RESEARCH PROGRAM IN FINANCE

WORKING PAPER NO. 26

A DISCRETE-TIME SYNTHESIS OF FINANCIAL THEORY:
PART III. EXTENSIONS AND PROSPECTIVE

by

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September 1974

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A DISCRETE-TIME SYNTHESIS OF FINANCIAL THEORY:
PART III: EXTENSIONS AND PROSPECTIVE

Mark Rubinstein

This paper is the third of a series of three that provide a synthetic treatment of financial theory, working from the context of a complete securities market. Closed-form solutions are derived for a full-fledged Arrow-Debreu economy incorporating uncertain noncapital income, many commodities, and explicit productive activities. To provide for commodity complementarities and yet obtain closed-form solutions, tastes for each commodity are represented by HARA utility functions which are additively and multiplicatively combined. In the corresponding separation properties and valuation equations, a purchasing power annuity replaces the simpler annuity of the previous papers. The derived financial measure of inflation is properly a ratio of the unweighted sums of the prices of all commodities for two different dates. The rate of growth of the money supply and the way money is utilized in the economy are shown to be capable of substantially altering the stochastic properties of nominal rates of return so that they may have little relationship to the stochastic properties of real rates of return. Closed-form Pareto-efficient producer decision rules are derived for an economy with composite additive logarithmic utility and stochastic Cobb-Douglas production functions. These decision rules are shown to have strong implications for the intertemporal stochastic pattern of security rates of return. The paper concludes by summarizing the results of the series and glancing at unsolved problems.

This paper is the third of a series of three¹ that provide a synthetic treatment of financial theory. The first paper provides a description of a simpler version of the economy analyzed here, as well as closed-form solutions for optimal decision and sharing rules under
various special cases. The second paper provides a derivation of several potentially empirically testable security valuation equations and analyzes the efficiency properties of the economies and their implications for corporate financial decisions and the dissemination of information. This paper continues with a description of an extended economy, modified to include uncertain noncapital income, many commodities, and explicit productive activities. Equilibrium measures of inflation are developed and shown to modify the separation properties and valuation equations of parts I and II. For a linear-logarithmic economy where stochastic elements enter through endowed resources and parameters in production functions, closed-form Pareto-efficient producer decision rules are derived and shown to have strong implications for the intertemporal stochastic pattern of security rates of return. The paper closes by summarizing the results of the series and assesses unsolved problems. Familiarity with the results and notation of the first two papers are presumed.

I. THE EXTENDED ECONOMY

The extended economy contains both consumers and producers. As in the full-fledged Arrow-Debreu economy, consumers are endowed with commodities with their availability conditioned on dates and states. Indexing commodities by \( k = 1, 2, \ldots, K \), let \( \{ x_{k0} \} \), \( \{ x_{ke} \} \) and \( \{ x_{ks,e} \} \) for each individual denote his endowed commodities to become available at dates \( t = 0 \), \( t = 1 \), and \( t = 2 \), respectively. Perfect and competitive
spot markets for commodities are opened at each date with prices \( \{p_{k0}\} \), \( \{p_{ke}\} \), and \( \{p_{ks,e}\} \), respectively. A consumer's initial wealth is then defined as

\[
W_0 = \sum_k p_{k0} \bar{c}_{k0} + \sum_{e,e} p_{ke} \bar{c}_{ke} + \sum_{e,e} p_{e} \sum_{s,e} p_{s} \sum_{k,ks,e} p_{ks,e} \bar{c}_{ks,e}
\]

where \( \{\bar{c}_{e}\}, \{\bar{c}_{s,e}\} \) has the same meaning as before; that is, 
\( \bar{c}_{0} = \sum_k p_{k0} \bar{c}_{k0} \), \( \bar{c}_{e} = \sum_k p_{ke} \bar{c}_{ke} \), and \( \bar{c}_{s,e} = \sum_k p_{ks,e} \bar{c}_{ks,e} \) may be interpreted as his endowed claims to dollar value consumption over dates and states. The consumer's problem is to choose his consumption of commodities over dates and states while satisfying his budget constraint so that if \( \{c_{k0}\}, \{c_{ke}\}, \{c_{ks,e}\} \) represent his choices

\[
W_0 = \sum_k p_{k0} c_{k0} + \sum_{e,e} p_{ke} c_{ke} + \sum_{e,e} p_{e} \sum_{s,e} p_{s} \sum_{k,ks,e} p_{ks,e} c_{ks,e}
\]

Rather than make these choices directly, a consumer is envisaged as first selling at each date his available commodities to producers who issue securities to him (i.e., state-contingent claims). Moreover, since the present value of his future available commodities is known, a consumer may pledge these as payment for securities which he may issue to other consumers. In brief, at date \( t = 0 \), a consumer chooses \( \{c_{k0}\} \) and \( \{W_e\} \) so that \( W_0 = \sum_k p_{k0} c_{k0} + \sum_{e} p_{e} W_e \), where \( \{W_e\} \) represent claims on wealth available at date \( t = 1 \). At date \( t = 1 \), if state \( e \) occurs, the consumer chooses \( \{c_{ke}\} \) and \( \{C_{s,e}\} \) satisfying \( W_e = \sum_k p_{ke} c_{ke} + \sum_{s,e} p_{s,e} C_{s,e} \). Finally, at date \( t = 2 \), if state \( s \) occurs, he spends his remaining wealth on commodities only so that \( C_{s,e} = \sum_k p_{ks,e} c_{ks,e} \).
A consumer's objective is to maximize the expected utility from his choice of commodities. To give structure to the problem we will analyze the solution for the HARA class of utility functions, permitting both modest complementarities and noncomplementarities among commodities. At each date available commodities are assigned to a group of complementary commodities. For example, butter and bread might be assigned to the "food group" while socks and shoes might be assigned to the "clothing group." The groups themselves are assumed noncomplementary. To capture this description, index commodity groups by \( n = 1, 2, \ldots, N \) and let subscript \( k_n = 1, 2, \ldots, K_n \) denote that commodity \( k_n \) belongs to group \( n \) containing \( K_n \) different commodities. Assume that the utility of each commodity is evaluated by elementary utility function \( u_{k_n}(c_{k_n}) \), where a subscript (omitted) permits a different elementary utility function for commodity \( k_n \) at each date. Assume \( u' > 0 \) and \( u'' < 0 \). The total utility of all commodities \( \{c_{k_n}\} \) consumed at any date is represented by \( \Sigma_n \nu_n \Pi_k u_{k_n}(c_{k_n}) \) where \( \{\nu_n \geq 0\} \) are nonnegative constants permitting different preferential weighting for different commodity groups (as well as dates). These weights are standardized so that \( \Sigma_n \nu_n = 1 \). This additive-multiplicative form of utility formalizes complementarities of commodities within groups and noncomplementarities among groups.4 Putting this together, a consumer's single-stage programming problem is to
\[
\max \sum_{n,n_0} \prod_{k_n} u_{k_n}(c_{k_n}) + \rho_1 \sum_{n} \sum_{n_1} \prod_{k_n} u_{k_n}(c_{k_n}) + \rho_1 \rho_2 \sum_{n} \sum_{n_2} \sum_{s} \sum_{e} \prod_{k_n} u_{k_n}(c_{k_n})
\]

subject to his budget constraint.

To derive closed-form solutions, further assume that \( u_{k_n}(c_{k_n}) \) lies within the HARA class. When \( B \neq 0,1 \) (generalized power utility),

\[
\prod_{k_n} u_{k_n}(c_{k_n}) = b(1 - b)_{\prod_{k_n}} \left( A + Bc_{k_n} \right)^{1-b_{k_n}}
\]

where \( \sum_{k_n} b_{k_n} = b \equiv B^{-1} \). The exponents \( b_{k_n} \) reflect the relative preferential weightings applied to different commodities within a commodity group; these weightings are nonstochastic but can depend on the date. However, irrespective of the date or commodity group, they are standardized to sum to the same constant \( b \). Differential preferences among commodity groups are represented by the multiplicative constants \( \{v_{nt}\} \). Since \( \sum_{n,t} = 1 \), differential preferences for total consumption at different dates are captured as before by patience rates \( \rho_1 \) and \( \rho_2 \).

To facilitate the solution of this problem in a manner which makes financial decisions (investment in securities) explicit, it is useful to solve the problem in four successive stages:
stage one: 
\[ \bar{u}_{nt}(C_{nt}) = \max_{\{c_{knt}\}} \Pi_k u_{knt}(c_{knt}) - \lambda_{nt} \left[ \Sigma_k p_{knt} c_{knt} - C_{nt} \right] \]

stage two: 
\[ U_t(C_t) = \max_{\{c_{nt}\}} \Sigma_n U_{nt}(C_{nt}) - \lambda_t [\Sigma_n C_{nt} - C_t] \]

stage three: 
\[ V_e(W_e) = \max_{C_e, \{c_{s,e}\}} U_e(C_e) + \rho_2 \Sigma_s \pi_s \pi_s e_s e_s (C_{s,e}) - \lambda_e [C_e + \Sigma_s e_s e_s - W_e] \]

stage four: 
\[ V_0(W_0) = \max_{C_0, \{W_e\}} U_0(C_0) + \rho_1 \Sigma e e e (W_e) - \lambda_0 [C_0 + \Sigma e e e e - W_0] \]

During stage one, for each date \( t \), commodity group \( n \) and commodity prices \( \{p_{knt}\} \), given the amount of wealth \( C_{nt} \) allocated to consumption of commodity group \( n \) at date \( t \), the consumer determines his optimal choice of commodities \( \{c_{knt}\} \). Knowing this, he can calculate his derived utility function \( \bar{u}_{nt}(C_{nt}) \) of expenditure on commodity group \( n \) at date \( t \). During stage two, using this derived function, for each date \( t \) given his total value of consumption \( C_t \), the consumer determines his optimal allocation of \( C_t \) among consumption in each commodity group. Knowing this, he can calculate his derived utility function \( U_t(C_t) \) of total consumption expenditure at date \( t \). He then proceeds as before in part I of this series of papers, calculating his derived utility of wealth function \( V_e(W_e) \) and finally using this to solve for his date \( t = 0 \) optimal
consumption-investment plan given his initial wealth \( W_0 \). Note that, since prices \( \{p_{k,t}\} \) are uncertain, \( \tilde{u}_{nt}(C_{nt}) \) and \( U_t(C_t) \) will in general be state-dependent utility functions. Therefore derived utility of wealth function \( V_e(W_e) \) will depend on state \( e \) not only through uncertain security prices but also through uncertain commodity prices.\(^5\)

When \( B = 1 \) (generalized logarithmic utility) or \( B = 0 \) (exponential utility), each commodity group \( n \) will be assumed to consist of one and only one commodity so that, in effect, each consumer has an additive utility function over both dates and commodities. In this case, the stage one and two programming problems reduce to

\[
U_t(C_t) \equiv \max_{\{c_{kt}\}} \sum_k u(c_{kt}) - \lambda_t \left[ \sum_k p_{kt} c_{kt} - C_t \right]
\]

where for \( B = 1 \), \( u(c_{kt}) = \ln(A + \frac{c_{kt}}{A}) \) and for \( B = 0 \), \( u(c_{kt}) = -ae^{-c_{kt}/A} \).

The \( n \) subscript has been omitted since it is redundant in these cases.

The effect of endogenous production decisions will be analyzed only for the case of additive logarithmic utility. To obtain closed-form results, production functions are assumed to be Cobb-Douglas. The aggregate amount \( x_{ke}^M \) of each commodity \( k \) produced as an output at date \( t = 1 \) if state \( e \) occurs is related to date \( t = 0 \) inputs by the function

\[
x_{ke}^M = \alpha_{ke} \Pi_h x_{hk0}^B_{he}
\]

where \( x_{hk0} \) is the amount of input \( h \) used to produce output \( k \) and
\[ \alpha_{ke} \text{ and } \beta_{hke} > 0 \] are exogenous production parameters reflecting exogenous technology and the vagaries of "nature." Similarly, for dates \( t = 1 \text{ and } t = 2 \),

\[ x_{ks,e}^{M} = \alpha_{ks,e}^{h} x_{hke}^{h} \]

If \( v_{h} = 0 \), then commodity \( h \) is nonconsumptive and used only for productive purposes. If \( \beta_{hk} = 0 \) for all \( k \), then commodity \( h \) is nonproductive and used only for consumptive purposes. If \( \beta_{hk} = 0 \) for \( h \neq k \) and \( \beta_{hk} = 1 \) for \( h = k \), then commodity \( h \) is simply being stored with depreciation factor \( \alpha_{k} \); and if also \( \alpha_{k} = 1 \), then storage is perfect.

The sum \( \Sigma_{h} \beta_{hk} \) measures the sign of returns to scale in the production of commodity \( k \); \( \Sigma_{h} \beta_{hk} < (\geq, >)1 \) as returns to scale are decreasing (constant, increasing).

Commodities are classified into two mutually exclusive and exhaustive types: primary resources, \( k = H + 1, \ldots, K \), are commodities which cannot be produced or stored and whose availability is exogenously determined in each period; produced commodities, \( k = 1, \ldots, H \), are commodities which must be produced from primary resources or previously produced commodities and whose availability is endogenously determined. Good examples of primary resources include the services of labor and land. In terms of the above production functions, \( \alpha_{k} = 0 \) (all \( h \)) for primary resource \( k \). By assumption, all endowed resources \( \{x_{k}^{M}\} \) are primary.\(^6\)

Therefore, if commodity \( k \) is not a primary resource, then \( x_{k}^{M} = 0 \).
At each date, the available aggregate amount of each commodity
h is allocated among aggregate consumption and production activities so that:

\[ X_{h0} = c_{h0} + \sum_k x_{hk0} \]

\[ X_{he} = c_{he} + \sum_k x_{hke} \quad \text{(if primary)}, \quad X_{he} = c_{he} + \sum_k x_{hke} \quad \text{(if produced)} \]

\[ X_{hs,e} = c_{hs,e} \quad \text{(if primary)}, \quad X_{hs,e} = c_{hs,e} \quad \text{(if produced)} \]

The only constraint placed on the organization of productive ac-
tivities is that it be production-efficient, so that by changing the deci-
sions made, no consumer can be made better off without some consumer
made worse off. Within this constraint, production may possibly be cen-
trally planned or decentralized through competitive firms. Since the
analysis of economic equilibrium will assume sufficient homogeneity across
consumers for the existence of a composite consumer (with additive loga-
ithmic utility), Pareto-efficient production decisions are derived simply
by maximizing the composite expected utility function subject to the above
Cobb-Douglas production functions and closure conditions. Rubinstein
(1974a) has shown that the production decisions so made are the same as
those chosen if production decisions were decentralized by competitive
value-maximizing firms.
II. OPTIMAL CONSUMER DECISION AND SHARING RULES

Since the method of solution required for the extended economy is in substance the same as used in part I, this section will confine itself to stating and interpreting results without indicating how they are derived. Successively solving the four-stage programming problem outlined in section I, for \( B \neq 0,1 \) the derived utility functions can be shown to be

\[
\bar{u}_{nt}(C_{nt}) \sim h_{nt} \frac{b}{1-b}(A\psi_{nt} + BC_{nt})^{1-b}
\]

\[
U_t(C_t) \sim h_t \frac{b}{1-b}(A\psi_t + BC_t)^{1-b}
\]

\[
V_e(W_e) \sim k_e \frac{b}{1-b}(A\phi_e + BW_e)^{1-b}
\]

\[
V_0(W_0) \sim k \frac{b}{1-b}(A\phi + BW_0)^{1-b}
\]

where

\[
h_{nt} \equiv \Pi_{k_n} \left( \frac{1-b_{k_n}}{p_{k_n}} \right)^{1-b} k_{nt}
\]

\[
h_t \equiv \sum_n (u_{nt} h_{nt})^B
\]

\[
k_e \equiv h_e + \sum_s p_{s.e} \left( \rho_{2_s.e} / p_{s.e} \right)^B s.e
\]

\[
k \equiv h_0 + \sum_e p_e (\rho_{1_e} / p_e) k_e
\]

\[
\psi_{nt} \equiv \sum_{k_n} p_{k_n} \psi_t \equiv \sum_n \psi_{nt}, \quad \phi_e \equiv \psi_e + \sum_s p_{s.e} \psi_{s.e}
\]

\[
\phi \equiv \psi_0 + \sum_e \phi_e
\]
Similarly, for $B = 1$

$$U_t(C_t) \sim \ln(A \psi_t + C_t)$$

$$V_e(W_e) \sim (1 + \rho_2)\ln(A \phi_e + W_e)$$

$$V_0(W_0) \sim (1 + \rho_1 + \rho_1 \rho_2)\ln(A \phi + W_0)$$

and for $B = 0$

$$U_t(C_t) \sim -e^{1/h_{\psi_e}} C_t / (A \psi_t)$$

$$V_e(W_e) \sim -\phi^{1/k_{\phi_e}} W_e / (A \phi_e)$$

$$V_0(W_0) \sim -\phi^{1/k_{\phi_e}} W_0 / (A \phi)$$

where, $\psi_t$, $\phi_e$ and $\phi$ are as previously defined and $h_t \equiv \Sigma_{k,pt} \ln(v_{kt} / p_{kt})$, $k_e \equiv \Sigma_{s,se} \ln(\rho_{s,se} / P_{s,se})$, $k \equiv \Sigma_{se} \phi_e \ln(\rho_{1,se} / P_e) + k_e$.

The derived functions, $V_e(W_e)$ and $V_0(W_0)$, are quite similar to their counterparts developed in part I. The essential difference lies in the impact of uncertain inflation on the revised state variables $\phi_e$ and $\phi$. To aid in interpreting these variables, it is convenient to select the "numeraire" so that $\psi_0 \equiv \Sigma_{k} k_{k_0} = 1$. Defining
\[ 1 + r_{It} = \frac{\psi_t}{\psi_{t-1}}, \phi_e \] and \( \phi \) may be rewritten as

\[ \phi_e = (1 + r_{Ie}) \left[ 1 + \sum_s p_{s,e}(1 + r_{I_s,e}) \right] \quad (\text{all } e) \]

\[ \phi = 1 + \sum_e \rho_e (1 + r_{I_e}) \sum_s p_{s,e} \left( 1 + r_{I_s,e} \right). \]

\( r_{Ic} \) has the natural interpretation of a rate of inflation and \( \phi - 1 \) may be interpreted as the date \( t = 0 \) price of a purchasing power annuity yielding equal certain real payments at all future dates. That is, the annuity yields \( 1 + r_{Ie} \) dollars at date \( t = 1 \) and \( (1 + r_{Ie})(1 + r_{I_s,e}) \) dollars at date \( t = 2 \). With uncertain inflation, the nominal payments of the annuity are uncertain but the real payments are certain.

The surprising feature of this annuity is its index for inflation. \( 1 + r_{Ie} \) is simply a ratio of the unweighted sum of prices of all commodities at date \( t \) to the unweighted sum of prices of all commodities at date \( t-1 \). That is, for the purpose of indexing a purchasing power annuity, the value of a basket containing one and only one of each available commodity should be used. Common indices of inflation such as the consumer price index or the implicit price deflator for GNP are weighted by the quantities of commodities consumed or produced. These indices have been used to measure not only the impact of inflation on real consumption and production but also on rates of return in securities markets.

We will show here that while these weighted indices may be appropriate for the former purpose, an unweighted index should be used when assessing the impact of inflation on security rates of return.
To see this intuitively, consider the rate of return a consumer requires to forgo present consumption. In the first period (one plus) this rate is \( 1 + r_F \equiv (\Sigma e P_e)^{-1} \), the riskfree rate of return on a short-term bond. Observe that \((1 + r_F)^{-1}\) is simply an unweighted average of state-contingent prices \(\{P_e\}\). In equilibrium, while the relative amounts of aggregate consumption available under each state \(e\) do not enter directly into the determination of \(r_F\), they enter indirectly through their effect on prices \(\{P_e\}\). Similarly, the relative supplies of commodities indirectly influence the rate of inflation \(r_{It}\) through their effect on commodity prices \(\{p_{kt}\}\). To weight these prices when computing rates of inflation would be double counting.

To measure the impact of inflation on real consumption a different index needs to be constructed. For example, suppose \(C_0\) is a consumer's optimal consumption expenditure at date \(t = 0\) with derived utility \(U_0(C_0)\). Let \(C_e\) be the nominal expenditure on consumption (assumed to be optimally allocated among commodities) at date \(t = 1\) with the same derived utility; \(^{10}\) that is, \(C_e\) is chosen so that \(U_0(C_0) = U_e(C_e)\). Define \(I_e \equiv C_e / C_0\) as the consumer's ideal inflation index with respect to date \(t = 0\) expenditure on consumption. It measures the number of dollars at date \(t = 1\) per dollar of consumption at date \(t = 0\) required to leave the consumer as well off as he was at date \(t = 0\). \(^{11}\) Consequently, \(I_e\) is defined to satisfy \(U_0(C_0) = U_e(C_0I_e)\). For \(b \neq 0,1\)

\[
\frac{b^b}{I-b} (A + BC_0)^{1-b} = \frac{b^b}{I-b} (Ae^e + BC_0I_e)^{1-b}.
\]
Solving for $I_e$, 

$$I_e = \left( \frac{1}{Ab/h_0^{1-B}} \right) \left[ \frac{1}{h_e^{1-B} - \psi \h_0^{1-B}} + \left( \frac{h_e}{h_0} \right)^{1-B} \right].$$

In general, then, the consumer's ideal inflation index is not independent of his beliefs and tastes and his optimal level of present consumption. Moreover, it is not generally equal to one plus the rate of inflation $r_{I_e}$. However, suppose between dates $t = 0$ and $t = 1$, while the absolute price level changes, relative prices remain the same. In this case, $p_{ke} = (1 + r_{I_e})p_{k0}$ for all commodities $k$. Moreover, suppose commodities are treated symmetrically by the consumer at dates $t = 0$ and $t = 1$; that is, suppose $b_{k,n0} = b_{k,n1}$ for all $k$ and $n$ and $\nu_{n0} = \nu_{n1}$ for all $n$. From the definitions of $h_0$, $h_e$ and $\psi_e$, it is easy to show that $h_e^{1-B} - \psi h_0^{1-B} = 0$ and $(h_e/h_0)^{1-B} = 1 + r_{I_e}$. Consequently, 

$I_e = 1 + r_{I_e}$ so that the ideal inflation index is the same as one plus the rate of growth of the absolute price level. More generally, one plus the rate of inflation measures the rate of growth of the absolute price level, while differences between the ideal inflation index and one plus the rate of inflation are the result of changes in relative prices and nonsymmetric treatment of commodities at different dates.

The derived utility functions may also be used to develop consumer decision and sharing rules in parallel with those in part I. For example, it can be shown that the optimal consumer commodity decision
rules are linear in total consumption expenditure for each date. As before, the optimal consumer consumption and investment decision rules are linear in wealth at each date. Since our primary concern is the effect of inflation on portfolio decisions and the rates of return of securities, only conditions for universal portfolio separation will be given.

**Theorem (universal portfolio separation):** If all consumers in the extended economy are the same except for their resources $W_0^i$ and taste parameter $A_1$, then at each date all consumers divide their wealth (after consumption) between two mutual funds: the market portfolio of all securities and a purchasing power annuity yielding equal certain real payments at all future dates. Moreover, the consumer consumption and investment sharing rules are stationary through time. In particular, for all $i$

$$C_0^i = \alpha^i_0 + \beta^i_0 C^M_0$$ and $$C_e^i = \alpha^i_e + \beta^i_e C^M_e$$ (all $e$)

$$W_e^i = \alpha^i_e + \beta^i_e W^M_e$$ (all $e$) and $$C_s.e = \alpha^i_s.e + \beta^i_s C^M_s.e$$ (all $e$ and $s$)

where

$$\alpha^i = \frac{A_M^i - A_{i,0}^M}{A_M^i + BW_{0}^i}$$ and $$\beta^i = \frac{A_1^i + BW_{0}^i}{A_M^i + BW_{0}^i}$$

In the extended economy, the purchasing power annuity is sufficient not only to hedge against all future shifts in investment opportunities, but also to hedge against future inflation. This is true even though there may be shifts in the relative price level and intertemporal stochastic dependence in the rate of inflation. Despite the increased complexity
of the extended economy, these simple separation properties are forthcoming.

III. VALUATION AND INFLATION

The valuation relationships developed in part II apply in the extended economy with minor modifications. Recall the two-parameter valuation theorem that with composite quadratic utility, in equilibrium at date \( t = 0 \)

\[
E(r_j) = r_F + \lambda_1 k(r_j, r_M) \text{Std } r_j + \lambda_2 k(r_j, r_N) \text{Std } r_j \quad \text{(all j)}
\]

where \( \lambda_1 \equiv \text{Std } W_1 / E[V'(W_1, \phi_1)] \) and \( \lambda_2 \equiv -A \text{ Std } \phi_1 / E[V'(W_1, \phi_1)] \).

From the proof of the theorem in part II, it is easy to see that this same equation applies in the extended economy, except that \( \phi_e \) and \( r_N \) are redefined in terms of a purchasing power annuity. For example, if the horizon \( T = 2 \), then

\[
r_{Ne} = \frac{\phi_e}{\phi - 1} = \frac{(1 + r_{le})(1 + \sum_{s} p_s e_s (1 + r_{Is.e}))}{\phi - 1}.
\]

If the intertemporal stochastic processes of nominal security rates of return and rates of inflation are each assumed independent then, equation (1) simplifies to

\[
E(r_j) = r_F + \lambda_1 k(r_j, r_M) \text{Std } r_j + \lambda_2 k(r_j, r_i) \text{Std } r_j \quad \text{(all j)}.
\]
If, instead, the rate of inflation is assumed certain, then $r_{Ne}$ in equation (1) is replaced by the rate of return of an annuity yielding equal certain nominal payments at all future dates. If neither of these assumptions are made, then $r_{Ne}$ must be interpreted as the rate of return on a purchasing power annuity.

However, the rate of inflation is not endogenously determined in equilibrium. Rather, at each date, the price of one commodity or $\sum_{k=1}^{K} x_{kt}$ may be chosen as "numeraire." To determine the stochastic process governing the rate of inflation, two missing elements must be added to the model: an exogenous money supply $M_t$ and its velocity $V_t$.

For example, consider the simplest case. At date $t = 0$ all consumers in the economy begin trading by transferring all their endowed resources to a central auctioneer who, at the equilibrium prices, supplies them with money. At this point, the central auctioneer holds all resources and consumers hold all the money supply. We can arbitrarily set the units of money so that $M_0 = 1$. Consumers then, at the equilibrium prices, exchange their money for the desired bundle of commodities and securities so that $V_0 = 1$. Consequently, at the end of trading at date $t = 0$, consumers hold all commodities and securities while the central auctioneer holds all money. In this simple example, money has been used trivially as a counting device. Since there are no exchange costs, consumers have no incentive to hold money so that the story has ended at date $t = 0$ with the central auctioneer holding all the money. Suppose that a similar method of trading is utilized at each date, the money
supply is held fixed by the central auctioneer at \( M_t = 1 \), and all consumers are informed of this monetary policy at date \( t = 0 \).

This basic monetary policy has strong implications for the nominal rate of return of the market portfolio. Since all commodities and securities are exchanged for money, from the classical Quantity Equation,

\[
M_0 V_0 = \sum_i \sum_k P_k c_k^{i} k_0 + \sum_i \sum_k P_k^{i} e^{i} e^{k} = W_0^M
\]

\[
M_1 V_1 = \sum_i \sum_k P_k c_k^{i} k_1 + \sum_i \sum_k P_k^{i} c_k^{i} e^{i} e^{s} = W_e^M \quad \text{(all } e)\]

\[
M_2 V_2 = \sum_i \sum_k P_k c_k^{i} k_2 + \sum_i \sum_k P_k^{i} c_k^{i} e^{s} = W_{s,e}^M \quad \text{(all } e \text{ and } s)\]

Since \( M_t \) and \( V_t \) are not stochastic, then \( W_e^M \) must be the same for all \( e \) and \( W_{s,e}^M \) must be the same for all \( e \) and \( s \). Moreover, since \( M_t = V_t = 1 \), then \( W_0 = W_e^M = W_{s,e}^M = 1 \). By definition, the nominal rate of return of the market portfolio \( r_{M_e} \equiv \frac{W_e^M}{W_0^M} - C_0^M \) and \( r_{Ms,e} \equiv \frac{W_{s,e}^M}{W_e^M} - C_e^M \). Since \( W_0^M \) and \( C_0^M \) are known at date \( t = 0 \) and \( W_e^M = 1 \), then the nominal rate of return of the market portfolio, \( r_{M_e} \), is certain at date \( t = 0 \). However, since \( C_e^M \) is uncertain at date \( t = 0 \), the nominal rate of return of the market portfolio for the next period, \( r_{Ms,e} \), is only known for certain at date \( t = 1 \). Of course, even though the product of one plus the real rate and one plus the rate of inflation (= one plus the nominal rate) is certain, these separate components are not. While the example is highly simplified, the moral of the story is that the stochastic properties of nominal rates
of return will be quite dependent upon monetary policy and the way money is used in the economy. The stochastic properties of real rates of return, which are of principal interest to economists, may be far removed from the stochastic properties of nominal rates of return. Unfortunately, nominal rates, not real rates, have provided the data for most empirical studies of security markets.

IV. OPTIMAL PRODUCER DECISION RULES

As indicated in section I, endogenous production decisions will be analyzed only for a production-efficient linear-logarithmic economy for which a composite individual can be constructed. In this case, at date \( t = 1 \) for each state \( e \), given consumption and production decisions made at date \( t = 0 \), production-efficient choices are determined by solving the following programming problem:

\[
\nu_e(x_{l_e}^M, \ldots, x_{k_e}^M) = \max_{\{c_{he}, \{x_{hke}\}} \sum_{h=1}^{K} \nu_h \ln c_{he} + \rho_2 \sum_{e} \sum_{k=1}^{K} \nu_{k2} \ln c_{k,s,e} \]

s.t. \( \ln x_{k,s,e}^M = \ln \alpha_{k,s,e} + \sum_{h=1}^{K} \beta_{h,s,e} \ln x_{hke} \)

and \( c_{k,s,e} = x_{k,s,e}^M / I \)

if \( k=1, \ldots, H \)

\( c_{k,s,e} = x_{k,s,e}^M / I \)

if \( k=H+1, \ldots, K \)

\( x_{h,e}^M = Ic_{he} + \sum_{k=1}^{K} x_{hke} \)

if \( h=1, \ldots, K \).
\( v_e(\mathbf{x}_{1e}, \ldots, \mathbf{x}_{Ke}) \) is the derived utility of arriving at date \( t = 1 \) with commodities \( \{\mathbf{x}_{be}^M\} \) available for consumption or production at that date. In turn, \( \{\mathbf{x}_{he}^M\} \) reflects the outcome of past consumption and production decisions.

Stating the problem in Lagrangian form, for each state \( e \)

\[
v_e(\mathbf{x}_{1e}^M, \ldots, \mathbf{x}_{Ke}^M) \propto \max_{\{c_{he}^M\}, \{x_{hke}\}} \sum_{h=1}^{K} v_{hl} \ln c_{he}^M + \rho_2 \sum_{s, s, e} \left[ \sum_{k=1}^{K} \beta_{hks, e} \ln x_{hks, e} \right] - \sum_{h=1}^{K} \lambda_{he} \left[ c_{he}^M + \sum_{k=1}^{K} x_{hke}^M - x_{he}^M \right]
\]

where several terms have been omitted since \( v_e(\cdot) \) is unique up to an additive transformation. Solving this problem, derived utility and optimal producer decision rules are

\[
v_e(\mathbf{x}_{1e}^M, \ldots, \mathbf{x}_{Ke}^M) \propto \sum_{h=1}^{K} \omega_{he} \ln x_{he}^M
\]

\( c_{he} = v_{hl} \omega_{he} x_{he} \) and \( x_{hke} = (\rho_2 \sum_{s, s, e} \beta_{hks, e}) \omega_{he} x_{he}^M \) (all \( h \) and \( k \) produced)

where \( \omega_{he} \equiv v_{hl} + \rho_2 \sum_{s, s, e} \beta_{hks, e} \).

Working backwards, at date \( t = 0 \), production-efficient choices are determined by solving the derived programming problem
\[ v_0(x_{10}^1, \ldots, x_{k0}^M) = \max_{\{c_{h0}, \{x_{hk0}\}} \sum_{h=1}^K c_{h0} \ln c_{h0} + \rho_1 \Gamma \pi \nu \left(x_{1e}^M, \ldots, x_{ke}^M\right) \]

s.t. \[ \ln x_{ke}^M = \ln \alpha_{ke} + \sum_{h=1}^K \beta_{hke} \ln x_{hk0} \quad \text{if } k = 1, \ldots, K \]

\[ x_{ke}^M = x_{ke}^M \quad \text{if } k = K+1, \ldots, K \]

\[ x_{h0}^M = 1c_{h0} + \sum_{k=1}^K x_{hke} \quad h = 1, \ldots, K. \]

Again, solving this problem, at the optimum

\[ v_0(x_{10}^1, \ldots, x_{k0}^M) = \sum_{h=1}^K \omega_{h0} \ln x_{h0} \]

\[ c_{h0} = \nu_0 \omega_{h0} x_{h0}^{-1-M} \quad \text{and} \quad x_{h0} = (\rho_1 \Gamma \pi \omega_{ke} \beta_{hke})^{1-M} \omega_{h0} \quad \text{all } h \text{ and } k \]

produced)

where \[ \omega_{h0} = \nu_{h0} + \rho_1 \Gamma \pi \sum_{k=1}^H \omega_{ke} \beta_{hke}. \]

To this point, it has merely been shown that closed-form solutions can be derived for Pareto-efficient aggregate consumption and production decisions in a multiperiod, stochastic linear-logarithmic economy. It remains to be seen how these decisions impact upon the intertemporal stochastic process of security prices. To simplify the following analysis, assume there is only one commodity whose aggregate quantity is represented by \( X \). The entire endowment of this commodity is assumed to be available
at date $t = 0$. In this case, the above relationships simplify to

$$c_0^M = x_0 (1 + \omega_0)^{-1} \quad \text{and} \quad c_e^M = x_e (1 + \omega_e)^{-1}$$

where $\omega_0 \equiv \rho_1 e (1 + \omega_e)^{\beta_e} \pi_{e} \beta_{e}$ and $\omega_e \equiv \rho_2 e \pi_{e} \beta_{e}s.e$. Using the production functions $x_e = \alpha_{e}(x_0 - c_0^M)^{\beta_e}$ and $c_{s,e}^M = \alpha_{s,e}(x_e - c_e^M)^{\beta_{s,e}}s.e$, it is possible to calculate $c_e^M$ and $c_{s,e}^M$ in terms of $x_0$ so that

$$c_e^M = \alpha_{e} (1+\omega_e)^{-1} \left[ \frac{\omega_0}{1+\omega_e} \right]^{\beta_e} x_0^{\beta_e}$$

and

$$c_{s,e}^M = \alpha_{s,e} \beta_{s,e} \left[ \frac{\omega_e}{1+\omega_e} \right]^{\beta_{s,e}} \left[ \frac{\omega_0}{1+\omega_e} \right]^{\beta_{s,e}} x_0^{\beta_{s,e}}$$

**Theorem** (intertemporal structure): In a single commodity economy, given that a composite individual can be constructed with logarithmic utility, production decisions are Pareto-efficient, and production functions are constrained to pure storage with an exogenous stochastic depreciation factor: the term structure is unbiased and the rate of return of the market portfolio follows a random walk if and only if the inverse of the depreciation factor is uncorrelated over time.

**Proof:** With pure storage, $\beta_e = \beta_{s,e} = 1$ so that $\omega_0 = \rho_1 (1 + \rho_2)$ and $\omega_e = \rho_2$. As a result, $1 + r_{c_e} \equiv c_e^M/c_0^M = \rho_1 \alpha_{e}$ and $1 + r_{c_{s,e}} \equiv c_{s,e}^M/c_e^M = \rho_2 \alpha_{s,e}$. Therefore, $\kappa \left[ (1 + r_{c_1})^{-1}, (1 + r_{c_2})^{-1} \right] = 0$ if and only if $\kappa \left[ \alpha_{1}^{-1}, \alpha_{2}^{-1} \right] = 0$. The conclusion follows immediately from the intertemporal structure theorem in part II. Q.E.D.
This theorem ties the classic hypotheses governing the inter-temporal structure of security rates of return back to the intertemporal stochastic characteristics of exogenous technological parameters of production functions, carrying the analysis in part II one step further. With logarithmic utility, pure storage production functions with uncorrelated depreciation factors produce an unbiased term structure and a random walk. The random walk phenomenon, empirically verified for many security markets, seems not therefore to be a byproduct of information-efficiency, as is usually claimed. It is rather a result of intertemporal random fluctuations of aggregate production opportunities.

V. PROSPECTIVE

The closing section of this series of papers begins by taking stock of how far we have come and ends with a glance at unsolved problems.

By working with the HARA class of utility functions and assuming security markets are exchange-efficient, it has been possible to generalize existing multiperiod consumption-portfolio theory to allow for an arbitrary discrete-time stochastic process of security prices and yet still obtain closed-form solutions. The HARA class achieves this end since, working backwards in time, the sequence of derived utility functions retain the same "form" and therefore are themselves members of the HARA class. By imposing well-chosen homogeneity conditions across individuals in the economy, explicit optimal sharing rules have been derived consistent with this multiperiod consumption-portfolio theory. These rules have been
used to develop a comparative statics analysis of resources, tastes, and beliefs. In particular, they were used to identify borrowers and lenders, to determine sufficient conditions for nonnegative consumption, to analyze the effects on choices of changes in absolute risk aversion, proportional risk aversion and patience, and to determine the size of side bets due to differences in beliefs. But perhaps most significant, the sharing rules were shown to imply portfolio separation properties which generalize existing results to a multiperiod setting. Invoking additional homogeneity conditions on patience, the natural multiperiod generalization of the 2-fund single-period separation theorem replaces the risk-free security with a risky annuity yielding equal certain payments at all future dates. These same separation properties also indicate the extent to which the complete markets context, introduced for convenience in modeling, can be relaxed. Indeed, it is safely conjectured that a complete market and an otherwise similar incomplete but exchange-efficient market are equivalent in the sense that they reach the same final allocations and have compatible price systems. This implies that an incomplete but exchange-efficient securities market can be modeled indirectly through a complete markets context. As I hope has been amply demonstrated, this context substantially simplifies the analysis.

By applying the previously developed multiperiod consumption-portfolio theory, it has been possible to generalize existing single-period security valuation models to their natural nontrivial multiperiod extensions, where the intertemporal stochastic process of security prices is
not exogenously restricted to a random walk. While other attempts in continuous-time have broken with the strict random walk assumption, unlike this series, they have not been free of other exogenously imposed restrictions on the stochastic process. In an intertemporal equilibrium model, this stochastic process is formally endogenous and to impose it exogenously may create inconsistency by overdetermining the model. While two types of simplifying assumptions are available in single-period models—restrictions on tastes and direct restrictions on probability distributions of end-of-period prices—only the former remains available in a multiperiod context. In the multiperiod two-parameter (quadratic utility) case, the first period expected rate of return of any security was shown to be a linear function not only of the risk-free rate and its covariance with the rate of return of the market portfolio, as in the single-period case, but also of its covariance with the rate of return of an annuity yielding equal certain payments at all future dates. Additionally, properly interpreted, the continuous-time solution derived elsewhere is equivalent to the limit of the discrete-time valuation equation as the trading interval approaches zero. Since no exogenous restrictions were imposed on the intertemporal stochastic process of security prices, an opportunity was presented to investigate under what conditions an unbiased term structure or a random walk will emerge as a natural result of equilibrium forces. These patterns of security price behavior were shown to depend crucially on the intertemporal stochastic process of aggregate consumption. Indeed, in the logarithmic utility case, these
are each verified if and only if the inverse rate of growth of aggregate consumption is uncorrelated over time. Unfortunately, such simple results do not carry over to other assumptions about tastes. These results, to my knowledge, represent the first attempt to link together the inter-temporal stochastic processes of real and financial variables in an equilibrium setting.

Summarizing existing work, necessary and sufficient conditions for the economic efficiency of perfect and competitive securities markets composed of nonsatiated, risk averse expected utility maximizers, were stated and given intuitive justification. The specific economies examined in this series are special cases. While the conditions for exchange- and production-efficiency were examined separately, they are quite similar. A third type of economic efficiency, with regard to the exchange and production of information about supply conditions, has received comparatively little attention in the literature. While concurring with the work of others that the production of information in a "pure exchange" economy is not Pareto-efficient, the full exchange of information is shown to be Pareto-efficient under realistic conditions. In particular, if security prices fully reflect the pool of all available information (weak information-efficiency), then the full exchange of information will make every individual better off (except those with consensus beliefs who, however, will be just as well off as before since prices will not change). Since some of the economies developed previously generate explicit sharing rules even in the presence of heterogeneous beliefs, it has been possible
to measure and separate the volume of trading arising from speculative (heterogeneous beliefs) and nonspeculative (heterogeneous resources and tastes) motives. If the securities market is, as most empirical work indicates, roughly weakly information-efficient, then almost all speculative trading is not only Pareto-inefficient but to the disadvantage of the individuals on both sides of a transaction. Unfortunately, analysis of the special economies developed in this paper, as well as the observation that most nonspeculative trading is accomplished through highly standardized packages of state-contingent claims such as bonds and mutual funds, suggests that most actual trading is speculative and therefore damaging to both parties.

The last task has been the extension of the previously developed closed-form portfolio and valuation results to a full-fledged Arrow-Debreu economy incorporating uncertain noncapital income, many commodities, and explicit productive activities. To provide for commodity complementarities and yet obtain closed-form solutions, tastes for each commodity have been represented by HARA utility functions, and for commodities consumed at the same date these separate functions have been simultaneously related multiplicatively for complementary commodities and additively for noncomplementary commodities. Again, working backwards, the sequence of derived utility functions retain the same "form" and therefore are themselves members of the HARA class. In the corresponding separation properties, a purchasing power annuity replaces the simpler annuity of the previous analysis. The real (used to determine the effect
of inflation on utility from consumption) and financial (used to determine
the payments of the purchasing power annuity) measures of inflation are
not generally the same. While the real measure (ideal inflation index)
depends in a complex way on the beliefs, tastes, and the initial level of
consumption and is different for different individuals, the financial
measure (rate of inflation) is simply a ratio of the unweighted sums of
the prices of all commodities for two different dates. As one would ex-
pect, in the corresponding valuation equations, the purchasing power an-
nuity again replaces the simpler annuity of the previous analysis. How-
ever, the rate of inflation used to adjust the purchasing power annuity
is not endogenously determined. It rather depends on an exogenous speci-
ification of the money supply and the way the economy utilizes money. In
the simple but basic illustration provided, the stochastic properties of
real and nominal rates of return are shown to have little relationship.

If production decisions are to be endogenously determined, it is
necessary to specify the technology and criteria for making decisions.
Using stochastic Cobb-Douglas production functions and assuming produc-
tion decisions are Pareto-efficient, closed-form producer decision rules
are derivable for composite additive logarithmic utility. In the single-
commodity case, financial stochastic processes mirror the real stochastic
process of the depreciation factor under pure storage. Empirically ob-
served random walks in security markets seem not therefore to be a by-
product of information-efficiency but rather a result of intertemporal
random fluctuations of aggregate production opportunities.
While some progress has been made toward supplying stochastic multiperiod closed-form solutions to a number of important problems in finance, many remain unsolved. For expository convenience, they will be grouped as follows:

1. other closed-form solutions to the same problems,
2. further extensions which may be conveniently handled,
3. costly information about supply conditions, and
4. costly information about demand conditions.

To obtain closed-form solutions, first it must be possible (a) to analytically solve the single-period consumption-portfolio problem. Second, it must be possible (b) to work backwards via dynamic programming to obtain a sequence of derived utility functions which permit analytic solutions to the sequence of single-period problems. Third, it must be possible (c) to solve the aggregation problem by imposing nontrivial (other than identical individuals) homogeneity conditions. Fourth, if production is introduced, grafting production functions on to composite utility functions must not destroy properties (a), (b), and (c). With this in mind, unfortunately I doubt that closed-form solutions which rely solely on restricting utility and production functions, significantly different from those presented in this series, can be derived. Attempts to do so quickly bring one face to face with simultaneous polynomial equations which have no solution. Economics, like physics, has its own share of three-body problems. For example, perhaps the most pressing need is to allow taste parameter \( B \) to either vary over time or across individuals. Again, intractable polynomials appear. The other alternative of restricting probability distributions falls afool of an earlier objection
that (except for the conditional distributions at the final date in a pure exchange setting) they are endogenous. However, with HARA utility, it may be possible to solve the aggregation problem with other cases of heterogeneous beliefs if a specific relationship among beliefs of different individuals is assumed.

While I am pessimistic about progress toward obtaining significantly different closed-form solutions, I believe there are several new elements which can be introduced into the model. First, some effort to unite the related literature on neoclassical economic growth with financial theory would seem to be worthwhile, since as Samuelson [1958] has cleverly shown, the rate of growth of the population influences the rate of interest. Moreover, to the degree this growth rate is uncertain, the future risk-free rate will be uncertain. A problem arises, however, in the microeconomic modeling of inheritance if members of a generation are permitted to be economically active prior to the receipt of their inheritance. In order to plan ahead, they would try to consider the size of their future inheritance, but this would depend on the choices of the still existing members of the previous generation. Uncertainty about these choices would be difficult to model.

Second, consumers and producers could be identified with countries and the stage set for an analysis of international diversification, determination of stochastic exchange rates in equilibrium, and international trade. Here, it would be useful to assume consumers only buy either domestically produced or imported goods and must use their own national
currency. Positive proportional transportation costs could be used to prevent complete specialization in production. Solnik [1974] has made some progress along these lines using a stochastic continuous-time model. As one would expect, security rates of return in any country are dependent upon economic developments elsewhere.

Third, the fiscal role of government as a redistributor of wealth and as a producer of public goods should provide rich ground for future work. Diamond and Mirrlees [1971] have provided an integrated single-period analysis under certainty, and it should be possible to incorporate it within the economic context provided in this series of papers and derive closed-form solutions.

A modest beginning has been made by Gonedes [forthcoming], Hirshleifer [1971], Rubinstein [1974b], Ohlson [1974], Arrow [1972] and others on the analysis of consumption, portfolio, and production decision with costly information about supply conditions. Gonedes has emphasized the feedback between investor's uncertain perceptions of the production of firms and their ability to influence those decisions by determining share prices. Hirshleifer has shown that under "pure exchange," the production of costly information about supply conditions is socially wasteful; and Rubinstein has demonstrated that it is generally possible to redistribute resources through a prior market for information so that after then trading in the securities market all individuals will be better off than without the prior market. Ohlson, for instantaneous trading horizons, has isolated conditions under which different individuals will have the same complete ordering of information
structures about supply conditions. Moreover, he has established relationships between an individual's absolute risk aversion and the value of an information structure. In discrete-time, Arrow has proven that the Shannon entropy measure of the "amount of information" is the expected gross value of perfect information if and only if an individual's tastes are logarithmic. However, none of these authors, nor anyone else to my knowledge, has succeeded in developing a model in which the cost of information about supply conditions is determined in equilibrium.

In the economies described in this series of papers, given the resolution of uncertainty about the supply of aggregate consumption at all dates, present and future prices and individual choice variables were taken as certain. Speaking loosely, to each level of aggregate supply there corresponded a unique and certain demand curve and hence a unique and certain set of prices. In a more general setting, even if supply uncertainty were resolved, lacking information about the resources, tastes, and beliefs of other individuals, an individual could not forecast demand and hence prices with certainty. This demand uncertainty will depend upon the capacity of the exchange technology to disseminate information about resources, tastes, and beliefs.

A perfect and competitive market performs this task efficiently through the price system, while maintaining the anonymity of buyers and sellers. In an imperfect but competitive market, linking up potential buyers and sellers uses up resources. The simplest version of the problem under uncertainty has been analyzed by Leland [1973] who considered
portfolio selection with proportional transactions costs. Unfortunately, the programming problem is complicated by absolute value notation. Moving a step further, Hahn [1971] has shown how exogenous proportional transactions costs can operate to shut down some futures markets since the present value of the transactions fee is lowered by postponing exchange. In contrast to the Arrow-Debreu economy, this necessitates the opening of markets in the future. Radner [1972] achieves a similar end by supposing that at any date consumers can only exchange contingent claims to events which they can both distinguish. As information becomes available at future dates, permitting a finer partition of the set of states, new markets will open. However, none of these authors, nor anyone else to my knowledge, has succeeded in developing a model in which transactions costs are determined in equilibrium. Even under certainty, where aggregate transactions cost is assumed to be a strictly concave function of trading volume, I can attest that closed-form solutions are difficult to obtain.

Not only would we like to explain the level of transactions costs but also the effect of these costs on the variety of securities available in the market. Presumably, from this would follow a better understanding of the role of money and financial intermediaries. If transactions costs per unit of volume are assumed to vary inversely with volume, securities for which volume is sufficiently low, in the absence of transactions costs, would tend not to be created. One way to identify the threshold transactions cost at which a security would be created would be to calculate the expected utility of individuals in a complete market and then in an
incomplete market with the security missing. It should then be possible to calculate the breakeven level of transactions cost. Hakansson [1974] is recently initiating work along these lines by determining, for an incomplete market, which new securities could be created so as to make every individual in the economy better off. His work has obvious implications for corporate capital structure decisions and financial intermediation.

Even in a perfect and competitive market, individuals have an incentive to learn more about demand conditions than the prices made available to them by the market. Individuals may benefit from obtaining direct information about the resources, tastes, and beliefs of other individuals. Using Jaffe's [1974] terminology, when such information is available, the market loses its anonymity and becomes personal. An individual may use this information to estimate how much others are willing to pay, how much he should buy or sell, or what (if there is uncertainty about quality) they are really selling. Insurance, loan, used car, and labor markets are real life examples of personal markets where the standard competitive analysis does not fully apply. The recently burgeoning literature dealing with this ramification of demand uncertainty has barely scratched the surface. 

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FOOTNOTES

1. See Rubinstein [1974c, 1974d].

2. For other theoretical papers modeling stochastic inflation see Hakansson [1969], Roll [1973], Long [1974], Fischer [1974], and Chen and Boness / in particular, the recent complementary paper by Litzenberger and Grauer [1974] is recommended for a more thorough analysis of commodity prices in equilibrium than that provided here. By contrast, this paper emphasizes the implications of properly measured inflation for portfolio separation and security valuation.

3. The linear-logarithmic economy described here is similar to the economy analyzed by Radner [1966], except stochastic elements have been introduced. In particular, initial resources (noncapital income) and parameters in production functions are assumed uncertain. These, in turn, create uncertainty in commodity and security prices. Moreover, the intertemporal pattern of initial resources and production parameters are allowed to follow the most generalized discrete-time stochastic process.

4. To compare the noncomplementarity of additive utility with the modest complementarity permitted by multiplicative utility, consider just two commodities, coffee ($k = 1$) and milk ($k = 2$). With additive utility tastes are measured by

$$\sum_{e \in E} \pi \left[ u_1(c_{1e}) + u_2(c_{2e}) \right] = \sum_{e \in E} \pi \cdot u_1(c_{1e}) + \sum_{e \in E} \pi \cdot u_2(c_{2e})$$
and with multiplicative utility, tastes are measured by

\[ \sum_{e} \pi \left[ u_1(c_{1e}) \cdot u_2(c_{2e}) \right]. \]

Assume utility functions are scaled so that \( u_k(c_{ke}) > 0 \).

1. For both additive and multiplicative utility, preferences over coffee lotteries are independent of a given fixed amount of milk received with the coffee.

To see this, if milk is fixed, then \( c_{2e} \equiv c_2 \) for all \( e \). Therefore, for additive utility tastes are measured by

\[ \sum_{e} \pi u_1(c_{1e}) + u_2(c_2) \sim \sum_{e} \pi u_1(c_{1e}) \]

and for multiplicative utility, tastes are measured by

\[ u_2(c_2) \sum_{e} \pi u_1(c_{1e}) \sim \sum_{e} \pi u_1(c_{1e}) \]

where \( \sim \) means "is equivalent up to an increasing monotonic transformation to."

2. For additive utility, preferences over coffee lotteries are independent of a given lottery of milk received with the coffee.

In this case, \( \sum_{e} \pi u_1(c_{1e}) + \sum_{e} \pi u_2(c_{2e}) \sim \sum_{e} \pi u_1(c_{1e}) \).

3. For multiplicative utility, preferences over coffee lotteries are not independent of a given (nondegenerate) lottery of milk received with the coffee.

In this case, for all \( \{c_{1e} \} \) there is no increasing monotonic transformation which can map \( \sum_{e} \pi \left[ u_1(c_{1e}) \cdot u_2(c_{2e}) \right] \) into \( \sum_{e} \pi u_1(c_{1e}) \). See Keeney [1974] and Pye [1972].
5. Fama [1970] has noted this dependency in a more general context.

6. The goal of obtaining closed-form solutions requires this assumption.

7. These closure conditions also apply to economies without endogenous production. In this case $x_{hk} = 0$ for all $h, k$ and all commodities are effectively primary.

8. This procedure, in effect, grafts production functions onto utility functions. Unfortunately, the goal of obtaining closed-form solutions for any given utility function severely restricts the allowable set of production functions. For example, for the additive logarithmic utility function analyzed in the text, the production function must either be Cobb-Douglas (log-linear) or pure storage (linear) to obtain closed-form solutions. With other members of the HARA class of utility functions, the only "interesting" production functions which yield closed-form solutions appear to be linear.

9. For notational convenience, the dependence of $h_{nt}$ and $\psi_{nt}$ on the state description, through their dependence on uncertain commodity prices, has been suppressed in the text.

10. When using derived utility functions for this purpose, they must be the exact functions derived and these may not be subject to additive transformations. This modification only affects the results for generalized logarithmic utility where
\[ U_e(C_t) = \ln(A \psi_t + C_t) + \sum_k v_{kt} \ln(v_{kt}/p_{kt}). \]

In this case, when \( A = 0 \), the ideal inflation index takes the simplest form

\[ I_e = \prod_k (p_{ke}/p_{k0})^{v_k}. \]

11. An alternative ideal inflation index is \( I_e^{-1} \) which measures the number of dollars at date \( t = 0 \) per dollar of consumption at date \( t = 1 \) required to leave the consumer as well off as he was at date \( t = 1 \).

12. Specifically, when \( B \neq 0,1 \), then at the optimum

\[ c_{kn} = \frac{Ab[(1-b)_{kn} \psi_{nt} - (1-b)p_{kn}]}{p_{kn}(1-b)} + \frac{1-b_{kn}}{p_{kn}(1-b)} c_{nt} \]

and

\[ c_{nt} = \frac{Ab[h_{nt} + \psi_{nt} C_{nt}]}{\sum h_{nt}} + \frac{h_{nt}}{\sum h_{nt}} C_t. \]

13. Although the role of production has been omitted from this example, its inclusion would not significantly alter the analysis.

14. Unfortunately, for other utility functions, this simple result does not obtain.

15. The word "safely" is used advisedly since Nils Nielson's doctoral dissertation [1974], which has recently come to my attention, proves the conjecture.
16. For example, see Rothchild [1973], Rothchild and Stiglitz [1974], Spence [1973], Jaffe [1974], and Rubinstein [1974b].
REFERENCES


