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THE SUPERFUND: EFFICIENT PATHS TOWARD
A COMPLETE FINANCIAL MARKET

by

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ABSTRACT

When investor preferences and beliefs are heterogeneous, there exist sequences of market structures, beginning with an incomplete market and ending with a complete financial market, such that each successive market structure dominates its predecessor in terms of investor welfare. Two established means whereby existing financial markets can be made "more complete" are security proliferation by firms and the expansion of option markets. This paper argues that the most direct method of improving market efficiency would be the establishment of a financial intermediary (Superfund), which would own a portion of the "market portfolio" and periodically issue fixed-term securities, each of which pays off if and only if a pre-specified "superstate" occurs. Under weak assumptions concerning investor beliefs and preferences and apparently innocuous assumptions with respect to investor willingness to process pre-decision information in great detail, the Superfund is shown to be able to bring about a "complete" financial market by issuing surprisingly few "supersecurities" and without encountering the usual problems associated with state identification and the "moral hazard."
I. INTRODUCTION AND SUMMARY

A financial market is complete when investors can, in effect, trade in Arrow-Debreu (1964, 1959) certificates; this would be the case when the number of linearly independent real-world securities is the same as the number of visualized states of the world and short sales are unrestricted. But the benefits of a complete market can also, under specialized conditions, be attained by trade in as few as a single instrument (issued by a mutual fund holding all "regular" securities). The case when two instruments, one riskless and one risky, are sufficient to achieve the benefits of a complete market has received particular attention in the literature; essentially, it holds under any one of the following conditions: (i) homogeneous probability beliefs and utility belonging to the HARA-class,¹ with the exponent the same for all investors (unless utility is negative exponential); (ii) homogeneous beliefs, normally distributed returns, and utility functions (at a minimum) defined on the whole line; or (iii) homogeneous probability beliefs, continuous-time decision-making by investors, and a stochastic process of the Itô variety. The reason is that in these cases all investors mix risky assets the same way.² Extensions incorporating two or more riskless assets of differing maturities have recently been made by Merton (1973) and by Rubinstein (1974).

The real world apparently lies somewhere between the highly restrictive conditions (i)-(iii) and a market which is sufficiently rich to be "complete" in the presence of significant diversity. But when investor preferences and beliefs exhibit substantial heterogeneity, incomplete markets are distinctly inferior, in a welfare sense, to "more complete" markets [Hakansson (1974)]. Thus, the strongest argument in favor of a complete financial market rests on the unique ability of such a market to achieve a Pareto-efficient allocation of resources among investors under heterogeneous preferences and beliefs. Despite
this decided virtue there are at least four important reasons why complete markets have not come into existence, giving rise to the view that the Arrow-Debreu model is "... in the nature of a magnificent tours de force enriching our insight, but with a somewhat strained relation to reality" [Koopmans (1974, p. 327)]. The first obstacle is the difficulty of reaching agreement (a priori) on a set of acceptable time-states. The second problem has to do with the subsequent determination of which state actually occurred, by no means a trivial task. The third obstacle haunting any implementation of a complete market has aptly been dubbed "the moral hazard": the temptation, not unknown in the insurance industry, to, by illicit means, bring about a particularly fortuitous state. The final major problem has to do with the enormity of the number of states, and consequently the number of securities, that, by most calculations, must be available for such a market to function properly.

This paper examines the implications surrounding the establishment of a new type of financial intermediary, which, for want of a better name, will be called the Superfund. In essence, the Superfund owns a portion of the "market portfolio" and periodically issues fixed-term securities, each of which pays off if and only if a pre-specified "superstate" occurs. Under weak assumptions about investor preferences and probability beliefs and apparently innocuous assumptions with respect to investor willingness to process pre-decision information in great detail, I show that the Superfund is, in principle, able to bring about a complete financial market (in the Arrow-Debreu sense) by issuing surprisingly few "supersecurities."

As a practical matter, it should have little difficulty in providing all the "advantages" of a complete financial market for a majority of investors, particularly, "small" investors. These advantages include "greater" economic welfare (by incorporating into the market, in a particularly efficient manner, all or the major profitable side-bet opportunities which are unresolvable within the
regular markets), the ability of investors to, in effect, borrow and lend at the same "riskless" rate and to make (implicit) pure short sales (without actually having to engage in borrowing or short-sale transactions), and virtual elimination of transaction costs for those who revise their portfolios fairly infrequently. Another noteworthy peculiarity of the Superfund is its special ability to apparently overcome or circumvent the four obstacles stated in the previous paragraph with ease.

The paper proceeds as follows. The bulk of the theoretical framework is developed in Part II, with the basic model specified in Section 2.1. Section 2.2 identifies and characterizes the equilibrium conditions when all securities are of the regular type and Section 2.3 when they are of the Arrow-Debreu variety. Section 2.4 defines the notion of equivalent markets and compares the two market structures via the Fundamental Implicit Price Theorem. The superfund is introduced in Section 2.5; in its ultimate (single-period) form, its presence is shown to result in a market structure which is equivalent to a complete financial market and when it holds a sufficiently large fraction of the market portfolio, all short selling (and borrowing) can be avoided (Theorem 2). Section 2.7 considers superfunds issuing less than a full array of Arrow-Debreu securities; the Basic Market Structure Dominance Criterion (Section 2.6) is shown to give a partial ordering of all possible superfunds. Finally, the conditions applicable to the present model under which Arrow-Debreu certificates can be consolidated are given in Section 2.8.

The ability to expand the opportunities of investors is the property of the superfund which is essentially responsible for its power to improve economic welfare. But welfare improvements are also available via security proliferation by firms and the expansion of option markets. Part III argues that, as a practical matter, these alternatives are not only limited in what they can ultimately
accomplish but are much more inefficient in reaching any given "level" of improvement. The superfund's relationship to regular mutual funds is also analyzed.

Part IV imposes two assumptions which essentially state that the amount of (probability) detail an investor is willing to process before making his decision is limited. These assumptions may be viewed as an attempt to implicitly "...add to our usual economic calculations an appropriate measure of the costs of information gathering and transmission" [Arrow (1974, p. 5)]. The implications of these premises are a drastic reduction in the number of "states" that need to be distinguished and virtual elimination of problems bearing on which states are relevant, on which state occurred ex post, and on the "moral hazard." An illustration is also included. Part V then turns to some of the pragmatic issues concerning return patterns, short sales, borrowing, probability estimation, transaction costs, administration, and the implications for the firm and for monetary policy. Part VI identifies the conditions under which the basic model considered can be embedded in an intertemporal setting without effect on the theorems presented. Finally, Part VII contains some concluding remarks.

II. THEORETICAL FOUNDATIONS

2.1. Basic Assumptions and Notation

We consider an ongoing economy in which the investors make decisions at the same (fixed) decision points. Our focus will be on an arbitrarily chosen single period. In line with the usual assumptions in this context, we postulate that all investment opportunities have (stochastically) constant returns to scale, are perfectly divisible, and can be sold short without restriction; transaction costs and taxes are ignored. The following basic notation will be used.

\[ s \] a basic state of the world at the end of the period

\[ n \] the number of basic states \[ s \]
the set of basic states \( S \) (i.e., \( S = \{s_1, \ldots, s_n\} \))

\( W_s \) total wealth available for allocation if state \( s \) occurs

\( I \) the number of investors \( i \)

\( y_i \) the resources of investor \( i \) available for investment (endogenous)

\( w_i \) end-of-period wealth of investor \( i \) (random variable)

\( d_{is} \) value assumed by \( u_i \) if state \( s \) occurs (investor \( i \)'s allocation if state \( s \) occurs, or his demand for state \( s \) Arrow-Debreu securities when only such securities are available)

\( \pi_{is} \) the probability assessment of investor \( i \) that state \( s \) will occur

\( u_i(w_i) \) the utility function of investor \( i \)

\( J \) the number of regular investment vehicles \( j \) (stocks, bonds, warrants, etc.) available in the market for the coming period

\( J^0 \) the number of continuing regular securities already present in investor portfolios \( (j^0 \leq J) \) as a result of previous decisions

\( a_{js} \) proceeds per share of security \( j \) if state \( s \) occurs \( (a_{js} \geq 0), \) all \( s, a_{js} > 0, \) some \( s, \) all \( j \)

\( J \) rank of matrix \( [a_{js}] \)

\( Z_j \) number of shares of security \( j \) outstanding in current period \( (Z_j > 0) \)

\( P_j \) price per share of regular security \( j \)

\( z_{ij} \) number of shares of security \( j \) demanded by investor \( i \) at beginning of period

\( z^0_{ij} \) number of shares of security \( j \) owned by investor \( i \) at end of previous period

\( r_i \) amount of investor \( i \)'s resources not in form of regular security holdings (withdrawals, additions)
\( \chi_is \) number of state \( s \) Arrow-Debreu securities, if available in conjunction with other securities, demanded by investor \( i \) at beginning of period \\
\( p_s \) price per share of the state \( s \) Arrow-Debreu security (if available) at beginning of period \\
\( M \) a market structure (the set of investment vehicles available in the market)

The investors' probability assessments are postulated to satisfy

\[
(1) \quad \pi_is > 0, \quad s = 1, \ldots, n, \quad \sum_{s=1}^{n} \pi_is = 1, \quad \text{all} \quad i.
\]

Thus, while we permit probability assessments to differ among investors, none assigns probability \( 0 \) to any of the identified states \( 1, \ldots, n \). Each investor is also assumed to be rational in the von Neumann-Morgenstern sense, with preferences that differentiate only with respect to his own end-of-period wealth; the utility functions \( u_i(w_i) \) are assumed to have the following properties:

\[
(2) \quad u'_i(w_i) > 0, \quad u''(w_i) < 0 \quad \text{for} \quad w_i > b_i > 0, \quad u'(b_i) = \infty, \quad i = 1, \ldots, I.
\]

We assume that total wealth is positive in each state, satisfying

\[
(3) \quad \sum_{i=1}^{I} b_i < w_s < W, \quad \text{all} \quad s,
\]

for some \( W \), and, for simplicity, that it is composed entirely of securities in positive supply, i.e.,

\[
(4) \quad z_j > 0, \quad \text{all} \quad j,
\]

some of which may be riskless. Thus, if the securities are of the regular variety, we obtain

\[
(5) \quad w_s = \sum_{j=1}^{J} a_{js} z_j = \sum_{i=1}^{I} d_is, \quad \text{all} \quad s,
\]
while if they are all of the $I$ Arrow-Debreu type, we have

\begin{equation}
W_i = \sum_{s=1}^{I} d_{is}, \quad \text{all } s.
\end{equation}

The rank $J$ of the $J \times n$ matrix $[a_{js}]$ clearly satisfies

\begin{equation}
J \leq \text{min}(J, n).
\end{equation}

Summing the components of initial wealth to be invested we obtain

\begin{equation}
y_i = \sum_{j=1}^{J} P_j z_{ij}^0 + r_i, \quad i = 1, \ldots, I.
\end{equation}

The numbers $r_i$, which represent net new investment by individual $i$ and are exogenous in the present model, are assumed to be consistent with positive prices in the budget constraint [see (13); when $J = J_0$, for example, this requires \( \sum_{i=1}^{I} r_i = 0 \)] and with

\begin{equation}
y_i > 0, \quad \text{all } i.
\end{equation}

The $r_i$ thus capture consumption-induced changes in the amount to be reinvested, the proceeds from liquidated firms, "new" capital going into new firms, and the expansion of existing firms without formally modeling the underlying process. Little is lost by this short-cut since our analysis is essentially one of comparative statics, and market structure changes frequently leave unchanged decisions not pertaining to security holdings. End-of-period wealth $w_i$, in the case of regular securities, is given by

\begin{equation}
w_i(s) = \sum_{j=1}^{J} a_{js} z_{ij}, \quad s = 1, \ldots, n; \quad i = 1, \ldots, I,
\end{equation}

while in an Arrow-Debreu world we obtain

\begin{equation}
w_i(s) = d_{is}, \quad s = 1, \ldots, n; \quad i = 1, \ldots, I.
\end{equation}
2.2. Market Structure ML: Regular Securities

In the remainder of this section we shall examine the welfare implications of various market structures, also referred to simply as markets, market situations, and market arrangements. For our purpose the focal point of a given market arrangement in these comparisons will be the set of instruments available for investment. Thus, even though in a larger sense the totality of the participants is an integral part of a complete description of a market arrangement \( M \), it is generally sufficient to think of the distinctive feature of \( M \) as simply the set of securities available for purchase.

When only regular securities are available, the resulting market situation may be described as

ML:

\[
\max \sum_{s=1}^{n} \pi_i s u_i^i \left( \sum_{j=1}^{J} a_{js} z_{ij} \right), \quad i = 1, \ldots, I
\]

subject to the budget constraints

\[
\sum_{j=1}^{J} P_j z_{ij} = \sum_{j=1}^{J} P_j^0 z_{ij}^0 + r_i^i, \quad i = 1, \ldots, I,
\]

where the prices \( P_1, \ldots, P_J \) are such that the optimal portfolios satisfy the market clearing conditions

\[
\sum_{i=1}^{I} z_{ij} = Z_j, \quad j = 1, \ldots, J.
\]

The equilibrium conditions for this case (EC1) consist of the following \( \times J + I + J \) equations in the \((I \times J + I + J)\) unknowns \( z_{ij}, \lambda_i, P_j \) (where the \( \lambda_i \) are Lagrange multipliers)

EC1:

\[
\sum_{s=1}^{n} \pi_i s u_i^i \left( \sum_{j=1}^{J} a_{js} z_{ij} \right) a_{js} = \lambda_i P_j, \quad j = 1, \ldots, J, \quad i = 1, \ldots, I;
\]
in view of assumptions (1)-(4). The assumptions also guarantee that a solution to ECl exists [Hart (1974)], but need not be unique, and that the equilibrium holdings $z'_{ij}$, multipliers $\lambda'_i$, and prices $P'_j$ satisfy

\begin{equation}
\sum_{j=1}^{J} P'_j z'_{ij} = \sum_{j=1}^{J} P'_j z'^0_{ij} + r_i, \quad i = 1, \ldots, I;
\end{equation}

\begin{equation}
\sum_{i=1}^{I} z'_{ij} = Z_j, \quad j = 1, \ldots, J
\end{equation}

However, in spite of (16), some of the $z'_{ij}$ will generally be negative, i.e., the equilibrium solution generally calls for short selling. Also, the equilibrium solution will in general be dependent on the previous holdings $z'^0_{ij}$ (as well as the $r_i$, the utility functions, and the probability assessments, of course).

2.3. Market Arrangement M2: Arrow-Debreu Securities

When only Arrow-Debreu securities are available in the current period, the market situation can be described as

M2:

\begin{equation}
\max_{\{d_{is}\}_{s=1}^{n}} \sum_{s=1}^{n} w_{is} u_i(d_{is}), \quad i = 1, \ldots, I
\end{equation}

subject to the budget constraints
\[ \sum_{s=1}^{n} p_s d_{is} = \sum_{s=1}^{n} p_s \sum_{j=1}^{\mathcal{J}} a_{js} x_{ij}^{0} + r_i, \quad i = 1, \ldots, \mathcal{I}, \]

where the prices \( p_1, \ldots, p_n \) are such that the optimal portfolio's satisfy the market clearing conditions

\[ \sum_{i=1}^{\mathcal{I}} d_{is} = W_s, \quad s = 1, \ldots, n. \]

The equilibrium conditions for this case (EC2) reduce the following \( \mathcal{I} \times n + \mathcal{I} + n \) equations in the variables \( d_{is}, \lambda_i, \) and \( p_s \):

EC2:

\[ \pi_{is} u'(d_{is}) = \lambda_i p_s, \quad s = 1, \ldots, n, \quad i = 1, \ldots, \mathcal{I}; \]

\[ \sum_{s=1}^{n} p_s d_{is} = \sum_{s=1}^{n} p_s \sum_{j=1}^{\mathcal{J}} a_{js} x_{ij}^{0} + r_i, \quad i = 1, \ldots, \mathcal{I}; \]

\[ \sum_{i=1}^{\mathcal{I}} d_{is} = W_s, \quad s = 1, \ldots, n. \]

Again, assumptions (1)-(3) and (9) insure that a solution, not necessarily unique, to EC2 exists [e.g., Arrow (1964)] and that the equilibrium holdings \( d''_{is} \), multipliers \( \lambda''_i \), and prices \( p''_s \) satisfy

\[ d''_{is} > b_{is} \geq 0, \quad s = 1, \ldots, n, \quad i = 1, \ldots, \mathcal{I}; \]

\[ \lambda''_i > 0, \quad i = 1, \ldots, \mathcal{I}; \]

\[ p''_s > 0, \quad s = 1, \ldots, n. \]

But, the most important property of M2 is that the allocations \( d''_{is} \) are Pareto-efficient "in the ultimate sense," i.e., not only are there no additional market trades that investors wish to engage in, but, in addition, there are no mutually beneficial nonmarket trades that investors desire with each other.\(^{11,12}\)
2.4. Equivalent Markets

In attempting to develop an ordering over market structures, we begin by establishing a notion of equivalent markets. A more complete analysis of equivalent markets is given in Håkansson (1974).

**Definition 1.** Market structure $M'$ is (strongly) equivalent to market structure $M''$ $(M' \sim M'')$ if, for all collections $^{13} \{\pi_{is}, u_i(w_i)\}$ satisfying (1) and (2), every equilibrium final allocation under $M'$ is also an equilibrium final allocation under $M''$, and conversely.

**Remark 1.** $M \sim M$.

Assume for a moment that $M_1$ has full rank $(\tilde{J} = J)$ and hence contains no securities issued by mutual funds or other pure holding companies with less than 100% ownership in each instrument held, and that $M_1'$ differs from $M_1$ only by the presence of such securities. Then the opportunities for choice of final allocations $d_{is}$ in $M_1'$ are clearly the same as in $M_1$ and, given that the shares of holding companies do not sell at premiums or discounts, we obtain

**Remark 2.** $M_1' \sim M_1$.

By our definition, it is not sufficient for two market arrangements to have the same equilibrium for only a narrow class of preferences and probability assessments but not for other classes. To be (strongly) equivalent, the two market arrangements must yield identical final allocations for any combination of preferences and probability assessments held by investors which satisfy the rather broad assumptions (1) and (2). Thus, even though $M_1$ and $M_2$ yield the same final allocations in special cases when $J = 1$ or 2, say, and $n$ is large [Rubinstein (1974)], $M_1$ (with $J = 1$ or 2) and $M_2$ (with $n$ large) are easily shown not to yield the same sets of equilibria in all cases and are therefore not equivalent.
The question that arises, then, is under what conditions, if any, ML and M2 are equivalent. We first observe that ML and M2 yield the same solution if and only if [see (16) and (23)]

\[ \sum_{j=1}^{J} a_{js} z_{ij}' = d_{is}' , \quad s = 1, \ldots, n , \quad i = 1, \ldots, I ; \]

in this case we also get

\[ \lambda' = \lambda'' \]

and

\[ p'_{j} = \sum_{s=1}^{n} p''_{s} a_{js} . \]

In view of (7), (26)-(28) always hold (for certain equilibrium pairs) if \( J = n \) [see also Cass and Stiglitz (1970, pp. 149-152)], i.e., if the number of (linearly independent) instruments traded in the market is equal to the number of states. That is, when \( J = n \), every equilibrium final allocation under M2 is also an equilibrium final allocation under ML; conversely, \( J = n \) implies that \( \pi_{is} u_{i}'(d_{is}')/\lambda_{i} \), \( s = 1, \ldots, n \), in (15) must be the same for all \( i \) for each equilibrium price structure \( \{p'_{j}\} \) so that every equilibrium final allocation under ML is also an equilibrium under M2. But if \( J < n \), there are (collections \( \{u_{i}(\omega_{i}), \pi_{is}\} \) satisfying (1) and (2) which produce) equilibrium allocations \( d_{is}'' \) which are simply not attainable [see (26)] under ML. Thus

**Remark 3.** \( ML \sim M2 \) if and only if \( J = n \).

The relationship between ML and M2 may also be studied by reference to (15) and (22), which, in turn, provide a simple and revealing means for proving that ML does not tend to a Pareto-efficient allocation unless (22) is satisfied for some \( p_{s} \) at \( d_{is} = d_{is}' \), \( i = 1, \ldots, I \). To this end, let (by reference to ECL)
\begin{equation}
\pi'_{i,s} = \frac{\pi_{i,s}'(d'_s) a_s}{\lambda'_i}, \quad s = 1, \ldots, n, \quad j = 1, \ldots, J, \quad i = 1, \ldots, I.
\end{equation}

By (15), we obtain

\begin{equation}
\sum_{s=1}^{n} \pi'_{i,s} = P'_j, \quad i = 1, \ldots, I, \quad j = 1, \ldots, J;
\end{equation}

\begin{equation}
P'_{i,s} = \frac{\pi'_{i,s}}{a_{i,s}} = \cdots = \frac{P'_{i,s,j}}{a_{i,s,j}}, \quad \text{all } i, j, s;
\end{equation}

and, rewriting (29),

\begin{equation}
\pi_{i,s}'(d'_s) a_s = \lambda'_i P'_{i,s}, \quad s = 1, \ldots, n, \quad j = 1, \ldots, J, \quad i = 1, \ldots, I.
\end{equation}

$P'_{i,s}$ may be viewed as that part of the total equilibrium price $P'_j$ of instrument $j$ which is attributed to state $s$ by individual $i$.

**Theorem 1.** *(The Fundamental Implicit Price Theorem.)* MI results in a Pareto-efficient allocation if and only if $P'_{i,s}$ given by (29) satisfies

\begin{equation}
P'_{i,s} = \frac{P'_{i,s,j}}{a_{i,s,j}}, \quad s = 1, \ldots, n, \quad j = 1, \ldots, J, \quad i = 2, \ldots, I.
\end{equation}

**Proof.** We show sufficiency first. (32) and (33) imply

\begin{equation}
\pi_{i,s}'(\sum_{j=1}^{J} a_{j,s,i} z'_{i,j}) a_s = \lambda_i P'_{i,j}, \quad \text{all } s, j, i,
\end{equation}

which, in view of (16)-(18) and the fact that $\sum_{j=1}^{J} a_{j,s,i} z'_{i,j}$ is independent of $i$, gives

\begin{equation}
P'_{j} = a_{j,s} \tilde{p}_s, \quad s = 1, \ldots, n, \quad j = 1, \ldots, J,
\end{equation}

for some $\tilde{p}_s$. But (34) and (35) imply

\begin{equation}
\pi_{i,s}'(\sum_{j=1}^{J} a_{j,s,i} z'_{i,j}) = \lambda_i \tilde{p}_s, \quad \text{all } s, j, i,
\end{equation}

and hence, in view of (1) and (2), any investor can become better off only by
new allocations which increase \( d'_{is} \) at prices less than \( \bar{p}_s \) and decrease \( d'_{is} \) at prices greater than \( \bar{p}_s \). Thus, every exchange resulting in a feasible allocation other than \( d'_{is} \) must make some investor worse off when (33) holds.

To prove necessity, suppose (33) does not hold for some investors in some state, e.g.,

\[(37) \quad p'_{i_1s_1} > p'_{i_2s_1}, \quad \text{some } j.\]

By (31), we obtain

\[(38) \quad p'_{i_1s_1} > p'_{i_2s_1}.\]

In view of (30) there is then some state \( s_2 \) such that

\[(39) \quad p'_{i_1s_2} < p'_{i_2s_2}.\]

By (2) and (15)-(18) there exist numbers \( \Delta_1, \Delta_2 > 0 \) and \( \bar{p}_l, \bar{p}_2 \) such that

\[(40) \quad p'_{i_2s_1} < \bar{p}_l < p'_{i_1s_1},\]

\[(41) \quad p'_{i_1s_2} < \bar{p}_2 < p'_{i_2s_2},\]

\[(42) \quad \pi_{i_1s_1} u'_{i_1} (d'_{i_1s_1} + \Delta_1) \geq \lambda_{i_1} \bar{p}_l,\]

\[(43) \quad \pi_{i_1s_2} u'_{i_1} (d'_{i_1s_2} - \Delta_2) \leq \lambda_{i_1} \bar{p}_l,\]

\[(44) \quad \pi_{i_2s_1} u'_{i_2} (d'_{i_2s_1} - \Delta_1) \leq \lambda_{i_2} \bar{p}_2,\]

\[(45) \quad \pi_{i_2s_2} u'_{i_2} (d'_{i_2s_2} + \Delta_2) \geq \lambda_{i_2} \bar{p}_2,\]

and

\[(46) \quad \Delta_1 \bar{p}_l = \Delta_2 \bar{p}_2.\]
After the even exchange with \( i_2 \) [see (46)] of \( \Delta_2 \) units of the state \( s_2 \) allocation \( d_{i_1 i_2}' \) at price \( \bar{p}_2 \) for \( \Delta_1 \) units of the state \( s_1 \) allocation at price \( \bar{p}_1 \), the increase in expected utility of investor \( i_1 \) is

\[
\Delta u_{i_1} \equiv \pi_{i_1 s_1} u_{i_1}'(x) \, dx - \pi_{i_1 s_2} u_{i_1}'(x) \, dx .
\]

Denoting the two integrals \( A \) and \( -B \), (1), (2), (42), (43), and (46) give

\[
A > \Delta_2 \bar{p}_2 > B
\]

or

\[
\Delta u_{i_1} > 0 .
\]

Similarly, for individual \( i_2 \),

\[
\Delta u_{i_2} = -\pi_{i_2 s_1} u_{i_2}'(x) \, dx + \pi_{i_2 s_2} u_{i_2}'(x) \, dx > 0 ,
\]

while for all others

\[
\Delta u_i = 0 , \quad i = i_1, i_2 .
\]

Thus, when (33) does not hold, there exist (even) trades not feasible under M1 after which some investors are better off and no investor is worse off than under allocation \( d_{i_1}' \). This completes the proof.

The following corollary is immediate:

**COROLLARY 1.** (33) holds if and only if (26)-(28) hold, i.e., if and only if M1 and M2 have the same solution(s), which is guaranteed for all collections \( \{\pi_{is}, u_i(w_i)\} \) satisfying (1) and (2) only if \( J = n \).
Since (33) generally does not hold in M1, it is apparent that in some sense M1 is inferior to M2. Before turning our attention to the problem of ordering market structures in more detail, we shall examine the notion of a "mixture" of M1 and M2.

2.5. Market Structure M3: Combining M1 and M2

Suppose fraction \( \delta \) (0 < \( \delta \) < 1) of all ordinary shares were owned by a fund, which in turn issues Arrow-Debreu securities. (To distinguish it from the usual mutual fund, such a fund will be called a superfund.) Thus the number of shares traded in the market would be \( J + n \); the supply of ordinary shares \( J \) would be \( (1 - \delta)Z_j \) and that of the state \( s \) Arrow-Debreu security \( \delta W_s \), since the market value of the fund in state \( s \) would be \( \delta W_s \). The market situation can now be described as

M3:

\[
\max_{\{z_{ij}, x_{is}\}} \sum_{i=1}^{n} \pi_{ui} \left( \sum_{j=1}^{J} a_{js} z_{ij} + x_{is} \right), \quad i = 1, \ldots, I,
\]

subject to the budget constraints

\[
\sum_{j=1}^{J} p_j z_{ij} + \sum_{s=1}^{n} p_s x_{is} = \sum_{j=1}^{J} p_j z_{ij}^0 + r_i, \quad i = 1, \ldots, I,
\]

where the prices \( p_1, \ldots, p_j, p_1, \ldots, p_n \) are such that the optimal portfolios satisfy the market clearing conditions

\[
\sum_{i=1}^{I} z_{ij} = (1 - \delta)Z_j, \quad j = 1, \ldots, J;
\]

\[
\sum_{i=1}^{I} x_{is} = \delta W_s, \quad s = 1, \ldots, n.
\]

The equilibrium conditions for this case (EC3) reduce to the following

\( I \times (J + S) + I + J + S \) equations in the variables, \( z_{ij}, x_{is}, \lambda_i, p_j \),
and $p_s$:

EC3:

(51) $\sum_{s=1}^{n} \pi_{is} u'_i(\sum_{j=1}^{J} a_{js} z_{ij} + x_{is}) a_{js} = \lambda_i^0 p_j$, $j = 1, \ldots, J$, $i = 1, \ldots, I$;

(52) $\pi_{is} u'_i(\sum_{j=1}^{J} a_{js} z_{ij} + x_{is}) = \lambda_i^0 p_s$, $s = 1, \ldots, n$, $i = 1, \ldots, I$;

(48) $\sum_{j=1}^{J} p_j z_{ij} + \sum_{s=1}^{n} p_s x_{is} = \sum_{j=1}^{J} p_j z_{ij}^0 + r_i$, $i = 1, \ldots, I$;

(49) $\sum_{i=1}^{I} z_{ij} = (1 - \delta) z_j$, $j = 1, \ldots, J$;

(50) $\sum_{i=1}^{I} x_{is} = \delta w_s$, $s = 1, \ldots, n$.

As before, assumptions (1)-(4) and (9) insure that a solution, not necessarily unique, to EC3 exists and that the equilibrium holdings $\bar{d}_{is}$, multipliers $\lambda_i''$, and prices $p_j''$, $p_s''$ satisfy

(53) $d_{is}'' > b_i \geq 0$, $s = 1, \ldots, n$, $i = 1, \ldots, I$;

(54) $\lambda_i'' > 0$, $i = 1, \ldots, I$;

(55) $p_j'' > 0$, $j = 1, \ldots, J$;

(56) $p_s'' > 0$, $s = 1, \ldots, n$.

But, by (29) and (52), $p_{ijs}'' = a_{js}p_s''$, $i = 1, \ldots, I$, so that by Corollary 1 every solution to EC2 is also a solution to EC3 and conversely, i.e.,

(57) $d_{is}'' = \sum_{j=1}^{J} a_{js} z_{ij}'' + x_{is}'' = d_{is}$,

(58) $\lambda_i'' = \lambda_i$,

(59) $p_s'' = p_s''$. 
\[ P''''_j = \sum_{s=1}^{n} p''''_s a_{js} \]

Formally, this gives

**Remark.** M3 \( \sim \) M2.

As in M1, however, it is entirely possible that \( z''''_{ij} < 0 \) and \( x''''_{is} < 0 \) for some \( i, j, \) and \( s, \) i.e., that for each optimal solution short selling is necessary for some individuals. We now show that short selling can be avoided by choosing \( \delta \) sufficiently large.

**Theorem 2.** If \( \delta < 1 \) is sufficiently large in M3, there exist solutions

\[ \left( \sum_{j=1}^{J} a_{is} z''''_{ij} + x''''_{is} = d''''_{is} \right) \]

such that

\[ z''''_{ij} \geq 0, \quad x''''_{is} \geq 0, \quad s = 1, \ldots, n, \quad j = 1, \ldots, J, \quad i = 1, \ldots, I. \]

In particular, (61) always holds if \( \delta \geq 1 - K \), where \( K > 0 \) is given by

\[ K = \min_{i,s} \frac{\sum_{s=1}^{n} d''''_{is}}{\sum_{s=1}^{n} W_s} \cdot \]

**Proof.** Let

\[ v_s = \frac{W_s}{\sum_{s=1}^{n} W_s}, \quad s = 1, \ldots, n; \]

\[ d''''_{is} = \sum_{s=1}^{n} d''''_{is}, \quad i = 1, \ldots, I; \]

and let \( K \) be defined as in (62). By (53) and (3),

\[ 0 < K \leq 1; \]

and, using (62),

\[ K d''''_{is} \leq d''''_{is}, \quad s = 1, \ldots, n, \quad i = 1, \ldots, I. \]

Now set
(66) \[ z_{ij}'' = K \frac{D_i''}{\sum_{s=1}^{n} W_s} z_j'' , \quad j = 1, \ldots, J , \quad i = 1, \ldots, I , \]

which gives, by (4),

(67) \[ z_{ij}'' > 0 , \quad j = 1, \ldots, J , \quad i = 1, \ldots, I , \]

and total demand of

(68) \[ \sum_{i=1}^{I} z_{ij}'' = KZ_j , \quad j = 1, \ldots, J . \]

By definition,

\[ x_{ij}'' = d_{is}'' - \sum_{j=1}^{n} a_{js} z_{ij}'' \]

\[ = d_{is}'' - KD_i'' v_s \]

\[ \geq 0 , \]

using (66), (5), (63), and (65). By (68), we set \( \delta = 1 - K \), which completes the proof.

The reader will recognize that (66) calls for purchasing the "market portfolio," which establishes

**COROLLARY 2.** The optimal allocation \( d_{is}'' \) is obtainable even if all complex securities not held by the Arrow-Debreu fund were owned by a regular mutual fund, i.e., a Pareto-efficient allocation in which all holdings are nonnegative is obtainable via ownership in two funds, each holding a fraction of all instruments, one issuing regular pro rata shares and the other issuing Arrow-Debreu securities.

On the basis of (67), the following may be separately noted.

**Remark 5.** If there is a riskless security in M3, a Pareto-efficient allocation of investment resources can be achieved without borrowing by any investor whenever \( \delta \) is sufficiently large.
2.6. Ordering Market Structures

Via Theorem 1 and Corollary 1, we observed that when \( J < n \), market structure \( M_1 \) is generally distinctly inferior to \( M_2 \), and hence, by Remark 4, to \( M_3 \), in that at least some investors under \( M_1 \) can make themselves better off, without causing anyone to become worse off, by engaging in exchanges not provided for by the market. Even if we did not wish to move all the way from \( M_1 \) to \( M_3 \), we might be interested in adding some new instruments to \( M_1 \), on the presumption that this would improve the general welfare. Clearly, the possibilities for new securities are staggering; at the same time the welfare effects of different proliferation patterns may well be quite diverse. Before considering specific suggestions for new instruments, therefore, some criterion for choosing among them appears most desirable. One such criterion was formulated in Hakansson (1974).

The basic market structure dominance criterion (BMSDC). Market structure \( M' \) is dominated by market structure \( M'' \) (\( M'' \) dom \( M' \)) if, in moving from every equilibrium under \( M' \) to every (different) equilibrium under \( M'' \), the change in expected utility of investor \( i \), \( \Delta u_i \), is such that

\[
\Delta u_i > 0 \quad \text{for some } i, \text{ for every collection } \{\pi_{is}, u_i(w_i)\}
\]

satisfying (1) and (2);

\[
\Delta u_i > 0 \quad \text{for all } i, \text{ for some collections } \{\pi_{is}, u_i(w_i)\}
\]

satisfying (1) and (2).

Paraphrased, the criterion says that if we compare equilibrium under \( M' \) with equilibrium under \( M'' \), then \( M'' \) dom \( M' \) if, under every conceivable situation consistent with our basic assumptions, at least some investors, and sometimes all investors, are better off under \( M'' \).

We observe that if \( M'' \) dom \( M' \), then we cannot also have \( M' \) dom \( M'' \).
[because (69) would then be violated for some collection \( \{i_s, u_i(w_i)\} \)].

Neither can we have \( M \) dom \( M \).

2.7 Market Structure \( M^4 \): Partitioning \( S \)

In the proof of Theorem 1, the kind of nonmarket exchange which investors engaged in to improve their lot involved private exchange of Arrow-Debreu securities. Of course, mutually beneficial side bets could also have been employed which involve more complex securities. In this section we consider a special class of such securities, namely, securities which pay off only if one of several states occur, in proportion to the quantity \( W_s \) for the states in question.

More specifically, let \( \tilde{S} = \{\tilde{s}_1, \ldots, \tilde{s}_K\} \) be a partition\(^{14}\) of \( S \). Now consider a "different" kind of superfund, one which owns proportion \( \delta \) of all regular securities and which issues one security for each subset of states, \( \tilde{s}_k, \; k = 1, \ldots, K \). Let the supply of supersecurities be \( \delta W_{\min} \leq \delta \min_s W_s > 0 \), i.e.,

\[
\tilde{Z}_j = \delta W_{\min}, \quad j = J + 1, \ldots, J + K.
\]

This gives

\[
a_{js} = \begin{cases} 
\frac{W_s}{W_{\min}}, & s \in s_k, \; j = J + k, \; k = 1, \ldots, K, \\
0, & \text{otherwise.}
\end{cases}
\]

The resulting market situation can be described as follows.

\( M^4 \):

\[
\max_{\{z_{ij}, \tilde{z}_{ij}\}} \prod_{i=1}^{n} \pi_i u_i \left( \sum_{j=1}^{J} a_{js} z_{ij} + \sum_{j=J+1}^{J+K} a_{js} \tilde{z}_{ij} \right), \quad i = 1, \ldots, I
\]

subject to the budget constraints
\( \sum_{j=1}^{J+K} P_j z_{ij} + \sum_{j=J+1}^{J+K} P_{j+1} \bar{z}_{ij} = \sum_{j=1}^{J} P_j z^0_{ij} + r_i, \quad i = 1, \ldots, I, \)

where the prices \( P_1, \ldots, P_J, P_{J+1}, \ldots, P_{J+K} \) are such that the optimal portfolios satisfy the market clearing conditions

\( \sum_{i=1}^{I} z_{ij} = (1 - \delta)Z_j, \quad j = 1, \ldots, J, \)

\( \sum_{i=1}^{I} \bar{z}_{ij} = \delta W_{\text{min}}, \quad j = J + 1, \ldots, J + K. \)

The equilibrium conditions for this case (EC4) reduce to the following:

\( I \times (J + K) + I + J + K \) equations in the variables \( z_{ij}, \bar{z}_{ij}, \lambda_i, P_j, \) and \( \bar{P}_j, \)

**EC4:**

\( \sum_{s=1}^{n} \pi_{is} u_i \left( \sum_{j=1}^{J} a_{js} z_{ij} + \sum_{j=J+1}^{J+K} a_{js} \bar{z}_{ij} \right) a_{js} = \lambda_i P_j, \quad j = 1, \ldots, J, \)

\( \sum_{s=1}^{n} \pi_{is} u_i \left( \sum_{j=1}^{J} a_{js} z_{ij} + \sum_{j=J+1}^{J+K} a_{js} \bar{z}_{ij} \right) a_{js} = \lambda_i \bar{P}_j, \quad j = J + 1, \ldots, J + K, \)

\( \sum_{j=1}^{J} P_j z_{ij} + \sum_{j=J+1}^{J+K} P_{j+1} \bar{z}_{ij} = \sum_{j=1}^{J} P_j z^0_{ij} + r_i, \quad i = 1, \ldots, I; \)

\( \sum_{i=1}^{I} z_{ij} = (1 - \delta)Z_j, \quad j = 1, \ldots, J; \)

\( \sum_{i=1}^{I} \bar{z}_{ij} = \delta W_{\text{min}}, \quad j = J + 1, \ldots, J + K. \)

Assumptions (1)-(4) and (9) again ensure that a (nonunique) solution to EC4 exists and that the equilibrium holdings \( \bar{a}_{is}, \) multipliers \( \lambda_i, \) and prices \( \bar{P}_j, P_j \) satisfy.
(79) \( \bar{d}_{is} > b_i > 0 \) \( , s = 1, \ldots, n \), \( i = 1, \ldots, I \) ;

(80) \( \bar{\lambda}_i > 0 \) \( , i = 1, \ldots, I \) ;

(81) \( \bar{P}_j > 0 \) \( , j = 1, \ldots, J \) ;

(82) \( \bar{F}_j > 0 \) \( , j = J + 1, \ldots, J + K \) .

It is readily verified that when \( K = 1 \), any solution to ECL also satisfies EC4, and conversely, and that when \( K = n \), any solution to EC4 satisfies EC3 and conversely. This gives

Remark 6. When \( K = 1 \) in \( M^4 \), \( M^4 \sim M^1 \) .

Remark 7. When \( K = n \) in \( M^4 \), \( M^4 \sim M^3 \sim M^2 \) .

In contrast to \( M^1 \), \( M^2 \), and \( M^3 \), \( M^4 \) is actually a set of market structures with a very large number of members. In comparing the members of \( M^4 \), we will require the following definition.

Definition 2. Let \( \bar{S}' = \{ \bar{s}'_1, \ldots, \bar{s}'_K \} \) and \( \bar{S}'' \) be partitions of \( S \). Then \( \bar{S}'' \) is strictly finer than \( \bar{S}' \) (\( \bar{S}'' \mathrm{ sf } \bar{S}' \)), if, for each \( \bar{s}'' \in \bar{S}'' \), there is an \( \bar{s}' \in \bar{S}' \) such that \( \bar{s}'' \leq \bar{s}' \) and \( K'' > K' \).

Our dominance criterion now makes possible the following partial ordering of the market structures \( M \in M^4 \).

Theorem 3. Let \( M', M'' \in M^4 \), where \( M' \), with rank \( R' \), is based on partition \( \bar{S}' \) of \( S \) and \( M'' \), with rank \( R'' \), is based on partition \( \bar{S}'' \) . Then \( M'' \mathrm{ dom } M' \) if and only if \( \bar{S}'' \mathrm{ sf } \bar{S}' \) and \( R'' > R' \).

Since Theorem 3 is a special case of Theorem 2 in Hakansson (1974), the proof is omitted.

Remark 8. \( M_2 \mathrm{ dom } M_1 \) if and only if \( J < n \).
Remark 9. If $M'' \text{ dom } M'$ and $M'' \text{ dom } M''$, then $M'' \text{ dom } M'$.

Remark 10. If $S'' \text{ sf } S'$ and $R'' = R'$, then $M'' \sim M'$.

Remark 11. Let $M \in M_4$ and have rank $R = \tilde{J}$ (the rank of $M_1$). Then $M \sim M_1$.

Remark 12. Let $M \in M_4$ and have rank $R > \tilde{J}$ (the rank of $M_1$). Then $M \text{ dom } M_1$.

Remark 13. Let $M \in M_4$ and have rank $R < n$. Then $M_3 \text{ dom } M$.

Remark 14. Let $M \in M_4$ and have rank $R = n$. Then $M \sim M_3 \sim M_2$.

The essence of Theorem 3 is twofold. First, to improve on market structure $M = M_1$ or $M \in M_4$, we need only find a strictly finer partition of $S$ such that the rank of $M$ is increased. Second, if neither $M'' \text{ sf } M'$ nor $M' \text{ sf } M''$, then regardless of the respective ranks of $M''$ and $M'$, there exist collections $\{w_i, u_i(w_i)\}$ satisfying (1) and (2) such that everyone is better off under $M''$ and other collections such that everyone is better off under $M'$—unless, of course, $M' \sim M''$.

Remark 8 reminds us that in moving from an incomplete to a complete, or to a "more complete," market some investors may become worse off. But all investors cannot become worse off—whereas all investors may become better off. This, then, is what we shall refer to as an improvement in economic welfare.

2.8. Security Consolidation

Finally, we state the following result.

**THEOREM 4.** In $M_2$, let $N$ be the number of distinct values assumed by $W_s$, $s = 1, \ldots, n$. Then the Arrow-Debreu certificates can, for every equilibrium price vector $p_1'', \ldots, p_n''$, be consolidated into $m$ trading instruments attached to superstates $t_k$, $k = 1, \ldots, m$, where
\[ N \leq m < n , \]
\[ \bigcup_{k=1}^{m} t_k = S , \]
\[ \bigcap_{k=1}^{m} t_k = \phi , \]

which yield the same allocation as before, i.e., the optimal allocation satisfies

\[ d''_{is_1} = d''_{is_2} , \text{ all } s_1, s_2 \in t_k ; \text{ all } k ; \text{ all } i ; \]

if

\[ w_{s_1} = w_{s_2} , \text{ all } s_1, s_2 \in t_k ; k = 1, \ldots, m ; \]

\[ \frac{\pi_{is_1}}{\pi_{is_2}} = \frac{\pi_{ls_1}}{\pi_{ls_2}} , \text{ all } s_1, s_2 \in t_k ; k = 1, \ldots, m ; i = 2, \ldots, I , \]

and at least one superstate \( t_k \) contains more than one elementary state \( s \).

Proof. In view of (1), (2), and (22)-(25), we obtain

\[ d''_{is_1} \gtrless d''_{is_2} \text{ if and only if } \frac{\pi_{is_1}}{\pi_{is_2}} \gtrless \frac{p_{s_1}}{p_{s_2}} . \]

Since, in equilibrium,

\[ \sum_{i=1}^{I} d''_{is} = w_s, s = 1, \ldots, n , \]

\( w_{s_1} = w_{s_2} \) implies that

\[ \sum_{i=1}^{I} d''_{is_1} = \sum_{i=1}^{I} d''_{is_2} . \]

(85)-(87) now imply
\[ d_{i s_1} = d_{i s_2}, \quad \text{all } s_1, s_2 \in t_k; \quad k = 1, \ldots, m; \quad \text{all } i; \]
i.e., that in equilibrium each investor holds an equal number of Arrow-Debreu certificates for each state belonging to superstate \( t \).

**Remark 15.** When \( \pi_{is} = \pi_s \), \( s = 1, \ldots, n \), \( i = 1, \ldots, I \), i.e., all investors have the same probability beliefs, then \( m = N \).

**Remark 16.** \[ p_t = \sum_{s \in t} p_s. \]

**Remark 17.** \[ \frac{p_{s_1}}{p_{s_2}} = \frac{\pi_{is_1}}{\pi_{is_2}}, \quad s_1, s_2 \in t_k; \quad k = 1, \ldots, m; \quad \text{all } i. \]

What Theorem 4 says is that a group of basic states \( s \) can be combined into a superstate \( t \), and a single superstate security substituted for all of the basic state securities in the group, if the following two conditions hold: (i) the supply of basic securities in the group is the same for each state, and (ii) the conditional probabilities of each \( s \) given \( t \), denoted \( \pi_{is|t} \), are the same for all investors. It is important to note that the preceding permits perfectly general (and diversified) preferences and nonhomogeneous probability assessments for all states (and, in particular, for the superstates). From another vantage point the theorem also says that two states cannot be consolidated into one simply because the total allocation \( W_s \) is the same for the two states—unless the probability assignments are homogeneous (Remark 15) or satisfy (85). Thus, even when all investors are concerned only about ending wealth, a larger number of states must be distinguished, and trading in a larger number of instruments is required, when probability assessments are nonhomogeneous than when they are homogeneous.
III. PRAGMATICS: SOME BASIC CONSIDERATIONS

3.1. Introduction

As a financial intermediary, the superfund differs from the usual mutual fund in two respects. First, in contrast to a mutual fund, a superfund (generally) owns the so-called market portfolio, i.e., it purchases the same percentage, e.g., 25%, of all the shares that are traded in the market.\textsuperscript{15} Second, while mutual funds issue only one kind of security, ordinary (common) shares, a superfund generally issues more than one kind of instrument and these instruments are different from the regular securities historically available in the market.\textsuperscript{16} These two features concerning investment and financing are not entirely independent. The ability to issue a full complement of what we will call group-state securities is contingent upon the superfund's holdings having a positive value in each state; moreover, the ability to make short sales (by individuals) unnecessary (to be discussed shortly) requires this value to be substantial in relation to the total market value of all securities for each state.

In Remark 2 (see, also, Remark 11), we observed that in a frictionless economy the existence of mutual funds and other pure holding companies (each holding 100% of at most one security) has no impact at all on economic welfare, the essential reason being that it does not expand the allocation opportunities faced by investors. The argument in favor of mutual funds, therefore, must rest entirely on their ability to reduce transaction costs and/or their ability to construct a more "suitable" portfolio for the investor. The evidence with respect to the first point is certainly more convincing than the evidence with respect to the second, in principle if not in practice [see, e.g., Jensen (1969)].

In contrast, Remark 12 shows that the presence of a superfund improves the
economic welfare in a frictionless economy whenever the investors' opportunity set is expanded. In addition, further improvements are achieved as finer and finer breakdowns of the group-states are made (Theorem 3) until in the limit the "ultimate" superfund makes possible the achievement of "maximal" welfare (Remark 14) by achieving, in effect, a complete financial market.

In evaluating the desirability of implementing a superfund, three questions appear most crucial:

(1) What alternatives to the superfund are available and what are their relative merits?

(2) How likely is the superfund to expand the opportunity set faced by investors?

(3) How would the superfund affect investor transaction costs?

It is apparent that these questions involve both theoretical and empirical considerations. We will address each in turn, the first two in this part and the third in Part V.

3.2. An Alternative

Expansion of the investors' opportunity set may also be accomplished by the creation of securities other than those issued by a superfund. In particular, as shown in Hakansson (1974), finer and finer subdivisions of a given firm's securities also improve the economic welfare whenever the rank of the old market structure is increased. Thus, the financing of a firm's expansion by bonds and warrants is better than using bonds alone—again assuming that the first alternative makes possible allocation opportunities not present under the second alternative.

In order to compare the relative merits of supersecurities versus regular security proliferation, we begin by making two key observations. First,
supersecurities have positive proceeds \( a_{js} \) in only a few states, regular securities in most or in all states. Second, the supply of a supersecurity is in every sense large, while that of a given regular security in a market in which the shares of many firms are traded is clearly small.

Since the optimal holdings in any given security, as a proportion of initial wealth \( y_i \), generally differ substantially among investors, the implication of the second property above is that a security in low supply will in equilibrium be sold short by many investors. Since we presently lack the facilities for pure short sales in securities markets, and the absence of pure short-sale opportunities adversely affects welfare, the creation of (low supply) regular securities, \textit{ceteris paribus}, appears less promising than the introduction of supersecurities, in the presence of which short sales, at a minimum, can be sharply reduced (Theorem 2). Turning to the first observation, when the rank of the regular securities market (i.e., of M1) is less than \( n \), we saw (in the proof of Theorem 1) that the inferiority of M1 in relation to a complete financial market (M2) is due to the existence of mutually beneficial side bets for which regular securities offer no resolution. As noted, either finer subdivisions of regular securities or the introduction of supersecurities generally make possible at least some of these side bets, thereby improving the economic welfare (Theorem 3). Since profitable side-bet opportunities are based on differences in the implicit prices \( P_{js} \) of individual states, supersecurities, for which \( a_{js} \) is positive for only a few states, appear to offer a more direct, and hence more efficient, means for quickly incorporating the major side-bet opportunities not resolvable within M1.

3.3. \textbf{Likelihood of Rank Increase}

As we have repeatedly stated, neither supersecurities nor subdivisions of regular securities will affect welfare unless they improve the opportunities
for final allocation, i.e., unless the rank of M1 is increased. We must therefore assess how likely a rank increase is as given sets of group-states and regular securities are further and further subdivided.

With respect to subdivisions of existing securities, it is clearly not easy, at least as a practical matter, to determine whether a proposed set of subdivisions of a given security can be duplicated by already existing securities. In any case, when there are many similar firms in the economy, the likelihood of a rank increase appears more limited; more importantly, the likelihood of a significant welfare improvement in this case seems much more remote. As a practical matter, there are also traditionally severe self-imposed limits on the number of distinct instruments issued by firms. This is true even though in the 1960's considerable ingenuity was applied to the design and marketing of new types of financial instruments.

On the other hand, that a finer breakdown of supersecurities should be accompanied by an increase in rank seems a foregone conclusion. Suppose a new (finer) supersecurity gives the over-all market return if that falls between 8 and 10%, inclusive. How could any conceivable combination of real-world regular securities and old supersecurities possibly duplicate that kind of return pattern, combined with, say, a return of 5%, if the market return falls outside the 8-10% range?

3.4. Options

While the preceding compared some important aspects of supersecurities to regular security proliferation, a similar comparison can be made with respect to options (puts and calls). Options differ from regular securities in that they are created by investors themselves—their very existence, in fact, testifies to the presence of side-bet opportunities not resolvable within the regular security market (M1). Schrems (1973) and Ross (1974) have argued that options
also provide an effective means whereby the efficiency of the regular
market can be improved. Based on our examination to this point, their effec-
tiveness would seem to fall in between that of regular security proliferation
and that of supersecurities. In particular, compared to supersecurities, the
supply of options will tend to be lower, they will tend to be attached to
"smaller" (individual firm) events, they have positive payoffs for a larger set
of basic (relevant) states, are generally attached to overlapping rather than
mutually exclusive states, and (as a result) possibly have a lower chance of
increasing the rank of available instruments. In any event, these differences
suggest that the degree of welfare improvement achievable with a given number
of supersecurities would tend to be greater than that obtainable with the same
number of options. Furthermore, as shown by Ross (1974), options may be
incapable of increasing the rank of the market to \( n \) (under heterogeneous
probability beliefs) whereas supersecurities always have that capability
(Remark 4)—this is true even when preferences are state-dependent. Options
are also subject to high transaction costs [see, e.g., Black and Scholes (1972)],
a characteristic which appears to be largely avoidable in conjunction with
supersecurities, as we will note in Section VI.

IV. TAKING COGNIZANCE OF THE NITTY GRITTY
OF INVESTMENT DECISION MAKING

4.1. Which Superfund?

Suppose, then, that we wish to consider the establishment of a superfund.
In so doing, two questions immediately arise: How could such a fund in fact be
implemented? Which superfund should be chosen among all possible such funds?
Let us begin by reviewing the situation surrounding the "ultimate" superfund,
i.e., one which always achieves a complete financial market by issuing one
security for each state $s$. As we have seen, the strongest argument in favor of a complete financial market rests on the special ability of such a market to achieve a Pareto-efficient allocation of resources among investors. Despite this clear-cut virtue there are, as noted in Section I, at least four important reasons why complete markets have not come into existence: the difficulty of agreeing \textit{(a priori)} on a set of acceptable time-states, the nontriviality of the subsequent determination of which state actually occurred, "the moral hazard," and the enormity of the number of states, and consequently the number of securities, that apparently would be involved. For example, if 1,000 of the regular instruments were capable of yielding one of 10 different proceeds at the end of the period, the number of states $n$ would be greater than 1,000.\textsuperscript{10}

Clearly, any attempt to grapple with these questions must take into account not only the ability of investors to deal with a huge state-space and extremely small probabilities, but also their \textit{willingness} to do so.

\textbf{4.2. Assumption One}

We now make the general supposition, based primarily on armchair observation, that the amount of detail an investor is willing to put up with in making his investment decision is limited. This general premise will be translated into two specific assumptions for the purpose of examining its implications in relation to the superfund.

The first assumption we make is

\[ \frac{\pi_{1s_1}}{\pi_{1s_2}} = \frac{\pi_{2s_1}}{\pi_{2s_2}}, \quad \text{all } s_1, s_2 \text{ such that } W_{s_1} = W_{s_2}, \quad i = 2, \ldots, I. \]

By Theorem 4, assumption (Al) implies that the Arrow-Debreu securities for all states with an equal supply of such securities can be combined into superstate securities with no effect on investor allocations or welfare. Thus, when (Al)
holds the "ultimate" superfund need only issue $N < n$ securities, one for each superstate $t$, i.e., one for each different possible value of end-of-period total wealth $W$.

The practical significance of moving from Arrow-Debreu securities to superstate securities should not be overlooked. First, recall that one of the stumbling blocks to implementing a complete financial market has been the difficulty of specifying meaningful state descriptions (how much detail?) and state-spaces $S$ even for single investors, not to mention the difficulties of reaching agreement among all investors. This seemingly unsurmountable difficulty has now been eliminated—the relevant states are identified with the different values of total end-of-period wealth $W$.

Second, even with agreement on some complex $S$, the determination of which state actually occurred might be easier said than done. Suppose, for example, that the earnings of several companies to the end of the period were part of the state description agreed upon. One problem clearly is that audited income statements are not available until months after the end of the period. And, as we know, even auditors make mistakes and are challenged, not to mention a host of other reasons for legitimate dispute. In contrast, the value of $W$ can be made available within a few minutes—it is simply the market value of all regular securities at the end of the period—and anyone can quickly verify it if he feels so compelled.

Third, we must consider the "moral hazard." Suppose, as in the previous illustration, the current earnings of some firms were part of the state description $s$. Then it would not be difficult, at least within a narrow range, for an investor to change the state that actually occurs, by sabotage or even legitimate means. For the investor to (significantly) influence the market value of all securities at the end of the period, while not impossible, is
clearly a much more difficult task. We shall return to this point shortly.

Finally, the number of possible values assumed by $W$, $N$, while conceivably quite large, can undoubtedly be expected to be substantially smaller than the number of states contained in any agreed upon complex set $S$.

Having observed that the value of Assumption One is considerable, we must now assess the price we have paid in making it. What (A1) says is that the conditional probabilities of each $s$ given $t$, denoted $\pi_{is|t}$, are the same for all investors. It is important to note that this permits nonhomogeneous assessments for all states, without any restriction at all on the probabilities assigned to superstates. Clearly, then, (A1) is much weaker than the assumption of homogeneous probability beliefs contained in most equilibrium models in finance, in particular, the capital asset pricing model, both the one-period version [Sharpe (1964), Lintner (1965), Mossin (1966)] and the continuous-time version of Merton (1973).

Ultimately, the realism of (A1) is, of course, an empirical matter. But it seems safe to conjecture that if $N$ (the number of distinct values of $W$) is large, the investor may neither have the time nor the inclination to go beyond an assessment of $\pi_{it}$—or, if he did, be willing to rely on the numbers he would thereby obtain.

4.3. Assumption Two

In the United States the numbers $W_t$ (the value of all regular securities if superstate $t$ occurs) are in the billions and trillions. We also know that most people are rather uncomfortable in working with large numbers, even if the task is only to attach small numbers (probabilities) to them. A more natural unit than total wealth is the rate of return or the wealth relative, $R_t$ (100% plus the rate of return), i.e.,
\[ R_t = \frac{W_t}{Y}, \quad t = 1, \ldots, N, \]

where \( Y \equiv \sum_{i=1}^{T} y_i \), total investor wealth of the beginning of the period.

Looking at the historical record since 1925, we find that the realized (annual) values of \( R_t \) have fallen in the range [Fisher and Lorie (1968)]

\[ .528 \leq R_t \leq 2.084. \]

But even with this background, it is, of course, not completely clear what the range of \( R_t \) in any given future period is other than that it is quite narrow (such as between 0 and, perhaps, 2).

Before worrying further about how to solve the range problem, let us consider another aspect of \( R_t \) (and the estimation of \( \pi_{it} \)). How many people would make a distinction between a return of 8.145% and 8.146%? Between 8.1% and 8.2%? Between 8% and 9%? My guess is that almost no one would make a distinction in the first case, almost everyone in the third one. And after once having had the experience of assessing, or attempting to assess, the probabilities of market returns to the nearest 1/1000 of 1%, even fewer would do it a second time. Thus, the following assumption is stated with some confidence:

\[ (A2) \quad \text{For each interval } \bar{R} \in (0, \infty) \text{ of length } L(\bar{R}), \text{ there exist numbers } \varepsilon > 0 \text{ and } \bar{r} \in \bar{R} \text{ such that whenever } L(\bar{R}) \leq \varepsilon, \text{ all investors would be willing to substitute the conditional distribution function } F_{\bar{R}}(r) \text{ for the conditional distribution function } F_{i\bar{R}}(r), \text{ where} \]

\[ F_{i\bar{R}}(r) \equiv \Pr_i \{ R_t \leq r | R_t \in \bar{R} \}, \quad i = 1, \ldots, I; \]

\[ F_{\bar{R}}(r) \equiv \begin{cases} 0, & \text{if } r < \bar{r} \\ 1, & \text{if } r \geq \bar{r} \end{cases}. \]
We maintain an open mind as to whether $\bar{R}$ might be the midpoint of the interval $\bar{R}$ or some other value.

It should be emphasized that the meaning of Assumption Two is not that the investor is willing to forego $F_{iR}(r)$ for $F_R(r)$ were $F_{iR}(r)$ available: rather, the assertion is that the investor is willing to use $F_R(r)$ instead of going to the trouble of developing $F_{iR}(r)$. Thus, (A2) is an attempt to give at least explicit recognition to the costs of performing economic calculations and gathering information prior to making the decision.

4.4. An Illustration

Casual observation suggests that the interval length for which investors are willing to employ $F_R(r)$ in lieu of $F_{iR}(r)$ is smaller for intervals $\bar{R}$ containing 1 and progressively larger as $\bar{R}$ moves away from the region near 1.

As a purely hypothetical illustration, it is conceivable, on the basis of (A1), (A2), and (88), that the state-space $S$ given in Table I might be agreed upon. Presumably state 121, for example, will be deemed to have occurred if the market portfolio return was greater than or equal to 14.875% and less than 15.25%, etc. Thus, if the supply of #121 securities is such that $1 per share can be distributed if the market portfolio return happens to be exactly 15%, then #121 will pay (if it pays at all) from $.9989 to $1.0022 per share, etc.

No hard evidence, of course, is available on the minimum number of securities that would be required for (A2) to be satisfied. But it is noteworthy that our example, which achieves a rather high degree of return discrimination, requires the issuance of only 216 securities by the superfund. Clearly, a finer breakdown of returns demands that a larger set of securities be issued.
### TABLE I

<table>
<thead>
<tr>
<th>State $\bar{s}$ (No.)</th>
<th>Return on market portfolio</th>
<th>State $\bar{s}$ (No.)</th>
<th>Return on market portfolio</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-50. % or less</td>
<td>120</td>
<td>14.75%</td>
</tr>
<tr>
<td>2</td>
<td>-49. %</td>
<td>121</td>
<td>15.00%</td>
</tr>
<tr>
<td>3</td>
<td>-48. %</td>
<td>122</td>
<td>15.5 %</td>
</tr>
<tr>
<td></td>
<td></td>
<td>123</td>
<td>16.0 %</td>
</tr>
<tr>
<td>40</td>
<td>-11. %</td>
<td>140</td>
<td>24.5 %</td>
</tr>
<tr>
<td>41</td>
<td>-10. %</td>
<td>141</td>
<td>25.0 %</td>
</tr>
<tr>
<td>42</td>
<td>-9.5 %</td>
<td>142</td>
<td>26.0 %</td>
</tr>
<tr>
<td></td>
<td></td>
<td>143</td>
<td>27.0 %</td>
</tr>
<tr>
<td>60</td>
<td>-5 %</td>
<td></td>
<td></td>
</tr>
<tr>
<td>61</td>
<td>0. %</td>
<td>215</td>
<td>99.0 %</td>
</tr>
<tr>
<td>62</td>
<td>0.25%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>63</td>
<td>0.50%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>64</td>
<td>0.75%</td>
<td>216</td>
<td>100.0 % or more</td>
</tr>
</tbody>
</table>

However, recall that before Assumptions One and Two were invoked, the number of state securities was visualized as being in neighborhoods like 1,000.10

4.5. **The Target Partition: $S^*$**

For purposes of discussion, we now make the following definition.

**Definition 3.** A partition $\bar{S} = \{\bar{s}_1, \ldots, \bar{s}_K\}$ of $S$ will be denoted $S^*$ if (i) all investors are willing to substitute $\bar{S}$ for $S$, (ii) $\bar{S}$ has the smallest $K$ (denoted $K^*$) of all $\bar{S}$ satisfying (i).

In view of (A2), a partition $S^*$ will exist but need not be unique.

If $K^*$ is small, which it may well be under Assumptions One and Two, the
answer to the question "Which superfund?" seems clear, namely, that superfund
which issues securities on the basis of partition $S^*$. Conceivably, $K^*$ may
be in the region from 100 to 1000. As a practical matter, of course, $S^*$ would
not be easy to determine. In real life, then, a partition $\tilde{S}$ for which (i) is
expected to hold for most investors might be chosen.

V. SOME IMPLICATIONS OF $S^*$

We now examine the implications of establishing a superfund based on parti-
tion $S^*$ (which will also be referred to as the Superfund). In practice, as we
have already noted, it may be difficult, of course, to determine $S^*$ so that a
partition $\tilde{S}$, which provides a complete financial market for a large number of
investors, might in fact be selected. In the latter case, the conclusions in
this and the following subsections would not necessarily apply to those inves-
tors for which $\tilde{S}$ is not sufficiently fine to give rise to a complete financial
market—but they do apply to all other investors (who are presumably in the
majority).

5.1. Attainable Return Patterns: Examples

It was pointed out in Part III that the strongest argument in favor of the
superfund as an alternative to regular security proliferation is its unique
ability to expand, in a particularly efficient manner, the return patterns
available to the investor. We now illustrate this point by some examples.

Suppose that there are 216 supersecurities, corresponding to the superstates
shown in Table I, and that their prices $p_{s^*}$, based on a supply of $\delta_{s^*}$
shares (as opposed to a supply of $\delta_{\text{min}}$ shares as in (71)), are those in
Table II—this gives a "riskless" rate of 7.34% (1/.9316).

Some examples of attainable return patterns are given in Table III.
### TABLE II

<table>
<thead>
<tr>
<th>State ( s^* ) (No.)</th>
<th>100 ( p_{s^*} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1,2,\ldots,41</td>
<td>.35</td>
</tr>
<tr>
<td>42,\ldots,60</td>
<td>.50</td>
</tr>
<tr>
<td>61</td>
<td>.75</td>
</tr>
<tr>
<td>62,\ldots,120</td>
<td>.80</td>
</tr>
<tr>
<td>121,\ldots,141</td>
<td>.50</td>
</tr>
<tr>
<td>142,\ldots,165</td>
<td>.24</td>
</tr>
<tr>
<td>166,\ldots,216</td>
<td>.10</td>
</tr>
</tbody>
</table>

\[ \sum_{s^*} 100 p_{s^*} = 93.16 \]

### TABLE III

<table>
<thead>
<tr>
<th>No.</th>
<th>Return pattern</th>
<th>Investment policy (( y_i = 10,000 ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>7.34% no matter what happens in market</td>
<td>Buy 10,734 shares of #1 through #216</td>
</tr>
<tr>
<td>2</td>
<td>0% if market goes down or is unchanged, 9.98% if market goes up</td>
<td>Buy 10,000 shares of #1 through #61, 10,998 shares of #62 through #216</td>
</tr>
<tr>
<td>3</td>
<td>0% if market goes up, 27.8% if market goes down or is unchanged</td>
<td>Buy 12,780 shares of #1 through #61, 10,000 shares of #62 through #216</td>
</tr>
<tr>
<td>4</td>
<td>-5% if market goes down 10% or more, 2% if market goes down less than 10% or up less than 15%, 29.64% if market goes up 15% or more</td>
<td>Buy 9,500 shares of #1 through #41, 10,200 shares of #42 through #120, 12,968 shares of #121 through #216</td>
</tr>
<tr>
<td>5</td>
<td>-60% if market goes down or is unchanged, 31.51% if market goes up</td>
<td>Buy 4,000 shares of #1 through #61, 13,151 shares of #62 through #216</td>
</tr>
<tr>
<td>6</td>
<td>13.23% if market goes up from 10 to 25%, 5% otherwise</td>
<td>Buy 11,323 shares of #101 through #141, 10,500 shares of all others</td>
</tr>
<tr>
<td>7</td>
<td>256.08% if market goes up 16%, 6% otherwise</td>
<td>Buy 35,608 shares of #123, 10,600 shares of all others</td>
</tr>
</tbody>
</table>
Policy No. 1 would be appropriate for an investor who wished to avoid "all" risk and would be a substitute for a riskless bond of the same maturity. Policy No. 2 might be chosen by an individual who is also very risk-averse but somewhat "optimistic" about the market's chances of going up; an even more "optimistic" investor might choose Policy No. 5. A "pessimistic" investor, on the other hand, might select something like Policy No. 3. An investor who believes a gain from 10 to 25% in the market is quite likely, but who is also very risk-averse, might invest according to Policy No. 6. In any case, it is evident that investors can invest according to their tastes and beliefs in a way which offers unlimited flexibility and requires only a modicum of information processing and verification compared to what is needed in regular markets: return patterns such as those given in Table III are, at least as a practical matter, simply not available in such markets even when supplemented with numerous options.

5.2. **Short Sales**

We now turn to an analysis of the impact on investor transaction costs caused by the presence of the type of new securities we are discussing. Transaction costs will be viewed in a broad sense: thus we will also examine how they are affected by changes in short-sales patterns, borrowing patterns, and the numerosity of securities.

By a direct application of Theorem 2, it follows that no security need be sold short in equilibrium in the presence of a Superfund holding a large enough proportion, $\delta$, of all regular securities outstanding. Even if a smaller $\delta$ is chosen, supersecurities need never be sold short and the need for short sales of regular securities would be considerably less than in the absence of a Superfund. Since real-world short sales practices, which require a deposit and severely restrict the investment of the proceeds, are distinctly inferior in an economic welfare sense to the pure short sales assumed in Section II, it is
clear that the availability of a Superfund makes possible a substantial reduc-
tion in the "transaction costs" associated with short sales by sharply reducing
these costs or entirely eliminating them.

5.3. Borrowing

Since borrowing is nothing but the short sale of a possibly riskless
security, the need for borrowing can also be sharply curtailed or eliminated if
a Superfund were available. In practice, borrowing investors typically pay a
premium over and beyond the rate received by saving investors; furthermore,
borrowing is usually restricted by the so-called margin requirement. Both of
these phenomena reduce economic welfare compared to what it would otherwise be
under assumptions (1) and (2).

To illustrate how borrowing can be avoided in the presence of a superfund
based on $S^*$, assume that investor $i$ wishes to buy $z_{12}$ shares of Co. $X$
stock at price $P_2$ on margin, borrowing $z_{11}$ dollars at 10% to complete the
purchase (thus, $\sum_{s^*} P_{s^*} = 1/1.10$). The proceeds on the stock in state $s^*$
are $a_{2s^*}^z$, giving him ending wealth $w_i(s^*) = a_{2s^*}^z z_{12} - 1.10z_{11} > 0$, on an
investment of $P_2 z_{12} - z_{11}$, where, by (60), $P_2 = \sum_{s^*} a_{js^*}^z P_{s^*}$. But he can
also obtain allocation $v_i(z^*)$ by purchasing $a_{2s^*}^z z_{12} - 1.10z_{11}$ shares of each
supersecurity $s^*$, which would cost him $\sum_{s^*} (a_{2s^*}^z z_{12} - 1.10z_{11}) P_{s^*} = P_2 z_{12} - z_{11}$,
or by acquiring $v_2 z_{12}$ shares of security 2 ($0 < v_2 < 1$) and $(1 - v_2) a_{2s^*}^z z_{12} -
1.10z_{11}$ shares of each supersecurity $s^*$, also for the same total cost.

In sum, the establishment of a superfund based on $S^*$ makes it possible
to eliminate, or at least reduce, certain socially "wasteful" costs such as the
costs connected with the approval and monitoring of short sales and margin loans,
margin calls, and bankruptcy resulting from investors overextending themselves.
In addition, investors will effectively be borrowing at the (riskfree) lending
rate of interest, which presumably would be equal to the rate on government
securities maturing at the end of the period, a possibility not presently available to most investors.

5.4. Estimating the Probabilities

In the presence of a superfund based on $S^*$, investor $i$ must assess the probability vector $\pi_{is^*}$. The components $\pi_{is^*}$ may be referred to as gross probabilities or macro-probabilities. It is not clear whether investors would be more comfortable estimating such probabilities directly or whether they would prefer building them up from estimates of the likelihood of detailed joint events affecting firms. The latter approach is, of course, enormously more laborious. The first approach is the one used in economic forecasting generally (in favor of the second), and investors may well feel more confident about aggregate estimates based on simple (incomplete) models than such estimates based on a more complex but less "reliable" model. In any case, the superfund permits the investor to by-pass the numbers $a_{js}$ —the estimation of which is no simple task—entirely.

5.5. The Administrative Framework: Iteration Zero

While a detailed discussion of how the Superfund might operate in reality is beyond the scope of this paper, the reader is entitled to some notion of how if might function as a basis for judging its real-world potential. The following possibilities are suggested only as points of departure in initiating a dialogue on the subject.

(1) The Superfund would issue supersecurities on an annual basis, e.g., at the opening of business each second Monday in January for a period ending at the close of business on Friday preceding the second Monday in January the following year.

(ii) The Superfund would be operated by an affiliate of the New York
Stock Exchange.

(iii) The Superfund would own up to perhaps 40% of the "market portfolio."

(iv) The composition of each year's "market portfolio" would be announced well in advance and remain fixed for the year. It would consist of a broad cross section of perhaps 2500 of the more actively traded instruments: stocks, bonds, warrants, government securities (but excluding less than 100% holding companies), whether traded in major, regional, or over-the-counter markets.

(v) The value of the Superfund at the end of the year would consist of the closing market values of the original instruments, plus dividends and the market value of instruments (or cash) received in exchange in the case of mergers, and from stock splits and stock dividends.

(vi) Individual investors would be permitted unlimited (initial) purchases of supersecurities. To make this possible, it may be necessary to restrict institutional investors from, say, placing more than 25% of their assets in supersecurities.

(vii) Investors holding supersecurities over their whole term would pay no commissions; instead, the cost of operating the Superfund would be subtracted, on a pro rata basis, from end-of-period "distributions."

(viii) During the year supersecurities would be bought and sold like other securities on a commission basis and would be listed on the NYSE.¹⁷

(ix) As a rule, initial offerings and final redemptions of supersecurities would be paid in cash; in the case of re-investment from one year to the next, cash exchanged with any investor would be limited to net additions or net withdrawals, if any.

(x) When the Superfund is expanding its assets prior to the beginning of the year (and, in particular, prior to the beginning of the first
year), payments by individual investors for supersecurities could not only be in the form of cash but in the form of regular securities (at least those included in the fund's "market portfolio" and in the form of shares in (specified) mutual funds). 18

(i) Mutual fund shares received from investors in payment for supersecurities would be redeemable in regular securities needed by the superfund and chosen from an "offer" list submitted by the mutual fund in question.

(ii) Cash and "excess" regular securities received by the Superfund for supersecurities would be exchanged (without commission) for "deficit" securities, at previous closing prices, with institutional investors submitting "accept" and "offer" lists. Any remaining "imbalance" in the superfund's holdings would be corrected by open market transactions.

(iii) Initial supersecurity prices would be determined by aggregation of investor demand schedules submitted for each supersecurity, the investors' payments, and perhaps the constraint \( \sum_{g^*} P_{g^*} = 1/(1 + r) \), where \( r \) is the current yield on a "risk-free" government security of comparable maturity.

As noted, the above should not be viewed as a concrete proposal but rather as suggestive of how the Superfund might operate. The details of how the Superfund might function are best left to be worked out by those more qualified for that task than I. But it seems clear that the Superfund, as conceived in this paper, is an implementable concept. And even if the cost of operating the
3.1. Introduction

As a financial intermediary, the superfund differs from the usual mutual fund in two respects. First, in contrast to a mutual fund, a superfund (generally) owns the so-called market portfolio, i.e., it purchases the same percentage, e.g., 25%, of all the shares that are traded in the market.\textsuperscript{15} Second, while mutual funds issue only one kind of security, ordinary (common) shares, a superfund generally issues more than one kind of instrument and these instruments are different from the regular securities historically available in the market.\textsuperscript{16} These two features concerning investment and financing are not entirely independent. The ability to issue a full complement of what we will call group-state securities is contingent upon the superfund's holdings having a positive value in each state; moreover, the ability to make short sales (by individuals) unnecessary (to be discussed shortly) requires this value to be substantial in relation to the total market value of all securities for each state.

In Remark 2 (see, also, Remark 11), we observed that in a frictionless economy the existence of mutual funds and other pure holding companies (each holding 100% of at most one security) has no impact at all on economic welfare, the essential reason being that it does not expand the allocation opportunities faced by investors. The argument in favor of mutual funds, therefore, must rest entirely on their ability to reduce transaction costs and/or their ability to construct a more "suitable" portfolio for the investor. The evidence with respect to the first point is certainly more convincing than the evidence with respect to the second, in principle if not in practice [see, e.g., Jensen (1969)].

In contrast, Remark 12 shows that the presence of a superfund improves the
Superfund were, say, $100 million per year, the transaction cost for "full-term" investors would be but a small fraction of 1%.

5.6. **Other Implications**

As one might suspect, the impact of a superfund based on \( S^* \) would reach well beyond investors (and the securities industry). For the firm, the implications are essentially twofold, touching upon two of its most crucial decisions: capital budgeting and the choice of financing.

Under the usual competitive market assumptions (in essence, that the firm's capital budget is not "big" enough to appreciably affect the prices \( p_{s^*} \)), the presence of the Superfund implies the following (Pareto-efficient) capital budgeting decision rule: Choose that project combination \( b \) which maximizes

\[
\sum_{s^*} v_{s^*}(b)p_{s^*},
\]

where \( v_{s^*}(b) \) is the net increase in the value of the firm at the end of the current period under project combination \( b \) if state \( s^* \) occurs. The practical significance of this implication should not be overestimated since the numbers \( v_{s^*}(b) \) are not easy to come by, especially for long-term projects. But (89) does indicate that the appropriate rule, in the presence of the Superfund, is a present value rule (which calls for maximizing the value of the firm) and does pinpoint attention on the critical quantities that are relevant for management to consider.

The essence of the Superfund is that it makes possible the "ultimate" allocation of resources among investors without having to rely on the allocation possibilities made available by individual firms via their capital structures (which are generally inadequate for the purpose—see Section 2.7, especially Remark 8). The implication of this is that the firm's capital structure becomes
irrelevant so that the firm gains nothing by issuing more than one security. However, there is generally no harm (in an allocation sense) caused by instrument proliferation by the firm.

The presence of the Superfund is also likely to increase the pressure for abolishment of regulated interest rate maxima on time and savings deposits. The reason for this is that financial institutions accepting (short-term) time deposits will face direct competition from the Superfund. Finally, since investors now can avoid borrowing (Section 5.3), the margin requirement, as an instrument of monetary policy, will lose some of its potency. Other ramifications resulting from implementation of the Superfund can also be cited but will, in the interest of space, be left for another occasion.

VI. MULTIPERIOD EXTENSIONS

Up to this point we have assumed a single-period model in which all investors have the same horizon. Let us now consider a (Fisharian) multiperiod consumption-investment model with a possibly random horizon. If consumers prefer more to less, are risk-averse, and have infinite marginal utility for a nonnegative level of consumption in each period, then their induced utility of wealth functions will satisfy (2) (written somewhat differently) in each period under weak conditions, [see, e.g., Fama (1970) and Sibley (1974)]. Moreover, the optimal strategy of each investor calls for him to review and (generally) revise his portfolio at the beginning of each period. This implies that (in the absence of exchange friction) all current investments, regardless of the expiration dates of the instruments traded, are made with respect to a horizon which falls at the end of the current period; all information about the future beyond that point is contained in the (currently applicable) induced utility of wealth function. As a result, if the decision points of all market participants coincide, a market structure in which single-period Arrow-Debreu
securities are issued sequentially as the periods arrive may be equivalent to one in which all maturities of Arrow-Debreu securities are issued at the beginning. The reason for this is twofold: The availability of Arrow-Debreu securities maturing beyond the current period need not change the opportunity set for the current period's investments; their impact is limited to the currently applicable induced utility of wealth functions, and this impact is based on the assumption that such securities will (still) be available in future periods.

This leads us to an important point: The preceding is, of course, true only if the current-period set of securities is in fact a "complete" set of securities. Whether this is so is determinable by reference to the induced (end-of-period) utility of wealth functions; if these functions depend only on wealth (as they typically do if future return distributions are perceived to be independent of previously realized returns and utility functions of consumptions are separable), then the superstates given by Theorem 4, identified one period at a time, are adequate if (85) holds. But if the currently applicable utility functions $V(c, (w, \hat{s}))$ depend on an end-period state description $(w, \hat{s})$ which goes beyond wealth $(w)$ [see, e.g., Hakansson (1971), Rubinstein (1974)], then the sequential issuance of securities based on superstates are sufficient to achieve a complete financial market only if the stronger condition

\[
(90) \quad \frac{\partial V(c, (w, \hat{s}_{i1}))}{\partial w} = \frac{\partial V(c, (w, \hat{s}_{i2}))}{\partial w} = \frac{\partial V(c, (w, \hat{s}_{i1}))}{\partial w} = \frac{\partial V(c, (w, \hat{s}_{i2}))}{\partial w}, \quad \text{all } s_1, s_2 \in \hat{s}_k, \text{ all } w > b_i, \quad k = 1, \ldots, m, \quad i = 2, \ldots, I,
\]

where $s_1 \in \hat{s}_{i1}$, $s_2 \in \hat{s}_{i2}$, all $i$, holds. However, the concept of a superfund is still viable if (90) is violated since supersecurities can, in principle, be issued for a state space of unlimited richness. One of the
unanswered questions is the extent to which significant welfare improvements can be obtained by breaking down the supersecurities defined in this paper into even "finer" instruments.

If the investment horizons among investors differ, then it would be necessary for the superfund to issue several maturities, one for each horizon (a somewhat analogous practice occurs in the option market). If (A1) or (90) holds for each horizon (whichever is applicable) along with (A2), the superfund would again, by sequential issuance of supersecurities applicable to currently relevant investor horizons, achieve all the benefits of a complete financial market. As a practical matter, determining the set of (exactly) relevant terms of supersecurities would not be easy, of course; and multiple-term securities would increase the number of securities in the market.

VII. CONCLUDING REMARKS

We have examined the primary implications that would result from the creation of a Superfund which would own a substantial portion of the "market portfolio" (say, from 10 to 40%) and (periodically) issue (single-term) securities for a set of pre-specified superstates (say, from 100 to 1000). Under assumptions which do not appear unrealistic with respect to preferences, probability beliefs, and the amount of detailed information an investor might be willing to process and develop before making his decision, we found that such a Superfund might well provide all the advantages of a complete financial market for a majority of investors. For these investors, primary benefit would be the availability of all relevant side-bet opportunities within the market, a clearly positive factor from the economic welfare viewpoint. But even for the remaining investors, the Superfund offers a much more efficient and reliable means of incorporating the major side-bet opportunities left unresolvable by conventional financial markets (evidence or the existence of such side-bet
opportunities is provided by the market in options) than security proliferation by firms and even the establishment of (more) sizeable option markets. Furthermore, the need for actual short sales and margin loans would be sharply reduced, eliminating much of the cost associated with credit approval, collateral requirements, margin account monitoring, margin calls, investor insolvency, etc; investors will, in effect, be able to make cost-free "pure" short sales and to borrow and lend at the "risk-free" lending rate. Investors who make their portfolio revisions relatively infrequently will also be able to virtually escape transaction costs. The main beneficiary would clearly be the small investor, who in the past has been forced to deal with highly disaggregated (and complex) information, limited opportunities for diversification and financing, and high transaction costs (per dollar of investment). While (no-load) mutual funds have made a high degree of diversification available to small investors at something on the order of a pittance, such funds are unable to provide the other advantages of the Superfund. And since the Superfund would take much of the wind out of the "increase-in-welfare" argument as a reason for security proliferation by firms, the establishment of a Superfund might actually reduce the total number of securities traded in the market, compared to what it otherwise would be, over a period of time—with superior welfare results.
1. The HARA-class (hyperbolic absolute risk aversion) consists of the following utility functions of wealth with the properties $u'(w) > 0$, $u''(w) < 0$ (the first over at least a finite range of positive wealth):

$$ u(w) = \begin{cases} 
\frac{1}{\gamma} (w + a)^{\gamma} & \gamma < 1 \quad \text{(decreasing absolute risk aversion)} \\
- \exp (\gamma w) & \gamma < 0 \quad \text{(constant absolute risk aversion)} \\
- (a - w)^{\gamma} & \gamma > 1, \text{ a large (increasing absolute risk aversion)}. 
\end{cases} $$


3. Two complications are circumvented by assuming strong rather than weak inequalities in (1). First, our expressions become less complex, with little qualitative change in the results. Second, we avoid confronting the possibility that state $s$ actually occurred even though some individuals assessed $\pi_{is}$ to be 0—which, in turn, raises issues beyond the scope of this paper.

4. A richer preference framework is considered in Section VI.

5. The first and second properties (more wealth preferred to less, nonsaturation, and risk aversion) require no comment. The third assumption (infinite marginal utility of wealth at some nonnegative level), while not difficult to defend on empirical grounds, also keeps the technicalities of our analysis to a minimum.

6. A riskless asset is one for which $a_{js} = a_j$, all $s$. While it is common to assume that the riskless asset, when it exists, is created by the investors themselves, i.e., that $Z_j = 0$, there is no loss of generality in assuming that some quantity has already been issued by, for example, a government agency.

7. An Arrow-Debreu security may be viewed as a special case of a regular security, that is, one for which $a_{js} = 1$, one $s$, $a_{js} = 0$, all other $s$.

8. Since last period's $w_i$ equals this period's $y_i - r_i$, and since $\Pr\{w_i > 0\} = 1$ in (the previous) equilibrium [see, e.g., (16)], this assumption is rather mild.
9. Even though, for given prices $P_j$, (15) and (13) have a unique solution for each $i$ if all securities are linearly independent [e.g., Hakansson (1970)].

10. Due to the presence of the $r_i$ terms, the equilibrium prices will be expressed in absolute terms, not just relative terms.

11. The term "constrained Pareto-efficiency" is sometimes used to denote allocations which are efficient relative to a given (constrained) set of allocation possibilities (such as that in ML) [see, e.g., Stiglitz (1972, p. 26)].

12. We will return to this point shortly.

13. By a collection $(\bar{v}_i, u_1(v_i))$ we mean the set of preferences and beliefs possessed by the collection of investors in the market ($i = 1, \ldots, I$).

14. $\bar{s} = \{\bar{s}_1, \ldots, \bar{s}_K\}$ is a partition of $S = \{s_1, \ldots, s_n\}$ if $\bar{s}_k \subseteq S$, $k = 1, \ldots, K$, $\bar{s}_j \cap \bar{s}_k = \emptyset$ if $j \neq k$, and $\bigcup_{k=1}^{K} \bar{s}_k = S$.

15. As a technical matter it is not necessary that the superfund purchase the "market portfolio," only that it own a substantial portion of the "wealth" in each state. But this requirement is most readily satisfied via ownership of the "market portfolio" and, as we will see, there are many pragmatic reasons for choosing this particular asset composition.

16. When $K = 1$, the superfund is "equivalent" to a regular mutual fund (see, also, Remarks 6 and 11).

17. Whether the opportunity to earn commissions on supersecurity transactions during the "year" will be sufficient inducement for "member firms" to provide "no-charge" exchanges "between years" [see (vii)] remains to be seen.

18. The "between years" investment and redemption transactions are probably best handled by the network of (NYSE) member firms, who, as a service, might be willing to accept an even broader class of instruments than those held by the Superfund in payment from investors.


