THE STRONG CASE FOR THE GENERALIZED LOGARITHMIC UTILITY MODEL AS THE PREMIER MODEL OF FINANCIAL MARKETS

by

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This paper begins by comparing the available well-developed microeconomic models in finance which recognize uncertainty. It is argued that models whose distinctive simplifying assumption restricts utility functions are superior to those which instead restrict probability distributions, both with respect to the realism of their assumptions and richness of their conclusions. In particular, the most successful model, based on generalized logarithmic utility (GLUM), is a multiperiod consumption/portfolio and equilibrium model in discrete-time which (1) requires decreasing absolute risk aversion; (2) tolerates increasing, constant, or decreasing proportional risk aversion; (3) assumes no exogenous specification of the contemporaneous or intertemporal stochastic process of security prices; (4) tolerates heterogeneity with respect to wealth, lifetime, time- and risk-preference and beliefs; (5) results in a complete specification of consumption/portfolio decision and sharing rules which include nontrivial multiperiod separation properties and explains demand for default-free bonds of various maturities and options; (6) leads to a solution to the aggregation problem; (7) results in a complete specification of the contemporaneous and intertemporal process of security prices which reveals necessary and sufficient conditions for an unbiased term structure and the market portfolio to follow a random walk as a natural outcome of equilibrium; (8) provides empirically testable hypotheses concerning relationships between the rate of growth of aggregate consumption, the rate of return of the market portfolio, and the rate of return of a perpetual default-free annuity which do not require inferences of ex ante beliefs from ex post data; (9) provides a nontrivial multiperiod extension of popular single-period security valuation models which is empirically testable; (10) yields a simple multiperiod valuation formula for an uncertain income stream even when this income is serially correlated over time; (11) results in the Pareto-efficiency of exchange arrangements and by the model's very construction makes clear just what kinds of securities guarantee efficiency; (12) insures that competitive value-maximising firms make Pareto-efficient production decisions; (13) yields necessary and sufficient conditions for the Pareto-efficiency of the dissemination of information and an exact measure of the ratios of the volume of speculative to non speculative trading over time; and (14) permits theoretical determination of what information is and is not reflected in security prices.

1. INTRODUCTION

In the microeconomic theory of finance, to develop detailed and/or empirically testable hypotheses recognizing uncertainty it has become customary to follow the advice of James Tobin who,
in summarizing almost two decades of research, wrote (1969, p. 13) \(^1\)

... it is very difficult to derive propositions that are simultaneously interesting and general. ... To get propositions with significantly more content than the prescription that the investor should maximize expected utility, it is necessary to place restrictions on his utility function or his subjective probability estimates ...

Theorists and empiricists, characteristically dividing in the United States along East-West lines, have developed both alternatives, resulting in the following model "types":

I. Restrictions on beliefs

   A. discrete-time normality model
   B. continuous-time lognormality model

II. Restrictions on tastes

   A. generalized power utility model
   B. exponential utility model
   C. generalized logarithmic utility model

\(^1\)Riskless arbitrage and welfare arguments constitute the chief exceptions to Tobin's statement. For example, see Modigliani and Miller (1952), Stiglitz (1969), Håkansson (1974), Arrow (1953), Leland (1973), Hirshleifer (1971), and Rubinstein (1975). Although, like Modigliani and Miller, Black and Scholes (1973) also use an arbitrage argument to deduce the relationship of the values of associated securities, they impose strong (though possibly realistic) restrictions on the stochastic process of equity prices.

\(^2\)Type I models are chiefly associated with the work of Markowitz (1959), Sharpe (1972), Lintner (1965), Mossin (1966), Fama (1972), Jensen (1972), Black (1972), Scholes (1972), Merton (1973), Long (1974), and Gonedes (1975).

\(^3\)Type II models owe their etiology to Arrow (1953), Debreu (1959), and Hirshleifer (1970) for their development of the complete markets economy, to the work of Arrow (1965) for his concepts of absolute and proportional risk aversion, to the work of Wilson (1968) for his aggregation and welfare analysis of HARA utility functions, to the work of Cass and Stiglitz (1970) for their isolation of the significance of HARA utility functions for portfolio separation, and to the work of Håkansson (1971) for his multiperiod portfolio analysis of constant proportional risk aversion under an arbitrary stochastic process of security prices. More recently, these strands have been tied together to model security market equilibrium in a single-period context by Rubinstein (1974a), Brennan and Kraus (1975), and Grauer and Littenberger (1974, 1975), and in a multiperiod context by Rubinstein (1974b) and Kraus and Littenberger (1975).

Type II models are distinguished from each other by the form of their assumed single-period utility of consumption functions. For

A. \(U(C_t) \sim \frac{b}{1-b} (A + BC_t)^{1-b} \quad (B \neq 0,1)\)

B. \(U(C_t) \sim -C_t/A \quad (B = 0)\)

C. \(U(C_t) \sim \ln(1 + C_t) \quad (B = 1)\)

where \(A\) and \(B\) are constants, \(b = B^{-1}\) and \(\sim\) means "is equivalent up to an increasing linear transformation to." These utility functions belong to and exhaust the HARA or linear risk tolerance class of tastes and represent the solutions to the differential equation

\[ -U'(C_t)/U''(C_t) = A + BC_t \]

for the given values of parameter \(B\).
On the basis of the number of published papers, a head count of currently active researchers, or classroom acceptance, Type I models would today be the overwhelming winners of a popularity contest. However, it is the thesis of this paper that Type II models, and in particular the generalized logarithmic utility model (GOLUM), are superior both in the realism of their assumptions and the richness of their conclusions.

Why, if this be true, are Type I models so popular? While a complete answer is difficult to provide, I believe a parallel can be drawn to many of the chief advances in the physical sciences. In the Aristotelian view, a body was naturally at rest so that once placed in motion it had to be accompanied by a mover at all times or otherwise it would stop. This view of motion was the critical obstacle on the road to modern science; the continuous motion of the heavenly bodies seemed to require unseen hands, a sublime Intelligence exercising masterly control over the universe. The idea of natural rest was replaced with the Galilean-Newtonian idea of inertia, that a body once set in motion would naturally continue its motion at a constant velocity along a linear path until something intervened to accelerate or decelerate it. Aristotelian celestial mechanics, viewing the sun as rotating about the earth, after a long struggle, has given way to the modern Copernican view. The great barrier that for so long delayed the development of modern chemistry, the idea of phlogiston, a substance that was lost to a body in the process of burning, has now become a relic of history, displaced by the idea that a burning body takes in, and not loses, a substance—oxygen. These great transpositions in the history of science which we find so acceptable today, nonetheless defeated the greatest intellects for centuries. Men were not lacking in evidence, but inherited habits of thought, which often extended beyond science proper to a worldview, caused them to cling stubbornly to superannuated ideas.

A similar, although obviously not as philosophically significant, flexibility in thinking is required in the shift from models of Type I to those of Type II. In the former case, consumers directly choose actual securities and measure their desirability by their impact on the mean and variance of portfolio return. Like the old Aristotelian theories, consumer behavior is interpreted in its most natural terms. In the latter case, like the modern physical theories, an inversion (literally inversion of a matrix) of view is required: consumers are interpreted as indirectly choosing state-contingent securities (portfolios of which constitute actual securities) and measure their desirability by their impact on a more general set of risk parameters. Such an inversion of view is justified if it can (1) explain the same phenomena as the old view, (2) provide a more elegant explanation, and

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4Hakanson (1975) provides his own answer to this question in a recent paper.
(3) explain other phenomena not capable of explanation under the old view. For example, it was not
that the Aristotelian earth-centered view was false, but rather that the implications of the heliocen-
tric view for the movements of other planets and stars commends it as superior. The remainder of this
section will contrast Type I and Type II models and sections following will develop GLUM since many of
its properties are not well known.

A list of assumptions we would hope would characterize a satisfactory microeconomic model
of financial markets might include:

Opportunities

*1. perfect and competitive financial markets
*2. choice and equilibrium over states
*3. choice and equilibrium over dates
4. discrete-time

Tastes

*5. rationality, non-satiation, risk aversion
5. decreasing absolute risk aversion
7. increasing, constant, or decreasing proportional risk aversion

Beliefs

8. arbitrary contemporaneous stochastic process of security prices
9. arbitrary intertemporal stochastic process of security prices

Heterogeneity

*10. wealth (composition and scale)
*11. risk-preference
12. time-preference
*13. lifetime
14. beliefs

Starred assumptions are currently shared by both Type I and Type II models, however the other desider-
atum are only simultaneously met by Type IIC model--GLUM.

Assumption 3 is met by Type II models without any unfortunate side effects, however it is met
for models of Type I at the price of severe restrictions on the contemporaneous and intertemporal
stochastic process of security prices. In particular, Fama (1972, chapter 8) has argued that the dis-
crete-time normality model applies in a multiperiod setting if the rates of return on all securities
both are normally distributed at each date and follow a (possibly nonstationary) random walk. First,
the inconsistency of the normality assumption with limited liability, unfortunate enough in a single-
period model, is even more damaging in a multiperiod model. Now we are asked to tolerate the distinct
prospect of trading securities at future dates with negative prices with the vision of consumers bank-
rupted in the first period carrying forward negative "wealth" continuing to have their credit honored
and therefore able to purchase a new portfolio and continue their participation in the securities
market. Second, if two- or more-period default-free pure-discount bonds exist, all securities cannot follow a random walk. For example, such a two-period bond (with an uncertain price at the end of the first period) must have negative serial correlation of its period-by-period rate of return. Third, as Hakansson (1975) observes, since the normal distribution does not reproduce itself multiplicatively, it cannot apply to returns on securities over two periods. However, by comparison, the chief virtue of the continuous-time lognormality model (13) is that it escapes these objections.

Assumption 4 is met by all models except Type II—the continuous-time lognormality model. First, since the time interval between dates can be made arbitrarily small in discrete-time models, they are in this respect of greater generality. Second, if the limit of consumption and portfolio decisions as the time interval approaches zero were the same as the decisions made in continuous-time, then it could be argued that while less general, at least the continuous-time results are "close" to the discrete-time results for sufficiently small but finite time intervals. However, as Jim Ohlson and Barr Rosenberg have mentioned to me, the limit to the discrete-time decisions as the time interval approaches zero is not generally the continuous-time decisions. For example, with constant proportional risk aversion, in discrete-time a consumer never borrows on net at the riskfree rate since to do so chances bankruptcy forcing expected utility to be negative infinite; however, some borrowing is frequently optimal in continuous-time. Third, Rosenberg has also observed that the volume of trading required to continuously maintain the optimal portfolio in continuous-time may be infinite for any finite length of time. This obviates an analysis of the relative sources of trading volume.

Assumption 6 is only violated by model IIB; however, GLUM is the only model which is inconsistent with constant and increasing absolute risk aversion. While the issue of the realistic sign of absolute risk aversion seems to be settled, the direction of proportional risk aversion remains an open question. Therefore, a satisfactory model should be tolerant of differing attitudes of proportional risk aversion (assumption 7). However, the discrete-time normality model is internally inconsistent with constant proportional risk aversion since in this case utility is undefined over negative levels of wealth yet such wealth levels are possible since normal distributions, regardless of high positive means, attach a finite probability to any finite negative level of wealth. Moreover, since both Type I models also attach a finite positive probability to any finite positive level of wealth,

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5 Hakansson (1975) observes that approximation arguments are not convincing. Suppose we argue that the probability of negative second period consumption, $c > 0$, is negligible. With a random walk, the probability of avoiding negative consumption over $t$ periods becomes about $(1-c)^t$, which may not be negligible since $\lim_{t \to \infty} (1-c)^t = 0$. 
they are inconsistent with most commonly used utility functions which exhibit nonsatiation and risk aversion. For example, the EARLS (linear risk tolerance) class, which contains all commonly used utility functions, is characterized by the differential equation \(-U'(\cdot)/U''(\cdot) = A + B(\cdot)\) where \(U\) is a utility function and \(A\) and \(B\) are fixed parameters. The discrete-time normality model is inconsistent unless \(B = 0\) (exponential utility) and the continuous-time lognormality model is inconsistent unless \(B = 0\) (exponential utility), \(B = 1\) (generalized logarithmic utility), or \(A = 0\) (constant proportional risk aversion).

The most telling dividing line between the models is assumptions 8 and 9: Type I models violate these requirements while Type II models do not. In an intertemporal equilibrium model, the stochastic process of security prices both across securities and over time is formally endogenous and should be derivable from the exogenous stochastic process of aggregate consumption or, with production, the exogenous stochastic process of parameters characterizing production functions. An intertemporal model which takes significant features of the stochastic process of prices as given, should not be regarded as a complete model of equilibrium. The discrete-time normality model adopts both the contemporaneous assumption of normality and the intertemporal assumption of a random walk. The continuous-time lognormality model, as developed by Merton (1973), is somewhat more general since it permits some nonrandomness in period-by-period riskfree rates but nonetheless exogenously ties the intertemporal stochastic process of all other securities to these riskfree rates. A second objection to the restrictions on the stochastic process of security prices imposed by Type I models is its unrealism. Many securities, such as bonds near maturity, warrants, puts and calls, and insurance, have rates of return not even well approximated by normality or lognormality.

Indeed, the imposition of exogenous restrictions on the stochastic process of security prices may produce absurd conclusions. For example, Rosenberg and Olson (1973) and independently Goldman have shown that the joint properties of identically distributed security rates of return over time and portfolio separation at each date (which characterize the simpler version of Merton's (1973) continuous-time lognormality model) force all risky securities in equilibrium at any date to have the same rate of return! In effect, all risky securities are identical from the point of view of an investor. As Bob Litterer has taught me, this degeneracy has a simple explanation. In the usual model, at each date the only state variables which affect portfolio choice are available wealth and rates of return on available securities. With portfolio separation, the composition of a consumer's portfolio of risky securities is only dependent on the joint distribution of security rates of return; but if this is identical at each date, then this composition must also be identical at each date.

Therefore, if consumer \(i\) bought amounts \(S_{jt-1}^i\) and \(S_{kt-1}^i\) of risky securities \(j\) and \(k\) at date \(t-1\) such that \(S_{jt-1}^i/S_{kt-1}^i = a_i^t\), then at date \(t\) he buys \(S_{jt}^i/S_{kt}^i = a_i^t\). If all consumers hold portfolios of risky securities with the same composition at any given date, then they all purchase the market portfolio. Consequently, \(a = a_i\) the same for all \(i\), and \(S_{jt-1}^M/S_{kt-1}^M = a\) where \(S_{jt}^M = e^{s_{jt}}\). Rates of return on the securities may be defined as \(1 + r_{jt} = S_{jt}^M/S_{jt-1}^M\) and \(1 + r_{kt} = S_{kt}^M/S_{kt-1}^M\). It therefore follows that whatever occurs, \(r_{jt} = r_{kt}\).
Assumption 14, heterogeneous beliefs, is only met by certain Type II models—exponential utility model and GLUM. WhileLintner (1969) has developed a hybrid model joining discrete-time normality with exponential utility which encompasses heterogeneous beliefs, this has only been at the cost of violating desideratum 6, decreasing absolute risk aversion. The chief drawback of the constant proportional risk aversion models (special cases of Type IIA) is the absence of heterogeneity with respect to beliefs as well as risk-preference. GLUM is singular in its capacity to cope with heterogeneous beliefs while not imposing patently unreasonable restrictions on tastes.

In addition to the realism of assumptions, the models can also be compared on the basis of the richness of their conclusions. A list of conclusions we would hope would characterize a satisfactory microeconomic model of financial markets might include:

Choice
1. complete specification of consumption/portfolio decision rules
2. complete specification of consumption/portfolio sharing rules

Equilibrium
3. solution of the aggregation problem
4. complete specification of contemporaneous security price relationships
5. complete specification of intertemporal security price relationships
6. simplicity and empirical testability of security price relationships

Efficiency
*7. Pareto-efficiency of exchange
*8. Pareto-efficiency of production (by competitive value-maximizing firms)
9. conditions for Pareto-efficiency of production and dissemination of information

Again, starred conclusions are currently shared by both Type I and Type II models; except conclusion 6, the others are not deductible from Type I models. Under Type II models, the generalized power utility model (IIA) yields all but conclusions 6 and 9; the exponential utility model (IIB) yields all but conclusion 6; only GLUM (IIIC) has sufficient fertility to yield them all.

Conclusions 1 and 2 are only partially reached by Type I models through their separation properties which imply that all consumers divide their wealth after consumption between the same two mutual funds, a risk-free security and the market portfolio. However, without forming a hybrid Type I-II model, this separation property has not been extended to infer whether a given consumer is a borrower or lender, let alone the magnitude of his borrowing or lending and as a result the amount of his consumption and his investment in the market portfolio. In contrast, to take the most successful

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Lintner's (1969) discrete-time normality-exponential utility model is the chief example of a hybrid model. However, as shown in Rubinstein (1974a), he gains little in explanatory power not available in a pure exponential utility model. Moreover, adding the normality assumption unnecessarily complicates his derivations. Merton (1971) also develops explicit consumption and portfolio decision rules combining the continuous-time lognormality model with HARA utility functions. Again as shown in Rubinstein (1974b), the lognormality assumption is unnecessary to the derivation of explicit multiperiod decision rules with HARA utility.
Type II model, with GLUM a consumer's portfolio choice at each date can be broken down into thirty-eight components, each of which expresses the extent his optimal portfolio differs from per capita holdings as a result of the deviation of his economic characteristics (wealth, risk-preference, time-preference, lifetime, and beliefs) from the average. Moreover, the portfolio separation property of GLUM, despite the permitted heterogeneity and the absence of restrictions on the intertemporal stochastic process of security prices, maintains the simplicity of the analogous results in Type I models. In addition, demand for bonds of various maturities can be inferred from GLUM, an issue which is closed to Type I models due to their strong random walk restrictions on the intertemporal stochastic process of security rates of return. Long's (1974) model is, however, an exception to this. Likewise, demand for "options" with payoffs under various conditions can be inferred from GLUM, another issue closed to Type I models due to their assumed homogeneity of beliefs.

Without creating a hybrid Type I-II model or assuming all consumers are identical, it has yet to be shown if there are other conditions under which the aggregation problem (conclusion 3) can be solved. Moreover, if restrictions on tastes required by Type II models are adopted, the additional restrictions imposed on beliefs by Type I models appear to be of no assistance in the solution of the aggregation problem. As a result of the ability of Type II models to solve the aggregation problem under nontrivial homogeneity conditions, they provide a complete endogenous specification of both the contemporaneous and intertemporal relationships among security prices (conclusions 4 and 5). In contrast, while the Type I models provide well-known contemporaneous linearity relationships between return and risk and precise measures of risk, neither the additive (riskfree rate) nor the multiplicative ("market price of risk") constants are determined from exogenous variables. Moreover, because of their strong exogenous restrictions on the intertemporal stochastic process of security prices, they are barred from developing nontrivial conditions for a random walk or unbiased term structure to be a natural outcome of equilibrium. In particular, the discrete-time normality model by assuming a random walk to start with, not only dodges the random walk question but also trivially forces an unbiased term structure. Indeed, one of the great merits of Type II models is their capacity to cope with these significant intertemporal pricing issues. However, among these models, only GLUM has relatively

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The aggregation problem is solvable in the weak form if it is possible to replace each heterogeneous economic characteristic with a homogeneous characteristic such that social choices are determined as before and, for all characteristics denominated in wealth, the homogeneous characteristic is an unweighted arithmetic average of the corresponding heterogeneous characteristic. The problem is solved in the strong form if the homogeneous characteristics so defined are not themselves functions of private or social choices. See Rubinstein (1974a) and section 3 of this paper.

For example, for the period-by-period rates of return of a two-period default-free bond to follow a random walk, these rates of return must be certain; but in this case the term structure is trivially unbiased to avoid riskless arbitrage.
simple conditions for these two classic hypotheses of the intertemporal stochastic process of security rates of return.

Other than its failure to incorporate heterogeneous beliefs, the chief drawback of the generalized power utility model (IIA) is the difficulty of empirically testing its security price relationships. While permitting heterogenous beliefs, the exponential utility model (IIB) has similar difficulties. Prospects are not entirely bleak, however. Their single-period forms can be tested and even used to infer social attitudes toward risk-preference and time-preference from the observed behavior of security prices.\(^{10}\) However, when embedded in a multiperiod setting, testing the first period valuation equation would appear to require either data on the rate of growth of aggregate consumption (currently only available at quarterly intervals in the United States) or exogenous restrictions on the intertemporal stochastic process of security prices, such as a random walk, which sufficiently insulate the first period valuation equation from consideration of future security rates of return.\(^{11}\) But similar assumptions on Type I models seem also required for their empirical testability. GLUM (IIC) is the only intertemporal model, which does not exogenously restrict the intertemporal stochastic process of security prices, with a first period valuation equation which can be tested without direct information on the rate of growth of aggregate consumption. Substituting in place of this is the rate of return on the market portfolio and the rate of return of a perpetual default-free annuity. Moreover, among Type I and Type II models, GLUM yields the simplest multiperiod valuation formula for an uncertain income stream, which can be written as the discounted (at appropriate risk-free rates) sum of explicitly calculated certainty equivalents of the uncertain income at each date, even if this income is serially correlated over time. GLUM also offers empirically testable hypotheses concerning the intertemporal structure of security prices. As a final empirical advantage, at least with respect to the contemporaneous relationship among the rate of growth of aggregate consumption, the rate of return of the market portfolio, and the rate of return of a perpetual default-free annuity, GLUM avoids the ex post-ex ante problem of empirical work. To test this relationship, it is unnecessary to measure (ex ante) beliefs.

While all Type I and II models are Pareto-efficient with respect to the variety of securities available (conclusion 7), certain popular generalizations of the discrete-time normality model are not. For example, in the absence of a riskfree security (Black (1972)), consumers generally will find

\(^{10}\)See Grauer (1975).

\(^{11}\)See Rubinstein (1974b).
exchange arrangements Pareto-inefficient since they would create a riskfree security if given the
chance. Likewise, in the discrete-time normality model with exponential utility to accommodate heter-
ogeneous beliefs, exchange arrangements are generally Pareto-inefficient unless a complete securities
market exists (i.e., an uncountably infinite number of securities to correspond with the uncountably
infinite number of states). Not only do Type II models avoid these difficulties but their very con-
struction makes clear just what kinds of securities would need to exist to guarantee Pareto-efficiency.
The mathematical development of these models begins with the assumption of a complete securities market
which "automatically" assures Pareto-efficiency. The subsequent generation of sharing rules then
reveals to what extent the securities market may be incomplete and yet remain Pareto-efficient. This
indicated the types of securities which are apt to be in greatest demand.

As a result of the work of Hirshleifer (1971) and Ng (1975), we now know that in any pure ex-
change economy with Pareto-efficient exchange arrangements and homogeneous beliefs, the production of
new information is Pareto-inefficient even if it is costless. However, the equally interesting ques-
tion of the dissemination of existing information in an economy with heterogeneous beliefs\(^\text{12}\) can obvi-
ously only be examined by models permitting belief heterogeneity, and this leaves only the exponential
utility model (IIB) and GLUM (IIC). Both models permit a comparative statics analysis of the effect
on consumer choices and equilibrium prices of moving from one set of heterogeneous beliefs to another
set of heterogeneous beliefs or to homogeneous beliefs. To see how serious are the effects of poor
information dissemination, it is also possible to precisely measure the nonspeculative (due to differ-
ences in wealth composition, wealth scale, risk-preference, time-preference, and lifetime) versus
speculative (due to differences in beliefs) sources of trading volume. Finally, unlike the other
models, IIB and IIC permit the explicit construction of consensus beliefs,\(^\text{13}\) a function of the different
beliefs of all consumers, which indicate what information is fully reflected in security prices.
Here GLUM has a slight advantage over the exponential utility model, since its consensus beliefs being
arithmetic rather than geometric weighted averages of beliefs across consumers, have a more natural
interpretation.

\(^{12}\)See Jaffe and Rubinstein (1975).

\(^{13}\)Consensus beliefs are those beliefs which, if held by all consumers in an otherwise similar
economy, would generate the same equilibrium prices as in the actual economy. See Rubinstein (1975).
2. THE GENERALIZED LOGARITHMIC UTILITY MODEL

For convenience in the development of the theory, we shall examine a three date \((t = 0, 1, 2)\) complete markets economy; neither assumption is required. The state at date \(t = 0\) is known with certainty; at date \(t = 1\) any one of \(E(e = 1, 2, \ldots, E)\) states can occur and at date \(t = 2\) any one of \(S(s = 1, 2, \ldots, S)\) states can occur. Each consumer, at date \(t = 0\) allocates his positive present wealth \(W_0\) among present consumption \(C_0 > 0\) and \(E\) "state-contingent claims" to wealth \(\{W_e\}\) at date \(t = 1\). Since \(P_e\) denotes the date \(t = 0\) present value of a unit of wealth received at date \(t = 1\) if and only if state \(e\) occurs, then \(W_0 \geq C_0 + \Sigma_e P_e W_e\). Similarly, at date \(t = 1\), if state \(e\) occurs and a consumer is alive, he allocates his wealth \(W_e\) among consumption \(C_e > 0\) at date \(t = 1\) and \(S\) "state-contingent claims" to consumption \(\{C_{s,e} > 0\}\) at date \(t = 2\). Since conditional on the occurrence of state \(e\), \(P_{s,e}\) denotes the date \(t = 1\) present value of a unit of consumption received at date \(t = 2\) if and only if state \(s\) occurs, then \(W_e \geq C_e + \Sigma_s P_s C_{s,e}\). On the other hand, if state \(e\) occurs and a consumer dies at date \(t = 1\), then \(W_e \geq C_e\) and he does not revise his portfolio. An exogenous counter \(\lambda\) will be used to denote whether a consumer is dead or alive at date \(t = 1\); if he is dead, \(\lambda = 0\) and if he is alive, \(\lambda = 1\). If he is dead, his lifetime \(T = 1\) and if he is alive, his lifetime \(T = 2\).

To provide for the endogenous determination of security prices, we append closure conditions that effectively take aggregate production decisions across dates and states as given. The population of the economy at each date is denoted by \(I_t (i = 1, 2, \ldots, I_t)\) where at date \(t = 0\) all consumers \((i)\) are alive, after consumption at date \(t = 1\), \(I_{t-1} = I_{t-1}^i\), and after consumption at date \(t = 2\), \(I_2 = 0\). Each consumer is endowed with resources \((\xi_0^i, (\xi_e^i), (\xi_{s,e}^i))\) so that more fundamentally,

\[
W_0^i = \xi_0^i + \Sigma_e P_e \xi_e^i + \Sigma_s P_s \Sigma_{s,e} \xi_{s,e}^i.
\]

Closure conditions then become

\[
E I_0^i = C_0^i = E_0^i, \quad E I_e^i = C_e^i = E_1^i, \quad E I_{s,e}^i = C_{s,e}^i = E_1^i.
\]

for all \(e\) and \(s\). Since the time-state distribution of resources across consumers is regarded as exogenous, the aggregate supply of resources across dates and states \(\{C_0^M, (C_e^M), (C_{s,e}^M)\}\) is also exogenous.\(^{14}\) As a result the period-by-period growth rates of aggregate consumption \(1 + r_C \equiv \frac{C_e^M}{C_0^M}\)

\(^{14}\) Rather than start with a given time-state distribution of endowed consumption across consumers, production decisions can be introduced which are capable of shifting aggregate consumption across dates and states. In this case, we would denote \((\xi_0^i, (\xi_e^i), (\xi_{s,e}^i))\) as the endowed resources of consumer \(i\). Each of \(J (j = 1, 2, \ldots, J)\) producers can transform resources from earlier to later dates
and \( 1 + r_{e,s} \equiv C_{s,e}^M / C_{e}^M \) are also exogenous. However, all wealth variables \( (W_{e}^0, W_{e}^1) \) are endogenous since they are endogenous functions of endogenously determined prices.

Since certain portfolios will play a significant role in the subsequent analysis, it is well to develop notation for them now. Aggregate wealth will be denoted by \( W_{e}^M = \sum_{e} W_{e}^M \) and \( W_{e}^M = \sum_{e} W_{e}^M \) for all \( e \), and the period-by-period rate of return on the market portfolio by \( 1 + r_{e} = W_{e}^M / (W_{e}^M - C_{0}^M) \) and \( 1 + r_{e,s} \equiv C_{s,e}^M / (W_{e}^M - C_{e}^M) \). In this context, \( (C_{0}^M, C_{e}^M, C_{s,e}^M) \) can be interpreted as net flows of cash from producers to consumers ("dividends"). A default-free first period bond has a certain rate of return defined by \( 1 + r_{e} \equiv (\Sigma_{e} P_{e})^{-1} \) and a default-free second period bond has a rate of return conditional on state \( e \) of \( 1 + r_{e} \equiv (\Sigma_{e} P_{e})^{-1} \). A two-period default-free pure discount bond has the certain compound rate of return of \((1 + r_{e})^2 \equiv (\Sigma_{e} P_{e})^{-1} \) and a rate of return in the first period conditional on state \( e \) of \( 1 + r_{e} \equiv (1 + r_{e})^2 / (1 + r_{e}) \). As a result, if the present value of a default-free annuity of one unit of wealth at date \( t = 0 \) and \( t = 1 \) is denoted by

\[
\Phi = 1 + \frac{1}{1 + r_{e}} + \frac{1}{(1 + r_{e})^2}
\]

and a default-free annuity of one unit of wealth at dates \( t = 1 \) and \( t = 2 \) is denoted by

\[
\Phi_{e} = 1 + \frac{1}{1 + r_{e}}
\]

then the first period rate of return of a default-free annuity purchased at date \( t = 0 \) and yielding one unit of wealth at dates \( t = 1 \) and \( t = 2 \) is

\[
1 + r_{e} = \frac{\Phi_{e}}{\Phi_{e-1}}.
\]

through a production function \( f_{i}(X_{0}^{i}, \{X_{j}^{i}\}, \{X_{s,e}^{i}\}) = 0 \) where by convention \( X \leq 0 \) represents an input and \( X > 0 \) an output. Although \( X_{0}^{i} \leq 0 \) and \( X_{s,e}^{j} \geq 0 \), \( X_{s,e}^{j} \) can have either sign. Consumer \( i \) is also endowed with shares \( \eta_{ij} \) of producer \( j \) such that \( 0 \leq \eta_{ij} \leq 1 \) and \( E_{i} \eta_{ij} = 1 \). Therefore, given production decisions, his endowed consumption would then be

\[
C_{0}^{i} \equiv \frac{X_{0}^{i}}{0} + E_{j} \eta_{ij} X_{0}^{j}, \quad C_{e}^{i} \equiv \frac{X_{e}^{i}}{e} + E_{j} \eta_{ij} X_{e}^{j}, \quad C_{s,e}^{i} \equiv \frac{X_{s,e}^{i}}{s,e} + E_{j} \eta_{ij} X_{s,e}^{j}.
\]

In this paper, although we take the decisions of producers as exogenous, its results would also apply if production decisions were endogenously determined.
We will also denote the present value of annuities maturing at a consumer’s death by

$$\phi_A = 1 + \frac{1}{1+r_F} + \frac{\lambda}{(1+r_F)^2} \quad \text{and} \quad \phi_{Ae} = 1 + \frac{\lambda}{1+r_{Fe}}.$$  

Each consumer is assumed to obey the Savage (1954) Axioms of Rational Choice: he has beliefs and tastes representable by probabilities and a utility function over lifetime consumption. That is, each consumer is assumed to maximize

$$E[\tilde{U}(C_0, C_1, C_2)] = \sum_{e} \pi_{e} \pi_{s,e} \tilde{U}(C_0, C_e, C_{s,e})$$

subject to his nonnegativity and budget constraints where $\pi_e > 0$ denotes the consumer’s subjective probability that state $e$ will occur and $\pi_{s,e} > 0$ denotes his state $e$ conditional subjective probability that state $s$ will occur. Additionally, $\tilde{U}$ is assumed to be strictly increasing in each of its arguments and strictly concave in the vector $(C_0, C_e, C_{s,e})$.

To this point (except for the finite number of states, the discreteness of time, and the non-negativity of consumption) the structure of the generalized logarithmic utility model is shared by all Type I and Type II models. GLUM draws its distinctive explanatory power by adopting three special assumptions surrounding consumer tastes:

1. **Intertemporal risk neutrality**: $\partial \tilde{U}(C_0, C_1, C_2)/\partial C_t = U_t'(C_t)$
2. **Separation of time- and risk-preference**: $U_t'(C_t)/U_{t-1}'(C) = \rho_t \quad (\text{all levels } C > 0)$
3. **Generalized logarithmic utility**: $U(C_t) = \ln(A + C_t)$.

Assumption (1) is common to the continuous-time lognormality model (1B) and all Type II models; assumption (2) is common only to Type II models; and assumption (3) distinguishes GLUM from other Type II models.

Interpreting assumption (1) verbally, the marginal utility of consumption at any date is independent of consumption at all other dates. As a consequence, a necessary and sufficient condition for assumption (1) is that there exist functions $\{U_t\}$ such that

$$\tilde{U}(C_0, C_1, C_2) = U_0(C_0) + U_1(C_1) + U_2(C_2).$$

---

$^{15}$As previously observed, the completeness of the securities market is not crucial to the conclusions of Type I or Type II models. Like the first stage of a moon rocket, it is very useful but will nonetheless be ejected once the payload has attained maximum thrust.
This implies two patterns of consumer behavior which if considered together may make this assumption reasonable. The first and most apparent is the independence of the utility of consumption at each date from consumption at other dates; for example, the utility added by consumption $C_1$ at date $t = 1$ is independent of the levels of consumption $C_0$ and $C_2$ at dates $t = 0$ and $t = 2$. Both habit formation and loss of enjoyment due to excessive past or excessive anticipated indulgence are not permitted. On the other hand, consider the following two lotteries over lifetime $T = 1$:

**Lottery 1:** $\left( \frac{1}{2} ; C_0 = a, C_1 = b \right)$ versus $\left( \frac{1}{2} ; C_0 = b, C_1 = a \right)$

**Lottery 2:** $\left( \frac{1}{2} ; C_0 = a, C_1 = a \right)$ versus $\left( \frac{1}{2} ; C_0 = b, C_1 = b \right)$

where $a < b$. A consumer who chooses the first lottery balances a poor year against a rich year regardless of which outcome occurs; a consumer who chooses the second lottery, either experiences a poor year every year or a rich year every year. The former consumer might aptly be termed "intertemporally risk averse" and the second consumer "intertemporally risk preferring." Richard (1975) has shown that a consumer will be indifferent between the two lotteries if and only if his utility function satisfies assumption (1). GLUM consumers are therefore "intertemporally risk neutral." On the basis of casual empiricism, Richard argues that consumers tend to be "intertemporally risk averse": they prefer a little now and more later, or vice versa, to the all or nothing alternative. However, as Nils Hakansson has suggested to me, the "intertemporally risk preferring" consumer has the advantage of avoiding severe jolts in his lifetime consumption pattern. How uncomfortable it must be for consumers stuck with the first lottery to flip in the second outcome from a high standard of living early in life to a much lower standard of living later in life. Similarly, under the first outcome, a consumer with a low standard of living early in life may be both psychologically and educationally unprepared for a much higher standard of living in later years. Taking all the arguments in this paragraph into account, perhaps the neutral position of assumption (1) is not unreasonable.

Assumption (2) permits the formal separation of the psychological notions of risk-preference (risk aversion) and time-preference (patience). Taken together, a necessary and sufficient condition for assumptions (1) and (2) is that there exist positive constants $\rho_t$ and a function $U$ such that

$$U(C_0, C_1, C_2) = U(C_0) + \rho_1 U(C_1) + \rho_2 U(C_2).$$

These constants express the relative preference for consumption at different dates. For example if $0 < \rho_1 < 1$, then the consumer prefers more consumption at date $t = 0$ than at date $t = 1$. In this instance, the consumer is characterized as "impatient" and if $\rho_1 > 1$ as "patient." Henceforth,
$(\rho_1, \rho_2)$ will be referred to as the measure of consumer time-preference. It will also be convenient to use the alternative notation $\delta_1 = \rho_2 (1+\rho_2)^{-1}$ and $\delta = \rho_1 (1+\rho_2) \left[ 1 + \rho_1 (1+\rho_2)^{-1} \right]^{-1}$. Risk aversion prevents excessive concentration of consumption at a single date; that is, as a result of the non-satiation and concavity properties of $\bar{U}$ it follows that $U' > 0$ and $U'' < 0$.\(^{16}\)

Assumption (3), no doubt the most controversial of GLUM, requires that all consumers behave as if they have generalized logarithmic utility functions. This assumption is not as restrictive as it might at first seem. First, all consumers must have decreasing absolute risk aversion. Second, since the heterogeneity of the model permits consumers to have taste parameters $A$ of different signs, it simultaneously tolerates quite diverse attitudes of proportional risk aversion:

<table>
<thead>
<tr>
<th>$A$</th>
<th>Proportional risk aversion</th>
</tr>
</thead>
<tbody>
<tr>
<td>positive</td>
<td>increasing</td>
</tr>
<tr>
<td>zero</td>
<td>constant</td>
</tr>
<tr>
<td>negative</td>
<td>decreasing</td>
</tr>
</tbody>
</table>

Henceforth, $A$ will be referred to as the measure of consumer risk-preference; the higher $A$, the "more risk preferring" the consumer. The graph of $U(C_t) = \ln(A + C_t)$ shows an everywhere strictly increasing and strictly concave curve with no upper or lower bound, with an intercept along the horizontal axis at $-A + 1$, and with $\lim_{C_t \to -A} \ln(A + C_t) = -\infty$.

\[ \ln(A + C_t) \]

\[ \begin{array}{ccc}
0 & & \\
-1 & & \\
-2 & & \\
-3 & & \\
\end{array} \]

\[ \begin{array}{ccc}
0 & & \\
1 & & \\
2 & & \\
\end{array} \]

\[ C_t \]

\[ -A \quad -A+1 \quad -A+e \quad -A+e^2 \]

\[ \lim_{C_t \to -A} \ln(A + C_t) = -\infty. \]

---

\(^{16}\) Roopmaa (1960) provides an axiomatic justification for the distinction between time- and risk-preference.
Regardless of the size of $A$, as $C_t$ increases a consumer's behavior becomes closer to the behavior of a consumer with constant proportional risk aversion since

$$\lim_{C_t \to \infty} \left[ \ln(A + C_t) - \ln C_t \right] = 0.$$  

Moreover, when $A = 0$ (or when $A$ is small relative to $C_t$), a consumer follows the famous Bernoulli-Laplace hypothesis about the marginal utility of income. Coincidentally, such a consumer also obeys the Weber-Fechner law of psycho-physics, that the marginal impact of a stimulus is inversely proportional to the intensity of the stimulus. Observe that $dk \ln C_t / dC_t = kC_t^{-1}$ where $k$ is a positive constant. More recently, Breiman (1961) has shown that such a consumer

1. maximizes the expected compound growth rate per dollar of investment;
2. minimizes the probability of ruin in the long run;
3. selects a portfolio strategy for which the probability that the compound growth rate per dollar of investment exceeds that generated by any other strategy approaches 1 as the number of periods goes to infinity;
4. reaches any given target wealth (after given consumption) level in the shortest possible expected time.

When $A \leq 0$, the consumer will never take the chance that $C_t \leq -A$ since such low levels of consumption have infinite disutility. Consequently, when $A \leq 0$, $-A$ may be interpreted as subsistence level consumption below which death occurs. When $A > 0$, the utility function by itself does not guard against zero or negative consumption and additional conditions are required to prevent a consumer from taking risks that may result in negative consumption.\(^{17}\)

---

\(^{17}\)Irrespective of the sign of $\tilde{A}$, a consumer's endowed wealth must be assumed large enough so that he is not forced to chance levels of consumption $C_t \leq -A$ at any date $t$. A necessary condition for $C_t^i > -A$, for all $i$ and $t$ is $C_t^N > -\sum_i A_i$ for all $t$. For any consumer, a sufficient condition for $C_t > -A$ for all $t$ is $\bar{C}_t > -A$ for all $t$. For any consumer, a necessary and sufficient condition for $C_t > -A$ for all $t$ is $W_0 > -A\Phi^*_\lambda$, where $A\Phi^*_\lambda$ is interpreted as the present value of $-A$ units of consumption at every date during the consumer's lifetime. Anyone for whom $W_0 \leq -A\Phi^*_\lambda$ will not be able to bear the pain of life and will either have committed suicide or died at birth of natural causes.

With this innocuous condition, for any consumer with $A \leq 0$ his feasible and therefore his optimal consumption choices are always above his subsistence level. However, for $A > 0$, although positive consumption at every date is feasible it will not always be optimal unless further conditions are added. These conditions can either (1) prevent any consumer from being too much of a non-conformist, or (2) reduce the differences in aggregate consumption among different dates and/or states. In an extreme case, if all consumers are identical then even if $A > 0$, feasible positive consumption at every date for all consumers is also optimal. For examples of much less stringent conditions, see footnote 33. Henceforth, we will assume that such conditions apply.
A feeling for the implications of generalized logarithmic utility for portfolio choice may be obtained from a simple example. Suppose a consumer with generalized logarithmic utility has $W$ and he can choose what proportion $1 - \alpha$ of his money to leave in cash and what proportion $0 \leq \alpha \leq 1$ to invest in a gamble yielding rate of return $r = 1$ with probability $0 < \pi < 1$ or rate of return $r = -1$ with probability $1 - \pi$. If the gamble is fair or unfavorable ($\pi \leq 1/2$), he leaves all his money in cash ($\alpha = 0$). If the gamble is favorable and a positive investment in it is sure to net the consumer more than $-A$, then he invests at least part of his money in the gamble ($\alpha > 0$). Moreover, if $A \leq 0$, then he never invests all his money ($0 < \alpha < 1$); but if $A > 0$ and the gamble is favorable enough, the consumer is tempted sufficiently to place all his funds at risk ($\alpha = 1$). It is easy to show that the following formula describes the optimal proportion $\alpha^*$ to invest in a favorable gamble:

$$
\alpha^* = \begin{cases} 
0 & \text{if } \tilde{\alpha} \leq 0 \\
\tilde{\alpha} & \text{if } 0 < \tilde{\alpha} < 1 \\
1 & \text{if } \tilde{\alpha} \geq 1
\end{cases}
$$

$$
\tilde{\alpha} = \frac{1 + A}{1 + \frac{A}{W}}(2\pi - 1)
$$

For a quick solution to the problem first solve for the optimal portfolio in terms of state-contingent securities and prices; second determine the state-contingent prices from actual security prices; and third convert the solution in terms of state-contingent securities to portfolio proportions of actual securities.

The programming problem is the Lagrangian expression

$$
\max \sum_{e} P_e \ln (A + W_e) - A \sum_{e} P_e W_e - W
$$

which yields the optimal decision rule: $W_e = (P_e / P_e) \ln (A + W_e) - A$ where $P_e = \sum_{e} P_e e_e$. In this example, there are two states and two actual securities; therefore the state-contingent prices can be calculated from prices of actual securities by solving the simultaneous equations

$$
\begin{align*}
1 &= P_1 + P_2 \\
1 &= P_1(2) + P_2(0)
\end{align*}
$$

so that $P_1 = P_2 = 1/2$. Substituting this into the optimal decision rule: $W_e = 2P_e (A + W_e) - A$ for $e = 1, 2$. Let $\alpha$ represent the proportion of initial wealth $W$ allocated to the risky security; therefore

$$
W_1 = (1 - \alpha)W + \alpha W(2) = (1 + \alpha)W \quad \text{and} \quad W_2 = (1 - \alpha)W + \alpha W(0) = (1 - \alpha)W.
$$

Selecting either state, say $e = 1$, and setting

$$
2P_e (A + W) - A = (1 + \alpha)W,
$$

it follows that $\alpha^* = \left(1 + \frac{A}{W}\right)(2\pi - 1)$ whenever $0 < \alpha^* < 1$.

Whenever the number of different actual ("complex") securities available is equal to the number of assumed states, a procedure similar to the above may be used to determine portfolio choices of actual securities. In practice, it is necessary to form a coarse partition of the number of states to make them equal to the number of securities available. However, with about ten or more securities, considering the difficulties of estimation, this may be a reasonable approach. With large numbers of securities, a computer is only required first to determine the solution $\left(P_e\right)$ to the simultaneous
The following table provides optimal proportions $\alpha^*$ for various levels of risk aversion $A$, wealth $W$, and belief $\pi$:

<table>
<thead>
<tr>
<th></th>
<th>$A = -1$</th>
<th>$A = 0$</th>
<th>$A = +1$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$1/2^*$</td>
<td>$1/2$</td>
<td>$1/2$</td>
</tr>
<tr>
<td></td>
<td>$1$</td>
<td>$2$</td>
<td>$4$</td>
</tr>
<tr>
<td>$W$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$W$</td>
<td>$1$</td>
<td>$2$</td>
<td>$4$</td>
</tr>
<tr>
<td>$W$</td>
<td>$4$</td>
<td>$2$</td>
<td>$1$</td>
</tr>
<tr>
<td>$W$</td>
<td>$1/2$</td>
<td>$1/2$</td>
<td>$1/2$</td>
</tr>
<tr>
<td>$\pi$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\pi$</td>
<td>$0.5$</td>
<td>$0.55$</td>
<td>$0.6$</td>
</tr>
<tr>
<td>$\pi$</td>
<td>$0.6$</td>
<td>$0.7$</td>
<td>$0.8$</td>
</tr>
<tr>
<td>$\pi$</td>
<td>$0.7$</td>
<td>$0.8$</td>
<td>$0.9$</td>
</tr>
<tr>
<td>$\pi$</td>
<td>$0.8$</td>
<td>$0.9$</td>
<td>$0.95$</td>
</tr>
</tbody>
</table>

*Utility undefined.*

Observe that, at the same level of wealth, consumers with $A < 0$ are more conservative than consumers with $A > 0$; and, in both cases as their wealth increases, their choices approach those made if $A = 0$.

Summarizing this section, the programming problem of every consumer $i$ in GLUM may be formally stated as:

$$\max_{C_i^0, (C_{i,e}^c)} \ln(A_i + C_i^0) + \rho_i c^i \ln(A_i + C_i^0) + \rho_i e_{s,e}^i \ln(e_{s,e}^i)$$

subject to $C_i^0 > 0$ for all $c$ and

$$C_i^0 + \sum_{e,s} C_{i,e}^c + \sum_{e,s} e_{s,e}^i C_{i,e}^c < C_i^0 = C_i^0 + \sum_{e,s} e_{s,e}^i + \sum_{e,s} e_{s,e}^i \lambda_i$$

where $\lambda_i = 0$ or $1$ and, if $\lambda_i = 0$ then $\rho_i = 0$; otherwise $\rho_i > 0$. Closure conditions require

$$\sum_{i=0}^M C_i^0 = C_i^0 \equiv \sum_{i=0}^M C_i^0, \quad \sum_{i=0}^M C_{i,e}^c = C_{i,e}^c \equiv \sum_{i=0}^M C_{i,e}^c, \quad \text{and} \quad \sum_{i=0}^M C_{i,s,e} = C_{i,s,e} \equiv \sum_{i=0}^M C_{i,s,e}$$

Equations $1 = \sum_{i=0}^M (1 + r_{i,j})$ for all $j$, and second to determine the solution for $\{\pi_j\}$ to the simultaneous equations $W = W_{i,m} \alpha_j (1 + r_{i,j})$ for all $e$, where $j(j = 1,2,\ldots,J)$ indexes available securities and $r_{i,j}$ is the rate of return of security $j$ associated with state $e$. 
for all $e$ and $a$. The subscript or superscript $i$ indicates that consumers may be different in every respect except they all have generalized logarithmic utility functions (but with different time- and risk-preference) and face the same prices—an implication of a competitive securities market. The problem confronting the theorist is to express the endogenous variables

\[
\text{choices: } \{c_t^1, \{c_{s,e}^1\}, \{c_{s,e}^i\}\} \quad \text{and prices: } \{p_e^1, \{p_{s,e}^i\}\}
\]

in terms of the exogenous variables

\[
\text{resources: } \{c_t^0, \{c_{s,e}^0\}, \{p_{s,e}^0\}\} \quad \text{and lifetime: } \{\lambda_e^i\},
\]

\[
\text{risk-preference: } \{\alpha_e^i\} \quad \text{and time-preference: } \{\rho_1^i, \rho_2^i\},
\]

\[
\text{beliefs: } \{\pi_{e,s}^i, \{\pi_{s,e}^i\}\}.
\]

3. FINANCIAL CHOICE

It is useful to view the consumer as solving a dynamic programming problem. That is, in place of solving the single-stage maximand of section 2, the consumer solves the following two-stage programming problem. First, at date $t = 1$, given state $e$ and wealth $w_e$, carried forward from the first period, the consumer

\[
V_e(w_e) = \max_{c_e, \{c_{s,e}\}} \ln(A + c_e) + \rho_2 \sum_{s,e} \ln(A + c_{s,e}) - \lambda_e[c_e + \sum_{s,e} c_{s,e} - w_e]
\]

where $\lambda_e$ is a state-dependent Lagrangian multiplier.\(^{19}\) Second, at date $t = 0$, knowing $V_e(w_e)$, the consumer

\[
V_0(w_0) = \max_{c_0, \{w_e\}} \ln(A + c_0) + \rho_1 \sum_{e} V_e(w_e) - \lambda [c_0 + \sum_{e} c_e - w_0].
\]

Following this pattern of analysis, the following optimal consumption and portfolio decision rules may be derived:

**Theorem (decision rules):** In the generalized logarithmic utility economy, the optimal consumption and portfolio decision rules at each date are linear in wealth at that date, and are otherwise independent of past decisions and the composition of wealth. Optimal consumption and portfolio decisions at any date also exhibit partial myopia in that they are dependent on security rates of return in future periods only through their dependence on future risk-free rates of return. In particular,\(^{20}\)

---

\(^{19}\)Recall from footnote 17 that we assume conditions on exogenous variables sufficient to insure $c_t^i > 0$ for all $i$.

\(^{20}\)In terms of "normal" income, $c_0 = \bar{A}[\phi_i^0(1-\delta) - 1] + (1-\delta)\Phi_i Y$ where $Y$ is that constant income received over a consumer's lifetime which has the same present value as initial wealth (i.e., $w_0 = \phi_i Y$).
\[ C_0 = A[\phi_e (1-\delta) - 1] + (1-\delta)W_0 \]

\[ W_e = A[(\pi / P_e)\phi_e \delta - \phi_e e] + (\pi / P_e)\delta W_0. \]

The derived utility of wealth functions also imply partial myopia. In particular,

\[ v_e(W_e) \approx (1 + \rho_2)\ln[A\phi_e + W_e] \]

\[ \nu_0(W_0) \approx (1 + \rho_1 + \rho_2\rho_3)\ln[A\phi + W_0] \]

where \( \approx \) means "is equivalent up to an additive transformation to."

**Proof:** Partially differentiating the first-stage maximand with respect to each choice variable, setting the derivatives equal to zero, and eliminating the Lagrangian multiplier

\[(1) \quad P_{s,e} (\alpha_e + C_{s,e}) = \pi_{s,e} \rho_2 (A + C_e) \quad (\text{all } s)\]

Summing over \( s, \) introducing the budget constraint and rearranging

\[(2) \quad A\phi_e + W_e = (1 + \rho_2)(A + C_e) \]

or alternatively stated

\[(3) \quad C_e = A[\phi_e (1 - \delta) - 1] + (1 - \delta)W_e. \]

Substituting this back into equation (1),

\[(4) \quad C_{s,e} = A[(\pi_{s,e} / P_{s,e})\phi_e \delta - A] + (\pi_{s,e} / P_{s,e})\delta W_e \quad (\text{all } s). \]

Now, substituting these expressions for \( C_e \) and \( (C_{s,e}) \) into

\[ v_e(W_e) \equiv \ln(A + C_e) + \rho_2 \sum_{s} \pi_{s,e} \ln(A\lambda + C_{s,e}) \]

and rearranging produces

\[ v_e(W_e) = (1 + \rho_2)\ln[A\phi_e + W_e] + \{\rho_2 \pi_{s,e} \ln(\pi_{s,e} / P_{s,e}) + \rho_2 \ln \rho_2 - (1 + \rho_2)\ln(1 + \rho_2)\}. \]

Since the last bracketed expression depends only on exogenous variables, the derived utility of wealth function \( v_e(W_e) \) is equivalent up to an additive transformation to the unbracketed term.\(^{21}\)

---

\(^{21}\) For the purpose of determining optimal decision rules, it is unnecessary to know the exact form of the derived utility of wealth function \( v_e(W_e) \) since, after differentiation to obtain optimal
derived expression for $V_e(W_e)$ is itself substituted into the second stage maximand, by partial differentiation it again follows that

\[(5) \quad P_e(A^e\lambda e + W_e) = \pi_e\rho_1(1 + \rho_2)(A + C_0) \quad (\text{all } e)\]

or alternatively, in conjunction with equation (2)

\[(1') \quad P_e(A + C_e) = \pi_e\rho_1(A + C_0) \quad (\text{all } e)\]

A similar analysis to the above then shows that

\[(2') \quad A^e\lambda + W_0 = (1 + \rho_1 + \rho_1\rho_2)(A + C_0)\]

\[(3') \quad C_0 = A[\phi_e(1 - \delta) - 1] + (1 - \delta)W_0\]

\[(4') \quad W_e = A[(\pi_e/P_e)\phi_e, \delta - \phi_e] + (\pi_e/P_e)SW_0 \quad (\text{all } e)\]

\[V_0(W_0) = (1 + \rho_1 + \rho_1\rho_2)\ln[A^e\lambda + W_0] + \left\{\rho_1(1 + \rho_2)\sum_e P_e\ln(\pi_e/P_e) + \rho_1(1 + \rho_2)\ln[\rho_1(1 + \rho_2)]\right\} - (1 + \rho_1 + \rho_1\rho_2)\ln(1 + \rho_1 + \rho_1\rho_2)\}

Q.E.D.

These decision rules yield immediate and sensible comparative statics results. The greater initial wealth $W_0$, the greater optimal consumption and portfolio holdings for each state since by definition $0 < \delta < 1$. Since \(\partial C_0/\partial \delta = -(A^e\lambda + W_0) < 0\) and \(\partial W_e/\partial \delta = (\pi_e/P_e)(A^e\lambda + W_0) > 0\), the more patient the consumer the less his initial consumption and the more his future consumption. Indeed, as a result of time-preference, he reduces his initial consumption by \(\delta(A^e\lambda + W_0)\) and adds this to his initial investment. His initial consumption, even if \(\delta\) is near one (extreme patience), has a greatest lower bound of \(-A\); his initial investment, \(\sum_e P_e W_e = A[\phi_e(\delta - 1) + 1] + SW_0\), even if \(\delta\) is near zero (extreme impatience), has a greatest lower bound of \(-A(\phi_e - 1)\). If \(\phi_e > (1-\delta)^{-1} = 1 + \rho_1 + \rho_1\rho_2\), then as risk-preference increases (i.e., \(A\) increases), initial consumption rises and initial investment falls; and vice versa. \(\phi_e > 1 + \rho_1 + \rho_1\rho_2\) means that the securities market applies a higher series of discount rates to future consumption than that implied by a consumer's time-preference. In particular, if \(\frac{1}{1+\tau} + \frac{1}{(1+\tau)^2} > \rho_1 + \rho_1\rho_2\), then with decreasing proportional risk aversion \((A < 0)\) a
consumer reduces his consumption but with increasing proportional risk aversion \((A > 0)\), he increases it. Finally, since \(W_e = (\pi_e^e/\bar{F}_e)\delta_{A(\Phi_\lambda + W_0) - A\Phi_{\lambda e}}\), the consumer invests more in claims to states he thinks more likely to occur or which have relatively cheap state-contingent claims.

Significantly, the derived utility of wealth functions are of the same "form" as the single-period utility function over consumption \(U(\tilde{\zeta}_e)\). To see this more clearly, consider a consumer who is alive at date \(t = 1\) \((\lambda = 1)\) with a constant rate of time-preference \((\rho_1 = \rho_2 \equiv \rho < 1)\); for him

\[
U(\tilde{\zeta}_e) = \ln(A + \tilde{\zeta}_e)
\]

\[
V_e(W_e) \approx (1 + \rho)\ln[A\Phi_e + W_e] \quad \text{and} \quad V_0(W_0) \approx (1 + \rho^2)\ln[A\Phi + W_0].
\]

This ensures that, even in an expanded setting with more than three dates, the sequence of single-period maximands is roughly similar. Indeed, if lifetime \(T = \infty\), it is easy to extrapolate from the above to show that \(V_0(W_0) = (1 - \rho)^{-1}\ln[A\Phi + W_0]\) where \(\phi\) now represents the present value of a perpetual annuity. If additionally at date \(t = 0\), the period-by-period spot and forward riskfree rates were equal, then \(\phi = (1 + r_F)/r_F\). Moreover, if future spot and forward riskfree rates were also equal to today's spot rate \(r_F\), then for any date \(t\), \(V_t(\tilde{\zeta}_e) = (1 - \rho)^{-1}\ln[A\Phi + \tilde{\zeta}_e]\); except for variations in wealth over time, the sequence of single-period maximands is exactly the same.\(^{22}\) At this point, it is also worth noting that since \(V'_t > 0\) and \(V''_t < 0\), then \(-V'_t(\tilde{\zeta}_e)/V''_t(\tilde{\zeta}_e) = \delta_{A(\Phi_\lambda + W_0)} = \tilde{\zeta}_e > 0\) for all dates \(t\).

The portfolio decision rules may be used to prove the following multiperiod separation theorem:

**Corollary** (portfolio separation). In the generalized logarithmetic utility economy, portfolio choices satisfy the generalized separation property: at each date, the consumer divides his wealth after consumption between

1. a default-free annuity terminating at his death, and
2. a risky portfolio the composition of which is independent of his wealth, time- and risk-preference, lifetime, and the returns of securities available at future dates.

In particular,

\[
W_e = (1 + (1 - \lambda)\tau_e + \lambda e\bar{\tau}_e)\{(W_0 - C_0) - \delta_{A(\Phi_\lambda + W_0)\} + (1 + \tau_F)\delta_{A(\Phi_\lambda + W_0)}
\]

for all \(e\), where \(1 + \tau_F \equiv \pi_e^e/\bar{F}_e\).

**Proof:** Assume the conclusion and use equation \((31)\) to substitute for \(C_0\). By canceling duplicating terms and rearranging, it is possible to derive the portfolio decision rule. This process can be repeated backwards. **Q.E.D.**

\(^{22}\) Note also that under these conditions, \(C_0 = (A/r_F)(1 - \rho(1 + r_F)) + (1 - \rho)W_0\).
This corollary generalizes the usual portfolio separation property to a nontrivial multiperiod setting in which none of the following assumptions have been made:

1. all security rates of return follow a random walk,
2. tastes are logarithmic ($\lambda = 0$),
3. lifetime is one period ($\lambda = 0$).

In any one of these special cases, the generalized separation property developed here collapses to more familiar single-period properties. With a random walk or a one-period lifetime, a consumer divides his wealth after consumption between a portfolio yielding the first period risk-free rate and a risky portfolio the composition of which is independent of his wealth. Similarly, with logarithmic utility, a consumer invests all his wealth after consumption in a single risky portfolio the composition of which is independent of his wealth. In either case, default-free long-term bonds have no raison d'etre; they are simply not needed by consumers to attain their optimal portfolio holdings. Therefore, since the default-free annuity can be regarded as an equally weighted portfolio of pure discount bonds of each possible maturity, one of the merits of the theory developed here is its ability to explain the demand for default-free bonds of varying maturities.

One other somewhat surprising result is the independence of the composition of the bond portfolio from the time pattern of time-preference ($\rho_1, \rho_2$), provided $\lambda = 1$. In the extreme case, if a consumer dies at date $t = 1$, then he only demands short-term bonds. Similarly, it can be shown if a consumer's utility were defined only over consumption at dates $t = 0$ and $t = 2$, then he would demand only long-term bonds. However, for anything short of these two extremes ($\rho_1 > 0$, $\rho_2 > 0$), a consumer balances his holdings of short- and long-term bonds equally. The "preferred habitat hypothesis" is therefore confirmed only for extreme forms of consumer tastes.

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23 Multiperiod separation results for the entire HARA class of utility functions may be found in Rubinstein (1974b).

24 However, in the exponential utility model (II3), Rubinstein (1974b) shows that a consumer's bond portfolio is sensitive to the time pattern of his time-preference. In this model, other things equal, a consumer with a greater rate of time-preference $\rho_2$ toward consumption at date $t = 2$ than average will tend to slant his bond portfolio toward date $t = 2$ maturities.

This sensitivity (model IIIB) or lack of sensitivity (model IIIC) of the demand for short-versus long-term bonds to the time pattern of consumer time-preference differs from the results of Stiglitz (1970). Using a three-date model, he examines the relationship between the term structure and an individual's demand for short- versus long-term bonds. At date $t = 0$, individuals choose between short- and long-term default-free pure discount bonds. At date $t = 1$, the previous short-term bonds mature and the previous long-terms now become the only existing short-term bonds. Consumers can revise their portfolios choosing between consumption and short-term bonds. At date $t = 2$, they consume their remaining wealth. Stiglitz introduces uncertainty by assuming the date $t = 1$ price of the long-term bonds is uncertain at date $t = 0$. His results differ because he admits no purely risky securities into his model. Had he done so, to the sources of demand for bonds he considers, he would have added their capacity as hedges against future shifts in the returns of risky securities. Moreover, without purely risky securities, if his model is carried to equilibrium by aggregating consumer
The portfolio separation corollary will hold even for a generalized logarithmic utility consumer embedded in an economy in which other consumers do not have generalized logarithmic utility, but differ arbitrarily (provided they are risk averse) in their tastes for consumption. However, when all consumers, as in GLUM, have generalized logarithmic utility, the following stronger separation property may be demonstrated:

**Corollary (universal portfolio separation).** In the generalized logarithmic economy, if all consumers have the same beliefs, portfolio choices satisfy the generalized universal separation property: at each date, each consumer divides his wealth after consumption between

1. a portfolio of default-free bonds of mixed maturities, the composition of which depends upon his wealth, risk-preference, time-preference and lifetime, and
2. the market portfolio containing all securities in proportion to their relative market values.

If, additionally, all consumers have the same lifetime, then the portfolio of default-free bonds reduces to a default-free annuity terminating at their death.

**Proof:** Since equation (5) holds for all consumers in the economy and by assumption all consumers have the same beliefs

\[(\pi_e' P_e)_{i} (1 + \rho_i^1) (A_i + C_0^1) = A_i [1 + \lambda_i (1 + r_{Fe})^{-1}] + W_i^e\]

for all \(i\) where \(\phi_{Ae} = 1 + \lambda (1 + r_{Fe})^{-1}\). Summing this over all \(i\)

\[(\pi_e' P_e)_{i} \sum_{i=1}^{i=n} (1 + \rho_i^1) (A_i + C_0^1) = (\sum_{i=1}^{i=n} A_i) + (1 + r_{Fe})^{-1} \sum_{i=1}^{i=n} \lambda_i A_i + \sum_{i=1}^{i=n} W_i^e.\]

By definition and closure, \(W_{e}^{Y} = \sum W_{e}^{Y}\) so that \((W_{e}^{Y} - C_0^Y) (1 + r_{Me}) = W_{e}^{Y}\). Similarly, by definition \(1 + r_{Fe} = \pi_e' P_e\), and \(1 + R_{Fe} = (1 + R_{F})^2 / (1 + r_{Fe})\). Letting \(K^{-1} = \pi_{i} (1 + \rho_i^1) (A_i + C_0^i)\) and using these relationships to substitute into the previous equation

\[(1 + r_{Fe}) = [K (A_i A_i) (1 + r_{F})^{-1}] (1 + r_{F}) + [K (\lambda_i A_i, A_i) (1 + r_{F})^{-2}] (1 + R_{Fe}) + [K (W_{0}^{Y} - C_0^Y) (1 + r_{Me})].\]

Therefore, there exist constants \(a, b, c\) such that in equilibrium \((1 + r_{Fe}) = a (1 + r_{F}) + b (1 + R_{Fe}) + c (1 + r_{Me})\) so that the risky portfolio, which features in the portfolio separation property, may be replaced by holdings of short- and long-term default-free bonds and the market portfolio of all securities. This added to the default-free annuity (i.e., short- and long-term default-free bonds) also held by each consumer implies the universal portfolio separation property. It is also easy to see from the above equation that if all consumers have lifetime \(T = 1\), then \(b = 0\) and they demand for bonds, the equilibrium intermediate price of the long-term bonds becomes certain. Not only does this force the term structure to be unbiased to prevent riskless arbitrage, but also implies that consumers, irrespective of their tastes, will be indifferent between short- and long-term bonds,
only hold short-term bonds and the market portfolio. Likewise, if all consumers have lifetime \( T = 2 \), then

\[ a(1 + r_{FE}) + b(1 + r_{NE}) = \left[K(\phi - 1)\sum_{t=1}^{T} A_t\right](1 + r_{NE}) \]

and they each hold a two-period default-free annuity and the market portfolio. Q.E.D.

Under the conditions of the corollary, all consumers find it sufficient to hedge against all future shifts in portfolio opportunities by borrowing or lending default-free bonds of various maturities. This result generalizes to any number of dates. For example, if all consumers live forever \((T = \infty)\), then each either borrows or lends a perpetual default-free annuity in addition to a long position in the market portfolio. Indeed, if the financial market is sufficiently rich in securities at date \( t = 0 \), then no consumer need ever revise his portfolio. In an extreme case, if a complete futures market existed at date \( t = 0 \) so that claims were not only available for each state \( e \) but also for each sequence of states \( e \) and \( s \), then irrespective of consumer tastes, no consumer will ever need to revise his portfolio. However, in GLM with homogeneous beliefs, a complete futures market is far from necessary. As long as at least default-free bonds exist of each maturity (the market portfolio exists trivially), then no portfolio revision will occur, consumers managing over time quite nicely off the interest income from the bonds and the dividend income from the market portfolio.

The behavior of consumers in GLM can also be usefully characterized by comparing their choices with those made by an appropriately defined "average" consumer. In particular, portfolio choices can be broken down into components, each describing the demand for particular securities created by the deviation of a particular economic characteristic from average. The decomposition of the molecule of portfolio choice into its constituent atoms lays bare the comparative statics of financial choice.

Meaningful "average" economic characteristics (in this case—wealth composition and scale, risk-preference, time-preference, lifetime, and beliefs) should satisfy certain properties. Let the economic characteristics of consumer \( i \) be denoted by the set \( \{x_i^c\} \) where subscript \( c \) indexes these characteristics. An "average" characteristic is denoted by \( x_c \) and its relationship to the underlying characteristics of consumers is described by the explicit function \( x_c = f_c(\{x_i^c\} \text{ for all } i) \). A consensus (or social) characteristic \( x_c \) is said to exist if it satisfies the following properties:

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25 Holdings of the market portfolio must be long (not short) when all consumers have the same beliefs since, from risk aversion, in equilibrium, the market portfolio is a favorable gamble relative to holding the bond portfolio and all risk averters accept a positive fraction of favorable gamble (see Arrow (1965)). In particular, from the portfolio separation corollaries, a consumer invests

\[ \delta_{t+1} \left[ A_{t+1} + \frac{K_{t+1} - C_{t+1}}{K_{t+1}} \right] \]

and

\[ -\frac{1}{\bar{v}_t} (\bar{v}_t^{t+1} \cdot n_t^{t+1} (x_{t+1}^c - A_t \hat{A}_{t+1} + \bar{v}_t^{t+1} > 0, \text{ this amount must be positive.} \]

26 Other interesting properties include non-dictatorship (no social choice depends solely on the characteristics of a group of consumers smaller than the population), positive responsiveness (an increase in a characteristic of any consumer affects social choices in the same direction as an increase in its corresponding consensus characteristic), and anonymity (any permutation of the same
1. **homogeneity:** if \( x^i_c \) is homogeneous, then \( x^i_c = x^i \).

2. **composability:** if \( c \) is denominated in units of wealth, then \( x^i_c = \mathcal{I}_c x^i_c / I \).

3. **consensus:** social choices (e.g. prices) are determined as if \( x^i_c = x^i_c \) for all \( i \).

If a composite characteristic is defined for all characteristics \( c \), then we shall say the weak aggregation problem is solved. A composite characteristic \( x^i_c \) is said to exist if, in addition to properties (1)-(3), it also satisfies:

4. **exogeneity:** \( x^i_c \) is not a function of private (e.g. consumption and portfolio) or social choices.

If a composite characteristic is defined for all characteristics \( c \), we shall say the strong aggregation problem is solved.

We will use GLUM to construct consensus and composite characteristics. However, since this type of analysis may be unfamiliar, I will first provide a simple illustration. Suppose each consumer in the economy maximizes the expectation of a single-period logarithmic utility function of wealth:

\[
\max_{\{w^i_e\}} \mathbb{E} \pi^i_e \ln W^i_e = \lambda_0 \left[ \mathbb{E} \pi^i_e W^i_e - W^i_0 \right].
\]

In this example, consumers are heterogeneous with respect to their resources (\( W^i_0 = \mathbb{E} \pi^i_e W^i_e \)) and beliefs \( \{\pi^i_e\} \). Partially differentiating with respect to \( W^i_e \), setting the derivative equal to zero, and eliminating the Lagrangian multiplier by introducing the budget constraint, produces the optimal decision rule:

\[
\mathbb{E}^i_e = \left( \frac{\pi^i_e}{\pi^i_0} \right) W^i_0
\]

for all \( e \). Since this holds for all \( i \), we can sum over all \( i \) and divide by \( I \) so that \( \mathbb{E} \mathbb{E}^i_e = \mathbb{E} \pi^i_e W^i_e / I \). By closure, since \( \mathbb{E} \mathbb{E}^i_e = \mathbb{E} \mathbb{E}^i_e W^i_e / I \), we can define average resources \( \mathbb{E} \pi^i_e W^i_e / I \) so that \( \mathbb{E} \pi^i_e W^i_e / I \). Now define average beliefs \( \pi^i_e \) such that \( \mathbb{E} \mathbb{E}^i_e = \mathbb{E} \mathbb{E}^i_e W^i_e / I \). Therefore, \( \pi^i_e = \left( \frac{\pi^i_0}{\pi^i_0} \right)^i \). Finally, define average wealth \( \mathbb{E} \mathbb{E}^i_e = \mathbb{E} \mathbb{E}^i_e W^i_e / I \) so that \( \mathbb{E} \mathbb{E}^i_e = \pi^i_0 \). From their definitions it is easy to verify these average variables satisfy properties

characteristic among consumers leaves social choices unchanged. Nondictatorship is satisfied by GLUM for all social choices (\( \{P^i_e\}, \{P^i_0\} \)). Even though a group of consumers die immediately after the first period, their demand for consumption at date \( t = 1 \) influences second period prices. This, as well as their influence on first period prices and therefore on the initial wealth of even long-lived consumers, also affects prices in the third period and beyond. However, even with endogenous production, their influence ebbs as they recede into the past. Indeed, with composite characteristics, they have no influence over prices after the second period. Positive responsiveness is satisfied by GLUM for all characteristics except lifetime. Anonymity is trivially satisfied by resources whenever the strong aggregation problem is solved; see Rubinstein (1974a, p. 233). More generally, if \( x^i_c = \mathbb{E} x^i_c / I \) or \( x^i_c = \left( \mathbb{E} x^i_c / I \right) \) and other characteristics are not a function of \( c \), then social choices are anonymous with respect to \( c \).
(i) and (ii) of consensus characteristics. To see they also satisfy (iii) observe that $P_{i} \pi_{e} = \pi_{e}^{i}$ for all $e$ together with $W_{0} = \sum_{e} P_{e} W_{e}$ are the necessary and sufficient conditions for an optimum that would have been derived had there existed only consumers with the specified consensus characteristics. Therefore, prices are determined as if $\{P_{e} = W_{e}^{i}\}$ and $\{\pi_{e} = \pi_{e}^{i}\}$ for all $i$.

However, composite beliefs do not exist because $\pi_{e}$ depends indirectly on prices through the dependence of initial wealth $W_{0}^{i}$ on prices. To obtain composite beliefs, assume additionally that there exist constants $\mu_{i} \equiv \frac{W_{0}^{i}}{P_{e}}$ for all $e$ and $i$. $\mu_{i}$ can be interpreted as a measure of the scale of wealth of consumer $i$, independent of the price system. If $\mu_{i} > 1$, a consumer is unambiguously "rich" and if $\mu_{i} < 1$ a consumer is unambiguously "poor." The composite scale of wealth $\mu \equiv \mu$ and $\Sigma_{i} \mu_{i} = I$. Clearly, since $W_{0}^{i} = \sum_{e} P_{e} W_{0}^{i}$ and $W_{e} = \sum_{e} P_{e} W_{e}$, then $W_{0}^{i} = \mu_{i} W_{0}$. With homogeneity of wealth composition, then composite beliefs $\pi_{e} = \Sigma_{i} \pi_{i}^{i} / I$.

Permitting heterogeneity with respect to wealth composition and scale, lifetime, risk- and time-preference, and beliefs:

**Theorem (weak aggregation):** In the generalized logarithmic utility economy, consensus characteristics exist for all exogenous variables. In particular,

resources: $c_{0} = \Sigma_{i} c_{0}^{i} / I$, $c_{e} = \Sigma_{i} c_{e}^{i} / I$, $c_{s,e} = \Sigma_{i} c_{s,e}^{i} / I$ (all $e$ and $s$)

risk-preference: $\lambda = \Sigma_{i} \lambda_{i}^{A_{1}} / (\Sigma_{i} A_{1} \neq 0)$, $\lambda = \Sigma_{i} \lambda_{i}^{A_{1}} / (\Sigma_{i} A_{1} = 0)$ (all $e$ and $s$)

Lifetime: $\lambda = \Sigma_{i} \lambda_{i}^{A_{1}} / (\Sigma_{i} A_{1} \neq 0)$, $\lambda = \Sigma_{i} \lambda_{i} / (\Sigma_{i} A_{1} = 0)$

Time-preference: $\rho_{1} = \Sigma_{i} \left( \frac{A_{1}^{i} + C_{0}^{i}}{(A + C_{0})^{i}} \right) \rho_{1}^{i}$

\[
\rho_{2} = \Sigma_{i} \left( \frac{\rho_{1}^{i}(A_{1}^{i} + C_{0}^{i})}{\rho_{2}(A + C_{0})^{i}} \right) \rho_{2}^{i}
\]

beliefs: $\pi_{e} = \Sigma_{i} \left( \frac{\rho_{1}^{i}(A_{1}^{i} + C_{0}^{i})}{\rho_{2}(A + C_{0})^{i}} \right) \pi_{e}^{i}$ (all $e$)

\[
\pi_{s,e} = \Sigma_{i} \left( \frac{\pi_{e}^{i}(A_{1}^{i} + C_{0}^{i})}{\rho_{2}^{i}(A + C_{0})^{i}} \right) \pi_{s,e}^{i}
\]

(all $e$ and $s$)

**Proof:** Summing equation (1') over all $i$ and dividing both sides by $I$:

$P_{e} \Sigma_{i} (A_{1}^{i} + C_{0}^{i}) / I = \Sigma_{i} \pi_{e}^{i} (A_{1}^{i} + C_{0}^{i}) / I$. 


Define average resources $C_i = \frac{E_i}{T_e} C_i^i$, average risk-preference $A = \frac{E_i}{T_e} A_i^i$, and average beliefs $\pi_e$ such that

$$P_e(A + C_e) = \pi_e \sum_i \rho_i(A_i + C_{0,i})/I.$$  

Therefore, $\pi_e = \sum_i \left( \frac{\rho_i(A_i + C_{0,i})}{\sum_i \rho_i (A_i + C_{0,i})} \right) \pi_e$. Define average time-preference $\rho_1$ such that

$$P_e(A + C_e) = \pi_e \rho_1 \sum_i (A_i + C_{0,i})/I.$$  

Therefore, $\rho_1 = \sum_i \left( \frac{A_i + C_{0,i}}{\sum_i \rho_i (A_i + C_{0,i})} \right) \rho_1$. Moreover, from the definitions of $A$ and defining average resources $C_{0,i} = \frac{E_i}{T_e} C_{0,i}$, it follows that

$$(a) \quad P_e(A + C_e) = \pi_e \rho_1 (A + C_{0,i})$$

in terms of the average variables. Combining (1), (2), and (5)

$$P_e \frac{E_i (A_i + C_{0,i})}{I} = \pi_e \sum_i \pi_e \rho_2 \rho_1 (A_i + C_{0,i}).$$

Summing over all $i$ and dividing both sides by $I$

$$P_e \frac{E_i (A_i + C_{0,i})}{I} = \pi_e \rho_2 \rho_1 (A_i + C_{0,i}).$$

Define average resources $C_{s.e} = \frac{E_i}{T_e} C_i^i / I$, average lifetime (provided $\Sigma_i A_i \neq 0$) $\Lambda = \frac{1}{T_e} \sum_i \frac{A_i}{\Sigma_i A_i}$, and average beliefs $\pi_{s.e} \pi_e$ such that

$$P_e \frac{E_i (A_i + C_{0,i})}{I} = \pi_{s.e} \pi_e \rho_2 \rho_1 (A_i + C_{0,i}).$$

Therefore, $\pi_{s.e} = \sum_i \left( \frac{\pi_{s.e} \rho_2 \rho_1 (A_i + C_{0,i})}{\sum_i \pi_{s.e} \rho_2 \rho_1 (A_i + C_{0,i})} \right) \pi_{s.e}$. Define average time-preference $\rho_2 \rho_1$ such that

$$P_e \frac{E_i (A_i + C_{0,i})}{I} = \pi_{s.e} \pi_e \rho_2 \rho_1 (A_i + C_{0,i}).$$

Therefore, $\rho_2 \rho_1 = \sum_i \left( \frac{\rho_i (A_i + C_{0,i})}{\rho_i \sum_i (A_i + C_{0,i})} \right) \rho_2 \rho_1$. Moreover, from the definitions of $A$ and $C_{0,i}$, it follows that

$$P_e \frac{E_i (A_i + C_{0,i})}{I} = \pi_{s.e} \pi_e \rho_2 \rho_1 (A + C_{0,i}).$$
Finally, combining this equation with equation (a)

\[(b) \quad P_{s,e}(\lambda + C_{s,e}) = \pi_{s,e}p_{2}(A + C).\]

From their definitions, it is easy to verify these average variables satisfy properties (i) and (ii) of consensus characteristics. To see they also satisfy property (iii), observe that equations (a) and (b) are the necessary and sufficient conditions for an optimum that would have been derived had there existed only consumers with the specified consensus characteristics. Therefore, prices \(\{p\}, \{P_{s,e}\}\) are determined as if \(\pi_{0} = C_{0}, \{\pi_{e} = C_{e}\}, \{\pi_{s,e} = C_{s,e}\}\), \(A = \lambda = \rho_{2} = \pi_{e} = \pi_{s,e}\) for all \(i\). Q.E.D.

**Corollary (strong aggregation):** In the generalized logarithmic economy, composite characteristics exist for all exogenous variables if any one of the following homogeneity conditions holds:

1. time-preference and beliefs are homogeneous;
2. resources are homogeneous in composition (but not necessarily in scale), the ratio of risk-preference to scale of resources is homogeneous, and lifetime is homogeneous;
3. resources are homogeneous in composition (but not necessarily in scale), and risk-preference is homogeneous and equal to zero.

In particular, in each case

**resources:** \(C_{0} = E_{1}c_{0}/I, \quad C_{e} = E_{1}c_{e}/I, \quad C_{s,e} = E_{1}c_{s,e}/I\) (all \(e\) and \(s\))

**risk-preference:** \(A = E_{1}A_{1}/I\) \((\Sigma_{1}A_{1} \neq 0)\), \(A = E_{1}(\lambda_{1}/A)A_{1}\) \((\Sigma_{1}A_{1} = 0)\)

**lifetime:** \(\lambda_{1} = E_{1}(\lambda_{1}/A)\lambda\) \((\Sigma_{1}A_{1} \neq 0)\), \(\lambda = E_{1}\lambda_{1}/I\) \((\Sigma_{1}A_{1} = 0)\)

and for case (1)

**time-preference:** \(\rho_{1} = \rho_{1}^{t} \quad \text{and} \quad \rho_{2} = \rho_{2}^{t}\) (all \(i\))

**beliefs:** \(\pi_{e} = \pi_{e}^{t} \quad \text{and} \quad \pi_{s,e} = \pi_{s,e}^{t}\) (all \(e, s,\) and \(i\))

and for cases (2) and (3)

**time-preference:** \(\rho_{1} = E_{1}\left(\frac{(1-\delta)}{1-\delta}\right)\rho_{1}^{t}\)

\(\rho_{2} = E_{1}\left(\frac{(\rho_{1}^{t}(1-\delta))}{\rho_{1}^{t}(1-\delta)}\right)^{t}\)

\(^{27}\)From the definition of consensus resource characteristics, the consensus budget constraint \(W_{0} = C_{0} + E_{e}s_{e}C_{e} + E_{s,e}s_{s,e}C_{s,e}\) is trivially satisfied.
beliefs: \[ \pi_e = \left( \frac{1}{\rho_1 (1-\delta_1) \rho_1 (1-\delta_1)} \right) \pi_e \quad (\text{all e}) \]

\[ \pi_{e.s} = \left( \frac{1}{\rho_0 \rho_1 (1-\delta_1) \rho_1 (1-\delta_1)} \right) \pi_{e.s} \quad (\text{all e and s}) \]

where \( 1 - \delta \equiv (1 + \rho_1 + \rho_1 \rho_2) \) and \( \mu_i = \frac{\tilde{c}_i}{c_0} = \frac{\tilde{c}_i}{c_s e} = \frac{\tilde{c}_i}{c_s e} \) for all e, s, and i.

**Proof:** To see that composite characteristics can exist, observe immediately for case (1) that each consensus characteristic is defined only in terms of exogenous variables; they are not dependent on prices. For case (2), there exists constants \( \mu_i = \frac{\tilde{c}_i}{c_0} = \frac{\tilde{c}_i}{c_s e} = \frac{\tilde{c}_i}{c_s e} \) for all e, s, and i. Additionally, there also exists a constant k such that \( A_i = k \mu_i \) for all i and \( \lambda_i = \lambda \) for all i.

Since equation \( (2') \) holds for all i,

\[ (A_i + c_i^2) = (1 - \delta_i) (A_i + \lambda_i + W_i^0) \]

and making appropriate substitutions consistent with case (2)

\[ (A_i + c_i^2) = (1 - \delta_i) \mu_i (k \lambda_i + W_i^0). \]

Using this relationship, a repetition of the above proof of the existence of consensus characteristics yields the composites defined in the theorem. For case (3), by following the same procedure, we derive similar composites except that since \( A_i = \lambda = 0 \) for all i. Observe that in this case it is unnecessary to assume homogeneous lifetimes. Q.E.D.

Since a complete set of consensus characteristics exists, we can meaningfully speak of a "consensus consumer" and imagine that prices are determined as if only consensus consumers existed. Since these consumers all make optimal choices according to equations (a) and (b), then these equations can be used to determine the relationship among prices in equilibrium. As demonstrated in section 4, these equations are especially useful for this purpose since they facilitate relating the price system \((p_e, \{p_{e.e}\})\) to aggregate consumption variables \((c_0^M = IC_0, \{c_e^M = IC_e\}, \{c_{s.e}^M = IC_{s.e}\})\) by virtue of properties (ii) and (iii).

However, since the consensus taste and belief characteristics are themselves functions of prices (through their dependence on \( c_i^2 \) and through its dependence on \( \phi_{i}^L \) and \( W_i^0 \) by equation \( (2') \)), the price relationships derived with equations (a) and (b) will not be complete. For example,
Type I models contain similarly incomplete price relationships since they fail to determine endogenously the riskfree rate and the "market price of risk."\(^28\) However, composite characteristics, by virtue of their independence from the price system, enable us to write prices as explicit functions only of exogenous variables, providing a complete set of price relationships.\(^29\) This will not be without the cost of imposing one of the set of strong homogeneity conditions described in the theorem. Fortunately, consensus statistics are sufficient to formulate most empirical hypotheses.

To highlight the comparative statics of the model, the consensus characteristics may be used to construct sharing rules which indicate how consumption or portfolio choices of a particular consumer deviate from the per capita choices of a consensus consumer. In contrast to Type I models where portfolio choices are broken down into two or at most three components,\(^30\) GLUM achieves thirty-eight components.

Since this type of analysis may be unfamiliar, I will again first provide a simple illustration. Suppose each consumer in the economy maximizes the expectation of a single-period logarithmic utility function of wealth. Then, from the previous simple aggregation example, \(p_{w_i} = \pi_{w_i} W_0\) for all \(i\) and for the consensus consumer \(\pi_w = \pi W_0\). Dividing these equations into each other,

\[
\begin{align*}
\frac{p_{w_i}}{\pi} &= \left(\frac{1}{\pi W_0}\right) W_0. \\
\text{If to the right-hand side we subtract and add the same number } \left(\frac{\pi W_0}{\pi W_0}\right) W_0, \text{ then}
\end{align*}
\]

\[
\begin{align*}
\frac{p_{w_i}}{\pi} &= \left(\frac{1}{\pi W_0}\right) W_0 + \left(\frac{\pi}{\pi W_0}\right) W_0. \\
\text{If again to the right-hand side we subtract and add } \left(\frac{\pi}{\pi W_0}\right) W_0, \text{ then}
\end{align*}
\]

\[
\frac{p_{w_i}}{\pi W_0} = \frac{1}{\theta} W_0, \\
\frac{p_{w_i}}{\pi W_0} = \frac{1}{\theta} W_0.
\]

\(^{28}\) Alternatively, since the "market price of risk" can be shown to be determined by the rate of return of the market portfolio and the riskfree rate, Type I models have little to say about the relationship of these important portfolios, other than \(E(W) > W_F\).

\(^{29}\) The existence of composite characteristics also has important welfare implications as developed in section 5.

\(^{30}\) For example, in the discrete-time normality model, as a slight extension of Rubinstein (1973), for all \(i\) and \(e\)

\[
w_i = \frac{(W_i - C_i)}{W_0} = \frac{\beta_i^{(2)}}{\theta_i} \frac{W_i}{W_0} + \theta_i \theta_i^{(2)}
\]

where \(\theta_i \equiv \frac{E(i)(W_i)}{E(i)(W_i)}\), \(\theta_i^{(2)} \equiv E_i^{(2)/2}\), and \(W_i \equiv E_i W_i / W_i\). Because of the random walk assumption, \(W_i\) is a state-independent derived utility of wealth function. This sharing rule implies that all consumers divide their wealth (after consumption) between a riskfree investment and the market portfolio. Note that \(\theta_i\) is itself indirectly a function of prices. Again Long's (1974) model is an exception which allows portfolio choice to be broken into several components. However, his choice of which state variables to exclude in his first period derived utility of wealth function is more or less arbitrary. By contrast, GLUM endogenously determines these state variables.
\[ W^t_e = \left( \frac{n^t_e - \pi^t_e}{\pi^t_e} \right) W^t_e + \left( \frac{n^t_e - \pi^t_e}{\pi^t_e} \right) W^t_e + \left( \frac{n^t_e - W^t_0}{W^t_0} \right) W^t_e + \left( \frac{n^t_e - W^t_0}{W^t_0} \right) W^t_e. \]

Finally, adding and subtracting \( W^t_0 \) to the right-hand side:

\[ W^t_e = \left( \frac{n^t_e - \pi^t_e}{\pi^t_e} \right) W^t_e + \left( \frac{n^t_e - \pi^t_e}{\pi^t_e} \right) W^t_e + \left( \frac{n^t_e - W^t_0}{W^t_0} \right) W^t_e + W^t_0. \]

Here the portfolio sharing rule can be broken into four components. \( \langle 0 \rangle \) is the (per capita) portfolio choice of the consensus consumer. \( \langle \omega \rangle \) is the additional (to \( \langle 0 \rangle \)) portfolio position taken by consumers with the same beliefs but more wealth than average. Observe that \( \sum_{e}^\omega (n^t_e - W^t_0) / W^t_0 = 0 \) so that under state \( e \) for every dollar a rich \( (n^t_e > W^t_0) \) consumer receives more than the average, a poor \( (n^t_0 < W^t_0) \) consumer sacrifices. Similarly, \( \langle B \rangle \) is the additional (to \( \langle 0 \rangle \)) portfolio position taken by consumers with the same wealth but with more optimistic beliefs than average. For optimists toward state \( e \), \( \pi^t_e > \pi^t_0 \) and for pessimists \( \pi^t_e < \pi^t_0 \). Moreover, if there is no correlation between the level of consumer wealth and consumer optimism (i.e., speaking loosely, if for every rich optimist there is also a rich pessimist), then from the definition of consensus beliefs, wealth effects cancel and \( \pi_e = \sum_e n^t_e / I \). In this case, since \( \sum_e (n^t_e - \pi_e^t) / \pi_e^t = 0 \), under state \( e \) every unit of wealth received by an optimist is lost by a pessimist. Optimists and pessimists can be interpreted as taking mutual and opposite speculative side bets (options) \( \langle B \rangle \) in addition to their state-dependent dividend (market portfolio) \( \langle 0 \rangle \). Finally, \( \langle BW \rangle \) may be interpreted as the additional (to \( \langle B \rangle \)) speculative side bet taken by those consumers with both different beliefs and wealth than average. For example, as we would expect, rich optimists make larger speculative side bets than poor optimists.31

Permitting heterogeneity with respect to wealth composition and scale, lifetime, risk- and time-preference, and beliefs:

Theorem (sharing rule): In the generalized logarithmic utility economy, with respect to the consensus characteristics defined in the weak aggregation theorem, the optimal portfolio sharing rule for the first period may be written as the per capita return from the market portfolio plus the sum of thirty-seven other components, each indicating the extent by which the portfolio of any consumer deviates from the consensus consumer. In particular, for any consumer \( i \) and for each state \( e \), his portfolio choice \( W^t_i \) is equal to the sum of:

31One of the disadvantages of the exponential utility model (IIIB), which also permits heterogeneous beliefs, is the insensitivity of speculative side bets to wealth. This property goes hand in hand with constant absolute risk aversion. See Rubinstein (1974b).
1. \( <W> \) \( W_e \)
2. \( <1W> \) \( + \left( W_0^4 - W_0 \right) [K] W_e \)
3. \( <1R> \) \( + \left( \delta_1 - \Lambda \right) \left[ \delta \Lambda K \right] W_e \)
4. \( <1T> \) \( + \left( \delta_1 - \delta \right) \left[ \delta^{-1} K \right] W_e \)
5. \( <1L> \) \( + \left( \Lambda_1 - \Lambda \right) \left[ P_{F,K} \right] W_e \)
6. \( <1MT> \) \( + \left( W_0^4 - W_0 \right) \left( \delta_1 - \delta \right) \left[ \delta^{-1} K \right] W_e \)
7. \( <1RT> \) \( + \left( A_1 - A \right) \left( \delta_1 - \delta \right) \left[ \phi \delta^{-1} K \right] W_e \)
8. \( <1RL> \) \( + \left( A_1 - A \right) \left( \Lambda_1 - \Lambda \right) \left[ P_{F,K} \right] W_e \)
9. \( <1TL> \) \( + \left( \delta_1 - \delta \right) \left( \Lambda_1 - \Lambda \right) \left[ \delta^{-1} P_{F,K} \right] W_e \)
10. \( <1RTL> \) \( + \left( A_1 - A \right) \left( \delta_1 - \delta \right) \left( \Lambda_1 - \Lambda \right) \left[ \delta^{-1} P_{F,K} \right] W_e \)
11. \( <2W> \) \( + \left( W_0^4 - W_0 \right) [K] A \)
12. \( <2R> \) \( - \left( A_1 - A \right) \left[ W_0 K \right] \)
13. \( <2T> \) \( + \left( \delta_1 - \delta \right) \left[ \delta^{-1} K \right] A \)
14. \( <2L> \) \( + \left( \Lambda_1 - \Lambda \right) \left[ P_{F,K} A^2 \right] \)
15. \( <2MT> \) \( + \left( W_0^4 - W_0 \right) \left( \delta_1 - \delta \right) \left[ \delta^{-1} K \right] A \)
16. \( <2RT> \) \( + \left( A_1 - A \right) \left( \delta_1 - \delta \right) \left[ \phi \delta^{-1} K A \right] \)
17. \( <2RL> \) \( + \left( A_1 - A \right) \left( \Lambda_1 - \Lambda \right) \left[ P_{F,K} A \right] \)
18. \( <2TL> \) \( + \left( \delta_1 - \delta \right) \left( \Lambda_1 - \Lambda \right) \left[ \delta^{-1} P_{F,K} A^2 \right] \)
19. \( <2RTL> \) \( + \left( A_1 - A \right) \left( \delta_1 - \delta \right) \left( \Lambda_1 - \Lambda \right) \left[ \delta^{-1} P_{F,K} A \right] \)
20. \( <3W> \) \( + \left( W_0^4 - W_0 \right) [K] A P_{F,e} \)
21. \( <3R> \) \( - \left( A_1 - A \right) \left[ W_0 K \right] P_{F,e} \)
22. \( <3T> \) \( + \left( \delta_1 - \delta \right) \left[ \delta^{-1} \Lambda \right] A P_{F,e} \)
23. \( <3L> \) \( - \left( A_1 - \Lambda \right) \left[ \left( \Lambda A_0 + W_0 \right) K \right] A P_{F,e} \)
24. \( <3MT> \) \( + \left( W_0^4 - W_0 \right) \left( \delta_1 - \delta \right) \left[ \delta^{-1} K A \right] A P_{F,e} \)
25. \( <3RT> \) \( + \left( A_1 - A \right) \left( \delta_1 - \delta \right) \left[ \phi \delta^{-1} K A \right] A P_{F,e} \)
26. \( <3RL> \) \( - \left( A_1 - A \right) \left( \Lambda_1 - \Lambda \right) \left[ \left( \Lambda A_0 + W_0 \right) K \right] P_{F,e} \)
27. \( <3TL> \) \( + \left( \delta_1 - \delta \right) \left( \Lambda_1 - \Lambda \right) \left[ \delta^{-1} P_{F,K} A^2 \right] P_{F,e} \)
28. \( <3RTL> \) \( + \left( A_1 - A \right) \left( \delta_1 - \delta \right) \left( \Lambda_1 - \Lambda \right) \left[ \delta^{-1} P_{F,K} A \right] A P_{F,e} \)
29. \( <B> \) \( + \left( \delta e \right) \left( \delta K^{-1} e \right) \)
30. \( <BW> \) \( + \left( \delta e \right) \left( W_0^4 - W_0 \right) \left[ \delta e \right] \)
31. \( <BR> \) \( + \left( \delta e \right) \left( A_1 - A \right) \left[ \phi \delta e \right] \)
32. \( <BT> \) \( + \left( \delta e \right) \left( \delta e \right) \left( K^{-1} e \right) \)
where the consensus absolute risk aversion of initial wealth \( K \equiv (A\phi_\lambda + W_0)^{-1} \),
\( W_e = C_e + P_s F_e e s, e, P_F \equiv (1+r_F)^{-2} \), and \( P_{F,e} \equiv (1+r_F e)^{-1} \).

**Proof:** Rearranging the portfolio decision rule (equation \( \delta' \)), \( p_e (A\phi_\lambda e + W_e) = \pi_e \delta (A\phi_\lambda + W_0) \). Since this holds for the consensus consumer as well as consumer \( i \),

\[
\frac{\pi}{\pi_e} = \frac{A_i}{A_j} \frac{(A\phi_\lambda e + W_e) - A_i \phi_\lambda e}{(A\phi_\lambda + W_0)^{-1}} \text{ where } K \equiv (A\phi_\lambda + W_0)^{-1} \]

The same method as in the example of adding and subtracting terms to the right-hand side of this equation will eventually produce the sharing rule. Since the algebra is very tedious, it will be left for the reader if he cares to verify the result. Q.E.D.

The components of the sharing rule have been coded to reflect both the type of security purchased and the motivation for the purchase. \( \langle 0 \rangle \) is again the portfolio choice of the consensus consumer. Components (2-10) coded with \( \langle 1 \rangle \) are the additional holdings of the market portfolio of consumer \( i \); components (11-19) coded with \( \langle 2 \rangle \) are his position in short-term default-free bonds; components (20-28) coded with \( \langle 3 \rangle \) are his position in long-term default-free bonds; and components (29-38) coded with \( \langle 4 \rangle \) are his position in options. Components coded with \( \langle 5 \rangle \) are nonzero due to atypical wealth; those with \( \langle 6 \rangle \) are nonzero due to atypical risk-preference; those with \( \langle 7 \rangle \) are nonzero due to atypical time-preference; and those with \( \langle 8 \rangle \) are nonzero due to atypical lifetime. All terms contained in brackets \( [\cdot] \) are nonnegative (assuming \( \lambda > 0 \)).

Focusing on the unadulterated components (2-5, 11-14, 20-24, 29-33) and including inferences from the optimal consumption sharing rule, the following table summarizes the comparative statics of financial choice for the generalized logarithmic utility economy:

---

32 The optimal consumption sharing rule is computed in a manner similar to the optimal portfolio sharing rule using the consumption decision rule (\( \delta' \)). It has been omitted to conserve space.
<table>
<thead>
<tr>
<th>Social Risk-preference</th>
<th>Wealth ($W_0$)</th>
<th>Risk-preference ($A$)</th>
<th>Time-preference ($\delta$)</th>
<th>Lifetime ($\lambda$)</th>
<th>Beliefs ($\pi_0$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Consumption</td>
<td>+   +   +</td>
<td>(-)  (0)  (+)</td>
<td>(-)  (0)  (+)</td>
<td>(-)  (0)  (+)</td>
<td>0   0   0</td>
</tr>
<tr>
<td>Market portfolio</td>
<td>+   +   +</td>
<td>?   ?   ?</td>
<td>(0)  (+)</td>
<td>-0  +</td>
<td>0   0   0</td>
</tr>
<tr>
<td>Short-term bonds</td>
<td>-0  +   +</td>
<td>-0  -   -</td>
<td>-0  +</td>
<td>-0  +</td>
<td>0   0   0</td>
</tr>
<tr>
<td>Long-term bonds</td>
<td>-0  +   +</td>
<td>-0  -   -</td>
<td>-0  +</td>
<td>-0  +</td>
<td>0   0   0</td>
</tr>
<tr>
<td>Options</td>
<td>+   +   +</td>
<td>+   +   +</td>
<td>+   +</td>
<td>+   +</td>
<td>+   +</td>
</tr>
</tbody>
</table>

**NOTE:** Social (consensus) risk-preference is measured by $A$ so that $A < 0$ is represented by a (-), $A = 0$ by a (0), and $A > 0$ by a (+). Similarly, the other minus (zeros, pluses) in the table indicate that given the social risk-preference indicated, less (same, more) of an item on the left margin of the table is held, if with respect to the corresponding characteristic on the top margin, a consumer is above the average (consensus) for the economy. The signs relating to options apply to an optimist; they are reversed for a pessimist. The implications of risk-preference for initial consumption can not be inferred without knowing the time-state path of aggregate consumption. The signs in the table presume $\lambda > 0$.

As one would expect, greater wealth permits more consumption, a larger investment in the market portfolio, and more extreme positions in options. However, greater wealth also leads to less investment in bonds of each maturity when the society exhibits decreasing proportional risk aversion in aggregate. Again this is to be expected since in the neutral case where all consumers have constant proportional risk aversion, there is no demand for default-free investments. As one would also expect, the more risk-prefering a consumer, the more he invests in the market portfolio, the less he invests in default-free bonds of each maturity, and the more extreme position he is willing to take in options. Likewise, the more patient a consumer, the less his consumption and the greater his investment in both the market portfolio and options. However, greater patience also leads to less investment in bonds of each maturity when the society exhibits decreasing proportional risk aversion. This also is to be expected since in the neutral case where all consumers have constant proportional risk aversion, there is no demand for default-free investments. When society exhibits constant proportional risk aversion, lifetime has no separate effect on consumer behavior other than its impact on time-preference (recall that if $\lambda_1 = 0$, then $\rho_2^1 = 0$ and if $\lambda_1 = 1$, then $\rho_2^1 > 0$). Of course, from our definition of options, differences in beliefs do not create demand for the market portfolio or default-free bonds.

The sharing rule is also consistent with our portfolio separation corollary with respect to lifetime: if $\lambda_1 = 0$ for all $i$, components (20-28) vanish and there is no demand for long-term bonds; likewise, if $\lambda_1 = 1$ for all $i$, components (11-19) and (20-28) merge so that all consumers invest in an equally weighted portfolio of short- and long-term bonds (an annuity) in addition to the market portfolio and options. Finally, the magnitude as well as the sign of each component can be inferred
from the strength of a consumer's nonconformist characteristics and the consensus variables in the brackets [·] of each component. For example, using components 3 and 4, other things equal, for the same percentage deviation in risk-preference as time-preference (i.e., $A_1 \delta - \delta_0 = \frac{A_1 - A}{A}$), in addition to the joint demand for the market portfolio (component 7), demand due purely to risk-preference is $A \Phi_1 K$ times the demand due purely to time-preference. When $A > 0$, then $0 < A \Phi_1 K < 1$ and time-preference is a more significant factor influencing demand for the market portfolio.33

4. FINANCIAL EQUILIBRIUM

With the weak aggregation problem solved, the chief obstacle blocking the analysis of security prices has been removed. We have only to use the first-order conditions for the consensus consumer, equations (a) and (b) and his budget constraints. In short, we can act as if all consumers are identical.

33Recall the previous analysis which demonstrated that if $A_1 \leq 0$ as long as $W_0^1 > -A_1 \Phi_1$, then positive consumption is assured at every date. However, this condition is not sufficient to guarantee positive consumption when $A_1 > 0$. In this case, the sharing rule can be used to bound the heterogeneity of consumers for whom $A_1 > 0$. To focus on the critical variables, suppose under certainty all consumers are identical except for their wealth and risk-preference and have lifetimes $\lambda_i = 0$. The sharing rules may then be simplified to

$$c_0^1 = \frac{A_0 W_0^1 + A_1 \Phi_0^1}{A_0 + \Phi_0^1} C_0 \quad \text{and} \quad c_1^1 = \frac{A_0 W_0^1 + A_1 \Phi_0^1}{A_0 + \Phi_0^1} C_1$$

for all $i$. Therefore, $c_0^1 > 0$ and $c_1^1 > 0$ if and only if $A_1 (\Phi C_0 - W_0) + W_0^1 (A+C_0) > 0$ and $A_1 (\Phi C_1 - W_0) + W_0^1 (A+C_1) > 0$. Since $\Phi C_0 - W_0 = P_F (C_0 - C_1)$ and $\Phi C_1 - W_0 = C_1 - C_0$, positive consumption is satisfied for $A_1 > 0$ if and only if

$$\frac{W_0^1}{A_1} \geq \max \left[ \frac{C_0 - C_1}{A + C_1}, \frac{P_F (C_1 - C_0)}{A + C_0} \right].$$

If aggregate consumption is equally spaced across time (i.e. $C_0 = C_1$), then positive consumption is always satisfied. With growing aggregate consumption (i.e. $C_1 > C_0$), we need only worry about the second term in the maximand. Since in equilibrium (see section 4), $P_F = \rho (A+C_0) / (A+C_1)$, then we require

$$\frac{W_0^1}{A_1} \geq \frac{\rho (C_1 - C_0)}{A + C_1}.$$ 

If the growth rate of aggregate consumption is denoted by $r$ so that $C_1 = C_0 (1+r)$ and time-preference is neutral so that $\rho = 1$, then this inequality is equivalent to

$$A_1 < \left[ 1 + \frac{A+C_0}{A+C_0} \right].$$

This permits risk-preference $A_1$ to even exceed $W_0^1$ in absolute magnitude even if most of the other
In this section, we shall use GLM to characterize three empirically oriented economic relationships:

(1) the form of the aggregate consumption function,

(2) the intertemporal stochastic relationships among returns on "basic" portfolios,

(3) the contemporaneous valuation formulas for arbitrary portfolios.

For some purposes, to simplify the nature of our empirical hypotheses, we will assume stationary social tastes. That is, the consensus consumer will be assumed immortal (T = \infty) and to have a constant rate of time-preference (\rho_t = \rho < 1 \text{ for all } t). This assumption for Type II models stands in direct contrast to its popular corresponding counterpart, stationary social beliefs, of Type I models. It is, of course, a matter of opinion which is apt to be closer to stationary, tastes or beliefs. However, despite Galbraithian chastisement, economists have traditionally regarded taste formation as beyond their purview. Like Archimedes, who would have moved the earth but for a place to stand, economists have argued that tastes—at least in the short run are relatively stationary compared to the macro variables of prices and output.

With stationary social tastes, equations (a) and (b) become

\begin{align*}
(a') & \quad P_e(A + C_e) = \pi_e \rho (A + C_0) \quad \text{for all } e \\
(b') & \quad P_{s,e}(A + C_{s,e}) = \pi_{s,e} \rho (A + C_e) \quad \text{for all } s \text{ and } e
\end{align*}

and the consensus consumer's budget constraints, require

\[ W_0 = C_0 + \sum_{e} P_e W_e \quad \text{and} \quad W_e = C_e + \sum_{s,e} P_{s,e} C_{s,e} \quad \text{for all } e. \]

Recall that private choices \( \{C_0, \{C_e\}, \{C_{s,e}\}\) are the per capita amounts and therefore exogenous to the economy; social choices \( \{P_e\}, \{P_{s,e}\}\) are the equilibrium prices in the economy with full heterogeneity; although social tastes represented by social time- and risk-preference (\rho, A) generally (if they are not composites) are not exogenous to the economy, nonetheless they are time-state independent constants; and social beliefs \( \{\pi_e\}, \{\pi_{s,e}\} \) are not only time-state dependent but also are those homogeneous beliefs which lead to the same equilibrium prices as the actual heterogeneous beliefs of consumers.

consumers in the economy are highly decreasingly proportional risk averse (i.e., A + C_0 is near zero) and the growth rate r of aggregate consumption is very high.
Using equations (a') and (b') and the consensus consumer's budget constraints,

**Theorem (aggregate consumption function):** In the generalized logarithmic utility economy with stationary social tastes, at each date per capita consumption is linearly related to the present value of a perpetual default-free annuity and per capita wealth by time-state independent constants. In particular, for all dates and states

\[ \widetilde{C}_t = a + b \widetilde{\phi}_t + c \widetilde{W}_t \]

where \( a = -A \) depends only on social risk-preference and has the opposite sign of social proportional risk aversion, \( c = 1 - \rho \) depends only on social time-preference and is between zero and one, and \( b = -ac \).

**Proof:** Following an analysis identical to the construction of the optimal decision rules for arbitrary consumer \( i \), for the consensus consumer at date \( t = 0 \)

\[ C_0 = A[\phi(1-\rho) - 1] + (1-\rho)W_0 \]

where, as a result of his immortality, \( 1 - \delta = (1+\rho^2+\ldots)^{-1} = 1 - \rho \) and

\[ \phi = 1 + \frac{1}{1+r_F} + \frac{1}{(1+r_F)^2} + \ldots \]

is interpreted as the present value of a perpetual default-free annuity yielding one unit of wealth now and at every date in the future. Similarly, at date \( t = 1 \), if state \( e \) occurs

\[ C_e = A[\phi_e(1-\rho) - 1] + (1-\rho)W_e \]

where \( \phi_e = 1 + \frac{1}{1+r_{Fe}} + \ldots \). More generally for any date \( t \)

\[ \widetilde{C}_t = A[\widetilde{\phi}_t(1-\rho) - 1] + (1-\rho)\widetilde{W}_t \]

where \( \widetilde{C}_t, \widetilde{\phi}_t, \) and \( \widetilde{W}_t \) are random variables depending on the state at date \( t \). Therefore,

\[ \widetilde{C}_t = -A + A(1-\rho)\widetilde{\phi}_t + (1-\rho)\widetilde{W}_t \quad \text{for all } t. \]

Q.E.D.

Since the observable variables \( (\widetilde{C}_t, \widetilde{\phi}_t, \widetilde{W}_t) \) in the aggregate consumption function are related linearly by time-state independent constants \( (\rho, A) \), it can be used to formulate empirical hypotheses, even though the constants \( \rho, A \) are not directly observable. First, the equation predicts that realized (ex post) per capita consumption, the present value of a perpetual annuity, and the corresponding per capita wealth will have the same exact linear relationship over time. Alternatively, correlation coefficient \( \kappa(C_t, A\phi_t + W_t) = 1 \), both for ex ante social prediction at the same date and ex post realization over time. This is, in effect, a life-cycle version of the aggregate consumption
function. Second, regressing realized values of \( \tilde{\phi}_t \) and \( \tilde{\omega}_t \) against \( \tilde{c}_t \) will determine constants \( a, b, \) and \( c \) which may be used to determine social time- and risk-preference (\( \rho, A \)). Third, measuring the realized values of \( \tilde{\phi}_t \) and \( \tilde{c}_t \) to a close approximation may be relatively easy compared with measuring \( \tilde{\omega}_t \). However, suppose a proportional proxy \( \tilde{\omega}_t' = k\tilde{\omega}_t \) for this variable can be obtained, where \( k \) is a time-state independent constant measuring the propensity of the proxy to underestimate the true \( \tilde{\omega}_t \). In this case, there still exists time-state independent constants \( a, b, c' \) such that \( \tilde{c}_t = a + b\tilde{\phi}_t + c'\tilde{\omega}_t' \) where \( A = -a, \rho = 1 + \frac{b}{a}, \) and \( k = -ac'/b \).

Unlike Type I models, GLUM also predicts relationships between the rates of return of "basic" portfolios: riskfree rates of return and the market portfolio. From equation (a'), in equilibrium

\[
P_e = (\lambda + C_0)\pi_e (\lambda + C_e)^{-1}.
\]

Summing this over \( e \)

\[
\frac{1}{1+\tau} \equiv \sum_e p_e = \rho \bar{E}\left[\frac{(\lambda + C_0)}{(\lambda + C_e)}^{\lambda}\right]
\]

where \( \bar{E} \) is an expectation operator. By equations (c') and (d')

\[
A + C_0 = (1-\rho)(\lambda + \tilde{\omega}_0) \quad \text{and} \quad A + C_e = (1-\rho)(\lambda + \tilde{\omega}_e)
\]

so that the above term in brackets \( [*] \) is equivalent to \( [(\lambda + \tilde{\omega}_0)/(\lambda + \tilde{\omega}_e)]^{\lambda} \). Since

\[
- \frac{\gamma''(\tilde{\omega}_0)}{\gamma''(\tilde{\omega}_e)} = A\pi + \tilde{\omega}_0 \quad \text{and} \quad - \frac{\gamma''(\tilde{\omega}_e)}{\gamma''(\tilde{\omega}_e)} = A\pi + \tilde{\omega}_e,
\]

this term can be interpreted as one plus the rate of change in social absolute risk aversion (in terms of wealth). This important variable will be denoted by \( (1+\tau)\gamma_e^{-1} \) and called the "Z factor," that is, \( 35 \)

34 As in Weber (1970), the distinctive feature of this version of the life-cycle hypothesis is the inclusion of interest rates in the consumption function, more in the classical rather than the Keynesian tradition.

35 One plus the rate of return of the market portfolio, as estimated by Standard and Poor's Composite Index of stocks or by the return on a joint stock and corporate bond portfolio such as constructed by Sharpe (1973), are candidate proxies \( \tilde{\omega}_t \). More traditional proxies can be found in the life-cycle consumption literature; see Ando and Modigliani (1963).

36 The chief empirical advantage of GLUM over all other Type II models is the simple linear relationship between absolute risk aversion in terms of consumption and absolute risk aversion in terms of wealth at the same date. That is, for all Type II models for all dates \( t \)

\[
- \frac{\gamma''(C_t)}{\gamma'(C_t)} = - \left( \frac{2C_t}{\tilde{c}_t} \right) \gamma''(\tilde{\omega}_t)
\]

This permits a direct translation from consumption variables to annuity and wealth variables. However,
Using this notation, \((1+r_F)^{-1} = \rho E[(1+r_Z)^{-1}]\). Extending this over many dates, for any date \(t\):

\[
1 + r_{Zt} = \frac{\Delta \hat{r}_t + \hat{\mu}_t}{\Delta \hat{r}_t + \hat{\mu}_t - 1}\]

\[
(1+r_{Zt})^t = (1+r_{Z1})(1+r_{Z2}) \ldots (1+r_{Zt}).
\]

Moreover, it is easy to show that in equilibrium the future riskless short-term rate given the state at date \(t\):

\[
(1+r_{Ft})^{-1} = \rho E[(1+r_{Zt})^{-1}]
\]

where the subscript of the expectations operator means expectations are taken with respect to social beliefs held at date \(t - 1\). Finally, the present riskless long rate through date \(t\):

\[
(1+r_{Fc})^{-t} = \rho E[(1+r_{Zc})^{-t}].
\]

These relationships can now be used to develop necessary and sufficient conditions for an unbiased term structure to emerge in equilibrium. The term structure of interest rates is said to be unbiased if an investment in a sequence of default-free short-term bonds (with proceeds at each date reinvested) up to any date \(t\) is expected to yield the same inverse compound rate of return as a single investment in a default-free long-term bond maturing at date \(t\).\(^{37}\) In a two-period economy, this implies and is implied by

\[
\frac{1}{(1+r_F)^2} = \frac{1}{1+r_F} E\left[\frac{1}{1+r_Z^2}\right].
\]

if the future marginal propensity to consume wealth \((\beta C_t / \beta w_t)\) is a random variable this muddies this translation for empirical purposes. The generalized logarithmic utility model is the only Type II model (in their multiperiod versions) for which \((\beta C_t / \beta w_t) = (1+r_{t+1} + r_{t+1} + r_{t+2} + \ldots)^{-1}\) is not random. Usually the MPC is a function of all time-state dependent prices and beliefs from date \(t + 1\) into the future, as well as time-preference and parameter \(\beta\). As a result MPC itself is a random variable depending on the state at date \(t\). Unless one cavalierly assumes MPC is not random (as would be the implication of assuming random walks for the rates of return on all securities) or confines himself to a single-period model, these other Type II models are difficult to adapt to current empirical methodology. See Rubinstein (1974b).

\(^{37}\)Rubinstein (1974b) compares this definition with other more popular definitions of an unbiased term structure.
From the definition of the first-period rate of return $r_N$ on a two-period default-free annuity, it is easy to show that the above equation holds (i.e. the term structure is unbiased) if and only if $r_N = E(r_N)$. With many periods, the term structure is unbiased if and only if the expected first-period rates of return on default-free annuities of successive maturities are the same.

The following theorem provides necessary and sufficient conditions for an unbiased term structure.

**Theorem (intertemporal structure)**: In the generalized logarithmic utility economy, the term structure is unbiased if and only if the $Z$ factor is serially uncorrelated. That is,

$$r_F = E(r_N) \quad \text{if and only if} \quad \kappa \left[ \frac{1}{(1+r_Z)^{1}}, \frac{1}{(1+r_Z^{2})^{1}} \right] = 0.$$

**Proof**: From previous analysis,

$$\frac{1}{1+r_F} = \rho E \left[ \frac{1}{1+r_Z} \right] \quad \text{and} \quad \frac{1}{1+r_F} = \rho E \left[ \frac{1}{1+r_Z^{2}} \right].$$

Therefore,

$$\frac{1}{1+r_F} E \left[ \frac{1}{1+r_Z} \right] = \rho^2 E \left[ \frac{1}{1+r_Z} \right] E \left[ \frac{1}{1+r_Z^{2}} \right].$$

Moreover, from equations (a') and (b')

$$\frac{1}{1+r_F} E \left[ \frac{1}{1+r_Z^{2}} \right] = \rho^2 E \left[ \frac{1}{1+r_Z} \right] E \left[ \frac{1}{1+r_Z^{2}} \right].$$

Summing this over all $e$ and $s$

$$\frac{1}{(1+r_F)^2} = \rho^2 E \left[ \frac{1}{(1+r_Z)(1+r_Z^{2})} \right].$$

As a result

$$\frac{1}{(1+r_F)^2} = \frac{1}{1+r_F} E \left[ \frac{1}{1+r_Z^{2}} \right] \quad \text{if and only if} \quad E \left[ \frac{1}{(1+r_Z)(1+r_Z^{2})} \right] = E \left[ \frac{1}{1+r_Z} \right] E \left[ \frac{1}{1+r_Z^{2}} \right].$$

Q.E.D.\(^{39}\)

\(^{38}\)To simplify presentation, this theorem has been developed with respect to the first two periods only and under stationary social tastes; neither are crucial to the results.

\(^{39}\)This makes use of the statistical property that two random variables are uncorrelated if and only if the expectation of their product equals the product of their expectations.
In the special case of socially constant proportional risk aversion (i.e. $A = 0$), GLM produces a surprisingly sharp portrait of the intertemporal stochastic process of basic portfolios.

**Corollary (logarithmic intertemporal structure):** In the generalized logarithmic utility economy with zero social risk-preference (constant proportional risk aversion), any one of the following statements is true if and only if the other statements are true:

1. The term structure is unbiased; that is,
   $$r_T = E(r_N).$$

2. The (inverse one plus) rate of return of the market portfolio is serially uncorrelated; that is,
   $$\kappa\left[(1+r_M)^{-1}, (1+r_M)^{-1}\right] = 0.$$

3. The (inverse one plus) rate of growth of per capita consumption is serially uncorrelated; that is,
   $$\kappa\left[(1+r_C)^{-1}, (1+r_C)^{-1}\right] = 0.$$

**Proof:** Since $1 + \tilde{r}_Z t = \frac{\bar{A}_{t+1} + \bar{W}_t}{\bar{A}_{t-1} + \bar{W}_{t-1}} = \frac{\bar{A} + \bar{C}_t}{\bar{A} + \bar{C}_{t-1}}$ and $A = 0$, then $1 + \tilde{r}_Z t = \frac{\bar{W}_t}{\bar{W}_{t-1}} = \frac{\bar{C}_t}{\bar{C}_{t-1}}$. From the definition of the rate of growth of aggregate consumption, $1 + \tilde{r}_C t = \frac{\bar{C}_t}{\bar{C}_{t-1}}$, so that $\tilde{r}_Z t = \tilde{r}_C t$. From the consumption decision rule, if $A = 0$, then $\bar{C}_{t-1} = (1-p)\bar{W}_{t-1}$. Therefore, since
   $$1 + \tilde{r}_N t = \frac{\bar{W}_t}{\bar{W}_{t-1}(\bar{C}_{t-1} - \bar{C}_{t-1})}, 1 + \tilde{r}_Z t = \rho(1+r_N t),$$

Q.E.D.

Whether or not (1) the term structure is unbiased or (2) the market portfolio follows a "random walk"\(^{41}\) depends critically on (3) the stochastic process governing aggregate consumption over time. Indeed, with social logarithmic utility only if this process is serially uncorrelated will the classical hypotheses be verified. The underlying real stochastic process of aggregate consumption carries directly over to the equilibrium financial stochastic process governing security prices. For example, it follows trivially from the theorem that if the (inverse one plus) growth rate of aggregate consumption is positively correlated over time, then the term structure will be biased with positive liquidity premiums. Moreover, although true of Type II models generally, the logarithmic utility case provides a simple illustration of the inference of the probability law governing financial processes.

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\(^{40}\) Note that $A = 0$ is consistent with heterogeneity with respect to risk-preference among consumers, as long as $\Sigma_i \frac{\lambda_i}{\lambda A_{i+1}} = 0$.

\(^{41}\) Strictly speaking, and as used in this paper, a random walk requires that the rate of return of the market portfolio be stochastically independent, not merely uncorrelated, over time.
from the probability law governing real processes. For example, since \(1 + r_{\text{Ce}} = \rho(1 + r_{\text{Re}})\), if \(1 + r_{\text{Ce}} \) were normally or lognormally distributed, then \(1 + r_{\text{Re}}\) would also be normally or lognormally distributed with greater mean and variance (since \(\rho < 1\)). Such an inference is the chief missing element from Type I models. 42

Like Type I models, GLUM also produces an explicit and relatively simple formula for the valuation of arbitrary portfolios of state-contingent claims (i.e., actual securities and portfolios). In addition, unlike Type I models, GLUM yields a convenient formula for discounting an uncertain stream of income.

Suppose actual ("complex") security \(j\) yields a total return (initial price plus capital gain plus dividend) of \(\bar{X}_{j,e}\) at date \(t = 1\) if state \(e\) occurs. Then the present value \(P_j\) of security \(j\) will be determined like any other portfolio of state-contingent claims by discounting its future income by the corresponding state-contingent prices; that is, \(P_j = \sum_{e} P_e \bar{X}_{j,e}\). Since, from equation (a''), \(P_e = \rho \bar{X}_{j,e}^{-1}\), then \(P_j = \rho \bar{X}_{j,1}^{-1}\). Since the expectation of the product of two random variables equals the product of their expectations plus their covariance, and from the definition of correlation coefficient,

\[
P_j = \rho \mathbb{E}(\bar{X}_{j,1}) \mathbb{E}[(1 + r_0)^{-1}] + \rho \mathbb{C}(\bar{X}_{j,1}, (1 + r_0)^{-1}) \text{Std} \bar{X}_{j,1} \text{Std} [(1 + r_0)^{-1}].
\]

By our previous analysis, \((1 + r_0)^{-1} = \rho \mathbb{E}[(1 + r_0)^{-1}], so that

\[
P_j = \frac{\mathbb{E}(\bar{X}_{j,1}) - \gamma \mathbb{C}(\bar{X}_{j,1}, (1 + r_0)^{-1}) \text{Std} \bar{X}_{j,1}}{1 + r_0},
\]

where \(\gamma = \text{Std} [(1 + r_0)^{-1}] / \mathbb{E}[(1 + r_0)^{-1}].

To extend this to valuation over two periods, suppose that \(\bar{X}_{j,e}\) takes its value from a dividend \(X_{j,e}\) at date \(t = 1\) and its value \(\bar{X}_{j,s,e}\) at date \(t = 2\). Then \(\bar{X}_{j,e} = X_{j,e} + \sum_{s} P_s \bar{X}_{j,s,e}\) so that \(P_j = \sum_{e} P_e X_{j,e} + \sum_{e} P_e \sum_{s} P_s \bar{X}_{j,s,e}\). Following a similar argument as in the single-period case, and then extending this to an infinite number of dates leads to the following theorem:

**Theorem (valuation):** In the generalized logarithmic utility economy with stationary social tastes, 43 the present price of any security (or portfolio of securities) equals the discounted

42Rubinstein (1974b) carries the exogenous source of uncertainty back even further to exogenous stochastic parameters in production functions.

43Again this theorem is little changed by nonstationary tastes.
sum at appropriate risk-free rates of its expected cash flows over time minus corresponding certainty equivalent factors reflecting risk aversion and uncertainty. In particular, for any security \( j \)

\[
P_j = \sum_{t=1}^{\infty} \frac{E(X_{1t}) - \gamma_t \gamma_t'(X_{1t} - (1+R_{zt})^{-t}) Std X_{1t}}{(1+R_{zt})^t}
\]

where

\[
1 + \tilde{r}_{zt} = (\tilde{A}_{zt} + \tilde{u}_t)/(\tilde{A}_{zt-1} + \tilde{u}_{t-1}) > 0
\]

\[
(1+\tilde{R}_{zt})^t = (1+\tilde{r}_z)(1+\tilde{r}_{zt})...(1+\tilde{r}_{zt}) > 0
\]

\[
(1+\tilde{R}_{zt})^{-t} = \rho_t E\left[(1+\tilde{R}_{zt})^{-t}\right] > 0
\]

\[
\gamma_t \equiv Std\left[(1+\tilde{R}_{zt})^{-t}\right]/E\left[(1+\tilde{R}_{zt})^{-t}\right] > 0.
\]

In addition, in terms of total return for the first period,

\[
E(r_j) = r_p + \gamma_t \gamma_t'(r_j - (1+r_z)^{-t}) Std r_j
\]

where \( 1 + r_{je} = \tilde{u}_j/P_j \).

In the special case of socially constant proportional risk aversion (i.e. \( A = 0 \)), since \( 1 + \tilde{r}_{zt} = \rho(1+\tilde{r}_{mt}) \) the valuation equation takes a surprisingly simple form, considering that it allows us to value an uncertain and possibly serially correlated stream of income.

**Corollary (logarithmic valuation):** In the generalized logarithmic utility economy with zero social risk preference (constant proportional risk aversion), for any security (or portfolio of securities) \( j \)

\[
P_j = \sum_{t=1}^{\infty} \frac{E(X_{1t}) - \gamma_t \gamma_t'(X_{1t} - (1+R_{mt})^{-t}) Std X_{1t}}{(1+R_{mt})^t}
\]

where

\[
1 + \tilde{r}_{mt} = \tilde{u}_t/(\tilde{u}_{t-1} \tilde{u}_{t-1})
\]

\[
(1+\tilde{R}_{mt})^t = (1+\tilde{r}_m)(1+\tilde{r}_{mt})...(1+\tilde{r}_{mt})
\]

\[
(1+\tilde{R}_{mt})^{-t} = E\left[(1+\tilde{R}_{mt})^{-t}\right]
\]

\[
\gamma_t \equiv Std\left[(1+\tilde{R}_{mt})^{-t}\right]/E\left[(1+\tilde{R}_{mt})^{-t}\right].
\]

If, additionally, it is assumed that the rate of growth of aggregate consumption follows a stationary random walk, then
\[
(1+r_{pt}^{-1}) = (1+r_p^{-1}) = E[(1+r_M^{-1})^{-1}]
\]

\[
\gamma_t = \sqrt{(\gamma^2 + 1)^t} - 1
\]

for all dates t where \( \gamma \equiv \frac{\text{Std}[(1+r_z^{-1})^{-1}]}{E[(1+r_M^{-1})^{-1}]} \).

The prospects for empirically testing the first period valuation relationship depend on estimating \( \tilde{r}_z \). Much depends on whether we assume the aggregate consumption function may be used to first obtain accurate estimates of social time- and risk-preference (\( \rho, \lambda \)) as well as our propensity \( k \) to underestimate per capita wealth. Following this approach, the realized values of \( \tilde{r}_z \) over time can be calculated by then measuring \( \tilde{\phi}_c \) and the proxy for \( \tilde{w}_t \). Empirical tests of Type I models also require a similar proxy for per capita wealth, such as the Standard and Poor's Composite Index for the "true" market portfolio. However, GLUM goes a step further by showing how to correct this proxy to make it a better estimate of the true economic variable. As a second empirical advantage, GLUM contains endogenous predictions of the nonstationary tendencies of \( \tilde{r}_z \). Since \( E[(1+r_z^{-1})^{-1}] = \left[ \rho(1+r_p) \right]^{-1} \), ex ante shifts in the mean of the \( Z \) factor can be anticipated by observing \( r_p \) at the beginning of each period. Furthermore, if it can be separately established that the term structure is unbiased, then \( (1+\tilde{r}_{zt})^{-1} \) is predicted to be serially uncorrelated over time. We may then be justified in postulating a distribution of \( (1+\tilde{r}_{zt})^{-1} \) with stationary shape and a shifting but predictable mean. On the other hand, if the term structure is positively biased, then \( (1+\tilde{r}_{zt})^{-1} \) will be positively correlated over time. We may then be justified in postulating a distribution of \( (1+\tilde{r}_{zt})^{-1} \) with a shifting variance predicted by the past observation of \( (1+\tilde{r}_{zt})^{-1} \) as well as a shifting mean predicted by observing \( \tilde{r}_{pt} \).

On the other hand, if good estimates of \( \rho, \lambda \) and \( k \) are not obtainable from the aggregate consumption function, then we can adopt more traditional though less satisfactory means of testing the first period valuation relationship. Type I models estimate the equivalent of \( \gamma \) by selecting a "basic" portfolio of securities and calculating \( \gamma \) in terms of it. If the basic portfolio is the market portfolio, then substituting \( j = M \) in the Type I valuation equation,

\[
E(r_j) = r_M + \gamma \text{Std}(r_J) \text{Std}(r_M), \quad \gamma = \frac{E(r_M) - r_M}{\text{Std}(r_M)}
\]

According to the theory, any portfolio (even a single-security) could have been selected to obtain an estimate of \( \gamma \); the market portfolio is selected in hope that the misspecifications in the equations of specific securities will cancel when they are aggregated to form this portfolio. The assumption of stationary social beliefs (relating to security rates of return) is then adopted to keep \( \gamma \) and the probability distribution of \( \tilde{r}_M \).
constant over time. At this point, the empirical analysis runs roughshod over the theory, since as shown in footnote 6, together with the separation property, this implies rates of return of all risky securities at the same date will be the same. Finally, it is assumed the financial markets are efficient in digesting information so that the social beliefs represent "unbiased" estimates of future realized rates of return. The joint effect of these assumptions is the validation of the procedure of aggregating ex post data to infer ex ante beliefs.

Frankly, as a theorist, I instinctively recoil at this brash methodology; however, this reaction may only spring from the natural proclivity of a theorist to exempt himself from matters of immediate practicality, in which a little dirt inevitably becomes lodged under one's fingernails. Brushing aside this propensity for the moment, adopting similar assumptions, the two now unknown parameters \( \gamma \) and \( A \) may be inferred from the two basic portfolios, \( j = M \) and \( j = N \). Substituting \( \tilde{r}_M \) and then \( \tilde{r}_N \) in the first period valuation relationship, it is easy to show that \( A \) is whatever value solves the equation

\[
\text{Cov} \left[ (E(r_M) - r_F) r_N - (E(r_N) - r_F) r_M, \ (A\phi_1 + w_1)^{-1} \right] = 0.
\]

With \( A \) and hence the probability distribution of the Z factor \( \tilde{r}_Z \) estimated, it is possible to estimate \( \gamma \). Stating this in terms of rates of return,

\[
A\phi + w_e = (w_0 - c_0) \left[ \tilde{A}(1+r_{Ne}) + (1+r_{Me}) \right]
\]

where \( \tilde{A} = A(\phi - 1)/(w_0 - c_0) \). Therefore, we can alternatively estimate \( \tilde{A} \) as the solution to

\[
\text{Cov} \left[ (E(r_M) - r_F) r_N - (E(r_N) - r_F) r_M, \ (\tilde{A}(1+r_{Ne}) + (1+r_{Me})^{-1} \right] = 0
\]

and then measure \( 1 + r_{Ze} \) by the proxy \( \tilde{A}(1+r_{Ne}) + (1+r_{Me}) \) since the effect of \( 1 + r_{Ze} \) on the single-period valuation relationship remains the same under a positive multiplicative transformation.

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44 In Rubinstein (1973), the intertemporal behavior of \( \gamma \) in the discrete-time normality model is analyzed implicitly presuming that the single-period valuation equation can be applied successively over time. The analysis suggests that \( \gamma = \left( E(r_M) - r_F \right) / \text{Std} \ r_M \) is unlikely to be an intertemporal constant. Other candidates in regression formulae such as \( E(r_M) - r_F \) or \( \left[ E(r_N) - r_F \right] / \text{Var} \ r_M \) are even less likely to be intertemporal constants.

45 Some simplification is possible when the term structure is unbiased. Then, \( r_F = E(r_N) \) and the equation simplifies to

\[
\text{Cov} \left[ r_N, \ (A(1+r_{Ne}) + (1+r_{Me})^{-1} \right] = 0.
\]

Unfortunately, even this equation has no analytic solution but may be solved, presuming a unique solution, numerically by computer.
It is unlikely that GLUM can prove an inferior empirical model to Type I models. These models, despite their strong assumptions governing the contemporaneous and intertemporal stochastic processes or prices and their assumption of homogeneous beliefs, yield surprisingly weak empirically testable conclusions. In essence, they only predict a linear relationship between expected return and risk (measured by covariance with the \( \bar{r}_H \)) and predict the intercept and slope terms.⁴⁶ They do not offer any prediction concerning aggregate consumption, the intertemporal stochastic process of security prices, or even the contemporaneous relationship between \( r_F \) and \( \bar{r}_H \). Add to this, as Kraus and Litzenberger (1975) have shown, that the logarithmic utility or "capital growth" model (\( A = 0 \)), which is a special case of GLUM, is empirically indistinguishable from Type I models with respect to the latter's linear risk-return relationship. This results from the high empirical correlation of the risk measures of the two models. In fact, the predictions of Type I models are so weak, they are likely to be equally well supported by alternative single-period Type II models.

5. FINANCIAL EFFICIENCY

The efficiency or welfare properties of a financial market may be usefully categorized under

(1) exchange: consumers are not motivated to create exchange arrangements not already provided by the market;

(2) production: competitive value-maximizing producers make Pareto-efficient production decisions;

(3) information: the acquisition and dissemination of information is Pareto-efficient.

A financial market can also be evaluated in terms of additional "ethical" efficiency criteria such as social rationality, nondictatorship, positive responsiveness, and anonymity.⁴⁷

The complete markets context in which GLUM has been developed is a powerful generator of financial theory and implies virtually no loss of generality relative to the current state of the art. For example, Type I models are revealed, through their separation properties, as thinly disguised versions of a complete market. Despite limitations of the complete markets approach revealed by increasing generalization, it remains the most powerful theoretical tool in the microeconomic theory.

⁴⁶ These comments do not strictly apply to Merton's (1973) continuous-time generalization where he allows nonrandomness in future risk-free rates, nor to Long (1974).
⁴⁷ See footnote 26.
⁴⁸ See Radner (1968).
of finance capable of being shifted comfortably between individual and aggregate levels of analysis, between conditions of certainty and uncertainty, and between static and dynamic settings with the fundamental theoretical elements exhibiting a continuity and a clarity otherwise unobtainable. Moreover, hopefully GLUM will forever displace the undeserved bugaboo that complete markets are incapable of rendering refutable empirical hypotheses.

Not only does the complete markets context of GLUM assure exchange-efficiency but it reveals through its sharing rule the minimum number of complex securities required for efficiency. In particular, if beliefs are homogeneous, then only default-free bonds of each maturity and a mutual fund representing the market portfolio need be available to assure efficiency; beyond these, no other securities will be in demand. With heterogeneous beliefs, options only need to be created for those states toward which there is disagreement; from the sharing rule, it is immediate if all consumers have the same beliefs for any state \( e \), then no options are bought or sold with respect to that state. More generally, for any subset of states if for any two states \( e \) and \( e' \) within the subset, the ratio of probabilities \( \frac{\pi_i^e}{\pi_i^{e'}} \), is the same for all consumers \( i \), then one complex security can replace the underlying options for states in this subset. 50

Since, in a complete market, every Pareto-efficient allocation can be reached by an appropriate redistribution of resources, the sharing rule reveals the necessary and sufficient conditions for exchange-efficiency even if a securities market were not utilized to effect exchanges. In particular, if a composite consumer existed, it is immediate that prices are insensitive to any redistribution of resources (which does not destroy the composite). Therefore, all components of the sharing rule are unchanged by a redistribution of resources except components \( \phi_0, \phi_6, \phi_{11}, \phi_{15}, \phi_{20}, \phi_{24}, \phi_{30}, \) and \( \phi_{34} \) which can be shown to sum to \( (W_0^1 - W_0^0)\pi_i^e P^{-1} \). If the sum of all the other components is represented by \( Q_i^e \), a number independent of the distribution of resources, then the portfolio sharing rule collapses to

49 Although a complete futures market has not been assumed, the opportunity to revise portfolios at future dates exactly compensates for the reduction in markets at any one date. See Arrow (1953).

50 Hakansson (1974) has proposed an interesting organization of the options market. Under certain conditions, there will be a one-to-one correspondence between the state of the world and the level of aggregate future wealth. In this case, the options market could be efficiently organized by the creation of a "superfund" which would sell state-contingent securities with payoffs for each possible level of aggregate future wealth. In practice, a proxy would need to be used such as a composite stock and bond index and securities would be issued with payoffs for discrete intervals of the index. For example, the superfund would market securities which would payoff if and only if the index were between 105 and 106 a year from the date of issue. Other securities would be issued covering other intervals.
\[ w_e^i = (w_0^i - w_0) \pi_e^i x_e + q_e^i \]  
(all \( e \) and \( i \)).

All Pareto-efficient allocations can be spanned by varying \( w_0^i \) from 0 to \( w_0^M \) in the first term.

Suppose a composite consumer exists and we try to answer the important social question of the optimal distribution of resources. Can we say that one distribution of resources \( \{w_0^i\} \) is preferred over all others? Clearly, consumers will disagree since any consumer \( i \) will vote for \( w_0^i = w_0^M \).

However, suppose we conceive of an "original situation" where a consumer is asked to choose between alternative distributions of resources without knowing in advance what his initial wealth will be once the distribution has been chosen. In particular, let \( \{w_0^k\} \) represent a feasible wealth distribution so that \( \sum w_0^k \leq w_0^M \) and let \( \pi_k^i \) be the subjective probability of consumer \( i \) that he will receive initial wealth \( w_0^k \). Therefore, judging among resources distributions, a consumer

\[
\max \sum_k \pi_k^i w_0^k (w_0^k) - \lambda_i \left( \sum_k w_0^k - w_0^M \right)
\]

where \( v_0^i \) is his derived utility of initial wealth function such that \( v_0^i > 0 \) and \( v_0^i < 0 \). If a composite consumer can be constructed such that prices are independent of the distribution of resources (as in GLM), then \( w_0^M \) is insensitive to the distribution of resources (i.e. \( \partial w_0^M / \partial w_0^k = 0 \)).

In this case, from the point of view of consumer \( i \), the optimum distribution of resources \( \{w_0^k\} \) satisfies

\[
\pi_k^i w_0^k (w_0^k) = \lambda_i \quad (\text{all } k).
\]

If all consumers do not know in advance what their initial wealth will be once the distribution has been chosen, then \( \pi_k^i = 1 \) for all \( k \) and \( i \) and \( v_0^i (w_0^k) = \lambda_i I \) so that \( w_0^k = w_0 \) for all \( k \) and \( i \).

We can summarize our conclusions concerning the efficiency of exchange arrangements by the following theorem:

**Theorem (exchange-efficiency):** In the generalized logarithmic utility economy, financial markets are exchange-efficient, requiring only default-free bonds of each maturity and the market portfolio if beliefs are homogeneous. If a composite consumer exists,

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51. The phrase "original situation" is taken from Rawls (1971). However, he defines it to mean a complete tabula rasa where no participant knows who (resources, lifetimes, tastes, beliefs) he will be. In my "original situation" participants know their (lifetimes, tastes, and beliefs) and are only uncertain as to their resources.

52. If the distribution of wealth, by affecting incentives, could influence production, then this is a poor assumption.
1. prices are insensitive to the distribution of resources;
2. all Pareto-efficient allocations are spanned by varying $w_0^i$ such that $0 \leq w_0^i \leq w_0^N$
in
$$w_e^i = \left( w_0^i - w_0 \right) \pi_e^i \pi_e^i + Q_e^i \quad (all \ e \ and \ i)$$
where $Q_e^i$ is independent of the distribution of resources;
3. in the "original situation" consumers will unanimously agree on an equal distribution of resources.

Not only does the complete markets context of GLUM assure exchange-efficiency, but it also assures production-efficiency. Moreover, if a composite consumer exists, his expected utility is a social welfare function which satisfies all but one (unrestricted domain) of the conditions for Arrow's (1951) impossibility theorem, and satisfies the Savage (1954) Axioms of Rational Choice, in particular the independence of tastes and beliefs (postulate four). As a result,

Theorem (production-efficiency): In the generalized logarithmic utility economy, financial markets are production-efficient. If a composite consumer exists, then competitive value-maximizing producers make the same production decisions as would have been chosen by using the social welfare function.

If now production decisions are regarded as exogenous, then it is well known that the social acquisition of new public information is Pareto-inefficient. However, if the effect on prices is ignored (or viewed as negligible), then from the exact derived utility of wealth functions (see the proof of the decision rules theorem), information systems will be ordered in preference by

$$\sum_z \pi_z \sum_e \pi_z \sum s \pi_z \sum e_z \ln \left( \frac{\pi_z \pi_z \sum e_z \ln (\pi_z \sum e_z / P_e) + \sum s \xi \pi_z \sum e_z \ln (\pi_z \sum e_z / P_e)}{\sum e \sum s \pi_z \sum e_z \ln (\pi_z \sum e_z / P_e)} \right)$$

where $z$ is a possible signal, $\pi_z$ is the prior probability of that signal $z$ will be received and $(\pi_z, \pi_z)$ are the revised posterior beliefs that states $e$ and $s$ will occur. Consumers with lifetime $\lambda = 0$ so that $\rho_2 = 0$, or more generally with the same $\rho_2^i$, will rank information systems $(\pi_z, \pi_z, \pi_z)$ in the same order.

The dissemination of existing private information is always Pareto-efficient if prices are not affected, in the extreme, homogeneous beliefs are formed and no speculative side bets are made.

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53 The word "competitive" in the definition of production-efficiency must be carefully defined. See Rubinstein (1974a) for this definition as well as a proof of the next theorem. Following Radner (1966), Rubinstein (1974b) develops closed-form production decision rules for logarithmic utility with Cobb-Douglas production functions containing random parameters.

54 With logarithmic utility, Arrow (1971) relates the expected gross value of perfect information to the Shannon entropy measure of the "amount of information," and Morris (1974) determines the maximum amount a consumer would pay for an information system.

55 See Jaffe and Rubinstein (1975).
The lost demand by optimists is just offset by the lost supply by pessimists. This emphasizes that the options market is a zero sum game which encourages consumers to slant their portfolio holdings away from perfect diversification. If prices are affected and a composite consumer exists, it is possible to identify who benefits and who loses from the publication of information. Even if the private information of an informed consumer is released without charge, he may still benefit if, for example, he is a borrower of short-term bonds and the information causes short-term interest rates to fall.

The consensus beliefs reflect what information (i.e. beliefs) are reflected in security prices. If the full dissemination of private information were to leave the consensus beliefs unchanged (i.e. if optimistic private information were offset by pessimistic private information), then we would say security prices fully reflect all available information. Nonetheless, without the release of the information, specific consumers will be holding nonoptimal portfolios with respect to beliefs based on all information, public and private. These nonoptimal portfolio holdings will be his speculative side bets. The extent of failure of the financial market to disseminate available information can be measured by the ratio of the volume of speculative (due to differences in beliefs) to nonspeculative (due to differences in wealth composition and scale, lifetime, and time- and risk-preference) trading. The portfolio sharing rule can be used to compare speculative (components 29-38) with nonspeculative demand (components 1-28 plus \( W_e - \bar{w}_e \)). If default-free bonds of each maturity and a mutual fund representing the market portfolio are available at date \( t = 0 \), then all trading volume at future dates is speculative. This results more generally from the relative ease with which standardized packages of securities can be created for nonspeculative purposes.

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56 However, if the existence of options provides an incentive for the private acquisition of information which affects production decisions, this will not be the case.

57 Ng (1975) has isolated similar effects of the social acquisition of good or bad news on the welfare of consumers.

58 See Rubinstein (1975). By this same line of reasoning, we can also precisely define what we mean by a consumer's subjective valuation of a security, namely, the value the economy would assign to the security if all consumers shared his beliefs.

59 Examples of calculations of the ratio of speculative to nonspeculative volume are contained in Rubinstein (1974b).
REFERENCES


