratios such that the resulting prices $p''_j$ (under $J''$) satisfy

\[
\sum_{j \in J''_{fc}} z''_j p''_j = \sum_{j \in J'_{fc}} z'_j p'_j, \quad f = 1, \ldots, F.
\]

Clearly, (19) merely requires that the total exchange by "even," i.e., that there is no effect on the firms' resources available for production activities. In certain cases we will make the following much stronger assumption.

**Price equivalence assumption.** Suppose there is a change from instrument set $J' = J^0$ to $J'' = J^I$ at time 0. Then instruments contained in investor endowments, but ceasing to exist at time 0, are compensated at price $p'_j$ per unit, where the $p'_j$ are equilibrium prices under $J'$. The price equivalence assumption cannot be meaningfully invoked unless (7) holds, i.e., unless capital structure "externalities" are absent. But even when (7) holds we will use the assumption sparingly; when we do employ it (19) will turn out to be satisfied.
III. FEASIBLE ALLOCATIONS

In this section, we characterize the sets of possible final allocations, \( \{d_{is}\} \), obtainable via different market arrangements. Recall that a market structure \( M(J) \) is any "full" set of instruments, i.e., any set of instruments \( J = \{j_1, \ldots, j_J\} \) satisfying (5). Thus, a market may have as few as one instrument. The set of final allocations, \( \{d_{is}\} \), obtainable via instrument set \( J \) will be denoted \( D(J) \), i.e.,

\[
D(J) \equiv \{d_{is} \mid \sum_{j=1}^{J} z_{ij} a_{js} = d_{is}, \quad i = 1, \ldots, I, \quad s = 1, \ldots, n, \quad \text{where} \quad \sum_{i=1}^{I} z_{ij} = Z_j, \quad j = 1, \ldots, J \}.
\]

The following definitions will prove useful.

**Definition 1.** Instrument set \( J'' \) is finer than instrument set \( J' \) \((J'' \preceq J')\) if both \( J' \) and \( J'' \) are full sets and if, for each \( j \in J' \), there exists a subset of instruments \( K_j \subseteq J'' \) such that \( f \) positive numbers \( d'_{kj} \)

\[
(20) \quad \sum_{k \in K_j} d'_{kj} a'_{ks} = a_{js}, \quad j = 1, \ldots, J', \quad s = 1, \ldots, n.
\]

**Definition 2.** \( J'' \) is strictly finer than \( J' \) \((J'' \preceq J')\) if \( J'' \preceq J' \) and \( J'' > J' \).

**Remark 1.** \( J' \preceq J \).

**Remark 2.** If (20) holds, then we must have \( Z_k' = d'_{kj}, \quad k \in K_j, \quad j = 1, \ldots, J' \).

We observe that "fineness" only induces a partial ordering on the set of \( J' \)'s. If a given firm has both stocks and bonds outstanding, the
resulting (market-wide) instrument set is strictly finer than if the same firm had only stock outstanding, ceteris paribus. The same would be true in comparing common and preferred stock financing with common stock alone but generally not in comparing common and preferred with common and bonds for any given firm; in the latter case, however, each set may possibly be finer than the other.

The following relationships between fineness and rank \( R \) of instrument sets are immediate.

**Remark 3.** \( J'' \Rightarrow J' \) implies \( R'' \geq R' \).

**Remark 4.** \( J'' \Rightarrow J' \) if \( J' = 1 \).

**Remark 5.** \( J'' \Rightarrow J' \) implies \( R'' \geq R' \).

Fineness and rank are intimately associated with the feasible allocations \( D \). The following significant relationships are readily verified.

**Lemma 1.** Let \( J'' \) and \( J' \) be full instrument sets such that \( J'' \Rightarrow J' \). Then any feasible allocation under \( J' \) is also feasible under \( J'' \), i.e., \( D(J') \subseteq D(J'') \).

**Corollary 1.** If \( J'' \Rightarrow J' \) and \( R'' = R' \), then \( D(J'') = D(J') \).

**Lemma 2.** \( D(J'') = D(J') \) if and only if \( R'' = R' \) and there exists some \( J \) of rank \( R' \) such that \( J \Rightarrow J' \) and \( J \Rightarrow J'' \).

**Remark 6.** Let \( D_n \) be the set of allocations attainable with an instrument set of (full) rank \( n \). Then \( D(J) \subseteq D_n \) for all \( J \).

**Remark 7.** Suppose \( R(J) < n \). Then \( D(J) \subseteq D_n \).
LEMMA 3. \( D(J') \subseteq D(J'') \) if and only if \( R'' > R' \) and there exists a \( J \) of rank \( R'' \) such that \( J \not\subseteq J'' \) and \( J \not\supset J' \).

LEMMA 4. Suppose \( D(J') \subseteq D(J'') \). Then there exists an instrument set \( J'' \) such that \( D(J''') = D(J') \), \( J'' \subseteq J''' \), and \( R(J''') = J'' \).

LEMMA 5. Consider \( J'' \) and \( J' \) where \( R'' \geq R' \). Then
\[
D(J') \cap D(J'') = D(J') \cup D(J'') \text{,}
\[
\text{if and only if there is no } J \text{ of rank } R'' \text{ such that } J \not\subseteq J' \text{, } J \not\supset J'' .
\]

Remark 8. Let \( J'' \) and \( J' \) be two instrument sets with ranks \( R' \) and \( R'' \), where \( 2 \leq R' \leq n - 1 \), respectively, such that there is no \( J \) of rank \( R' \) for which \( J \not\subseteq J' \), \( J \not\supset J'' \). Then there exist final allocations under \( J'' \) which are infeasible under \( J' \).

Lemma 5 may be illustrated by a simple example:

\[
\begin{array}{cccc}
\text{J'} & \text{J''} \\
\hline
\text{a}_{j_1} & \text{a}_{j_2} & \text{a}_{j_3} & \text{a}_{j_4} \\
\hline
j & 1 & 2 & 3 & Z_1 \\
1 & 1 & 2 & 2 & 10 \\
2 & 2 & 2 & 3 & 10 \\
\end{array}
\quad
\begin{array}{cccc}
\text{W}_s & 30 & 40 & 50 \\
\end{array}
\]

For instance, an allocation of \((15, 18, 24)\) for the three states is attainable with \( J' \) (buy 3 and 6 shares, respectively, of the two instruments) but not with \( J'' \). Conversely, an allocation of \((15, 18, 33)\) is feasible under \( J'' \) (buy 9 and 3 shares, respectively, of the two instru-
Lemma 3 tells us that a sure way to increase the final allocation possibilities is to make a finer and finer breakdown of existing instruments into an ever larger set of linearly independent securities. But as Remark 8 reminds us, sheer numbers are not the end-all: two instruments alone may be able to accomplish some of the things that a million (linearly independent) instruments cannot. The relation \( \leq \) among feasible allocation sets \( \{d_{ij}\} \) induces only a partial ordering of instrument sets and hence market structures.

IV. STRONGLY EQUIVALENT MARKETS

When is it reasonable to suggest that two market structures are equivalent? We adopt the following criteria.

**Strong basic market structure equivalence criterion.** Market structure \( M' \) is strongly equivalent to market structure \( M'' \) \( (M' \sim M'') \) if, for all collections \( \{\pi_{is}, u_i(w_i)\} \) satisfying (1) and (2), every equilibrium final allocation under \( M' \) is also an equilibrium final allocation under \( M'' \), and conversely.

**Weak basic market structure equivalence criterion.** Market structure \( M' \) is weakly equivalent to market structure \( M'' \) \( (M' \text{ equiv } M'') \) if, in moving from equilibrium under \( M' \) to equilibrium under \( M'' \) (and conversely), the change in expected utility of investor \( i \), \( \Delta u_i \), is, for all collections \( \{\pi_{is}, u_i(w_i)\} \) satisfying (1) and (2), either (i) zero for all investors or (ii) positive for some investors and negative for some (other) investor(s).

**Remark 9.** \( M \sim M' \).

Since no one equilibrium (as we will also see in Remark 11) is "superior" to any other in a given market, that is, no equilibrium leaves some investors better off unless some others are left worse off compared to their welfare in
any other equilibrium in that market, strong equivalence implies weak equivalence. But the converse does not hold (see, e.g., Theorems 4 and 5). In this section we focus our attention on strong equivalence.

By our definition, it is not sufficient for two market arrangements to have the same equilibrium for only a narrow class of preferences and probability assessments but not for other classes.\footnote{12} To be strongly equivalent, the two market arrangements must yield identical final allocations for any combination of preferences and probability assessments held by investors which satisfy the rather broad assumptions (1) and (2)—this is the reason for calling the criterion "basic." The reasons for choosing the class of probability assessments and preferences satisfying (1) and (2) (as opposed to a narrower class) will be elaborated in Section VIII.

Given our criterion, then, the question that arises is: under what conditions are two market systems, \( M' \) and \( M'' \), strongly equivalent? We first observe that \( M' \) and \( M'' \) yield the same equilibrium allocations [see (12)-(14)] if and only if

\[
\sum_{j=1}^{I'} a'_{ij} z'_{ij} = \sum_{j=1}^{I''} a''_{ij} z''_{ij} = d_{is}, \quad i = 1, \ldots, I, \quad s = 1, \ldots, n.
\]

A necessary condition for an equilibrium under \( J' \) to be an equilibrium under \( J'' \), and conversely, is

\[
D(J') = D(J'').
\]

To see this, suppose there are allocations \( \{d'_{is}\} \notin D(J'') \) [this requires \( R(J'') < n \)]. Then we can always find probabilities \( \pi_{is} \) and preferences \( u_i(\nu_i) \) satisfying (1) and (2) such that some \( \{d''_{is}\} \notin D(J'') \) is an equilibrium under \( J' \) (for further details, see proof of Lemma 8 in the Appendix).
Turning to the sufficiency of (22), we observe that when (22) holds, $R' = R''$ by Lemma 2, and there exists a market structure $M^*$ of rank $R'$ such that $M^* \not< M'$, $M^* \not< M''$. Then, invoking the price equivalence assumption, an equilibrium final allocation(s) under $M^*$ is also an equilibrium final allocation under $M'$ (and $M''$) if we set

$$
\begin{align*}
(23) & \quad \sum_{j=1}^{J'} z'_{ij} a_{js} = \sum_{j=1}^{J''} z''_{ij} a_{js} = \sum_{j=1}^{J^*} z^*_{ij} a_{js}, \quad \text{all } i, s; \\
(24) & \quad \lambda^*_i = \lambda^*_{i}, \quad \lambda^*_i \quad \text{i} = 1, \ldots, I; \\
(25) & \quad p^*_j = \sum_{k \in K_j^*} d^*_j p^*_{k}, \quad j = 1, \ldots, J'; \\
(26) & \quad p''_j = \sum_{k \in K_j^*} d''_{j} p^*_{k}, \quad j = 1, \ldots, J'';
\end{align*}
$$

where the numbers $d^*_{k}$ are given by (20), which is readily verified by reference to (12)-(14) and Remark 2; the existence of \{z'_{ij}\} (and \{z''_{ij}\}) satisfying (23) (and conversely) follows from the fact that $D(J') = D(J'') = D(J^*)$. But

$$
(27) \quad R^* = R' = R'', \quad M^* \not< M', \quad M^* \not< M'',
$$

also implies that every solution to $M'$ and every solution to $M''$ must also be a solution to $M^*$, because, for each $i$, every row in (14) under $M^*$ is obtainable as a linear combination of rows in (14) under $M'$ (and $M''$); their prices $p^*_{k}$ are such that (24)-(26) and (12) hold. Thus we can state

**Theorem 1.** $M'' \sim M'$ if and only if $D'' = D'$, i.e., if and only if $R'' = R'$ and there exist some $J$ such that $J \not< J''$, $J \not< J'$, and $R(J) = R'$, provided that the price equivalence assumption holds.

Equations (25) and (26) give...
COROLLARY 2. If $M' \sim M''$, then $V'_f = V''_f$, $f = 1, \ldots, F$, i.e., the value of each firm (for a given final equilibrium allocation) is the same in both markets.

We now illustrate the theorem with some examples (in which the price equivalence assumption is assumed to hold). The following definition will be useful.

**Definition 3.** Instrument $j$ is reproducible within instrument set $J$ if there exist numbers $c_k$ such that

$$a_{js} = \sum_{k \in J, k \neq j} c_k a_{ks}, \quad s = 1, \ldots, n,$$

i.e., if it is obtainable as a linear combination of other instruments in $J$. If an instrument is not reproducible, it will be termed nonreproducible or unique.

**Remark 10.** If $M' \sim M''$, then $M'$ and $M''$ have the same number of unique instruments (but the converse is not true).

**Illustration 1.** Consider two market structures $M'$ and $M''$ containing only common stock and riskless bonds. Then $M' \sim M''$ (as long as both $M'$ and $M''$ contain some bonds or a riskless asset can be created by investors).

**Illustration 2.** Consider market structure $M'$ in which firm $A$ has common stock, (risky and/or riskless) bonds, and risky preferred stock outstanding, the latter being reproducible. Let $M''$ be the same as $M'$ except that the preferred stock of firm $A$ has been exchanged for common stock (or new or existing bonds, or common stock and riskless bonds if such bonds already exist). Then $M'' \sim M'$. 
Illustration 3. Consider market structure \( M' \) in which firm \( A \) owns \( x_B \% \) of the common stock of firm \( B \) and \( x_C \% \) of that of firm \( C \), where \( 0 < x_B < 100 \), \( 0 \leq x_C < 100 \). Let \( M'' \) be the same as \( M' \) except that firm \( A \) has decreased its holdings in firm \( B \) to \( y_B \% \) and increased its holdings in firm \( C \) to \( y_C \% \), where \( 0 \leq y_B < x_B \), \( x_C < y_C < 100 \). Then \( M'' \sim M' \).

Illustration 4. Consider market structure \( M' \) in which there are riskless instruments but no mutual funds. Let \( M'' \) be the same as \( M' \) except for the presence of mutual funds, each of which owns 100% of at most one risky instrument and issues only common shares and possibly riskless bonds. Then \( M'' \sim M' \).

The preceding examples should be noted as much for what they do not say as for what they do say. Different capital structures involving common stock and risky bonds and/or other risky instruments generally do not give rise to (strongly) equivalent markets, neither does the acquisition by one firm of 100% of the shares of another firm (even if it was previously 90%-owned) or a partial or full spinoff of the shares of a previously wholly owned company.

As the reader has undoubtedly discovered, Illustration 1, in particular, has an intimate relationship to Proposition I of Modigliani and Miller (1958): the value of the firm is independent of its capital structure. The MM analysis, and the subsequent extension by Stiglitz (1969, 1974), was limited to common stock and debt instruments; the crucial role of the risklessness of the debt for the proposition to hold was especially stressed by Stiglitz. Stiglitz also noted certain extensions incorporating risky debt (1969), but these did not go as far as Theorem 1, and other instruments and situations, as depicted in Illustrations 2, 3 and 4, were not explicitly considered.
But there is a much more fundamental relationship between MM Proposition I and Theorem 1. The proposition deals with the (equilibrium) market values of firms, Theorem 1 with equilibrium final allocations, $d_{is}$, to investors. Do these two different concepts lead to the same result, i.e., do strongly equivalent markets imply the same sets of equilibrium market values of firms, and are all markets consistent with a set of firm equilibrium prices strongly equivalent? The answer to the first part is in the affirmative, as shown by Corollary 2. But the answer to the second part of our question is negative, as we will show in the next section (Corollary 5). The remainder of our comparisons with the MM results will therefore be postponed to that section (and to Section VIII).

V. ORDERING MARKET STRUCTURES

The essence of Theorem 1 is that two markets are strongly equivalent, based on a definition involving expected utility via equilibrium allocations, if and only if the feasible (final) allocations they make possible are identical. We now examine weak equivalence and establish a criterion for ranking non-equivalent markets which is based more directly on welfare and show that it gives the same partial ordering of market structures as the relation "$c" on the set of feasible allocations $D$.

The basic market structure dominance criterion (BMSTC). Market structure $M'$ is dominated by market structure $M''$ ($M''$ dom $M'$) if, in moving from every equilibrium under $M'$ to every (different) equilibrium under $M''$, the change in expected utility of investor $i$, $\Delta u_i$, is such that

\begin{equation}
\Delta u_i > 0 \text{ for some } i, \text{ for every collection } \{\pi_{is}, u_i(w_i)\}
\end{equation}

satisfying (1) and (2);
(29) \( \Delta u_i > 0 \) for all \( i \), for some collections \( \{ \pi_{is}, u_i(w_i) \} \)
satisfying (1) and (2).

Paraphrased, the criterion says that if we compare equilibrium under \( M' \)
with equilibrium under \( M'' \) and find that they differ, then \( M'' \) dom \( M' \) if,
under all conceivable tastes and beliefs consistent with our basic assumptions,
at least some investors, and sometimes all investors, are better off under \( M'' \).

We observe that if \( M'' \) dom \( M' \), then we cannot also have \( M' \) dom \( M'' \)
(because (28) would then be violated for some collection \( \{ \pi_{is}, u_i(w_i) \} \)).

Neither can we have \( M \) dom \( M' \).

We begin by establishing a number of subsidiary results.

**Lemma 6.** Let \( u(x) \) be strictly concave and let \( J \) be an instrument
set of rank \( R(J) = J \). Then the functions \( f_i(z) \) given by

\[
(30) \quad f_i(z_1, \ldots, z_J) = \sum_{s=1}^{n} \pi_{is} u_i \left( \sum_{j=1}^{J} z_j a_{js} \right), \quad i = 1, \ldots, I,
\]

are strictly concave in \( z \).

A proof may be found in Hakansson (1970). The following corollary
follows immediately from the fact that every hyperplane lies above a
strictly concave function.

**Corollary 3.** Let \( z'' \neq z' \) and let

\[
(31) \quad \Delta j = z''_j - z'_j, \quad j = 1, \ldots, J,
\]

\( \Delta u = f(z'') - f(z') \),

\[
(33) \quad q_j = \frac{\partial f(z_1, \ldots, z_J)}{\partial z_j} \bigg|_{z_j = z''_j}, \quad j = 1, \ldots, J,
\]
where \( f \) is any function given by (30) and rank \( R(J) = \bar{J} \). Then there exist numbers \( \Delta u_j \) such that

\[
\sum_{j=1}^{J} \Delta u_j = \Delta u,
\]

(34)

\[
\Delta u_j > \Delta_j q_j, \quad j = 1, \ldots, \bar{J}.
\]

(35)

COROLLARY 4. Let \( R(J) < \bar{J} \). Then \( f(z) \) given by (30) is such that for any pair \( z' \neq z'' \), either

\[
\frac{f(z' + z'')}{2} > \frac{1}{2} f(z') + \frac{1}{2} f(z''),
\]

(36)

or

\[
f\left(\frac{z' + z''}{2}\right) = f(z') = f(z'').
\]

(37)

Definition 4. When (37) holds, \( z' \) and \( z'' \) will be said to be equivalent allocations; when (36) holds (which requires \( z' \neq z'' \)), \( z' \) and \( z'' \) will be said to be non-equivalent allocations.

Definition 5. Let \( J_1, J_2, \ldots \) be a sequence of instrument sets for which

\[
D(J_1) = D(J_2) = \ldots.
\]

Then the minimal instrument set of this sequence, \( J_{\text{min}} \), is a member for which \( R(J_k) = \bar{J}_k \).

We now give an important intermediate result.

LEMMA 7. Let \( J' \) and \( J'' \) be instrument sets such that

\[
D(J') \leq D(J'')
\]

(38)
and let the price compensation assumption hold. If the equilibrium allocations \( \{z'_{ij}\} \) and \( \{z''_{ij}\} \), respectively, are non-equivalent for some \( i \), then

\[
\Delta u_i > 0 \quad \text{for some } i,
\]

where

\[
\Delta u_i = \sum_{s=1}^{n} \pi_{is} \left[ u_i \left( \sum_{j=1}^{J''} a''_{ij} z''_{ij} \right) - u_i \left( \sum_{j=1}^{J'} a'_{ij} z'_{ij} \right) \right], \quad i = 1, \ldots, I,
\]

i.e., at least some investors are better off under \( J'' \).

The proof is given in the Appendix.

Observing that \( D(J) \subseteq D(J') \), Lemma 7 enables us to make

Remark 11. If market structure \( M(J) \) has more than one equilibrium final allocation \( \{d^x_{is}\} \), then each equilibrium makes some investors better off and some worse off (measured in welfare) compared to every other (non-equivalent) equilibrium final allocation.

Recalling the definitions of strong and weak market equivalence, we also obtain

Remark 12. If \( M'' \sim M' \), then \( M'' \) equiv \( M' \), i.e., every equilibrium under \( M'' \) which makes some investors better off than they are in equilibrium under \( M' \) also makes some (other) investors worse off.

Lemma 7 insures that all investors are never worse off under \( J'' \) than they are under \( J' \) --as long as \( D(J') \subseteq D(J'') \). On the other hand, they may all be better off under \( J'' \), as we now show.

LEMM 8. Let \( J' \) and \( J'' \) be full instrument sets such that

\[
D(J') \subset D(J'')
\]
and let the price compensation assumption hold. Then there exist collections \( \{\pi_{is}, u_i(w_i)\} \) satisfying (1) and (2) such that

\[
\Delta u_i > 0, \quad i = 1, \ldots, I,
\]

where \( \Delta u_i \) is given by (40), i.e., all investors may be better off under \( J'' \) than under \( J' \).

The proof is given in the Appendix; the two equilibria constructed in the proof are such that (A.18) and (A.19) hold, i.e., such that the value of each firm remains unchanged as we move from one market structure to the other. This leads to the following important observation.

**COROLLARY 5.** Let \( M' \) and \( M'' \) be markets such that \( \mathcal{D}(J') \subset \mathcal{D}(J'') \), let the price compensation assumption hold, and let \( E' \) and \( E'' \), respectively, denote equilibria in the two markets. Then there exist collections \( \{\pi_{is}, u_i(w_i)\} \) satisfying (1) and (2) such that (a) the value of each firm in \( E' \) is the same as in \( E'' \), and (b) every investor is better off under \( E'' \) than under \( E' \), i.e., (42) holds.

The essence of Corollary 5 is that it points out that the relationship between the value of the firm and stockholder welfare is not as direct as one might like and as, from all appearances, is generally believed. We return to this question in Section VIII.

**LEMMA 9.** Let \( J' \) and \( J'' \) be full instrument sets with ranks \( R' \) and \( R'' \), respectively, where \( R' \leq R'' \), such that there is no \( J \) of rank \( R'' \) for which \( J \not\subset J' \), \( J \not\subset J'' \), and let the price compensation assumption hold. Then there exist collections \( \{\pi_{is}, u_i(w)\} \) satisfying (1) and (2) such that

\[
\Delta u_i > 0, \quad i = 1, \ldots, I,
\]
and other collections for which

\[(43) \quad \Delta u_i < 0, \quad i = 1, \ldots, I, \quad \]

where \( \Delta u_i \) is given by \((40)\).

The proof is similar to that of Lemma 8 and is therefore omitted.

Lemmas 7, 8, and 9 now yield

**THEOREM 2.** Under the price compensation assumption, \( M'' \) dom \( M' \), i.e., if and only if \( D' \subseteq D'' \), i.e., if and only if \( R'' > R' \)

and there exist some \( J \) of rank \( R \) such that

\[(45) \quad R = R'', \quad J \not\subseteq J', \quad J \not\subseteq J''. \]

**THEOREM 3.** Under the price compensation assumption, \( M'' \) equiv \( M' \) if and only if \( D' = D'' \), i.e., if and only if \( R'' = R' \) and there exists some \( J \) of rank \( R \) such that \( R = R', \quad J \not\subseteq J', \quad J \not\subseteq J'' \).

**Remark 13.** If \( M'' \) dom \( M' \) and \( M''' \) dom \( M'' \), then \( M'''' \) dom \( M' \).

In words, Theorem 2 says that if \((44)\) and \((45)\) hold, then every equilibrium in \( M'' \) always leaves at least some investors better off than they would be under any equilibrium in \( M' \); furthermore, (there are probability beliefs \( \pi \) and preferences \( u_i(w_i) \) satisfying \((1)\) and \((2)\) such that,) for some equilibrium pairs, all investors are better off under \( M'' \) than under \( M' \).

And \((44)\) and \((45)\) are not only sufficient but necessary conditions for the preceding to hold.
VI. APPLICATIONS

We now illustrate Theorem 2 with some examples.

Illustration 5. Consider market structure $M'$ in which firm A has common stock and convertible or risky nonconvertible bonds, both unique, outstanding and in which riskless bonds are present. Let market structure $M''$ be the same as $M'$ except that the bonds of firm A have been exchanged for common stock alone or common stock and riskless bonds. Then $M'$ dom $M''$.

Illustration 6. Consider market structure $M'$ in which firm A has risky nonconvertible bonds outstanding. Let $M''$ be the same as $M'$ except that the risky bonds of firm A have been divided into two issues, which are both risky and unique and one of which is subordinated to the other. Then $M''$ dom $M'$. (If the senior issue were to be riskless and other riskless bonds are available in $M'$ (and hence $M''$), then we would have $M'$ equiv $M''$ or $M'' \sim M'$.)

Illustration 7. Consider market structure $M'$ in which firm A has numerous (nonconvertible) bond issues, risky convertible or nonconvertible preferred stock, and common stock outstanding, where the two stock issues are unique. Let $M''$ be the same as $M'$ except that the preferred stock has been exchanged for common stock. Then $M'$ dom $M''$.

Illustration 8. Consider market structure $M'$ in which the common stock of both firm A and of firm B is unique. Let $M''$ be the same as $M'$ except that (100% of) the common stock of firm B has been exchanged for common stock in firm A (in a merger without "synergy"). Then $M'$ dom $M''$. (If less than 100% of the common stock of firm B were acquired by firm A, then $M'$ equiv $M''$ or $M' \sim M''$.)

Illustration 9. Consider market structure $M'$, which contains riskless
bonds, in which the common shares of firms A and B are unique, and in which holding company H, among other investments, owns 100% of the common stock of firm A. Let market structure M" be the same as M' except that holding company H has increased its ownership of firm B common to 100% in an exchange involving its own shares and riskless bonds. Then M' dom M".

At this point several comments are in order. First, as Illustrations 8 and 9 show, 100% mergers are generally harmful to economic welfare, at least in the absence of synergy. Conversely, spinoffs of 100%-owned firms and the separate incorporation of "unique" divisions, followed by the selling of at least some shares to the public, are generally beneficial to economic welfare. These results are at odds with most conclusions in the literature, which hold that nonsynergistic mergers lead to "equivalent markets" [e.g., Mossin (1966), Mueller (1969), Levy and Sarnat (1971), Rubinstein (1974)]. But the reason for the discrepancy is very simple: in the cited studies, all investors hold all productive assets in the same ratios, a result which, as noted in Section I, obtains when homogeneous probability beliefs and certain restricted preference classes are assumed on the part of investors. When this assumption is relaxed, the 100% nonsynergistic merger may make every investor worse off—even when the value of the merged firm equals the sum of the values of the old firms.

To attain a welfare improvement, in the sense of the Basic Market Structure Dominance Criterion, a given firm must increase the number of its nonreproducible instruments. But this step, as we have seen, is not sufficient: (41) must also hold. Thus, if a firm can attain an increase in the number of unique instruments it has outstanding by, for example, issuing common stock and risky bonds in exchange for preferred stock, then the result will be one of four possibilities:
(a) all investors will be worse off,
(b) some investors will be worse off and some better off,
(c) there is no change, or
(d) all investors will be better off.

Only if the feasible allocations truly expand can possibility (a) be ruled out (see Lemma 9).

In recent years, instruments possessing a conversion privilege have become increasingly popular among corporations. In particular, bonds and preferred stock are issued with the understanding that they are exchangeable into common stock at a prespecified rate. Unfortunately, the welfare implications of the conversion privilege are much more ambiguous than they are for debt-equity decisions and mergers, for example, as the following example shows (see, also, Illustrations 5 and 7). The reason is that the addition of a conversion feature does not generally improve the fineness of existing instruments if bonds are already risky; and the rank of the market may either decrease, remain the same, or increase.

Illustration 10. Consider market structure $M'$ in which firm $A$ has common stock and risky bonds outstanding. Let $M''$ be the same as $M'$ except that the bonds of firm $A$ have a conversion privilege. Then one of the four possibilities in (46) will hold in comparing $M'$ to $M''$; the rank of $M''$ will be smaller than, equal to, or larger than that of $M'$ without discernible pattern.

On the other hand, the presence of warrants can be seen to be frequently beneficial to investor welfare. Consider the case in which warrants are issued in conjunction with bonds in lieu of bonds alone, as is often the case in practice.
Illustration 11. Consider market structure $M'$ in which firm A has common stock and riskless bonds outstanding. Let $M''$ be the same as $M'$ except that the bonds of firm have been replaced by bonds and warrants, where the warrants are unique. Then $M''$ dom $M'$.

We now examine the extent to which the preceding results are affected when corporate income taxes are introduced.

VII. RECOGNIZING CORPORATE INCOME TAXES

The great majority of studies dealing with the firm's capital structure decision consider only the taxless case—notable exceptions are Modigliani and Miller (1963) and Baumol and Malkiel (1967). The reasons why corporate income taxes matter, of course, is that interest paid on debt is a deductible expense in virtually all countries, while dividends are viewed as distributions of income. Thus, the corporate tax bill can be reduced by increasing "leverage," ceteris paribus.

The presence of corporate taxes divides our analysis naturally into two cases. Suppose we consider only capital structure (market structure) changes which do not alter the quantities of debt securities or, more precisely, interest-bearing securities outstanding. Then $b_{f s}$ in (7) may be interpreted as the assets of firm $f$ in state $s$ after taxes; it is clearly unaffected by any capital structure change not altering the quantity and nature of bonds outstanding. Thus, in this case, all of our analysis in Sections IV and V goes through for comparisons not involving changes in the quantity of bonds. As a result, Illustrations 2 (if preferred is exchanged for common only), 3, 4, 6, 7, 8, 9 (if the exchange involves only common stock), and 10 are valid even in the presence of corporate income taxes; only Illustrations 1, 5, and 11 (and "parts" of Illustrations 2 and 9) are affected.
by that tax. Thus, for purposes of examining the impact of corporate income
taxes, it is sufficient to assume that all instruments are either common
stock or bonds.

In order to accomplish a meaningful analysis of the corporate income tax,
we must rule out gratuitous "welfare" improvements of the money-pump variety:
a decrease in total corporate income taxes resulting from the use of increased
leverage must presumably be compensated for elsewhere. Since our model is
limited to financial markets, the simplest way to cope with this problem is to
assume that the corporate tax structure is automatically adjusted so as to
keep total corporate income taxes, in each state of the world, independent of
the capital structures employed by firms.

For this purpose, we assume a linear corporate tax structure in which
the tax payments of firm $f$ are

$$
(47) \quad t \left( X_f(s) - i_f \right) + e_T, \quad f = 1, \ldots, F,
$$

where $t$ is the corporate tax rate, $X_f(s)$ is taxable income of firm $f$
in state $s$ before interest, $i_f$ is firm $f$'s interest deduction, and
$e_T$ is a constant depending on total interest deductions. If total corporate
taxes in state $s$ are

$$
(48) \quad \sum_{f=1}^{F} t X_f(s) = A,
$$

where $A$ is a (large positive) number, then choosing $e_T$ such that

$$
(49) \quad e_T = \frac{t}{F} \sum_{f=1}^{F} i_f \quad - A
$$
preserves equality between (47), summed over all firms, and (48) for every
capital structure and every state $s$. The reader has undoubtedly noted
that (47) has a strong similarity to the present corporate tax formula in
the United States.\textsuperscript{16}

Consider market structure $M'$ (involving common stock and riskless bonds). Let $M''$ be the same as $M'$ except that firm 1 has increased its (riskless) debt, i.e., $i''_1 > i'_1$, $i''_f = i'_f$, $f = 2, \ldots, F$. Thus, by (49),

\begin{equation}
(50) \quad e''_T > e'_T, \quad e''_T - e'_T < t(i''_1 - i'_1).
\end{equation}

For each firm we now obtain, for each state $s$,

\begin{equation}
(51) \quad b''_{fs} - b'_{fs} \begin{cases} 
   e'_T - e''_T + t(i''_1 - i'_1) > 0, & \text{firm 1,} \\
   e'_T - e''_T < 0, & \text{firms 2, \ldots, F.}
\end{cases}
\end{equation}

Thus, (7) does not hold, i.e., (51) gives

\begin{equation}
(52) \quad \sum_{j \in f, s} a''_{js} Z''_j > \sum_{j \in f, s} a'_{js} Z'_j, \quad f = 1, \quad f = 2, \ldots, F \} \quad s = 1, \ldots, n,
\end{equation}

even though, in view of (48) and (49)

\begin{equation}
(53) \quad \sum_{f=1}^{F} \sum_{j \in f, s} a''_{js} Z''_j = \sum_{f=1}^{F} \sum_{j \in f, s} a'_{js} Z'_j = W_s, \quad s = 1, \ldots, n.
\end{equation}

Since $a''_{js} - a'_{js}$ is independent of $s$ [see (51)] and (53) holds, $M'$ and $M''$ offer the same allocation possibilities, i.e.,

\begin{equation}
(54) \quad D(J') = D(J'').
\end{equation}

But in view of (52), (54) no longer implies that $M' \sim M''$ since the price equivalence assumption cannot be invoked under "externalities." However, strong market equivalence still obtains under certain conditions. To show this, let $\{d'_{is}\}$ be a final equilibrium allocation under $M'$ with prices $P'_j$ and multipliers $\lambda'_i$. Then there exist prices $P''_j$ such that (13)-(14) are satisfied for final allocation $\{d'_{is}\}$ and $\lambda'_i$; in particular, $P''_j$ given by
(55) \[ p''_j = p_j + (a''_j - a'_j)k' \] if \( j \) is a common stock,
and
\[ \sum_{j=1}^{J''} p''_j z''_j = \sum_{j=1}^{J'} p'_j z'_j, \]
where \( k' > 0 \) is the equilibrium price of $1 in each state under \( M' \).

But in view of (51) and (55), (12) will not be satisfied for all collections \( \{\pi_{Is}, u_i(w_i)\} \) unless the initial holdings, \( z_{ij}^0 \), of risky instruments are held in the same ratios by all investors, i.e.,
\[ \frac{z_{ik}^0}{z_{ij}^0} = \frac{z_{j'}^0}{z_{ij}^0}, \quad i = 2, \ldots, I, \]
all risky instruments \( k \) and \( j = 1, \ldots, J^G = j' \).

The preceding result may be summarized as

THEOREM 4. Let \( M' \) consist of common stock and riskless bonds and assume a tax structure given by (47) and (49). Let \( M'' \) be the same as \( M' \) but with different debt-equity ratios. Then \( M' \sim M'' \) if and only if initial endowments satisfy (57).

In view of (55) and (56) we obtain

COROLLARY 6. Under the assumptions of Theorem 3 and (57), a firm can increase its value by increasing its debt-equity ratio but only at the expense of other firms' values; stockholder welfare is unaffected.

Theorem 4 and Corollary 6 are based on the strong basic market structure equivalence criterion. By (54) and Theorem 3 we can also state the following weak market equivalence result.

THEOREM 5. Let \( M' \) consist of common stock and riskless bonds, assume a tax structure given by (47) and (49), and let the price compensation assumption hold. Let \( M'' \) be the same as \( M' \) but with different
debt-equity ratios. Then $M' \equiv M''$, i.e., some investors are better off in equilibrium (measured in expected utility) under $M''$ only if other investors are worse off, i.e., corporate income taxes have only redistributive effects.

As in the case of (57), Corollary 2 no longer holds:

**Remark 14.** Under corporate taxes of form (47) and (49), the value of the firm is not independent of the debt-equity ratio even for riskless debt.

In comparing risky debt to riskless debt in the presence of corporate taxes, Theorem 2 applies directly. Thus we conclude this section with

**Illustration 12.** Consider market system $M'$, with a tax structure given by (47) and (49), in which firm $A$ has common stock and risky (with respect to principal only) bonds outstanding, both unique, and in which riskless bonds are present. Let $M''$ be the same as $M'$ except that the bonds of firm $A$ are riskless. Then $M' \succ M''$.

**VIII. DISCUSSION**

Having presented some of the implications of the market equivalence and market dominance criteria proposed in this paper, we now return to a discussion of the main assumptions used in our basic model and their limitations, particularly in an intertemporal setting. We also examine briefly the relationship of the criteria to production decisions, review further the firm value/capital structure relation, note the applicability of the criteria to options, and comment on their apparent real-world message.

**Robustness of Assumptions**

The Basic Market Structure Dominance Criterion (BMSDC) presumes essentially complete heterogeneity among investors, assuming only that they assign positive probability to each state, prefer more wealth to less, and are risk-averse with respect to wealth. While the first assumption seems
basic in any general theory of equilibrium, it is noteworthy only because most studies in finance have assumed homogeneous beliefs. The second and third assumptions are highly robust in that they hold for the induced utility of wealth functions in each period in general multiperiod models of risk-averse consumers facing the linear technology generally observed in financial markets [e.g., Fama (1970)].

A question that naturally arises is whether further assumptions might reasonably be imposed on the utility functions that would lead to a more "complete" ordering of market structures than the partial ordering induced by BMSDC. As Mossin (1968) and Merton (1971) have noted, only the HARA-class (hyperbolic absolute risk aversion)\(^{17}\) leads to predictable induced utility of wealth functions—and even members of this class may fail to do so if nonnegative consumption is to be observed [Makinson (1971)]. Thus, even in an economy with stationary risk attitudes with respect to consumption, the applicable induced utility of wealth functions are highly unpredictable and are likely to vary considerably from period to period. However, one property of investor preferences which is widely believed to prevail is that of decreasing absolute risk aversion (i.e., \(-u''(w)/u'_1(w)\) is decreasing in \(w\)) [Arrow (1963), Pratt (1964)], at least based on single-period reasoning. Its applicability to induced utility of wealth functions in multiperiod consumption models has been examined by Neave (1971). And this property, added to the other two, does give rise to a more "complete" (partial) ordering of probability distributions (third-order stochastic dominance) than risk aversion and monotonicity (second-order stochastic dominance) alone [Bawa (1974)] under homogeneous probability beliefs. But adding decreasing absolute risk aversion to BMSDC induces the same ordering of markets as BMSDC alone. This is due to the offsetting impact of the heterogeneity of the probability assessments. There is some question whether
meaningful orderings "between" that developed in this paper and that obtained under homogenous beliefs and the highly restrictive "constant mix of risky assets" assumption, in which all market structures are equivalent in the presence of a riskless asset [Rubinstein (1974)], are available.

While the ordering of markets induced by BMSDC, as we have seen, is highly robust for fixed coefficients $a_{js}$, these coefficients will change from period to period, especially for common stock and other residual instruments. For a particular firm, then, an instrument set which dominates another in one period need not do so in the following period, e.g., an instrument may be unique in one period but not in the next. But dominance cannot be reversed over time if instruments remain unique; the principles of a single period may therefore serve well over time, also. This question will be analyzed further in what follows.

Stability Across Production Decisions

The model presented in this paper has not explicitly considered investment and production decisions since the coefficients $a_{js}$ have been taken as fixed. We have already noted that different coefficients, even if the number of securities remains the same, generally leads to different partial orderings of market structures. We now identify conditions under which a given ordering of capital structures is stable across production decisions.

The key to this stability property is (45). Let $\Delta a_{js}$ be the change in $a_{js}$, $j = 1, \ldots, \tilde{J}$, $s = 1, \ldots, n$, resulting from a given change in production plans and denote by $J_A$ the "resulting" set of instruments. Suppose $J'' \subseteq J'$ and $J''$ has rank $R''$, i.e., there exists a set $J$ of rank $R'' > R'$ such that $J \not\subseteq J''$, $J \not\subseteq J'$. Then, by Theorem 2 the
production change preserves dominance if \( J'' \) dom \( J' \), i.e., if there exists a \( J' \) of rank \( R'' \geq R' \) such that

\[
J' \preceq J'' \quad J' \preceq J''
\]

A sufficient, but not necessary, condition for this to be true is that the rows of \([\Delta a'_{js}']\) and of \([\Delta a''_{js}']\) are obtainable as linear combinations of the rows of \([a_{js}']\) of set \( J' \), i.e., there exist numbers \( c'_{jk} \) and \( c''_{jk} \) such that

\[
\Delta a'_{js} = \sum_{k=1}^{J'} c'_{jk} a'_{ks}, \quad j = 1, \ldots, J', \quad s = 1, \ldots, n, \tag{58}
\]

\[
\Delta a''_{js} = \sum_{k=1}^{J''} c''_{jk} a''_{ks}, \quad j = 1, \ldots, J'', \quad s = 1, \ldots, n; \tag{59}
\]

because (59) implies \( R' = R'' = R' = R'' \), and (58). Equation (59) is a slight generalization of the "unanimity condition" of Eeckern and Wilson (1974), that is, the condition under which investors, presently in equilibrium, will react unanimously to a proposed change in production plans. Thus, BMSDC exhibits a certain stability across production changes, in particular proposed production changes about which investors are unanimous under \( J \) (and hence \( J'' \) and \( J' \)).

**The Value of the Firm and Stockholder Welfare**

In the recent literature in finance, the firm's objective has overwhelmingly been taken to be the maximization of its market value. It is well known that this objective is in the owners' interest under certainty and when financial markets are complete and competitive under uncertainty. This objective is also consistent with investor welfare under highly restrictive assumptions when markets are incomplete, especially markets which, as a result of these assumptions, are equivalent to complete markets [e.g., Rubinstein (1974)]. But the value maximization proposition has been shown to be increasingly
vulnerable in incomplete markets in general [e.g., Hirshleifer (1970, p. 275), Wilson (1972), Leland (1973), and Ekmang (1974)]. This paper casts further doubts on value maximization as an appropriate objective of the firm, with particular reference to capital-structure decisions. In particular, we observed the following: (a) that a change in capital structure may make every investor better (or worse) off without change in the equilibrium value of any firm (Corollary 5); (b) that a firm may be able to increase its market equilibrium value, by increasing leverage, without effect on the welfare of any investor (Corollary 6); and (c) that the use of (riskless) debt to increase the firm's equilibrium market value, if it changes allocations at all, will improve the welfare of some investors only at the expense of others (Theorem 5).

Options

The preceding analysis was based on the assumption that all instruments were issued by firms and government agencies \( Z_j > 0 \), all \( j \). But in practice, instruments are also issued by investors themselves (e.g., puts and calls); their impact has been analyzed by Schrems (1973) and by Ross (1974). The framework developed in this paper clearly applies to such instruments also. In particular, whether the availability of a given put or call results in a dominating market structure is determinable by reference to Theorem 2. Option markets, of course, are characterized by relative "thinness" and high transaction costs [Black and Scholes (1972)]. Our disregard of transaction costs is therefore of greater significance in connection with options than in relation to other instruments. A means of improving the efficiency of financial markets, alternative to both security proliferation by firms and options and without many of the drawbacks of these vehicles, is the Superfund [Hakansson (1974)].
Concluding Comments

This paper has developed a partial ordering of incomplete financial markets, an ordering which shows considerable robustness with respect to preference assumptions and production decisions. Loosely speaking, the following general conclusions emerge, with and without corporate income taxes: security proliferation of itself is of little virtue; the issuance of risky debt, preferred stock, and warrants, and the attachment of a conversion privilege to riskless bonds tend to be beneficial to investor welfare; the convertibility feature of risky bonds and of preferred stock is of ambivalent value; 100% "nonsynergistic" mergers tend to reduce investor well-being; investor welfare effects are generally not reflected in the values of firms: the change from an incomplete to a richer (or even fully complete) financial market need not change the equilibrium value of a single firm even when such a change makes every investor better off.

APPENDIX

Proof of Lemma 7: By Definition 5 and Theorem 1, we can, since we are measuring utility differences, substitute \( \bar{J}_{\min}'' \) for \( J_{\min}'' \); the equilibrium conditions for \( J_{\min}'' \) are, by (12)-(14), substitution of the equivalent utility function \( \lambda_i'' u_i'(w_i) \) for \( u_i'(w_i) \), and the price compensation assumption

\[
(A.1) \quad \sum_{s=1}^{n} \pi_is_i' \left( \sum_{j=1}^{n} s_j''' z_{ij}' \right)s_j''' = B_j'', \quad j = 1, \ldots, \bar{J}_{\min}'' , \quad i = 1, \ldots, I ;
\]

\[
(A.2) \quad \sum_{j=1}^{\bar{J}_{\min}''} p_j'' z_{ij}'' = \sum_{j \in \bar{J}_{\min}''} p_j' z_{ij}''' + \sum_{j \notin \bar{J}_{\min}''} p_j' z_{ij}'', \quad i = 1, \ldots, I ;
\]

\[
(A.3) \quad \sum_{i=1}^{I} z''_{ij} = z_j'', \quad j = 1, \ldots, \bar{J}_{\min}'' .
\]
By (38) and Lemma 1, there exists an allocation $\tilde{z}''_{ij}$ under $J''_{\text{min}}$ such that (see (10))

\[(A.4) \quad \tilde{a}''_{is} = c'_{is}, \quad i = 1, \ldots, I, \quad s = 1, \ldots, n, \]

i.e., which is the same as that achieved by $J'$ via equilibrium allocation $\{z'_{ij}\}$. Thus, we need only compare the set of equilibrium allocations $\{z''_{ij}\}$ with the set $\{\tilde{z}''_{ij}\}$.

Let, for any pair $\{z''_{ij}\}, \{\tilde{z}''_{ij}\}$,

$$\Delta_{ij} = z''_{ij} - \tilde{z}''_{ij}, \quad \text{all } i, j.$$  

We have by (13),

\[(A.5) \quad \sum_{i=1}^{I} \Delta_{ij} = 0, \quad j = 1, \ldots, J''_{\text{min}}, \]

and since by assumption $z''_{i} = \tilde{z}''_{i}$ for some $i (i \in \overline{I})$,

\[(A.6) \quad \Delta_{ij} = 0, \quad \text{some } i, j.\]

Let $J^+_{i}$ be the set of indices $j$ such that $\Delta_{ij} > 0$ and $J^-_{i}$ the set such that $\Delta_{ij} \leq 0$ (clearly $J^+_{i} \cup J^-_{i} = J''_{\text{min}}, \text{ all } i$). By Corollary 3, there exist numbers $\Delta u^+_{ij}$ and $\Delta u^-_{ij}$ such that

\[(A.7) \quad \sum_{j \in J^+_{i}} \Delta u^+_{ij} + \sum_{j \in J^-_{i}} \Delta u^-_{ij} = \Delta u_{i}, \quad i = 1, \ldots, I; \]

$$\Delta u^+_{ij} > p''_{ij} \Delta_{ij} > 0, \quad j \in J^+_{i}, \quad i \in \overline{I},$$

\[(A.8) \quad \Delta u^-_{ij} > p''_{ij} \Delta_{ij}, \quad j \in J^-_{i}, \quad i \in \overline{I};\]

\[(A.9) \quad \Delta u^+_{ij} = \Delta u^-_{ij} = 0, \quad \text{all } j, i \notin \overline{I}.\]

Summing over $i$ we get, using (A.5), (A.6), and (17)
\[
\sum_{i=1}^{I} \Delta u_{i,j}^+ > P'' \sum_{i=1}^{I} \Delta u_{i,j}^- \quad \text{if} \quad j \in J^+ \equiv P'' \Delta_j > 0 \\
\sum_{i=1}^{I} \Delta u_{i,j}^- > P''(-\Delta_j) \\
\]

Summing over \( j \) and using (A.7) we obtain

\[
\sum_{i=1}^{I} \Delta u_{i}^- > 0 ,
\]

which implies (39), completing the proof.

**Proof of Lemma 8.** To prove the lemma for any \( J' \) and \( J'' \) satisfying (41) it is sufficient, by Lemma 4 and Theorem 1, to choose \( J' \) and \( J'' \) such that

\[
R(J') = \bar{J}' ,
\]

and

\[
a'_{js} = a''_{js} = a_{js}^- , \quad s = 1, \ldots, n , \quad j = 1, \ldots, \bar{J}'-1
\]

\[
a''_{\bar{J}''-1,s} + a''_{\bar{J}'s} = a'_{\bar{J}''s} , \quad s = 1, \ldots, n ,
\]

where both instruments \( \bar{J}''-1 \) and \( \bar{J}' \) are unique.

Let \( \{ d'_{is} \} \) (or \( \{ \sigma_{ij}, \lambda'_{i}, P'_j \} \)) represent an equilibrium under \( J' \).

Our approach will be to construct an equilibrium \( \{ d''_{is} \} \) (or \( \{ \sigma''_{ij}, \lambda''_{i}, P''_j \} \)) under \( J'' \) which is infeasible under \( J' \) and which makes every investor better off in terms of expected utility. In fact, we shall construct the new equilibrium in such a way that it will have the same price structure as the old one, i.e., such that

\[
d''_{is} = d'_{is} + \Delta_{is} , \quad \text{all } i,s ;
\]
\[ z''_{ij} = z'_{ij} + \bar{A}_{ij}, \quad \text{all } i, \ j = 1, \ldots, J''-2; \]
\[ z''_{i,j''-1} = z'_{i,j''} + \bar{A}_{i,j''-1}, \quad \text{all } i; \]
\[ z''_{i,j''} = z'_{i,j''} + \bar{A}_{i,j''}, \quad \text{all } i; \]
\[ \lambda''_{i} = \lambda'_{i}, \quad \text{all } i; \]
\[ p''_{j+1} = p'_{j}, \quad \text{all } i; \]
\[ p''_{j''-1} = p'_{j''}; \]

Consider the system of \( n + J'' + 1 \) equations and \( 2n \) inequalities,
\[ \sum_{s=1}^{n} \Delta_{s} a_{js} = 0, \quad j = 1, \ldots, J''-2; \]
\[ \sum_{s=1}^{n} \Delta_{s} a''_{j''-1,s} = \Delta, \quad \Delta > 0; \]
\[ \sum_{s=1}^{n} \Delta_{s} a''_{j''s} = -\Delta; \]
\[ \sum_{j=1}^{J''} \bar{A}_{j} a''_{js} = c_{s} \Delta_{s}, \quad s = 1, \ldots, n; \]
\[ \sum_{s=1}^{n} u_{s} c_{s} \Delta_{s} = 0; \]
\[ c_{s} > 0, \ u_{s} > 0, \quad s = 1, \ldots, n; \]

in the \( n + J'' + 2n \) unknowns \( \Delta_{s}, \bar{A}_{j}, u_{s}, \) and \( c_{s} \). Since \( a''_{js} > 0 \) for some \( s \), (A.20)-(A.22) requires
\[ \Delta_{s} > 0 \text{ some } s, \Delta_{s} < 0 \text{ some } s, \]

and since \( \Delta_{s} \) appears on the right side of (A.23), (A.20)-(A.23) does have a solution such that (A.24)-(A.25) holds. Furthermore, if \( \{\Delta_{s}, \bar{A}_{j}, u_{s}, c_{s}\} \)
is a solution to (A.20)-(A.25), then \( \{ k \bar{a}_s, k \bar{a}_j, \bar{u}_s, c_s \} \) is a solution to (A.20)-(A.25) with the right side of (A.21)-(A.22) multiplied by \( k \), for any \( k \neq 0 \). We are now ready to put the numbers \( \Delta_s, \bar{a}_j, \bar{u}_s, \) and \( c_s \) to work.

In the equations (12-14) applied to \( j' \), choose collection \( \{ \pi_{is}, u_i(w_i) \} \) such that

\[
(A.27) \quad \pi_{is} u_i' \left[ \sum_{j=1}^{j'-1} z_{ij}' a_{js} + z_{ij}' \bar{a}_{js} \right] = \bar{u}_s + k_i \Delta_s ,
\]

\[ \quad s = 1, \ldots, n , \quad i \in I^+ ; \]

\[
(A.28) \quad \pi_{is} u_i'(d_i') = \bar{u}_s - k_i \Delta_s ,
\]

\[ \quad s = 1, \ldots, n , \quad i \in I^- ; \]

where the \( k_i \) are numbers such that

\[
(A.29) \quad k_i > 0 , \quad \text{all } i ;
\]

\[
(A.30) \quad \sum_{i \in I^+} k_i + \sum_{i \in I^-} k_i = 0 ,
\]

\[
(A.31) \quad \bar{u}_s + k_i \Delta_s > 0 , \quad \text{all } i,s ;
\]

and \( I^+ \) and \( I^- \) partition all investors into two nonempty subsets.

Applying (A.27) and (A.28) to (12)-(14), we obtain, using (A.12) and (A.20)-(A.22), the equilibrium conditions

\[
(A.32) \quad \sum_{s=1}^{n} \bar{u}_s a_{js} = \lambda_j' p_i' , \quad \text{all } i , \quad j = 1, \ldots, j'-1 ;
\]

\[
(A.33) \quad \sum_{s=1}^{n} \bar{u}_s a_{js}' = \lambda_j' p_i' , \quad \text{all } i ;
\]

\[
(A.34) \quad \sum_{j=1}^{j'} z_{ij} p_j' = \sum_{j=1}^{j'} z_{ij} p_j' + r_i' , \quad \text{all } i ;
\]
(A.35) \[ \sum_{i=1}^{i} z_{ij} = z_{j'}, \quad j = 1, \ldots, J'; \]

for instrument set \( J' \). Note that the allocation \( \{a_{is}'\} \) does not satisfy (14) under \( M'' \); for instrument \( J''-1 \) we get, from (A.27) and (A.21),

\[ \sum_{s=1}^{n} u_{s} a_{j''-1, s} = \lambda_{i}'' P_{j''-1} + \begin{cases} -k_{1} A \quad \text{i} \in I^{+}, \\ k_{1} A \quad \text{i} \in I^{-}, \end{cases} \]

and similarly for \( J'' \).

Now consider \( J'' \). Set

(A.36) \[ \Delta_{is} = \begin{cases} k_{1} c_{s} A \quad \text{i} \in I^{+}, \\ -k_{1} c_{s} A \quad \text{i} \in I^{-}, \end{cases} \quad \text{all } s; \]

(A.37) \[ \tilde{\Delta}_{ij} = \begin{cases} k_{1} \tilde{A}_{j} \quad \text{i} \in I^{+}, \\ -k_{1} \tilde{A}_{j} \quad \text{i} \in I^{-}, \end{cases} \quad j = 1, \ldots, J''; \]

(A.38) \[ \Delta u_{is} = \pi_{is} (u_{i}(a'') - u_{i}'(a')) = \begin{cases} -k_{1} A \quad \text{i} \in I^{+}, \\ k_{1} A \quad \text{i} \in I^{-}, \end{cases} \quad \text{all } s. \]

Note that \( \Delta_{is} \) and \( \Delta u_{is} \) are of opposite sign, as required by (1) and (2).

Using (A.38) and (A.27)-(A.28), the equilibrium conditions (14) for \( J'' \) become

(A.39) \[ \sum_{s=1}^{n} \pi_{is} u_{i}(a'') a_{js} = \sum_{s=1}^{n} u_{s} a_{js} = \lambda_{i}'' P_{j''}, \quad j = 1, \ldots, J'', \quad \text{all } i; \]

Thus, by (A.12), (A.32), and (A.33) we obtain (A.17)-(A.19). Applying (A.19) to the left side of (12) gives, using (A.14)-(A.16), (A.37), (A.39) (A.23)-(A.24), and the price compensation assumption
\[ \sum_{j=1}^{\bar{J}''} z_{ij}^p p_j^" = \sum_{j=1}^{\bar{J}''} \left( z_{ij}^- \pm k_i \bar{\Delta}_j \right) p_j^" + \left( z_{ij}^+ \pm k_i \bar{\Delta}_j^" \right) p_j^" \]

\[ = \sum_{j=1}^{\bar{J}''-2} z_{ij}^- p_j^" + z_{ij}^+ \left( p_j^" - p_j^" \right) + \frac{k_i}{\bar{\lambda}_i} \sum_{s=1}^{\bar{J}''-1} \sum_{j=1}^{\bar{J}''} p_j^" \bar{\Delta}_j \]

\[ = \sum_{j=1}^{\bar{J}'} z_{ij}^- p_j^" = \sum_{j=1}^{\bar{J}'} z_{ij}^- p_j^" + r_i, \quad \text{all } i. \]

Finally, (13) becomes, using (A.14)-(A.16), (A.37), (A.30), and (A.35)

\[ \sum_{i=1}^{I} z_{ij}^" = \sum_{i=1}^{I} z_{ij}^- + \sum_{i \in I^+} k_i \bar{\Delta}_j + \sum_{i \in I^-} -k_i \bar{\Delta}_j \]

\[ = \sum_{i=1}^{I} z_{ij}^- = z_j^", \quad i = 1, \ldots, \bar{J}''-1 ; \]

\[ \sum_{i=1}^{I} z_{ij}^" = z_j^", \quad i = I. \]

Consequently, \( \{d_{is}^"\} \) or \( \{z_{ij}^", \lambda_i^", P_j^"\} \) given by (A.13)-(A.19) and (A.36)-(A.38) gives an equilibrium under \( J'' \) whenever \( \{d_{is}'\} \) or \( \{z_{ij}', \lambda_i', P_j'\} \), given (A.27) and (A.28), produces an equilibrium under \( J' \).

It remains to show that (42) holds for the two equilibrium allocations we have constructed. By a generalization of Corollary 3, we obtain, using (A.13), (A.39), (A.26), and (A.36)

\[ (A.40) \quad \Delta u_{is} + \bar{v}_s \bar{\Delta}_s = \begin{cases} k_i \bar{v}_s \bar{c}_s \bar{\Delta}_s, & i \in I^+ \\ -k_i \bar{v}_s \bar{c}_s \bar{\Delta}_s, & i \in I^- \end{cases}, \quad \text{all } s, \]

where the \( \Delta u_{is} \) are numbers satisfying

\[ \sum_{s=1}^{n} \Delta u_{is} = \Delta u_i, \quad i = 1, \ldots, I, \]

with \( \Delta u_i \) defined by (40). But by (A.40) and (A.24)

\[ \Delta u_i = \sum_{s=1}^{n} \Delta u_{is} + k_i \sum_{s=1}^{n} \bar{v}_s \bar{c}_s \bar{\Delta}_s = 0, \quad i = 1, \ldots, I, \]

which completes the proof.
Footnotes

1. The HARA-class (hyperbolic absolute risk aversion) consists of the following utility functions of wealth with the properties $u'(w) > 0$, $u''(w) < 0$ (the first over at least a finite range of positive wealth):

$$u(w) = \begin{cases} \frac{1}{\gamma}(w+a)^\gamma & \gamma < 1 \text{ (decreasing absolute risk aversion)} \\ -\exp(\gamma w) & \gamma < 0 \text{ (constant absolute risk aversion)} \\ -(a-w)^\gamma & \gamma > 1, \text{ a large (increasing absolute risk aversion).} \end{cases}$$


3. Two complications are circumvented by assuming strong rather than weak inequalities in (1). First, our expressions become less complex, with little qualitative change in the results. Second, we avoid confronting the possibility that state $s$ actually occurred even though some individuals assessed $\pi$ to be 0—which, in turn, raises issues beyond the scope of this paper.

4. The first and second properties (more wealth preferred to less, nonsaturation, and risk aversion) require no comment. The third assumption (infinite marginal utility of zero wealth), while not difficult to defend on empirical grounds and not essential, keeps the technicalities of our analysis to a minimum.

5. A riskless asset is one for which $a_{js} = a_j$, all $s$. While it is common to assume that the riskless asset, when it exists, is created by the investors themselves, i.e., that $\pi_j = 0$, there is no loss of generality in assuming that some quantity has already been issued by, for example, a government agency.

6. Thus, government agencies are also referred to as "firms."

7. Options which are not issued by firms but created by investors (puts and calls) will be considered in Section VIII.

8. $\tilde{s} = \{\tilde{s}_1, \ldots, \tilde{s}_K\}$ is a partition of $S = \{s_1, \ldots, s_n\}$ if $\tilde{s}_k \subset S$, $k = 1, \ldots, K$, $\tilde{s}_j \cap \tilde{s}_k = \emptyset$ if $j \neq k$, and $\bigcup_{k=1}^{K} \tilde{s}_k = S$. 
9. Since last period's \( w_i \) equals this period's \( y_i - r_i \), and since \( \Pr\{w_i > 0\} = 1 \) in (the previous) equilibrium [see (15)], this assumption is rather mild.

10. Due to the presence of the \( r_i \) terms, the equilibrium prices will be expressed in absolute terms, not just relative terms.

11. By a collection \( \{\pi_i, u_i(w_i)\} \) we mean the set of preferences and beliefs possessed by the (collection of) investors in the market \( (i = 1, \ldots, I) \).

12. Recall Section I in which we gave conditions insuring the equivalence of all market structures.

13. The term "constrained Pareto-efficiency" is sometimes used to denote allocations which are efficient relative to a given (constrained) set of allocation possibilities [see, e.g., Stiglitz (1972, p. 26)].

14. Since investors in the present model make no new decisions until the end of the period, the warrants presumably will not be exercised in the interim.

15. Recall that the corporate tax deduction for interest is based on effective interest paid (computed by amortization of premiums and discounts) rather than the nominal rate of interest.

16. At this writing the first $25,000 of corporate income is taxed at a 22% rate and any excess at a rate of 48%. Thus, \( t = .48 \) and \( e_T = (.22 - .48) 25,000 \) or \( -6,500 \). In the case of losses, carryback and carryover provisions of the Internal Revenue Code tend to uphold the approximate validity of (47). While the number \( e_T \) has not been changed in recent years in response to changes in corporate debt policy, there is no intrinsic reason why it could not be used in this manner.

17. See footnote 1.

18. For options, the total supply \( Z_j \) would be zero. As noted earlier (footnote 5), this involves no loss in generality when added to (4).
REFERENCES


