DIRECT EVALUATION AND CORPORATE FINANCIAL THEORY

by

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ABSTRACT

This paper presents a comprehensive approach to stock value maximizing corporate policy, based on direct inferences from observed security prices. Firms are assumed to operate in a competitive securities market where positive returns are positively valued, and the consequences of their decisions are uncertain and extend through time. Potential investment projects are evaluated by a comparison with a portfolio of marketed securities with similar returns. When no such equivalent portfolio exists, one can systematically search for two portfolios that will bound the value of the project as tightly as possible. Consideration of the risk-free markets available at future dates permits a representation of project returns by a set of "discounted values," and an assessment of the possible future interest rates can improve the bounds on value. When direct evaluation is applied to financing decisions, the invariance of the value of the firm to purely financial operations is seen to be extremely general. But the direct evaluation approach also clarifies the distributional implications of financing decisions. It is shown that, even in the absence of taxes, the firm's stockholders will generally not be indifferent to its financial policies.

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I. INTRODUCTION.

Perhaps the most difficult aspect of corporate decision-making is putting a value on potential projects and claims whose future returns through time are uncertain. To make decisions that are in the interests of their stockholders, firms must be able to project the changes in value that would arise from alternative decisions.

In a world of certainty it is relatively simple to assess the value of any sequence of future returns. But uncertainty vastly complicates valuation procedures. How does one evaluate uncertain returns through time? One method, which is currently in wide use, suggests raising the rate at which expected future earnings are discounted, to reflect the firm's "cost of capital". A more sophisticated alternative is valuation through economic analysis of investors' preferences and expectations. Perhaps the most prominent theory of this type is the "Capital Asset Pricing Model" (CAPM), developed by Sharpe [18], Lintner [11] and others.

Both "risk discounting" and CAPM have serious shortcomings.
Simple examples show that the use of a risk-adjusted cost of capital of the firm is meaningful only under stringent and generally unrealistic circumstances. The CAPM assessment of values is based on the assumptions that everyone shares the manager's probabilities about the returns of all assets, and that every individual's preferences depend only on the mean and variance of his portfolio's returns. Furthermore, CAPM is a "one-period" model which is difficult to extend to a multi-period world.\footnote{The CAPM valuation for time 0 depends on the assessment of the price \( P_1 \) at time 1, and attempts to evaluate \( P_1 \) by CAPM must be based not only on the future risk-free interest-rate and the future "price of risk", but also on the assessment of the asset's price at time 2, and so forth. It is not clear how this sequential assessment converges and how practical the procedure really is.}

The approach to corporate decisions suggested in the present paper is based on direct inferences from existing security prices. It is argued that a substantial part of the valuation needed to determine management action can be derived directly from observed prices, can be exercised with reasonable accuracy in a multi-period framework, and need not depend upon rigid assumptions about investors' preferences and their probability assessments.

This paper is intended as an overview of the direct evaluation approach as applied to corporate decisions. For completeness, it draws from many of the earlier studies of the "state-of-the-world" description of uncertainty.
developed by Arrow and Debreu and subsequent "state-preference" models of securities markets. Many aspects, however, are to the best of our knowledge presented here for the first time.

The mathematical aspects of direct evaluation are all very basic. Although the implementation for large decision problems will usually require the use of a computer, there are today widely available ready-to-use computer codes for all the problem formulations (such as linear equations and linear programming) suggested in this paper.

The structure of the paper is as follows. Section II shows how some simple and intuitively appealing principles give rise to a fundamental valuation theorem. Section III gives guidance in selecting investment projects on the basis of the valuation theorem, and the procedures are made more explicit in section IV. Section V indicates the approximations that are involved when the fit of the data is less than perfect. Sections VI and VII show how the evaluation can be extended to a wider class of investment projects by placing bounds on value and by an explicit consideration of risk-free investments available at future dates. Section VIII applies the direct evaluation approach

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2As an incomplete sample in alphabetical order, consider Arrow [1], Beja [2], Debreu [4], Ekeren and Wilson [5], Hirschleifer [7], Leland [9], Myers [14], Radner [16], Rubinstein [17].
to financing decisions. The invariance of the value of the firm to purely financial operations is shown to be extremely general. But our direct evaluation approach also clarifies the distributional implications which financial operations may have. A simple example shows that, even in the absence of taxes, the firm's stockholders will generally not be indifferent to the financial policies of the firm.

II. BASIC PRINCIPLES OF VALUATION AND DECISION.

We begin with a set of marketed securities. Included in this set are stocks, bonds, options, and all other financial claims which are traded and have a market price. Investors select personal portfolios of these securities. We shall assume that investors are small relative to the market, securities are perfectly divisible, and transactions costs are negligible.

The extent of the existing market for securities is critically important for valuation theory, because it indicates the range of possible portfolios available to investors. The basic notion used to represent this extent is the "span" of the market.

Definition: The span of a given set of securities is the set of all portfolios which can be formed with these securities. We include the possibility of portfolios with short positions. The span of all traded securities is called the span of the market.
The market value of a portfolio is simply the sum of the market values of the securities in that portfolio, with short positions taken with a negative sign.

We now make a simple assumption about the relationship between securities or portfolios and their market values.

**Positive Evaluation:** Let \( j \) and \( k \) represent two securities or portfolios. If the returns to security \( j \) are at least as great as the returns to security \( k \) at all times and under every circumstance, then the market value of security \( j \) is at least as great as the market value of security \( k \).

In short, the assumption says that non-negative cash flows have non-negative market values — "to get more, you must pay more".³

An immediate implication of the positive evaluation assumption is given by the following important principle.

**Equivalence Principle:** any two portfolios with identical returns must have the same value.⁴

The problem of corporate financial theory is not to value currently traded securities, but to estimate the value of

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³This is similar to Hakansson's [6] "no easy money condition".

⁴Investors' preference for higher returns over lower returns is the essence of both positive evaluation and the equivalence principle, and (given the "free disposal" of financial income) the two are in fact interchangeable.
untraded prospects or claims. A prospect is defined as a sequence of uncertain future cash flows through time. Can we say anything about the value of untraded prospects if they were to be made available to the market? The answer is yes, if we add one further assumption.

**Competitive Markets:** The market values of currently traded securities will not be changed by the introduction of a new claim or prospect lying within the span of the market.

By "lying within the span of the market" we mean that the new claim has returns or prospect precisely the same as some portfolio composed of currently traded securities. The competitive markets assumption seems reasonable if the returns of the new claim are small relative to total returns in the market.

Our assumption can now be used to prove the following fundamental valuation theorem.

**Valuation Theorem:** Let \( Y \) be a prospect whose returns are identical to the returns of a portfolio of currently traded securities. Let \( x_1, x_2, \ldots, x_J \) be the number of shares of each security (1 through J) in the identical portfolio, and let \( p_1, p_2, \ldots, p_J \) be the current price per share of each security. Let \( w(Y) \) denote the value of the prospect \( Y \) if it were marketed. Then \( w(Y) \) must satisfy

\[
w(Y) = \sum_{j=1}^{J} x_j p_j
\]

That is, the value of the prospect is equal to the value of
the portfolio yielding identical returns. The theorem follows directly from the competitive market assumption and the equivalence principle.

Our interest in valuation of unmarketed prospects is not purely academic. Rather, we assume that potential values have direct relevance to the welfare of stockholders and the behavior of managers. This relevance is captured by the following assumption.

Value maximization: The firm chooses between alternative decisions so as to maximize the value of the equity of current common shareholders.

Stock value maximization is known to be in the interests of all common shareholders when the perfectly competitive markets assumption holds and the decision alternatives do not change the span of the market.\(^5\)

III. PROJECT SELECTION AND INVESTMENT DECISIONS.

The simple principles of the preceding section can be used to guide the selection of investment projects. An investment project has two components: its prospect, and its initial cost (known with certainty). Similarly, a security has a prospect and a current market price.

It is conventional wisdom that in the absence of taxes and

\[^5\text{See Leland [8] and Ekern and Wilson [5].}\]
reorganization costs capital budgeting and financial decisions can be separated. In section VIII we show that this is not true in general. To separate the discussion of capital budgeting and defer the analysis of financing decisions, we shall assume for now all-equity firms with internal financing. This implies that all returns and costs of investment accrue only to the current holders of common stock. The change in the value of equity will therefore be the value of the project's prospect less its initial cost.

To indicate the usefulness of our approach, let us first consider an investment which is in the same risk class as some marketed security. An investment is said to belong to the same risk class as a security if its returns are identical to the returns derived from holding some number of shares (say x) of the security. That is Y=xZ, where Y is the prospect of the investment and Z is the prospect of the security. If the price of the security with prospect Z is P, then by the valuation theorem we have immediately w(Y)=xP. The investment project with prospect Y and initial cost I should be undertaken if and only if

\[ w(Y) = xP \geq I. \]

Compare this with the traditional "net present value" (NPV) method using the firm's cost of capital as the risk-adjusted discount factor. This clearly applies only if the project is in the firm's risk class (it would certainly not apply to projects which are much more "risky" than the firm's present activities). If this is true, it is first necessary to assess the firm's "cost of capital". To do this, one must substitute numerical surrogates \( z_1, z_2, \ldots \), called the "expected" values, for
the uncertain cash flows \( z_1, z_2, \ldots \), representing the firm's future returns, and solve for \( r \) in the non-linear equation
\[
P = \sum_t \frac{z_t}{(1+r)^t}.
\]
One must then similarly compute the project's "expected" future returns \( y_1, y_2, \ldots \) and accept the project if and only if
\[
I < \sum_t \frac{y_t}{(1+r)^t}.
\]
It is seen that the traditional evaluation of NPV by the firm's cost of capital is unnecessarily complicated. It depends on a number of superfluous variables \(^6\) and involves considerable computations. On the other hand, when traditional NPV computation using the firm's cost of capital is at all applicable direct evaluation gives an immediate answer.

Suppose now that the investment project with prospect \( Y \) is not in the same risk class as any security. Nonetheless, we can still value the project if it lies within the span of the market. In this case the project's returns are identical to some portfolio with \( x^1, \ldots, x^J \) shares of securities with prospects \( z^1, \ldots, z^J \). This means
\[
Y = \sum_{j=1}^{J} x^j z^j \tag{1}
\]
and by the Valuation Theorem \( w(Y) = x^j p^j \), where \( p^j \) is the price of security \( j \). As before, the investment project should be undertaken if and only if
\[
w(Y) = \sum_j x^j p^j > I. \tag{2}
\]

When CAPM is used for project evaluation, the same conclusions will be reached, because in that case
\[
E[Y] = \sum x^j E[z^j] \quad \text{and} \quad \text{Cov}[Y, R_M] = \sum x^j \text{Cov}[z^j, R_M]
\]
where \( R_M \) is the market's rate of return. Note, however, that in our approach the conclusion is derived directly

\(^6\) In particular, it is not at all clear what the "expected" future returns really mean. If they are mathematical expected values, it is not clear on what probability assessments they are based, and whether there is any consensus with respect to these probabilities.
from observed prices. It is independent of any probabilities used to assess expected values and covariances, and does not require measurement of "the market price of risk" as CAPM does.

IV. PROJECT SELECTION: HOW TO USE DIRECT EVALUATION.

The preceding section has shown how projects within the span of the market can be directly evaluated. But to value an arbitrary project we must be able to determine whether it lies within the span of the market, and what the composition of the portfolio with identical returns is. In this section, we outline a method whereby these questions can be answered.

First, we must be more precise in specifying what we mean by a prospect. Prospects, and the relationship between them, can in principle be described by a joint probability distribution. This is quite awkward when the number of uncertain cash flows under consideration is substantial. It is much more convenient to describe the whole set of joint relationships by specifying separately the relation of each uncertain cash flow to a common basic structure. This common reference structure is the set of possible future "events". For each time \( t \), let there be \( k_t \) events \( e_{it} \), \( i=1,\ldots,k_t \). Each event at \( t \) represents a basic "scenario" for that time. How detailed these scenarios should be and, accordingly, how many alternative basic events should be considered for every time \( t \), is a matter of model construction and will be left open at this point. A
sequence of future events is called "the (future) state-of-the-world".  

For any prospect \( z \), let \( z_{it} \) denote the returns generated by the prospect in event \( e_{it} \) at time \( t \). The description \( \{z_{it}\} \) will be shown below to suffice for direct evaluation. A strength of our approach is that neither our definitions nor our results require specification of subjective probabilities (as does CAPM). Of course, we do not rule out the fact that managers and shareholders may have subjective probabilities. Rather, our results hold no matter what these probabilities might be.

Let \( z^1, ..., z^J \) denote the prospects associated with the \( J \) existing securities. The problem of identifying a portfolio of these securities whose returns are identical to the project's returns \( Y \) can then be formulated as the problem of finding \( x^j, j=1, ..., J \) that solve the following system of linear equations

\[\text{(Equations)}\]

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7The number \( M \) of possible states of the world is generally different from the total number \( K \) of possible events. \( M \) is bounded above by, but need not equal, the product \( k_1k_2k_3... \) as some combinations of events may be considered impossible. In a model with "complete recall", where all past history is considered a relevant part of the description of events, \( M \) is equal to \( k_T \) where \( T \) is the horizon of the model.

8The extent of consensus with respect to \( z_{it} \) as a fixed number will be left an open question. See, however, Beja [2] for an indication that the state description may indeed be sufficiently flexible so that \( z_{it} \) can be interpreted as the expected return in event \( e_{it} \).  

11
\[ y_{it} = \sum_{j=1}^{J} x_{j}^{i} z_{it}^{j} \quad t=1, \ldots, T \quad i=1, \ldots, k_{t} \]  

With computational procedures (such as linear programming) which are today widely available in computer coded form, it is fairly easy to find a solution (or alternatively to indicate that one does not exist) even if the number of future events (and thus the number of equations) is fairly large.

Consider now the problem of selecting an investment plan, possibly involving more than one out of a set of \( N \) feasible projects \( y^{1}, \ldots, y^{N} \). If the projects are mutually unrelated and are all within the span of existing markets, then equations (3) and (2) can be recursively applied to determine the acceptability of each project separately. From a purely computational viewpoint, however, it may be advisable to solve simultaneously for the whole optimal investment plan even when separate evaluation is permissible. We must then solve the following linear programming problem, where \( s^{n} \) denotes the "scale" at which project \( n \) is implemented:

Choose \( x^{j}, j=1, \ldots, J \) and \( s^{n}, n=1, \ldots, N \)

to maximize \[ \sum_{j=1}^{J} x^{j} p^{j} - \sum_{n=1}^{N} s^{n} \ln \]  
subject to \[ \sum_{j=1}^{J} x^{j} z^{j}_{it} - \sum_{n=1}^{N} s^{n} y^{n}_{it} = 0 \quad t=1, \ldots, T \]  
\[ i=1, \ldots, k_{t} \]  
\[ 0 \leq s^{n} \leq 1 \quad n=1, \ldots, N \]  

This problem is called "the basic planning problem", or in short "problem P". When the conditions for separate
evaluation apply, the optimal values of \( s^n \) will be either 0 or 1, and the above formulation of problem \( P \) may be used even if arbitrary choices of scale are not permissible. The computational effort in solving problem \( P \) is little more than that of solving equations (3) for one project, with a very small sensitivity to the number \( N \) of projects.

Problem \( P \) can be modified to allow for substantial interdependencies between projects, and for restrictions on the permissible choices of scale in implementation (which may become relevant with project dependence). For each project, identify a finite number of feasible scales, to be considered as different and mutually exclusive projects. Associate with each of the "new projects" a different index number \( n \) and modified \( y^n \) to allow for the appropriate scale. Let \( N' \) be the thus extended number of projects. Problem \( P \) is then modified to problem \( IP \) - for planning with interrelated projects - by replacing \( N \) with \( N' \) in (P-a), replacing constraint (P-b) with

\[
 s^n = 0 \text{ or } 1 \text{ for } n=1,\ldots,N \tag{IP-b}
\]

and adding the constraints

\[
 \sum_{n} y^n s^n < 1 \tag{IP-c}
\]

whenever the projects in the set \( A \) are mutually exclusive, and

\[
 s^n - s^n' > 0 \tag{IP-d}
\]

whenever undertaking project \( n \) is a prerequisite for undertaking project \( n' \).

Problem \( IP \) is of the "mixed integer programming" type. The fact that a feasible solution with \( s^n=0 \) for all \( n \) always exists both in problem \( P \) and in problem \( IP \) reflects the feasibility of "doing nothing", so that if the best project has a negative net present value none will be undertaken.

It may be both conceptually and computationally desirable to separate the "evaluation" part in problems \( P \) and \( IP \).

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9 Although commercial computer codes for mixed integer programming problems are also widely available, they are computationally far less efficient than codes for linear programming and even for "pure" integer programming (especially of the 0-1 variables type).
- involving a comparison with a portfolio of existing securities - and the selection part. A separation of this kind is indeed possible, because as is well known\(^{10}\) equilibrium prices in perfect capital markets must be consistent with an underlying system of "elementary prices". The elementary prices, denoted by \( q_{it} \), \( t=1,\ldots,T \) \( i=1,\ldots,k_t \) can be interpreted as "the price today for a claim to one money unit receivable at time \( t \) contingent on event \( e_{it} \)"., and they solve the system of linear equations

\[
p^j = \sum_{i,t} z_{it}^j q_{it} \quad j=1,\ldots,J
\]

(4)

The solution need not be unique, but all solutions are fully equivalent for the evaluation of streams within the span of the market. When the solution is unique, the market is said to be "complete with respect to the model" (as characterized by the structure of events \( e_{it} \)). Any prospect \( Y \) of future returns within the span of the market can be evaluated by

\[
w(Y) = \sum_{i,t} Y_{it} q_{it}
\]

(5)

This simple computation can be repeated for all potential investments, while equations (4) need be solved only once. With no interdependencies, each project \( Y^N \) is accepted if and only if \( I^N \leq w(Y^N) \).

If there is some dependence between projects, we must choose \( s^N \) \( n=1,\ldots,N \) to solve

\[
\max \sum_{n} s^N (\sum_{i,t} Y_{it} q_{it} - I^n)
\]

subject to constraints (IP-b), (IP-c) and (IP-d).

Note that in this variation the elementary prices have replaced the portfolio weights \( x^j \). The programming

\(^{10}\)Cf. Beja [2] and Leland [8].
problem does not involve constraints (P-a) and is of the "zero-one" or "boolean" type.\textsuperscript{11} On the other hand, it should be also noted that while the solution to equations (3) also indicates whether each project is in the span of the market, the evaluation through elementary prices does not. Problems P and IP indicate if there is an acceptable investment plan within the span of the market, but not whether all projects are in the span (which is relevant for the applicability of value maximization). It follows that the programming formulations outlined above are more attractive when it is relatively easy to determine independently if project returns are spanned by the market. When the market is complete with respect to the adopted model formulation, this is of course no problem, because all prospects are in the market's span.

V. INCONSISTENCIES AND MARKET IMPERFECTIONS.

The fit between a model description and observed data cannot be expected to be always perfect, and this may be reflected by "inconsistencies" in observed prices. These inconsistencies could be attributed to transactions costs, or also to disagreements about the appropriate description of uncertainty and of prospects. Observed prices may then contradict the equivalence principle, and there may be two distinct solutions \( \{ x^j \} \) and \( \{ \bar{x}^j \} \) to equations (3), such that for some \( Y \)

\[ \sum_j x^j p^j < I < \sum_j \bar{x}^j p^j. \]

On the basis of \( \{ x^j \} \) the project is rejected, while on the basis of \( \{ \bar{x}^j \} \) it is accepted. Since obtaining all feasible

\textsuperscript{11} Besides the general advantage of decomposing a large problem into two smaller ones, this decomposition is especially attractive because pure zero-one programming problems are the easiest to solve of all programming problems involving some integer constraints, and are by far easier than mixed integer programming problems.
solutions to a system of equations is much more difficult than obtaining one solution, the investment decision using equations (3) may depend on the arbitrary solution obtained. With problems P and IP, the selected solution is considered "acceptable" if there exist some \( \{x^j\} \) such that the value of the objective-function \((P-o)\) is positive, regardless of whether some other set of values exists such that \((P-a)\) is again satisfied, but with a negative value for \((P-o)\). It may be considered undesirable to undertake a plan which is marginally acceptable by some evaluation if by other evaluations based on the same market data it is clearly unacceptable. The preferred approach may in this case be to base the decision on some "average" evaluation.

When there are inconsistencies in observed prices, the system of equations (4) is unsolvable. An "average" evaluation may be provided by some "approximate solution" to this system. Since a set of \(q\)'s that satisfies (4) for all assets \(j\) does not exist, the average solution must not attempt to satisfy the equation accurately for some \(j\) and ignore the equations for other \(j\)'s. Rather, it would attempt to simultaneously approximate all equations (perhaps with some unequal relative importance) as much as possible. Such an approximation is given by the following:

choose \(\hat{q}_{it}, \, t=1,...,T \quad i=1,...,k_t\)

to minimize \(\sum_j a^j (\sum_i t z_{it} \hat{q}_{it} - p_j)^2\) \hspace{1cm} (7)

where \(a^j > 0\) is a "weight coefficient" indicating the relative
importance attached to the data on asset $j$. $a^j_1$ could be higher with securities for which more reliable data is available, or ones which represent higher market values. When considerations of this kind are taken as only marginal, it is of course most convenient to let simply $a^j_1 = 1$ for all $j$. The procedure is a (weighted or unweighted) "least squares" approximation, for which efficient computer codes are again widely available.\textsuperscript{12} The minimizing $\{\tilde{a}_{it}\}$ are then used as approximate evaluators in equation (5) or the optimization problem (6).

\textbf{VI. INDIRECT EVALUATION.}

Thus far, we have developed valuation procedures for investment projects within the span of the market. We now turn to the case where the returns of the specific investment project are distinctly different than the returns of any portfolio of market securities. Under slightly stronger assumptions, however, we shall indicate that a direct extension of our earlier techniques can give guidance on the selection of investment projects not within the market's span.

What we require is that the assumptions of Positive Evaluation, Competitive Markets, and Value Maximization, presented in section II, hold even when considering projects

\textsuperscript{12}These are most frequently used for multiple regression analysis.
not within the market's span.\textsuperscript{13} Given that these assumptions continue to hold, consider a portfolio \( \{x_j^i\} \) whose returns never exceed the project's returns \( Y \), i.e.
\[ Y_{it} \geq \sum_j x_j^i z_{it}^j \text{ for all } i \text{ and } t. \]

By Positive Evaluation we know that the value of the portfolio \( x \) cannot exceed the value of the project:
\[ w(Y) \geq \sum_j x_j^i p_j^i. \]

Similarly, if we can find a portfolio \( \bar{x} = \{\bar{x}_j^i\} \) such that
\[ Y_{it} \leq \sum_j \bar{x}_j^i z_{it}^j \text{ for all } i \text{ and } t \]
then
\[ w(Y) \leq \sum_j \bar{x}_j^i p_j^i. \]

That is, the value of the project must lie between the values of the portfolios \( x \) and \( \bar{x} \). Clearly, we want the bounds to be as tight as possible, in order to identify the value of the project as closely as possible. Thus a pair of linear programming problems suggest themselves:

Max \( \sum_j x_j^i p_j^i = \bar{v} \)
subject to \( \sum_j x_j^i z_{it}^j \leq Y_{it} \) \( t=1,...,T \) \( i=1,...,k_t \)

Min \( \sum_j x_j^i p_j^i = \underline{v} \)
subject to \( \sum_j x_j^i z_{it}^j \geq Y_{it} \) \( t=1,...,T \) \( i=1,...,k_t \)

The solutions \( \bar{v} \) and \( \underline{v} \) will bound the value of the investment

\textsuperscript{13} Value Maximization is a stronger assumption in this case, because the set of portfolios available to stockholders when the highest-valued decision alternative is chosen will not necessarily include the sets of portfolios that would be available to them under all other alternatives. There is no longer complete stockholder unanimity about the preferred decision, but in its absence value maximization seems at least an institutionally plausible policy.
project. Value Maximization will then direct us to accept the project if $V > I$,
reject the project if $V < I$.

The method does not determine a decision if $I$ lies between $V$ and $\bar{V}$.

It can readily be verified that if the project is within the span of the market (and there are no inconsistencies in observed prices) then $V = \bar{V}$, and the technique gives an exact valuation. Indeed, since it may be difficult to determine whether a project lies within the span of a large set of securities, it may be useful to start with problems (8) and (9); if the optimal solutions are the same, spanning of the project is verified.

When the project returns $Y$ are not spanned by the market, $Y$ cannot be accurately evaluated by the elementary prices $q_{it}$; even if a solution to equations (4) exists it will not be unique, and different solutions will generally induce different evaluations. If there are no inconsistencies in observed prices, then $V = \min \sum_{it} q_{it} Y_{it}$ and $\bar{V} = \max \sum_{it} q_{it} Y_{it}$ over all $\{q_{it}\}$ that solve (4). If inconsistencies exist, it may happen that $V > \bar{V}$. The decision is still clearcut when $I$ is either above or below both $V$ and $\bar{V}$, but otherwise approximations such as (7) must again be used.

Using bounds in the search for an acceptable investment plan is also straightforward. The equality constraints (P-a) in problems P or IP are simply replaced by the inequality constraints

$$\sum_{j=1}^{J} x^j z^j_{it} - \sum_{n=1}^{N} s^ny^N_{it} \leq 0 \quad t=1,\ldots,T \quad i=1,\ldots,k_t \quad (P-A)$$

and all other elements of the problem formulations are left unchanged. The problems then search for the most valuable
portfolio of existing securities which is "dominated" by a feasible investment plan. An investment plan which thus dominates the most valuable portfolio is then given by the solution for possible action. If its cost is less than the value of the dominated portfolio, the plan should be undertaken.

VII. EVALUATION WITH A SEQUENCE OF MARKETS.

If the model specifies many more events than currently traded securities, the bounds \( \underline{V} \) and \( \overline{V} \) generated by the solution of (8) and (9) for some projects may be not sufficiently tight to render a clearcut "accept" or "reject" decision. In this section, we suggest a method for incorporating into the evaluation procedure anticipations about future risk-free interest rates (which may be based on more than current price data). Rather than restrict the evaluation to comparisons involving only markets available to investors at the decision date, we can then also consider the markets which may exist in various circumstances at future dates. These considerations permit prospects to be represented by their discounted values of returns across states. In circumstances noted below, such a representation may permit a simpler and more exact indirect evaluation of prospects.

In what follows, we shall assume that markets for short-term default free money are available at all time points under
consideration. We assume further that the future risk-free one-period interest-rate depends only on the "event" which obtains at that time. The one-period risk-free interest-rate prevailing at time \( t \) if event \( e_{it} \) obtains is denoted by \( r(e_{it}) \). Recall that a "state of the world" was defined as a (possible) sequence of events through the horizon. Let \( E^m \) \( m=1, \ldots, M \) denote the set of possible states. If event \( e_{it} \) is the \( t \)th event in state \( E^m \) we say that \( e_{it}=E^m_t \), and \( r(E^m_t), r(E^m_2), \ldots \) is the sequence of future one-period risk-free interest-rates in state \( E^m \).

Consider now taking a prospect \( Y \), and at each time reinvesting the cash flows which occur in the risk-free asset. The resulting prospect will yield cash flows only at the terminal date. The amount of cash that the resulting prospect will finally yield will depend upon the entire sequence of events. That is, the resulting prospect will pay zero at all periods until the terminal date, and then an amount which depends upon \( E^m, m=1, \ldots, M \).

If, instead of receiving the prospect \( Y \), an investor receives the prospect yielding the above end-period future

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14 This is an assumption about the description of events adopted in the model, and not an assumption on interest rates or the environment. It simply states that two future contingencies which involve different short-term risk-free interest-rates must be specified as different "events" within the framework of the model. For a detailed and explicit model involving interest-rates contingent on future events, see Beja [3].
values, he could recreate $Y$ by borrowing at the riskless rate the amounts reinvested above.\footnote{This amount of riskless borrowing will be possible if the "end-period prospect" is pledged as collateral.} Since one prospect can thus be converted costlessly into the other, the investor will be indifferent as to which prospect he has. Thus, for purposes of choice, we need only focus on the pattern of future values across states. The "future value" $Y_T(E^m)$ in state $E^m$ is given by

$$Y_T(E^m) = \sum_{t=1}^{T} Y_{it}(m) (1+r(E^m_t))(1+r(E^m_{t+1})) \ldots (1+r(E^m_{T-1}))$$

where $it(m)$ is the pair that characterizes the event which prevails at time $t$ if the state of the world is $E^m$. Similarly, we can define a "discounted value" vector $Y_\theta$ by

$$Y_\theta(E^m) = \sum_{t=1}^{T} Y_{it}(m)/(1+r(E^m_\theta))(1+r(E^m_{T-1}))$$

Since $Y_\theta$ and $Y_T$ are directly related by

$$Y_T(E^m) = Y_\theta(E^m)(1+r(E^m_\theta))(1+r(E^m_T)) \ldots (1+r(E^m_{T-1}))$$

it is clear that all investors are indifferent between two prospects which have identical discounted value vectors.\footnote{Note, however, that while the complete state-dependent description of the discounted value is sufficient for the investors' evaluation of the project, the probability distribution of the discounted value (as indeed also the probability distribution of returns) is not sufficient to determine the valuation (cf. Beja [2]).} Any project therefore can be accurately represented by the discounted values across states.

If the project $Y$ is within the span of existing markets, the discounted value vector $Y_\theta$ can be reduced further to a
single market value, as there exist \( x^j, j = 1, \ldots, J \) such that
\[
y_0(E^m) = \sum_{j=1}^{J} x^j y^j_0(E^m) \quad m = 1, \ldots, M
\]
and thus \( w(y) = \sum_{j=1}^{J} x^j p^j \).

If \( y \) is outside the span of the market, we can again compute lower and upper bounds as in problems (8) and (9) by replacing the \( K \) constraints on event-contingent returns by \( M \) constraints on state-dependent discounted values. When \( M < K \), the dimensionality of the problem is reduced.

If the markets at time 0 are sufficiently diversified, at least some of the future interest rates may be implicitly defined by the system of time 0 prices. For example, let \( A(it) \) denote a collection of events at \( t+1 \) such that one of these events occurs if and only if event \( e_{it} \) occurs at \( t \), and let
\[
Q_{it} = \sum_{i' \in A(it)} q_{i', t+1}
\]
Suppose that in the solution of equations (4) for the elementary prices the values of \( q_{it} \) and \( Q_{it} \) for some event \( e_{it} \) are uniquely defined, though \( q_{i', t+1}, i' \in A(it) \) may or may not be unique. Then equilibrium conditions imply that
\[
1 + r(e_{it}) = q_{it}/Q_{it}
\]
(for otherwise there would exist opportunities for profitable arbitrage).

The major importance of evaluation by the discounted value vector lies in the ability to make some inferences about future interest rates even when their values are not fully implicit in the system of present prices. In this case, the bounds given by problems (8) and (9) with constraints on state-dependent discounted values may be considerably tighter than the corresponding bounds with constraints on event-contingent returns. Information about future interest-rates is then equivalent to knowing the market prices of more securities than are currently traded.
Indeed, an explicit value may sometimes be stated even for prospects which are not spanned by existing securities, because (10) may be solvable even when (3) is not. This is clearly not a direct evaluation, and would be inappropriate if the project returns were spanned by existing securities. But in the absence of sufficient data for direct evaluation, it may be advisable to use whatever general knowledge is available about the economically relevant properties of future events, to obtain an improved indirect evaluation. A constant risk-free interest-rate \( r \) is an obvious first approximation.\(^{17}\)

\[\text{VIII. FINANCING DECISIONS.}\]

The "valuation theorem" contains the basic "irrelevance of purely financial operations to the value of the firm" which is perhaps the cornerstone of modern financial theory.\(^{18}\) Furthermore, the direct evaluation approach and the explicit specification of the way returns depend on future events allows for explicit consideration of the distributional implications of financial decisions. It will be shown below.

\[^{17}\text{Assumptions on the values of future interest rates contingent on various possible future events may be much more acceptable than many of the assumptions underlying most of the currently popular evaluation models. In particular, note that in estimating the market parameters for CAPM, stationarity of the risk-free interest rate is almost invariably assumed.}\]

\[^{18}\text{Originating with the path-breaking papers of Modigliani and Miller on the structure of capital of the firm [13] and on dividend policy [12].}\]
that, even in the absence of taxes, financial policy is generally highly relevant for the values of the various securities that comprise the firm's financing.

Since "the irrelevance theorem" follows immediately from the direct evaluation principle that \( w(Y) = w(Y^A) + w(Y^B) \) whenever \( Y = Y^A + Y^B \), we shall first briefly review some of its more interesting implications (most of which are today well-known). Direct evaluation not only provides a concise exposition, but also helps clarify how general the irrelevance property really is. Indeed, much of the early controversy concerning some of these implications centered around unnecessarily restrictive assumptions involved in the original exposition. In what follows, we assume only the absence of taxes and reorganization costs (which are actually a special case of transactions costs). The valuation theorem then implies that the total value of the firm is insensitive to any of the following facets of corporate policy.

Corporate portfolio selection and diversification of corporate activity.

Since the value of the firm's portfolio or the contribution of any investment plan to the value of the firm is equal to the sum of the values of the components in the portfolio or investment plan, it follows immediately that the total value is independent of the relationship between the returns generated by the separate components. Merger is of course but a special case of this (it can always be viewed as an acquisition of one firm by the other, and thus as a "financial investment"). Therefore, the above also indicates that a non-synergistic merger cannot enhance the
overall value of the firms involved.\textsuperscript{19}

**Insurance of corporate activity.**

Suppose $Y$ is the prospect of future returns generated by an insurance policy with some specified coverage, and let $H$ denote the required premium (which may also extend through time). The insurance market is clearly part of the overall financial markets, and therefore if $Y$ is traded for $H$ it follows by definition that the market value of $Y$ is the market value of $H$. Insurance is thus no more than an exchange of two equally-valued prospects, with no net effect on the value of the firm. An immediate corollary of this is that the operations of insurance companies (as indeed of all financial intermediaries) cannot be meaningfully analyzed in the context of value maximization in a perfect market. The analysis of issues such as "reinsurance policy" or "balanced risk-portfolio" cannot be restricted to "risk-sharing" considerations, and must explicitly take into account such factors as taxes, transactions costs, information etc.\textsuperscript{20}

**Dividend policy - external vs. internal financing of investments.**

Suppose that the firm can raise in the capital markets the amount $I$ it needs for investment by issuing some new securities that will generate the prospect $H$ of future returns to the new security-holders. The amount $I$ raised in the market must by definition equal the value $w(H)$ of the promised returns $H$. It follows that the net contribution to the total value of the firm is $I - w(H) = 0$. The value of the firm with reinvestment of earnings must equal the value of the firm with (higher) initial dividend payments combined with external financing.\textsuperscript{21}

**The structure of capital of the firm.**

Suppose the firm's financing is composed of $N$ types of securities, and let $Z_j$ denote the prospect of future returns

\textsuperscript{19}On these issues, see Myers [15] and Levy and Sarnat [10].

\textsuperscript{20}The underlying reason is that investors can hedge their own holdings and are therefore not too sensitive to the firm's risks. They need not actually undertake privately an equivalent insurance, but can in fact attain the desired result by a proper diversification of their portfolios.

\textsuperscript{21}See Miller and Modigliani [12] for the original exposition of this important result.
on all securities of type j. Then

$$w(z^1)+w(z^2)+\ldots+w(z^N) = w(z^{1+}z^{2+}\ldots+z^{N+})$$

and as long as the prospect of overall returns $$z^{1+}\ldots+z^{N+}$$ is unchanged the overall value of the firm is independent of its capital structure.\(^{22}\)

The "irrelevance" of corporate financial policy may be easily misinterpreted. It is most tempting to conclude that in a value maximization context shareholders and management are indifferent between any two alternative financing decisions (at least in the absence of taxes). This interpretation is wrong, because alternative financing decisions may have different effects on the values of the different types of securities outstanding, and will thus have distributional implications. The interests of common shareholders may contradict the interests of bondholders, preferred shareholders etc. The existing institutional framework associates "ownership" (and voting) with common stocks. An institutionally consistent management policy that maximizes the value of the firm's currently outstanding common stock is not indifferent to the financing decision.

The point is best illustrated by an example, which for expository convenience will be extremely simplified. Suppose that the management of some firm can decide on a major expansion of the firm's activities, which will increase the firm's overall returns under all contingencies at all future

\(^{22}\)See Modigliani and Miller [13] for the original exposition.
time points by 20%, and wishes to determine how much at most would it be advisable to invest for this venture. The firm has outstanding one million shares of common stock currently selling for $20 per share, and $10 million pure discount bonds with one year to maturity. The current one-year risk-free interest rate is 6%, but the bonds' current yield is 11%, because it is realized that under certain circumstances the firm will be able to pay only 50 cents to the dollar on its currently outstanding liabilities. The total value of the firm is thus $29 million, and one may tend to conclude that management should be willing to pay up to $5.8 million for a 20% expansion, regardless of the method through which this expansion is financed. But the investment will change the value of the outstanding bonds, and the amount and direction of this change will depend on the way the investment is financed. The change in the value of common stock will be the change in the value of the firm less the change in the value of the bonds.

To evaluate the implications of the investment on currently outstanding securities, we summarize the relevant information implicit in market prices, and get a partial solution of equations (4). Let $Q_D$ denote the sum of $q_{i1}$'s for events $e_{i1}$ in which the firm defaults, and $Q_{D*}$ the corresponding sum when it does not. Then equations (4) reduce to
\[ 10Q_D + 5Q_D = 9 \]
\[ 1.06Q_D + 1.06Q_D = 1 \]

which is solved by

\[ Q_D^* = 0.856610 \]
\[ Q_D = 0.086780 \]

Suppose the firm issues $5 million more bonds of identical standing. Then even if the firm's probability of default does not increase, the value of the currently outstanding debt must fall, because in the event of default the firm will be able to pay only $6 million (120% of $5 million) on all bonds. The total value of the bonds will be

\[ 15 \times 856,610 + 6 \times 86,780 = 13,369,830 \]

Currently outstanding bonds will then be only a 10/15 fraction of the total, and their value must fall from $9 million to $8,913,220. Stockholders would be in this case willing to invest up to $5,886,780 to undertake the investment.

Now suppose that holders of the currently outstanding bonds are protected against a dilution of their claim by a covenant which prohibits the issue of any more debt with equal (or superior) standing. If the investment is undertaken, the value of these bonds will rise to

\[ 10 \times 856,610 + 6 \times 86,780 = 9,086,780 \]

and stockholders would be unwilling to invest if the initial cost exceeds $5,713,220.
Similar situations may exist with other securities (e.g. preferred stock, convertibles etc). The irrelevance of financial policy and the separation between investment and financing decisions is valid when the values of all high-priority securities remain unchanged (e.g. when default free bonds remain default free). The following general conclusion emerges: even in the absence of taxes, common stockholders prefer that investments be financed with securities of the highest possible priority. This tendency is often restricted by arrangements designed to protect the holders of high priority securities. But as a consequence of these arrangements, profitable investment opportunities may be forfeited. A careful statement of the separation between investment and financing decisions is the following. As long as the value of a project exceeds its initial cost, there is in principle a financing arrangement that would benefit the holders of some securities without hurting the holders of any other securities. If contractual (or legal) provisions preclude financing arrangements that benefit common shareholders (who have the controlling power), potentially profitable investments will be rejected. This will result not only in a loss to current shareholders, but also in an overall social loss.
REFERENCES.


