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by

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THE INTERNATIONAL CAPITAL ASSET PRICING MODEL IN
DISCRETE TIME

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I. Introduction

This paper develops a discrete time analogue to Solnik's International Capital Asset Pricing Model (1974) and compares the derivation to that of modeling real returns on risky assets in an inflationary environment. In addition, this paper explores the relationship between the two international pricing models and shows how the forward exchange market can be easily simulated within the discrete time model.

One of the principal problems in the derivation of an International Capital Asset Pricing Model, ICAPM, is the handling of multiplicative random variables. These arise because each investor views the returns on foreign risky assets through the veil of uncertain returns on the value of foreign exchange. Whereas this problem disappears in continuous time, it remains a difficulty in discrete time.

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1This problem was discussed by Stevens (1971).
Central to the Solnik analysis are the assumptions of continuous time and the existence of a constant investment opportunity set. The first assumption of continuous time implies that an investor's demand for an asset is determined simply by a function of its first and second moments of returns. The second assumption of a constant investment opportunity set allows his intertemporal demand analysis to collapse into a demand analysis which is myopic. That is, the demands are structurally similar to a one-period problem over an infinitesimal time interval. The thrust of this result is similar to that of Fama (1970), who showed that the multiperiod problem using normally generated returns and a constant investment opportunity set collapses to a single period mean-variance problem. Merton (1973) has shown that within a purely domestic capital market and with continuous time and a constant investment opportunity set, one can derive the same capital asset pricing model as in discrete time with quadratic utility and a one-period trading horizon. This, therefore, suggests that it may be possible to derive Solnik's ICAPM in a one-period model with discrete time and quadratic utility.

There are four important results. First, even if the forward rate is an unbiased predictor of the future spot rate, an investor may still

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2 Solnik assumed that the instantaneous expected return (per unit of time) and standard deviation of returns on the i-th country's market portfolio, and the i-th country's riskless rate remained constant through time (Solnik, 1974, p. 504).

3 To be sure, Merton also derives important equilibrium results when the assumption of a constant investment opportunity set is dropped.
find it optimal to purchase uncovered foreign fixed interest securities. Second, Solnik's simple ICAPM cannot be derived in discrete time without making an assumption similar to that invoked in modeling real returns in an inflationary environment. Third, the paper stresses that a simple ICAPM can be developed if and only if the returns on the exchange rates are uncorrelated with returns on risky assets. Fourth, starting from the complete discrete time model, Solnik's ICAPM can be obtained in the limit as the period spacing approaches zero. This derivation alone provides a pedagogical motivation for this paper.

The paper is divided into four sections. In section II the demand equations for assets are developed in discrete time under the assumptions of a one-period model and quadratic utility. Section III introduces an approximation which simplifies the problem and allows for the derivation of a simple ICAPM. Section IV shows how the demand equations given in section II change as the limiting case of continuous time is approached. Finally, section V contains a summary.

II. The Optimal Demand for Foreign and Domestic Assets over a Discrete Time Period

A. Assumptions

Let us make the following assumptions about the nature of the capital market:

1) The capital markets are always in equilibrium; there is no trading at nonequilibrium prices.
2) The capital markets are perfect, with no transactions costs, taxes, or capital controls. In particular there are no constraints on international capital flows.

3) In each country there exists a market for borrowing and lending at the riskless rate.

4) In each market, it is possible to sell assets short.

Further, let us make the following assumptions about the behavior of the investors:

5) Investors are risk averse, and evaluate portfolios according to predictions about future risk and expected return.

6) Investors are price takers.

7) Investors have homogeneous expectations about exchange rate changes and asset returns.

8) Investors maximize the expected utility of their wealth at the end of one investment period where their wealth is denominated in their own domestic currency.

9) Investors' consumption is limited to their own country.

Except for assumption 8, these are the same as those used by Solnik in the derivation of his model. Although the forward exchange market is not formally introduced it will be shown later that forward exchange contracts can be simulated from existing assets so long as the realistic assumption of covered interest rate parity holds. Finally, each investor is concerned about the value of his portfolio at the end of the investment period denominated in his domestic currency. This nationalistic objective
ignores the international composition of his consumption basket. Inci-
dently, these assumptions are consistent with a world of pegged or flexi-
ble exchange rates.

B. Definition of terms

Let there be $n$ countries, $i = 1, n$, and let there be one risky asset (the market portfolio) and one riskless asset per country. Further let,

$$\tilde{I}_i = \text{the return relative on country i's risky asset from the viewpoint of an investor in that country. A tilde above a letter indicates a random variable.}$$

$$\tilde{I}_i^k = \text{the return relative on country i's risky asset from the viewpoint of an investor in country k. Throughout this paper the following convention is used. The subscript refers to the country in which the asset is located, and the superscript refers to the investor's country of residence. For subsequent clarity of presentation, the superscript will be omitted when } k = i, \text{ so that instead of } \tilde{I}_i^i, \text{ just } \tilde{I}_i \text{ will be used.}$$

$$\tilde{f}_{ik} = \text{the exchange rate between countries } i \text{ and } k \text{ in terms of units of country } k \text{ per unit of country } i \text{ (i.e. it takes } \tilde{f}_{ik} \text{ currency units of country } k \text{ to buy one currency unit of country } i).$$

$$\tilde{g}_i^k = \text{return relative obtained by holding country i's currency from country k's viewpoint. Suppose that an investor from country k takes } (\tilde{f}_{ik})^0 \text{ units of currency } k \text{ at } t = 0 \text{ and buys one unit of currency } i. \text{ At } t = 1 \text{ that one unit of country i's currency}$$
would be worth \( (f_{ik})^1 \) units of country k's currency. The investor from country k would have obtained a return relative of \((f_{ik})^1/(f_{ik})^0\). If, on the other hand, the investor had purchased the risky asset of country i at \( t = 0 \), his return relative over the investment period would be \( \tilde{r}^k_i = (\tilde{r}_{ik})^1_1/(f_{ik})^0 = \tilde{r}^k_i \).

\( R^k_i \) = return relative on the riskless asset in country i from country i's point of view.

\( \tilde{R}^k_i \) = return relative on the riskless asset in country i from country k's point of view. Note that \( \tilde{R}^k_i = \tilde{r}^k_i R^k_i \).

\( W_0^k \) = the initial wealth of the investor from country k.

\( W_1^k \) = the wealth of the investor from country k at the end of the period.

Now that the nomenclature has been introduced, it is possible to show that so long as covered interest rate parity holds the formal exclusion of the forward market does not reduce the investor's opportunity set, because forward exchange contracts can be simulated from the existing assets. At \( t = 0 \), the investor could contract to sell one unit of country i's currency forward to receive \((f_{ik})^F \) units of country k's currency at \( t = 1 \). Alternatively he could borrow \( 1/R^k_i \) units of country i's currency, convert them immediately into units of country k's currency at the spot rate \((f_{ik})^0 \), and invest the proceeds in the riskless asset of country k. At \( t = 1 \), the investor pays out one unit of country i's currency in both cases, and receives \((f_{ik})^F \) and \((f_{ik})^0 B_k^k/R^k_i \) units of country k's currency in the first and second cases, respectively. When covered interest rate
parity holds, \( (f_{ik})^F = (f_{ik})^O B_k / B_i \), and these two strategies are identical.

The wealth of an investor from country \( k \) at the end of the period is given by, \(^4\)

\[
W_i^k = W_0^k \left[ \sum_{i=1}^{n} \frac{k_i k}{i} + \sum_{i=1}^{n} v_i B_i \right] \tag{1}
\]

with

\[
\sum_i v_i^k + \sum_i \omega_i^k = 1 \tag{2}
\]

where \( w_i^k \) = proportion of initial wealth of the investor from country \( k \) invested in the risky asset of country \( i \).

\( v_i^k \) = proportion of initial wealth of the investor from country \( k \) invested in the riskless asset of country \( i \).

The following mathematics are simplified by making the following transformations.

Let

\[
x_i^k = \omega_i^k \tag{3}
\]

and

\[
y_i^k = v_i^k + \omega_i^k \tag{4}
\]

so that now \( \sum_i y_i^k = 1 \). \tag{5}

Consequently equation (1) becomes:

\[
W_i^k = W_0^k \left[ \sum_{i=1}^{n} x_i^k B_i (1 - B_i) + \sum_{i \neq k} v_i^k B_i B_i + v_k^k B_k \right]. \tag{6}
\]

\(^4\)From this point, the tildes indicating random variables have been omitted to make the exposition clearer.
But by definition \( g_k = 1 \), and by substituting \( y_k = 1 - \sum_{i \neq k} y_i \) we derive,

\[
\begin{align*}
\omega_1^k &= \omega_0^k \left[ \sum_{i=1}^{n} x_i g_1^k (I_i - B_i) + \sum_{i \neq k}^{n} y_i^k (g_i^k B_i - B_k) + B_k \right].
\end{align*}
\]

(7)

C. Demand equations for assets

It is assumed that each investor maximizes the expected utility of his end-of-period wealth, i.e., \( \text{MAX} \mathbb{E} \left[ U(W_1^k) \right] \), using the quadratic utility function:

\[
U(W_1^k) = \omega_1^k - \frac{1}{2} \omega_1^k < \frac{1}{2b}.
\]

(8)

Consequently the first-order conditions for expected utility maximization are given by:

\[
\begin{align*}
\frac{\partial \mathbb{E} \left[ U(W_1^k) \right]}{\partial x_i^k} &= \mathbb{E} \left[ g_1^k (I_i - B_i) - 2b\omega_1^k x_i^k (I_i - B_i) \right] = 0 & i = 1, \ldots, n
\end{align*}
\]

(9)

\[
\begin{align*}
\frac{\partial \mathbb{E} \left[ U(W_1^k) \right]}{\partial y_i^k} &= \mathbb{E} \left[ g_i^k B_i - B_k - 2b\omega_1^k (g_i^k B_i - B_k) \right] = 0 & i = 1, \ldots, n \quad \text{if} \quad i \neq k
\end{align*}
\]

(10)

\[5\] There is a misprint on page 506 in Solnik (1974): \( y_k^k \) does not equal \( 1 - y_1^k \).
These first-order conditions can be greatly simplified by using vector notation. Throughout this paper the following notational convention will be used: upper-case letters with a tilde beneath (A) will denote matrices; lower-case letters with a tilde beneath (a) will denote column vectors; letters with no tilde beneath will denote scalars.

\[
\begin{bmatrix}
  k & (I - B_k) \\
  \vdots & \\
  k & (I - B_n)
\end{bmatrix}
\]

\[m \times n \quad (n \times 1)\]  

\[
\begin{bmatrix}
  k & (I - B_k) \\
  \vdots & \\
  k & (I - B_n)
\end{bmatrix}
\]

\[p \times (n-1) \quad (n-1) \times 1\]  

\[
\begin{bmatrix}
  k \\
  \vdots \\
  k
\end{bmatrix}
\]

\[x \times n \quad n \times 1\]  

\[
\begin{bmatrix}
  k \\
  \vdots \\
  k
\end{bmatrix}
\]

\[y \times (n-1) \quad (n-1) \times 1\]  

\[
\begin{bmatrix}
  k \\
  \vdots \\
  k
\end{bmatrix}
\]

Consequently

\[
\omega^k = w_0^{k} \left[ m \times k + p \times y + B_k \right]
\]
and equations (9) and (10) can be rewritten as

\[
E \left[ m^{k} - 2b w_{1}^{k} \right] = 0 \quad (16)
\]

\[
E \left[ p^{k} - 2b w_{1}^{k} \right] = 0 \quad (17)
\]

Combining equations (16) and (17) and substituting for \( w_{1}^{k} \) we obtain:

\[
\begin{bmatrix}
E(m^{k}) \\
E(p^{k})
\end{bmatrix} = 2b w_{0}^{k} \begin{bmatrix}
E(m m' k x + m p k' y + m B_{k}) \\
E(p m' k x + p p k' y + p B_{k})
\end{bmatrix} \quad (18)
\]

which can be rearranged to yield

\[
\begin{bmatrix}
E(m^{k}) \\
E(p^{k})
\end{bmatrix} = \begin{bmatrix}
x^{k} \\
y^{k}
\end{bmatrix} \quad (19)
\]

where

\[
A = \begin{bmatrix}
\frac{k^{k} k'}{m m'} & \frac{k^{k} k'}{m p} \\
\frac{k^{k} k'}{p m'} & \frac{k^{k} k'}{p p'}
\end{bmatrix} \quad (20)
\]

and

\[
T^{k} = \frac{1 - 2b w_{0}^{k} B_{k}}{2b w_{0}^{k}} \quad (21)
\]

By matrix inversion we can solve for \( x^{k} \) and \( y^{k} \) so that

\[
\begin{bmatrix}
x^{k} \\
y^{k}
\end{bmatrix} = T^{k} A^{-1} \begin{bmatrix}
E(m^{k}) \\
E(p^{k})
\end{bmatrix} \quad (22)
\]
Obviously, the solution values for \( x^k \) and \( y^k \) will depend on the nature of the matrix \( A \). Because \( \mathbb{E}(p_{k,k'}) \) and \( \mathbb{E}(m_{k,k'}) \) are functions of both \( g_i^k \) and \( I_i \), both \( x_i^k \) and \( y_i^k \) will depend on both \( g_i^k \) and \( I_i \). If we assume that \( \mathbb{E}(g_i^kI_i) = \mathbb{E}(g_i^k)\mathbb{E}(I_i) \), and \( \mathbb{E}(g_i^k) = 1 \), then the proportion of investor \( k \)'s initial wealth invested in country \( i \)'s risky stock is given by:

\[
x_i^k = T_k \left[ \sum_{j=1}^{n} s_{ij} (\mathbb{E}(I_j) - B_j) + \sum_{j=n+1}^{n+k-1} s_{ij} (B_j - n - B_k) + \sum_{j=n+k+1}^{2n} s_{i,j-1} (B_j - n - B_k) \right] \tag{23}
\]

where \( s_{ij} \) is the \( ij \)th element of \( A^{-1} \).

A similar expression can be derived for the proportion of investor \( k \)'s initial wealth invested in country \( i \)'s riskless bonds, i.e., \( y_i^k - x_i^k \).

It is interesting to note that, if covered interest rate parity holds, the covered holdings of foreign bonds are identical to holdings of domestic bonds, and \( y_i^k - x_i^k \) indicates the proportion of investor \( k \)'s initial wealth invested in domestic and covered foreign bonds. If \( y_i^k - x_i^k > 0 \) then investor \( k \) has decided it is optimal to have uncovered holdings of foreign bonds. This could occur, under the above assumptions, even if the forward rate was an unbiased predictor of the future spot rate, i.e., \( (f_{ik})^F = \mathbb{E}(f_{ik})^1 \). Practitioners in international finance often assert that the investor should always cover in this case, but such an assertion ignores any possible reduction in overall portfolio variance by holding exposed foreign bonds.
The complexity of these demand equations precludes the derivation of a simple ICAPM. It will be shown that in order to derive such a model in discrete time it is necessary for matrix \( A \) to be block diagonal and the next section introduces an assumption which produces this result.

III. The Discrete Time Analogue of Solnik's ICAPM

A. An approximation

To derive a simple ICAPM, the expressions for the rates of return on foreign assets have to be simplified to reduce the complexities introduced by multiplicative random variables. A similar problem was encountered in the studies of 'riskless' and risky returns in an inflationary environment.

Fama (1975) showed that although the real return on a 'riskless' treasury bill, \( \bar{r}_t \), is related to the nominal return \( R_t \) and the rate of change in purchasing power, \( \bar{\Delta}_t \), by:

\[
1 + \bar{r}_t = (1+R_t)(1+\bar{\Delta}_t) = 1 + R_t + \bar{\Delta}_t + R_t\bar{\Delta}_t
\]

(24)

the expression could be rewritten as

\[
\bar{r}_t = R_t + \bar{\Delta}_t
\]

(25)

because "in monthly data, \( R_t \) and \( \bar{\Delta}_t \) are close to zero."\(^6\)

Chen and Boness (1975) made a similar assumption in deriving a capital asset pricing model under uncertain inflation. Although the real

\(^6\)Fama (1975), p. 270.
rate of return on a risky asset \( j \) in time period \( t \), \( \tilde{r}_{jt} \), is related to the nominal rate of return \( \tilde{R}_{jt} \) and the random rate of inflation \( \tilde{R}_{at} \) by:

\[
1 + \tilde{r}_{jt} = \frac{(1+\tilde{R}_{jt})}{(1+\tilde{R}_{at})} = 1 + \tilde{R}_{jt} - \tilde{R}_{at} - \tilde{R}_{jt}\tilde{R}_{at} + \tilde{R}_{at}^2 + \ldots \quad (26)
\]

they assumed that this could be approximated by:

\[
\tilde{r}_{jt} \approx \tilde{R}_{jt} - \tilde{R}_{at}. \quad (27)
\]

A similar assumption will be made in this paper. First, let us rewrite \( I_i^k \) and \( B_i^k \) as follows:

\[
I_i^k = (1+h_i^k)(1+R_i) = 1 + h_i^k + R_i + h_i^kR_i \quad (28)
\]

\[
B_i^k = (1+h_i^k)(1+D_i) = 1 + h_i^k + D_i + h_i^kD_i \quad (29)
\]

where \( R_i \) is the percentage return (expressed as a fraction) on country \( i \)'s risky asset from country \( i \)'s point of view;

\( h_i^k \) is the percentage return (expressed as a fraction) on holding the currency of country \( i \) from country \( k \)'s point of view;

and \( D_i \) is the percentage return (expressed as a fraction) on the riskless asset in country \( i \) from country \( i \)'s point of view.

---

\( ^7 \) Chen and Boness (1975), p. 471, footnote 3.
In the spirit of Fama, Chen, and Boness, the final terms in equations (28) and (29) will be ignored, and it will be assumed that:

\[ I_i^k = 1 + h_i^k + R_i \]  \hspace{1cm} (30)

\[ B_i^k = 1 + h_i^k + D_i \]  \hspace{1cm} (31)

On substituting these revised expressions for \( I_i^k \) and \( B_i^k \) into (1) we obtain:

\[ w_i^k = w_0^k \left[ \sum_{i=1}^{n} \left( I_i^k \right) + \sum_{i=1}^{n} \left( B_i^k \right) \right] \]  \hspace{1cm} (32)

Making the same transformations as given in equations (3), (4), and (5), and knowing that \( h_i^k = 0 \) by definition, equation (32) can be rewritten as:

\[ w_i^k = w_0^k \left[ \sum_{i \neq k} \left( I_i^k \right) + \sum_{i \neq k} \left( B_i^k \right) + (1 + D_k) \right] \]  \hspace{1cm} (33)

8 Incidentally this approximation is not without historical precedence in international finance. Aliber (1973) states that the interest rate parity theorem could be given by:

\[ e^* = D_k - D_i \]  \hspace{1cm} (i)

where \( D_k - D_i \) is the interest agio, and \( e^* \), the exchange agio, equals

\[ \frac{\left[ (f_{ik}^F - (f_{ik})^0) \right]}{(f_{ik})^0} \]. In its full form, the interest rate parity theorem states that:

\[ (1 + D_k) = (1 + D_i) \left[ (f_{ik}^F) / (f_{ik})^0 \right] \]  \hspace{1cm} (ii)

which can be rearranged to yield

\[ e^* = D_k - D_i - D_i e^*. \]  \hspace{1cm} (iii)
B. Demand equations for assets

Assuming again that the investor from country $k$ has a quadratic utility function and maximizes the expected utility of end-of-period wealth, the first-order conditions are

$$\frac{\partial E[u(w^k_i)]}{\partial x^k_i} = W^k_0 E[(R_i - D_i) - 2bw^k_i(R_i - D_i)] = 0 \quad i = 1, n \quad (34)$$

$$\frac{\partial E[u(w^k_i)]}{\partial y^k_i} = W^k_0 E[(h^k_i + D_i - D_k) - 2bw^k_i(h^k_i + D_i - D_k)] = 0 \quad i = 1, \ldots, n \quad i \neq k.$$  

(35)

Again these equations can be simplified considerably by rewriting them in vector form, where $x^k$ and $y^k$ are as defined before and

$$\gamma = \begin{bmatrix} R_1 - D_1 \\ \vdots \\ R_n - D_n \end{bmatrix} \quad (n \times 1)$$  

(36)

Note that $\gamma$ is independent of the investor's country of residence, and therefore has no superscript.

$$\delta^k = \begin{bmatrix} h^k_1 + D_1 - D_k \\ \vdots \\ h^k_n + D_n - D_k \end{bmatrix} \quad (n-1) \times 1$$  

(37)
Consequently,

\[
\begin{align*}
\omega_{1}^{k} = \omega_{0}^{k} \left[ \gamma_{k}^{x} x_{k}^{y} + \delta_{k}^{x} \gamma_{k}^{y} + (1 + D_{k}) \right]
\end{align*}
\]  

(38)

and equations (34) and (35) can be summarized by

\[
E \begin{bmatrix}
\gamma \\
\delta
\end{bmatrix} = 2b \begin{bmatrix}
E(\omega_{1}^{k} \gamma) \\
E(\omega_{1}^{k} \delta)
\end{bmatrix}
\]  

(39)

Substituting for \( \omega_{1}^{k} \) we get,

\[
E \begin{bmatrix}
\gamma \\
\delta
\end{bmatrix} = 2b \omega_{0}^{k} \begin{bmatrix}
E\left( \gamma_{k}^{x} x_{k}^{y} + \gamma_{k}^{x} \delta_{k}^{y} + \gamma(1 + D_{k}) \right) \\
E\left( \delta_{k}^{x} x_{k}^{y} + \delta_{k}^{x} \delta_{k}^{y} + \delta(1 + D_{k}) \right)
\end{bmatrix}
\]  

(40)

which can be rearranged to yield,

\[
c^{k} \begin{bmatrix}
E(\gamma) \\
E(\delta)
\end{bmatrix} = E \begin{bmatrix}
x^{k} \\
y^{k}
\end{bmatrix}
\]  

(41)

where

\[
c^{k} = \frac{1 - 2b \omega_{0}^{k} (1 + D_{k})}{2b \omega_{0}^{k}}
\]  

(42)

and

\[
E = \frac{\gamma_{k}^{x} \gamma_{k}^{y} \delta_{k}^{x} \delta_{k}^{y}}{(2n-1)(2n-1)}
\]  

(43)
The matrix \( B \) has a very interesting structure. On substituting for \( \gamma \) and \( \tilde{\phi}_k \), and assuming, for simplicity, that \( E(h^k_1) = 0 \) and \( \text{cov}(R_1, h^k_1) = 0 \) appendix I shows that we obtain:

\[
C^k_z = \left[ z z' + \Delta \right] \begin{bmatrix} x^k_1 \\ y^k_1 \end{bmatrix} \tag{44}
\]

where

\[
\begin{bmatrix} x_1 \\ \vdots \\ x_n \\ y_k \end{bmatrix} = \begin{bmatrix} \alpha_1 - D_1 \\ \vdots \\ \alpha_n - D_n \\ D_1 - D_k \\ \vdots \\ D_n - D_k \end{bmatrix}
\tag{45}
\]

\[
\alpha_1 = E(R_1) \tag{46}
\]

\[
\Delta = \begin{bmatrix} \Sigma & 0 \\ 0 & \phi^k \end{bmatrix} \tag{47}
\]

\( \Sigma \) = the variance-covariance matrix of returns (in discrete time) on the risky assets, with each return being denominated in the currency of the asset country.

and \( \phi^k \) = the variance-covariance matrix of returns (in discrete time) from holding foreign currencies from k's viewpoint.

Consequently, the system of equations summarized in (44) can be solved for \( x^k_1 \) and \( y^k_1 \) where
\[
\begin{bmatrix}
X^k \\
Y^k \\
Z^k
\end{bmatrix} = C^k [zz' + \Delta]^{-1} z
\]
\[
= C^k [z^{-1} 0] [0 (\phi^k)^{-1} z]
\]
(48)

where
\[
C^k = C^k \frac{|\Delta|}{|\Delta + zz'|}
\]
(49)

Thus the demand for the risky asset of country \(i\) is
\[
d^k_i = \omega^k_0 x^k_i = \omega^k_0 C^k \sum_j s_{ij} (\alpha^j - D^j)
\]
(50)

where \(s_{ij}\) are elements in \(\Sigma^{-1}\) and the demand for the riskless asset of country \(i\) is
\[
e^k_i = \omega^k_0 (y^k_i - x^k_i) = \omega^k_0 C^k \sum_{j \neq k} \eta^k_{ij} (D^j - D^k) - d^k_i
\]
(51)

where \(\eta^k_{ij}\) are elements in \((\phi^k)^{-1}\).

Except for the initial constant term \(\omega^k_0 C^k\), these demand equations are identical to those of Solnik and lead to his separation theorem:

All investors will be indifferent between choosing portfolios from the original assets or from n+1 funds, where a possible choice for those funds is:
- the world stock market portfolio (hedged against exchange risk)
- the \(n\) bonds of each country.
The proportion of the risky fund invested in asset $i$ is:

$$\frac{\sum_{j=1}^{n} s_{ij}(\alpha_j - D_j)}{\sum_{i=1}^{n} \sum_{j=1}^{n} s_{ij}(\alpha_j - D_j)}.$$ 

C. An ICAPM in discrete time

Since the demand function for stocks is separable from that for bonds, it is possible to derive an equilibrium relationship between the expected returns on equity assets directly from equation (50). From this equation the aggregate demand function for risky assets is given by:

$$\sum_{k} d_{i}^{k} = \frac{1}{A} \sum_{j=1}^{n} s_{ij}(\alpha_j - D_j)$$

where

$$\frac{1}{A} = \sum_{k} w_{0c}^{k}$$

The supply of the risky asset from country $i$ is equal to $q_{i}M$ where $M$ is the total market value of the world's risky securities. Equating demand and supply we obtain

$$q_{i}M = \sum_{k} d_{i}^{k} = \frac{1}{A} \sum_{j=1}^{n} s_{ij}(\alpha_j - D_j)$$

which as shown in appendix 2 can be converted into the following equation:

$$\alpha_i - D_i = MA \sum_{j=1}^{n} q_{i} \sigma_{ij}.$$
If we define

$$\alpha_M = \sum_i q_i \alpha_i$$  \hspace{1cm} (56)

$$D_M = \sum_i q_i D_i$$  \hspace{1cm} (57)

$$\sigma_{iM} = \sum_j q_i \sigma_{ij}$$  \hspace{1cm} (58)

Then

$$\alpha_i - D_i = MA_{iM}$$  \hspace{1cm} (59)

$$\alpha_M - D_M = MA \sum q_i \sigma_{iM} = MA \sigma_M^2$$  \hspace{1cm} (60)

Combining equations (59) and (60) we obtain the discrete time analogue of Solnik's international capital asset pricing model

$$\alpha_i - D_i = \frac{\sigma_{iM}}{\sigma_M^2} (\alpha_M - D_M).$$  \hspace{1cm} (61)

IV. The Relationship of the Discrete Time Model to the Continuous Time Model

Using the demand equations introduced in section II and without having to make the assumption given in section III A, the continuous time model of Solnik can be derived in the limit when the time interval is infinitesimal. Let the price dynamics for $R, h, k,$ and $D_i$ be given by
the following expressions: 9

\[ R_i = \alpha_i t + \sigma_i \sqrt{t} \ Z_i \]  

(62)

where \( t \) = some arbitrarily small length of time

\( \alpha_i \) = the instantaneous expected rate of return

\( \sigma_i \) = the instantaneous standard deviation of returns

\( z_i \) is a random variable, independent of all states, with \( E(z_i) = 0 \) and \( \text{var}(z_i) = 1 \)

\[ h_i^k = \mu_i^k t + \delta_i^k \sqrt{t} \ Z_i^k \]  

(63)

where \( \mu_i^k \) = the instantaneous expected rate of return

\( \delta_i^k \) = the instantaneous standard deviations of returns

\( z_i^k \) is a random variable, independent of all states, with \( E(z_i^k) = 0 \) and \( \text{var}(z_i^k) = 1 \)

\[ D_i = \lambda_i t \]  

(64)

where \( \lambda_i \) = the instantaneous rate of return on the riskless asset.

In the limiting case of continuous time, terms of higher order than \( t \) can be ignored and, so long as it is assumed that the returns on risky assets are uncorrelated with the returns on the exchange rate, 10 i.e.,

9 Note that equations (62), (63), and (64) correspond to Solnik's equations on pages 503 and 504 (Solnik [1974]). The symbol \( t \) (and not \( dt \) as in Solnik [1974]) will be used for simplicity of exposition. With some abuse of language, \( t \) can therefore be read as "length of trading interval."

10 The importance of this assumption can be seen by considering the submatrix \( E(p_{mn} \text{ } \text{ } k') \) in \( A \) when the time interval is infinitesimal.

The \( ij \)th element of this \((n-1)xn\) submatrix is \( E[g_i^k (I - B_i)](g_j^k (I - B_j)) \) which on substituting for \( g_i^k \), \( B \), and \( I \) becomes \( E[h_i^k (I + D_i) + (D_i - D_k)] \), \( (R_i - D_i) + h_j^k (R_j - D_j) \). Finally, by using the expressions for \( R_i \), \( h_i^k \), and
\( E(Z_i^* Z_j^*) = 0 \), the demand equations given in (22) can be simplified considerably to yield:

\[
\begin{bmatrix}
\mathbf{x}^k \\
\mathbf{y}^k
\end{bmatrix} = \mathbf{t}^k \begin{bmatrix}
\Sigma^{-1} \\
0
\end{bmatrix} \begin{bmatrix}
\begin{pmatrix}
\alpha - \lambda \\
\mu^k + \lambda - \lambda_k
\end{pmatrix}
\end{bmatrix}
\]

(65)

where \( \Sigma = \| \sigma_{ij} \|_{nxn} \) the instantaneous variance-covariance matrix of security returns. Note that \( \sigma_{ij} = E(\sigma_i^* Z_i Z_j) \).

\( \phi^k \) is the \((n-1) \times (n-1)\) instantaneous variance-covariance matrix of returns on the exchange rate from country \( k \)'s viewpoint. Note that the \( ij \)th element of this matrix is \( \delta_{ij}^k E(Z_i^* Z_j) \).

\( \alpha - \lambda = \begin{pmatrix} \alpha_1 - \lambda_1 \\ \vdots \\ \alpha_n - \lambda_n \end{pmatrix} \)

(66)

D. given in equations (62), (63), and (64), respectively, and ignoring all the terms of higher order than \( t \), it is found that all the terms can be approximated to zero except that one derived from \( E(h_{iR}^k) \) which is:

\[
E(h_{iR}^k) = E\left( (\mu_i^k + \delta_i^k \sqrt{E} Z_i^*) (\alpha_i + \delta_i^k \sqrt{E} Z_j^*) \right)
\]

\[
= \delta_{ij}^k \sigma_{ij} E(Z_i^* Z_j)
\]

(1)

Now, equation (i) is equal to zero if and only if \( E(Z_i^* Z_j) = 0 \). This is equivalent to assuming that the returns on the risky assets are uncorrelated with the returns on the exchange rate. Whereas it is possible that \( E(Z_i^* Z_j) = 0 \), \( i \neq j \), we cannot know a priori that \( E(Z_i^* Z_j) = 0 \).
and
\[ \begin{bmatrix} \mu^k + \lambda - \lambda_k \\ \mu_i + \lambda_i - \lambda_k \\ \mu_n + \lambda_n - \lambda_k \end{bmatrix} = \begin{bmatrix} \mu^k + \lambda - \lambda_k \\ \mu_i + \lambda_i - \lambda_k \\ \mu_n + \lambda_n - \lambda_k \end{bmatrix} \] (67)

Because the inverse of \( A \) becomes block diagonal in continuous time the demand equations are simplified considerably. The demand by an investor from country \( k \) for the risky asset of country \( i \) is

\[ d_k^i = W_0^k x_i^k = W_0^k k \sum_j s_{ij}(\alpha_j - \lambda_j) \] (68)

where \( s_{ij} \) are elements in \( \Sigma^{-1} \) and the demand by an investor from country \( k \) for the riskless asset of country \( i \) is

\[ e_i^k = W_0^k (y_i^k - x_i^k) = W_0^k k \sum_{j \neq k} \eta_{ij}(\lambda_j - \lambda_k + \mu_j^k) - d_k^i \] (69)

where \( \eta_{ij} \) are elements in \( (\phi^k)^{-1} \). Except for the initial constant term \( W_0^k k \) these demand equations are identical to those of Solnik, and can be used to derive his ICAFPM.

The advantage of using continuous time is that the problem of multiplicative random variables disappears as terms of higher order than \( t \) are ignored.

V. Summary

Under the assumptions of no restrictions on international capital flows and consumption limited to the investor's own country, this paper
considers the derivation of an ICAPM in discrete time. To avoid the complexities introduced by multiplicative random variables and to obtain a simple ICAPM, it is necessary to make an assumption similar to that made in studies of real returns on riskless and risky assets during inflation. However, when the period spacing is infinitesimal such an assumption is unnecessary and in the limiting case of continuous time the problem of multiplicative random variables disappears and Solnik's model is derived.
Appendix 1: The Structure of Matrix B Given in Equation (43)

Assume \( E(h_{ik}^k) = 0 = \text{cov}(R_i, h_{ik}^k) \) :

\[
B = \Sigma + \begin{bmatrix}
[\alpha_1 - \overline{D}_1] & \ldots & [\alpha_n - \overline{D}_n] \\
\vdots & \ddots & \vdots \\
[\alpha_n - \overline{D}_n] & \ldots & [\alpha_n - \overline{D}_n]
\end{bmatrix}
\begin{bmatrix}
[\overline{D}_1] & \ldots & [\overline{D}_n] \\
\vdots & \ddots & \vdots \\
[\overline{D}_n] & \ldots & [\overline{D}_n]
\end{bmatrix}
\Sigma^k + \begin{bmatrix}
[\overline{D}_1] & \ldots & [\overline{D}_n] \\
\vdots & \ddots & \vdots \\
[\overline{D}_n] & \ldots & [\overline{D}_n]
\end{bmatrix}
\begin{bmatrix}
\end{bmatrix}
\]

Let \( \begin{bmatrix} \alpha - \overline{b} \\ \alpha - \overline{b} \end{bmatrix} \)

\[
\begin{bmatrix}
[\alpha - \overline{b}] \\
\vdots \\
[\alpha - \overline{b}]
\end{bmatrix}
\]

\[
\begin{bmatrix}
D_1 - \overline{D}_k \\
D_2 - \overline{D}_k \\
\vdots \\
D_n - \overline{D}_k
\end{bmatrix}
\]

Then \( B = \begin{bmatrix}
(\alpha - \overline{b})(\alpha - \overline{b})' \\
(\alpha - \overline{b})(\alpha - \overline{b})'
\end{bmatrix}
\begin{bmatrix}
\Sigma & 0 \\
0 & \phi^k
\end{bmatrix}
\]

Let \( \begin{bmatrix} \alpha - \overline{b} \\ \alpha - \overline{b} \end{bmatrix} \)

\[
\begin{bmatrix}
\alpha - \overline{b} \\
\alpha - \overline{b}
\end{bmatrix}
\]

Then \( B = [zz' + \Delta] \) where \( \Delta = \begin{bmatrix}
\Sigma & 0 \\
0 & \phi^k
\end{bmatrix} \)
Appendix 2: Proof that \[ MA \sum_{j=1}^{n} q_j \sigma_{ij} = \alpha_i - D_i \]

From equation (54) we know that

\[ q_j = \frac{1}{AM} \sum_{k=1}^{n} s_{jk} (\alpha_k - D_k) \]

where \( s_{jk} \) are elements in \( S^{-1} \).

Now \( S^{-1} = \frac{S^{+}}{|S|} \)

where \( S^{+} \) is the adjoint of matrix \( S \), i.e., the transpose of the matrix obtained from \( S \) by replacing each element in \( S \), \( \sigma_{ij} \), by its cofactor \( S_{ij} \). Therefore \( S^{+} = \| S_{ij} \| \).

Consequently \[ MA \sum_{j=1}^{n} \sigma_{ij} q_j = \sum_{j=1}^{n} \sigma_{ij} \sum_{k=1}^{n} s_{jk} (\alpha_k - D_k) \]

\[ = \frac{1}{|S|} \sum_{j=1}^{n} \sigma_{ij} \sum_{k=1}^{n} (\alpha_k - D_k) S_{kj} \]

\[ = \frac{1}{|S|} \left\{ \sum_{j=1}^{n} \sigma_{ij} S_{1j} (\alpha_1 - D_1) + \sum_{j=1}^{n} \sigma_{ij} S_{2j} (\alpha_2 - D_2) + \ldots + \sum_{j=1}^{n} \sigma_{ij} S_{nj} (\alpha_n - D_n) \right\} \]

But \[ \sum_{j=1}^{n} \sigma_{ij} S_{kj} = 0 \quad k \neq i \]

\[ = |S| \quad k = i \]

Therefore \[ MA \sum_{j=1}^{n} \sigma_{ij} q_j = \frac{1}{|S|} |S| (\alpha_i - D_i) = \alpha_i - D_i . \]
References


