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Risk in Linear Risk
Tolerance Economies

by

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BETA AS A MEASURE OF RISK IN LINEAR RISK
TOLERANCE ECONOMIES

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I. Introduction and Summary

In this paper a computational approach is employed to examine several issues that may provide an explanation for some of the nonsupportive results found in recent empirical studies of the (mean variance) capital asset pricing model ((MV)CAPM). When the limited liability of financial assets is recognized the derivation of the (MV)CAPM cannot be based on the assumption of a joint normal probability distribution of security returns. In this case, if all investors are rational in the von Neumann-Morgenstern sense, the mean variance assumptions imply that all investors have quadratic tastes, which in turn implies that risky assets are inferior goods. However other simple two mutual fund CAPMs, consistent with discrete-time multiperiod decision making and in which risky assets are normal goods, can be derived from the power linear risk tolerance (LRT) utility functions, where the power is less than one and all individuals exhibit either constant or decreasing proportional risk aversion.

Given the less than full-fledged empirical support for some of the positive implications of the (MV)CAPM and the desirability of having a theory of capital asset pricing where risky assets are normal goods, we pose two closely related questions. First, what does the plot of points in expected return-beta space look like for different LRT economies and specific *ex ante* return distributions, in particular some constructed from real-world *ex post* realizations? This question (and the second) is posed on a theoretical level and attacked by numerical means. As a handy frame

of reference the reader may wish to view the analysis as a sequel to Sharpe (1970), chapter 5. In chapter 5, Sharpe analytically derives the linear expected return-beta tradeoff for the (MV)CAPM while in this paper we employ numerical means to determine the pattern of the expected return-beta plots for the broader class of power utility LRT CAPMs. Implicit in the question is whether the expected return-beta plots for hypothetical power LRT economies more closely resemble the plots found in the empirical studies of the MV CAPM than the straight-line relationship predicted by the MV model. Second, as beta is no longer the sole measure of risk in a power LRT model and a preference for positive skewness can be shown for investors with power tastes, can we better explain the structure of security prices in a power LRT economy with a three-parameter model (which includes a systematic skewness as well as a systematic covariance parameter)?

The paper is divided into four sections. In section II the major findings of the empirical tests of the (MV)CAPM are reviewed together with possible reasons that have been offered to explain the discrepancy between the theoretical predictions and the empirical findings. We focus on a suggestion by Hakansson (1974) who notes that because of the nature of investor tastes which must be assumed in order to derive the (MV)CAPM we might turn to a different CAPM based on more realistic investor tastes.

In section III the two-date power utility LRT models are formulated in a way that allows us to address the questions pertaining to the shape of the expected return-beta plot and the possibility of a three-parameter model providing a more accurate positive theory of security pricing. In

addition, a number of LRT valuation equations are reviewed as they provide a basis for part of the analysis and the interpretation of some of the results of this study.

In the fourth section the two questions are examined. We found that for a highly skewed two risky security example and various power utility economies the slope of the estimated security market line (SML) was negative. With *ex post* realizations as estimates of the *ex ante* return distribution the plots for the power utility LRT economies displayed the basic patterns found in the empirical studies--on average the plots were flatter than predicted by the (MV)CAPM but on occasion were too steep. Thus, the power utility LRT CAPMs may offer a more consistent explanation of the pricing of risky securities than does the explanation offered by the (MV)CAPM. Turning to the second question as to whether a three-parameter model may provide a more accurate description of security pricing in power LRT economies and abstracting from special cases we found that, in the majority of cases examined, the two-parameter model predicted the actual expected returns better than the three-parameter model.

II. Review of the Empirical Evidence

The disparity between theoretical predictions and empirical evidence. In the decade since the inception of the (MV)CAPM a number of empirical studies have been undertaken to test the positive implications of the theory. Recent empirical evidence has not given full-fledged support to the (MV)CAPM's explanation of the determination of risk premiums

on risky securities. Instead of finding that portfolios were plotting on an upward-sloping SML

$$(1) \quad E(r_j) = r + [E(r_M) - r]\beta_j$$

as predicted by the (MV)CAPM, a number of studies have reported that, on the average, low beta portfolios are earning higher average returns and high beta portfolios lower average returns than the mean variance theory predicts. In addition empirical estimates of the SML have shown that, at least in the period since 1940, on average the intercept is systematically greater than the riskfree rate. Even worse in some subperiods the slope of the SML has actually been found to be negative.¹

Possible explanations. Several attempts have been made to explain these results. Friend and Blume (1970) put forth several tenable reasons for the observed biases, which include the inability of investors to borrow large amounts of money at the same riskfree rate at which they can lend, and deficiencies in the return-generating models which are required to translate *ex ante* expected returns and "risks" into *ex post* realizations. Black, Jensen, and Scholes (1972) noting that the estimated intercept of the SML has on average been systematically above the riskfree rate, suggested that a two-factor (consisting of a market factor and a zero beta factor) model is generating security returns. Blume and Friend (1973) suggest that the assumption of a perfectly functioning short-sales mechanism required for the Black-Jensen-Scholes argument to hold may be an even more tenuous assumption than the assumption that investors can borrow or

lend unlimited amounts at a riskless rate of interest. They again consider the possibility of the inadequacies of the return-generating mechanism without reaching a definite conclusion and propose that the major difficulty may be a partially documented, but rationally inexplicable, segmentation of the markets for bonds and common stocks. On the other hand, Chen and Boness (1974) have suggested that the results may be caused by stating investor expectations of returns in nominal terms while completely ignoring uncertain inflation.

An alternative explanation. In this paper a different possibility is explored. Because of the nature of investor tastes that must be assumed in order to derive the (MV)CAPM it has been suggested that we might turn to a different CAPM based on more realistic investor tastes. As noted by Hakansson (1974) in the presence of limited liability² the (MV)CAPM is consistent with the expected utility theorem only if all investor preferences are quadratic, i.e., investor i 's utility function is of the form:

$$(2) \quad u_i(w) = b_i w - w^2 \quad \text{or} \quad - (a_i - w)^2,$$

where a_i is large.

If one makes the usual assumptions of homogeneous probability beliefs and unlimited borrowing and lending at the riskfree rate, other CAPMs can be derived for the considerably broader class of linear risk tolerance utility functions for which the separation property holds, namely,

$$(3) \quad u_i(w) = 1/\gamma(w + a_i)^\gamma, \quad \gamma < 1;$$

$$(4) \quad u_i(w) = - (a_i - w)^\gamma, \quad \gamma > 1 \quad \text{and} \quad a_i \text{ large};$$

$$(5) \quad u_i(w) = -\exp(a_i w), \quad a_i < 0;$$

where the parameters a_i are allowed to vary among investors to reflect differing degrees of risk aversion, but a common γ value must be shared by all investors.³

A CAPM based on any of the polynomial families of utility functions in equation (4) (where a family corresponds to a given γ value) will imply that risky assets are inferior goods, while a model based on any of the power families of utility functions contained in equation (3) will imply that risky assets are normal goods. Thus, a CAPM based on any of the power families in equation (3) would seem to be the more likely model. The complicating factor is: which family, or risk tolerance parameter γ , provides the "correct or best" model?

Because of the separation property exhibited by the LRT utility functions a linear (although not necessarily mean variance efficient) relationship continues to hold in expected return-standard deviation space for all the LRT CAPMs but the linear relationship in expected return-beta space holds only when all investors have quadratic tastes.⁴ Thus, for any of the other utility functions in the LRT class the plot in expected return-beta space produces a scatter which may contain negative or flat "relationships" such as have been found in the empirical studies cited earlier. This observation leads to the two questions posed in the introduction.

III. The Two-date Market Equilibrium Models

In this section two key ideas are developed that play an important role in the analysis that follows. First, the market equilibrium conditions are developed for the power utility LRT CAPMs and it is shown that the relative values of securities in the market portfolio can be found from a simple transformation of the solutions to just one expected utility problem. Second, a number of LRT valuation equations are reviewed that will aid in understanding and interpreting the results of this study.

A. The Market Equilibrium Conditions for Power Utility LRT CAPMs

For simplicity we focus on a two-date wealth accumulation model.⁵

The following notation is employed:

z_1^i the dollar amount lent by individual $i, i = 1, \dots, I$, at date 0, where negative values indicate borrowing.

z_j^i the dollar amount invested by individual i in security $j, j = 2, \dots, J$, at date 0, where negative values indicate short sales.

$w_0^i = \sum_{j=2}^J z_j^i + z_1^i$, the wealth of individual i at date 0.

r_{je} one plus the rate of return on risky security j if state of nature e occurs.

r one plus the rate of return on the riskless security.

$w_{1e}^i = \sum_{j=2}^J z_j^i r_{je} + z_1^i r$ or $\sum_{j=2}^J z_j^i (r_{je} - r) + w_0^i r$, the wealth of

individual i at date 1 if state of nature (event) e occurs,
 $e = 1, \dots, E$.

π_e the assumed homogeneously held probability belief that state of
 nature e will occur.

S_{j0} the aggregate present value of security j .

S_{j1e} the aggregate future value for security j if state of nature
 e occurs.

$u_i(\cdot)$ utility of wealth function of individual i .

Each individual chooses the dollar amounts to invest in the risky
 securities by maximizing his expected utility of future wealth

$$\max_{\{z_j^i\}} \sum_e \pi_e \left[u_i \left(\sum_{j=2}^J z_j^i (r_{je} - r) + w_0^i r \right) \right].$$

Consider the power utility functions in equation (3) and note that w_0^i and
 a_i are nonrandom. This means that they can be moved outside the expecta-
 tion operator to yield a new maximization problem equivalent to

$$(6) \quad \max_{\{v_j\}} \sum_e \pi_e \frac{1}{\gamma} \left(\sum_{j=2}^J v_j (r_{je} - r) + r \right)^\gamma, \quad \gamma < 1,$$

where

$$v_j = \frac{z_j^i}{w_0^i + \frac{a_i}{r}},$$

which holds for all investors in the economy.⁶ The optimal investment policies are determined by the set of equations:

$$(7) \quad z_j^{i*} = v_j^* \left(w_0^i + \frac{a_i}{r} \right) \quad \begin{array}{l} i = 1, \dots, I, \\ j = 2, \dots, J, \end{array}$$

and
$$z_1^{i*} = w_0^i - \left(w_0^i + \frac{a_i}{r} \right) \sum_{j=2}^J v_j^* \quad i = 1, \dots, I,$$

while the market clearing conditions

$$(8) \quad v_j^* \sum_{i=1}^I \left(w_0^i + \frac{a_i}{r} \right) = S_{j0} \quad j = 2, \dots, J,$$

$$\sum_{i=1}^I w_0^i - \sum_{i=1}^I \left(w_0^i + \frac{a_i}{r} \right) \sum_{j=2}^J v_j^* = S_{10},$$

close out the model.

In general the relative values of the risky securities will depend on the risk tolerance parameter γ and the individual parameters, the a_i 's. But by writing the individual decision problem in terms of the $\{v_j\}$ the relative values of the risky securities in different LRT economies may be calculated by solving (6) for the $\{v_j\}$ and then forming the set of

ratios,⁷ the $\left\{ \frac{v_k}{\sum_{j=2}^J v_j} \right\}$. Further, these ratios may be calculated without

knowledge of the distribution of initial wealth $\{w_0^i\}$, or individual taste

parameter $\{a_i\}$, or even the number of investors in the economy. And once these relative market values of the risky securities are known we may address the questions posed earlier.

B. The LRT Valuation Equations

Rubinstein (1974) has shown that whenever a composite individual can be constructed then in equilibrium a number of valuation equations obtain. The general LRT valuation equation is

$$(9) \quad E(r_j) = r + \lambda \kappa(r_j, -u'(w_1)) \sigma(r_j) \quad \text{for all } j$$

where

$$\lambda \equiv \frac{\text{Std}[u'(w_1)]}{E[j'(w_1)]} > 0,$$

and $\kappa(\cdot)$ indicates the correlation coefficient while $\text{Std}(\cdot)$ indicates standard deviation. Alternatively, this result can be stated in multi-parameter form

$$(10) \quad E(r_j) = r + \sum_{n=2}^{\infty} \lambda_n \sigma_n(r_j, r_M) \quad \text{for all } j$$

where M is the market portfolio of all securities

$$\lambda_n \equiv \frac{-u^{(n)}(w_0)^{n-1}}{(n-1)! E[u'(w_1)]} \quad \text{for all } n \geq 2$$

$u^{(n)}$ is the n^{th} derivative of u evaluated at $E(w_1)$,

w_0 is wealth after consumption and

$$\sigma_n(r_j, r_M) \equiv E[(r_j - E(r_j))(r_M - E(r_M))^{n-1}] \quad \text{for all } j \text{ and } n.$$

Clearly (9) and (10) provide a general statement of valuation for the LRT economies but (9) is stated in terms of unobservable variables and (10) leaves us with little idea of the importance of higher-order moments in explaining expected returns. Nor, more basically, is it apparent what economic meaning attaches to fourth- and higher-order moments. Thus, as a preference for positive skewness can be shown for those with power utility functions Kraus and Litzenberger (1972) have argued that we might better produce a positive theory of security pricing by assuming investor preferences are defined over the first three moments exclusively. The three-parameter valuation equation is given by⁸

$$(11) \quad E(r_j) = r + \lambda_1 \beta_j + \lambda_2 \theta_j \quad \text{for all } j$$

where

$$\beta_j \equiv \frac{\text{Cov}(r_j, r_M)}{\text{Var}(r_M)} \quad \text{is the systematic covariance of security } j;$$

$$\theta_j \equiv \frac{\text{Cov}(r_j, r_M)}{\text{Skewness}(r_M)} = \frac{E((r_j - E(r_j))(r_M - E(r_M))^2)}{\text{Skewness}(r_M)}$$

$$\lambda_1 = \left[\frac{E(r_M) - r}{\beta_z} \right] \quad \text{is the market price of beta reduction;}$$

$$\lambda_2 = \left[E(r_M) - r - \left(\frac{E(r_z) - r}{\beta_z} \right) \right] \quad \text{is the market price of systematic skewness;}$$

$E(r_z)$ = the expected return on the zero θ portfolio;

and

β_z = the β of the zero θ portfolio.

These valuation equations are extensively employed in conducting and interpreting the analysis that follows.

IV. An Examination of the Expected Return-beta Relationship and the Three-parameter Model

A. A Three-security Example⁹

Suppose that we have a series of LRT economies where individuals share homogeneous beliefs that the joint equilibrium distribution of security returns is as presented in table 1. Optimal solution values $\{v_2^*, v_3^*\}$ for the expected utility problem investors are assumed to solve are presented in table 2. Because the two risky securities are the only risky securities in the economy and all investors mix their portfolio of risky assets in the same proportions the market portfolio will be composed of a

weighted average, i.e., $\left\{ v_2^* / \sum_{j=2}^3 v_j^*, v_3^* / \sum_{j=2}^3 v_j^* \right\}$, of these two securities.

And as we would expect, the final two columns of table 2 show that the relative composition of the market portfolio changes for different economies. Knowing the relative composition of the market portfolio allows us to calculate the joint distribution of the returns on the two risky securities and the market portfolio, and hence we can calculate the actual moments of the market portfolio and the actual expected returns, systematic skewness of the two securities in each economy.¹⁰

TABLE 1
THE RETURN DISTRIBUTION

Security	Return	State		
		1	2	3
1	r	1.05	1.05	1.05
2	r_{2e}	1.50	0.00	1.50
3	r_{3e}	1.00	1.00	2.50
Probability	π_e	.80	.10	.10

$E(r_2) = 1.35$ $Var(r_2) = .2025$ $Skew(r_2) = -2.667$
 $E(r_3) = 1.15$ $Var(r_3) = .2025$ $Skew(r_3) = +2.667$
 $Cov(r_2, r_3) = .0225$

TABLE 2
SOLUTION VALUES AND THE COMPOSITION OF THE OPTIMAL PORTFOLIO

Power of Utility Function	Solution Values			Composition of Optimal Portfolio	
	v_2	v_3	$\sum_{j=2}^3 v_j$	$\frac{v_2}{\sum_{j=2}^3 v_j}$	$\frac{v_3}{\sum_{j=2}^3 v_j}$
2.0	--	--	---	.813	.187
.5	.811	1.689	2.500	.324	.676
.25	.693	1.808	2.501	.277	.723
.1	.640	1.490	2.130	.300	.700
.033	.62	1.32	1.94	.320	.680
0.	.607	1.26	1.867	.325	.675
-.1	.58	1.08	1.66	.349	.651
-1.0	.38	.46	.84	.452	.548

The expected return-beta plot. For any of the eight LRT economies considered here we can plot four points in expected return-beta space-- those of the riskfree security, the two risky securities, and the market portfolio of risky securities. These actual expected return-beta points are shown in the following figures as dots. Figure 1 presents the plot for quadratic utility; and as expected the securities plot exactly on an upward sloping SML.

Figure 2 contains the plot for the logarithmic utility economy.¹¹ There are two ways of viewing the diagram. First, we can fit a least squares regression through the plot of risky securities and determine whether the risky securities lie on a positively sloping line that passes through the riskless return. Notice that the risky assets fall exactly on this line.¹² However, the slope is negative, and the intercept far above the riskless return. The result should not be overly surprising as the LRT valuation equations (1), (9), (10), and (11) showed that β is only the correct measure of risk if investor tastes are quadratic. At the same time, it should cause some concern to those who, in a less formal manner, have argued that one if not the key result of the theory is the formal justification of a positive expected return-risk tradeoff for risk-averse investors. As this example clearly shows for a whole series of risk-averse investors, there can be a negative *ex ante* equilibrium expected return-risk (as measured by β) tradeoff for risky securities. We also might note that this negative relationship is consistent with some of the plots found in the empirical studies cited earlier. Second, we can draw a line through the two points determined by the riskless asset and the market portfolio

and determine where the risky securities lie relative to this relation (which incidently corresponds to the SML). Here we notice, again consistent with the empirical evidence, that the low (high) security earns a higher (lower) expected return than predicted by the (MV)CAPM.

A three-parameter model as the determinant of the structure of security prices. To review briefly, Kraus and Litzenberger assume investor preferences are defined only over the first three moments, derive a three-parameter valuation equation, and attempt to test it empirically as a positive theory of security pricing. We take a different tack and ask: given specific return distributions and hypothetical power utility LRT economies, where all investors act exactly as the theory postulates, does a three-parameter rather than a two-parameter model appreciably better explain the structure of security prices? And more specifically, in light of the multiparameter valuation equation (10), is a three-parameter model sufficient at some level of approximation to explain the structure of security prices? In this two-risky-security example the answer to both questions is in the affirmative. Predicted expected returns can be calculated for the two- and three-parameter models by first calculating betas, systematic skewnesses and their respective prices for the two risky securities in the various economies and then plugging these values into the respective valuation equations (1) and (11). For the two-parameter, or beta, model the expected return predictions can be read directly from the SML in the figures, i.e., for a given value of β read up to the SML to find the predicted expected return. For the three-parameter, or skewness, model the

predicted expected return is shown on the diagrams by a cross. Both predictions can be compared to the actual expected return depicted by a dot. As a means of facilitating a comparison of the predictive ability of the two models in more complicated examples we have calculated the ratio of the squared error in the prediction of expected returns by the skewness model to the squared error in the prediction of expected returns by the beta model. A ratio of zero indicates that the skewness model is essentially correct, while ratios greater than one indicate that the beta model is predicting expected returns more accurately. Figure 2 shows that the two-parameter model predicts (explains) the actual expected returns very poorly for logarithmic tastes while the three-parameter model predicts the actual expected returns exactly.

B. Examples Based on Ex Ante Return Distributions Estimated From Ex Post Data

Estimating the return distributions and calculating the optimal portfolio holdings. The joint return distributions were estimated in discrete nonparametric form. Essentially they are joint frequency distributions. Thus, if a distribution was constructed from E observations on a vector of $J-1$ risky securities (portfolios) each of the E observations on the $J-1$ component vector of security returns was assigned a probability estimate of occurrence of $1/E$.

A given return distribution together with the return on a riskless asset was taken to be an estimate of investor beliefs. The weights in the market portfolio, or alternatively the proportions in which investors held

the risky securities, were then computed by forming the set of ratios $\left\{ \frac{v_1^*}{\sum_{j=2}^J v_j^*} \right\}$ where the $\{v_j^*\}$ are the solutions to equation (6) derived by numerical means.¹³

The data. A primary concern in constructing the return distributions from *ex post* data is that it can be argued that these distributions may in some sense approximate real world *ex ante* beliefs. To provide robustness of this sort we varied the assumed length of the holding period, the time interval over which returns were measured and the method of grouping securities into portfolios.

The common stock data consisting of annual, quarterly, and monthly returns and annual market values were obtained from the CRSP and Compustat data bases. The sample period was constrained by the availability of market value data. As a result returns were measured over the 1946-71 period and market values from 1953 to 1971. The universe of stocks for which data were continuously available (ranging from 295 to 362 companies) were divided into five portfolios based on a beta ranking.

The annual return data from portfolios of bonds and stocks were taken from studies by Fisher and Weil (1971) and Fisher and Lorie (1968), respectively, and covered the period 1926 to 1965. The returns from treasury securities were taken to be the riskless rate of interest.¹⁴

As stated, a primary purpose of the paper is to determine the shape of the expected return-beta plot for power utility LRT economies. On one level the enquiry as to the shape is from a purely theoretic

perspective. However, from a slightly different point of view, we might wish to know if the power utility plots are more consistent with the plots found in the empirical studies than the straight-line trade off predicted by the (MV)CAPM. To gain insight into the question we present a subset of the plots actually analyzed. There are three basic types of plots. The power plots are constructed for square root, logarithmic, and the more risk averse -5 power utility economies. For comparative purposes two other types of plots are presented. First, are plots for a quadratic utility or MV economy, where the SML always obtains. And second, more in the spirit of comparison with the empirical studies, plots are presented where the market portfolio is taken to be the weighted average of the actual market values of the portfolios analyzed (whether or not these actual market value weights matched any expected utility predicted weights).

As a final observation on the general nature of the plots note that data depicting shifts in the return distribution and the weights used in calculating the market portfolio are shown in the top corner of the figures and that some of the calculated expected utility weights are negative. Securities with negative weights are "over priced" as everyone in the economy would wish to sell them short. Thus, we do not have an equilibrium solution for the specified tastes and beliefs. Nonetheless, it is still interesting to ask what the plots look like. After all, in the empirical studies the surrogate used for the market portfolio, whether it be Fisher's, the Standard and Poor's, or the Dow Jones Index, is not necessarily a positive weighted average of the securities used in the study. However positive weights are certainly a desirable feature. Therefore, a number of examples,

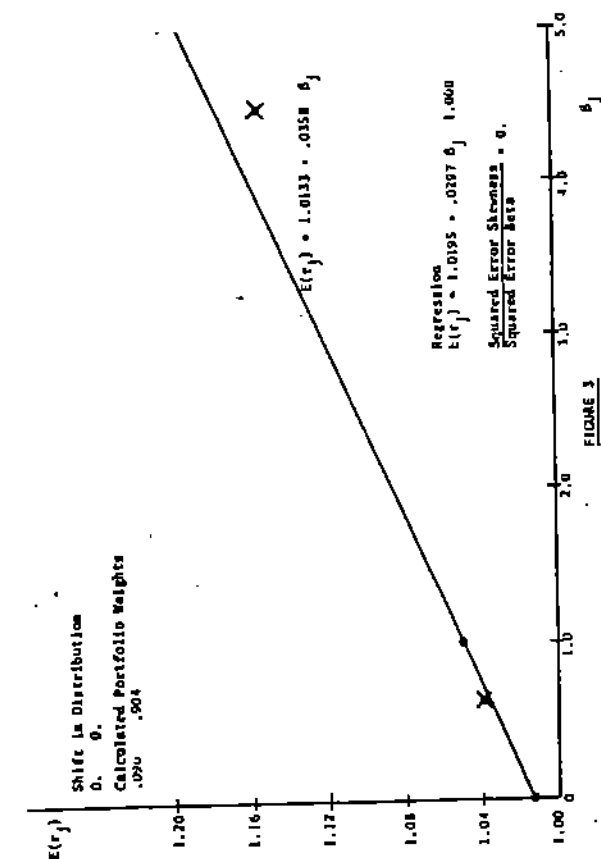
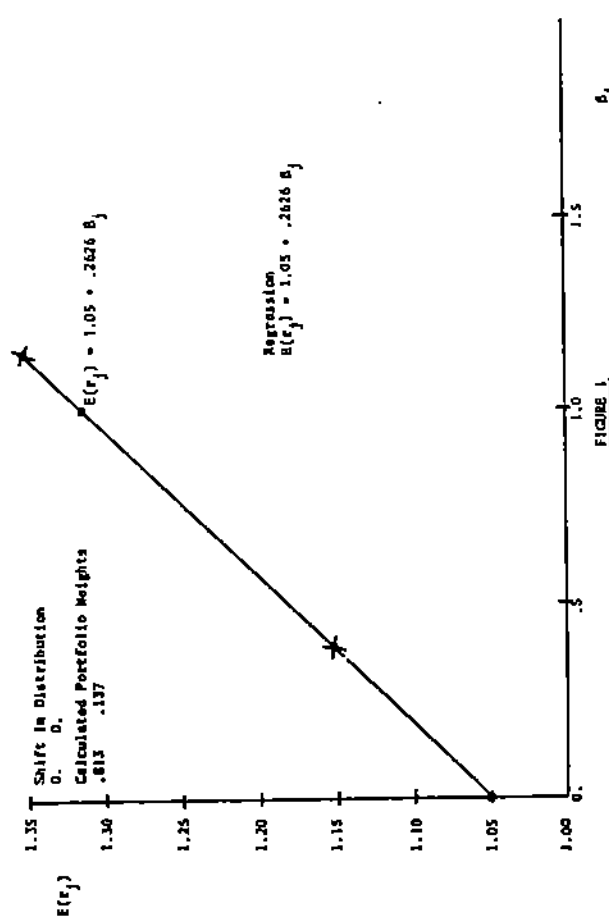
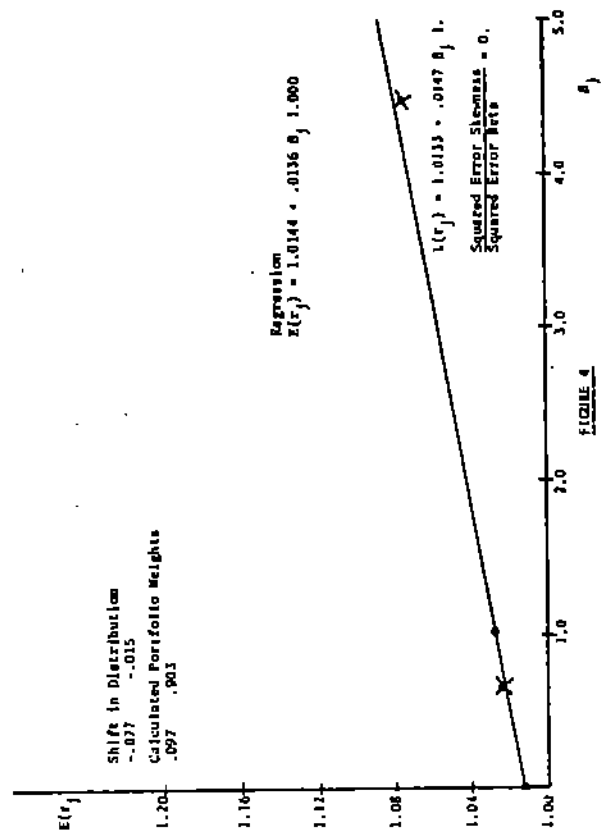
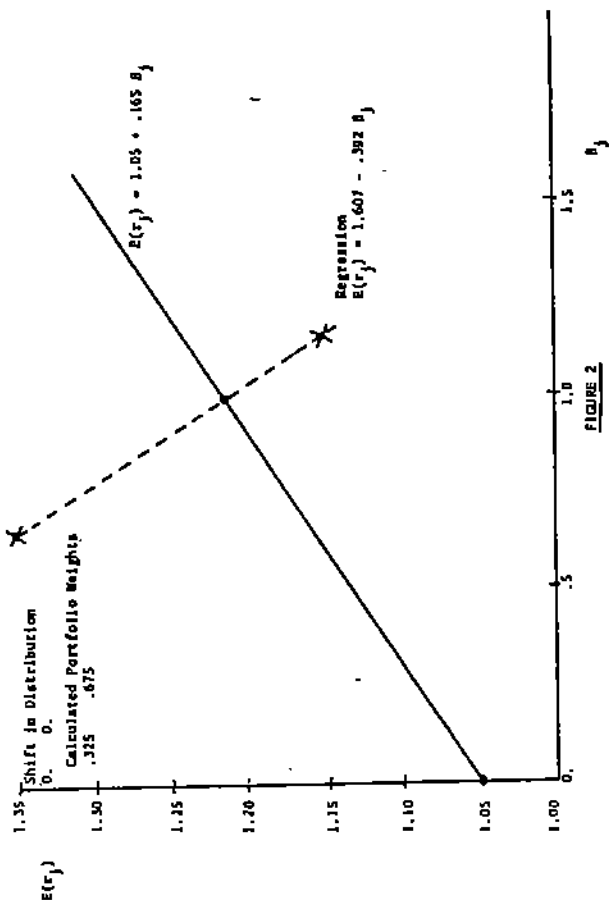


FIGURE 1
Bond Stock Portfolios Annual Data Distribution 1926-1965
1954 Riskfree Return
Square Root Utility

FIGURE 2
Bond Stock Portfolios Annual Data Distribution 1926-1965
1954 Riskfree Return
Square Root Utility

FIGURE 3
Bond Stock Portfolios Annual Data Distribution 1926-1965
1954 Riskfree Return
Square Root Utility

FIGURE 4
Bond Stock Portfolios Annual Data Distribution 1926-1965
1954 Riskfree Return
Square Root Utility

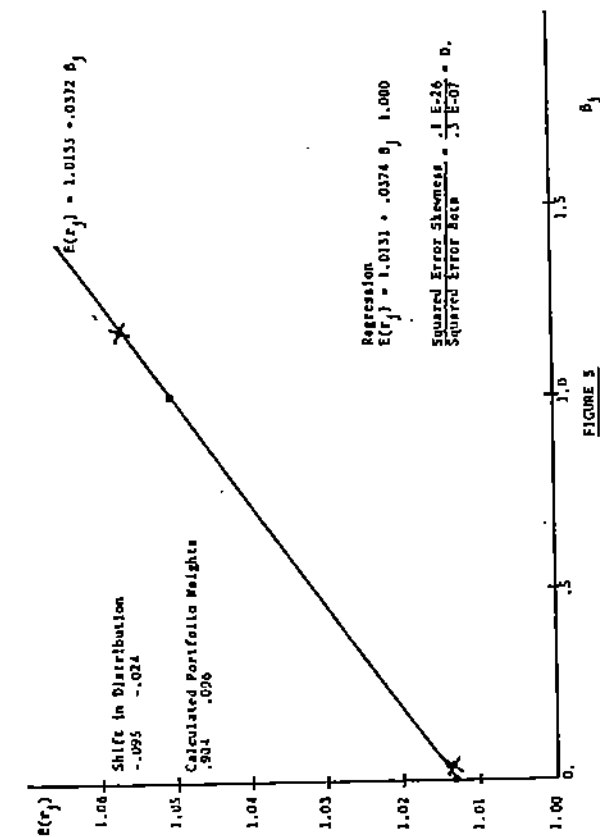


FIGURE 5
Bond Stock Portfolios Data Annual Distribution 1926-1965
1954 Risk-free Return Square Root Utility

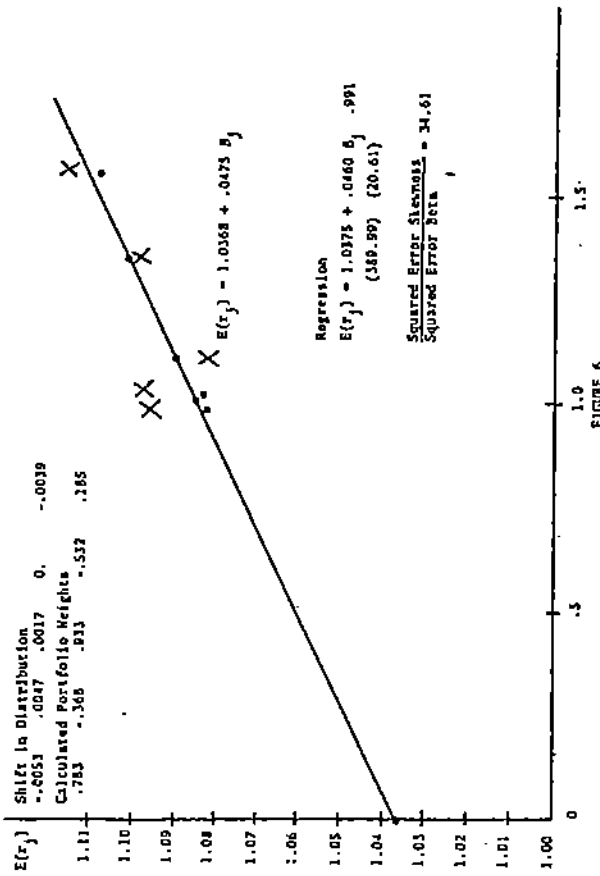


FIGURE 6
Five Beta Portfolios Annual Data Distribution 1946-1971
1964 Risk-free Return Logarithmic Utility

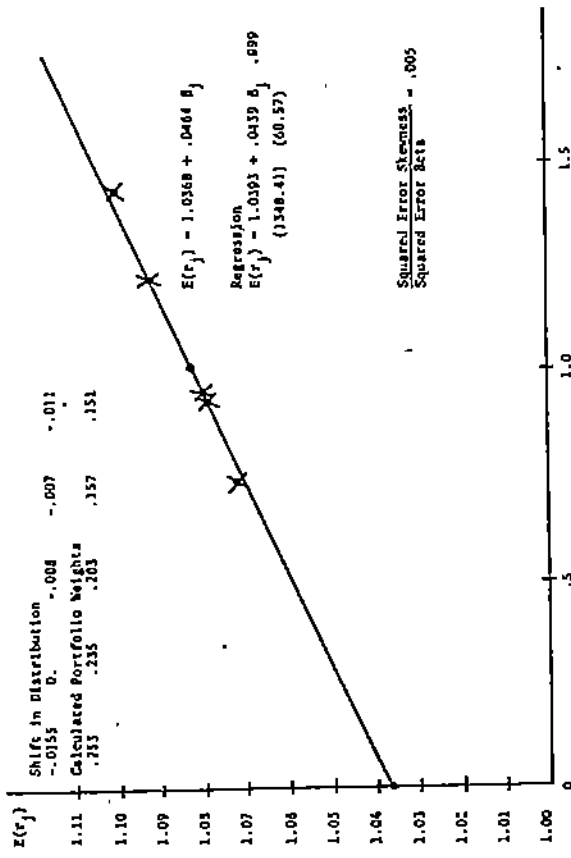


FIGURE 7
Five Beta Portfolios Annual Data Distribution 1946-1971
1964 Risk-free Return Logarithmic Utility

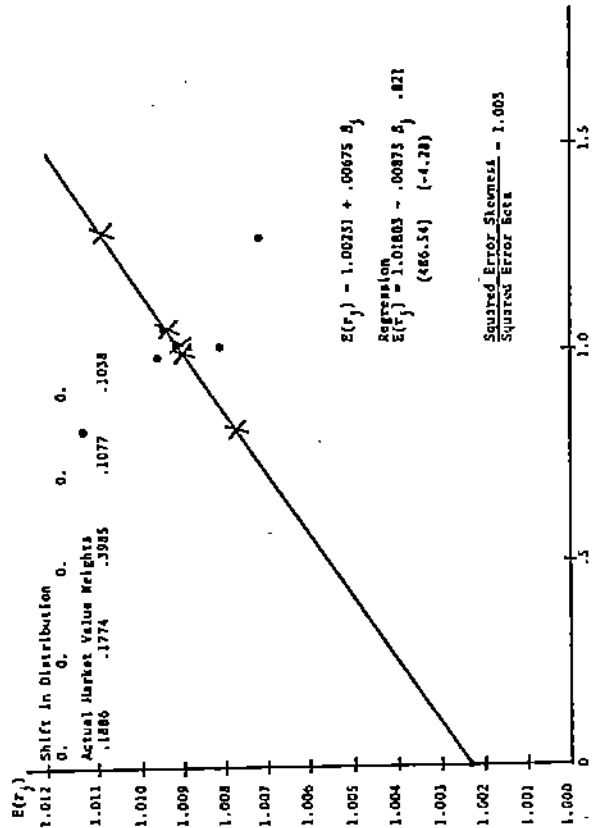


FIGURE 8
Five Beta Portfolios Monthly Data Distribution 1959-1963
1959 Risk-free Return Logarithmic Utility

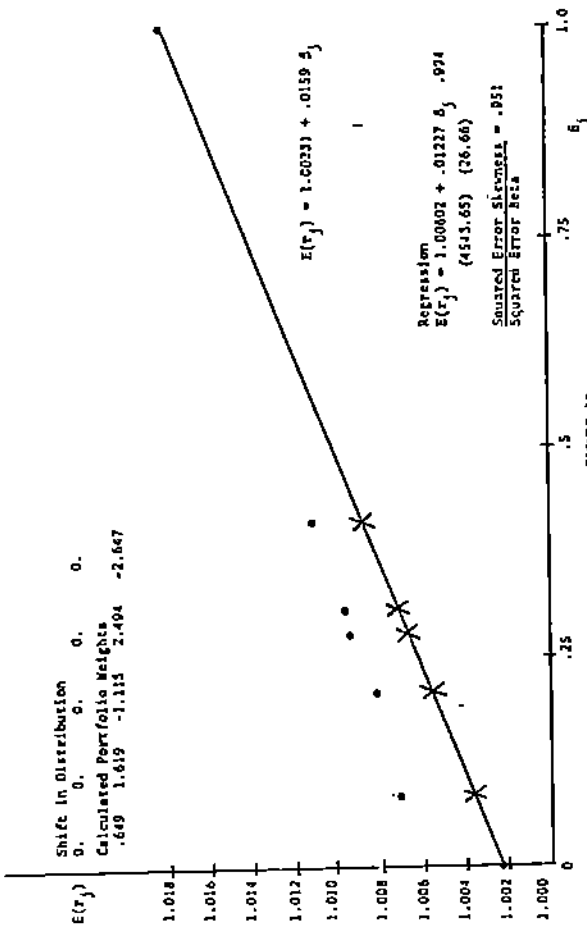


FIGURE 9
Five Beta Portfolios Monthly Data Distribution 1959-1963
1959 Riskfree Rate

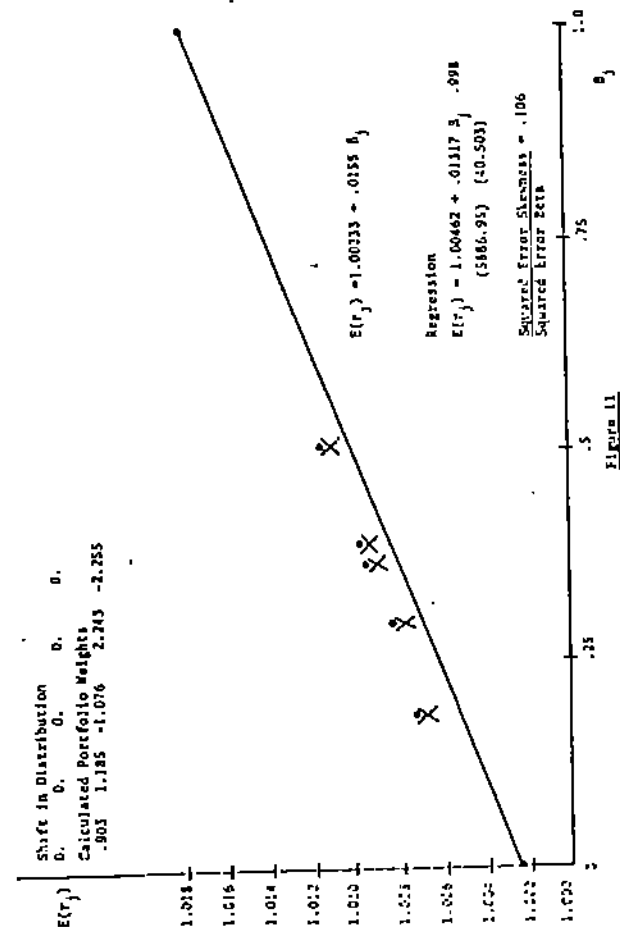


FIGURE 10
Five Beta Portfolios Monthly Data Distribution 1959-1963
1959 Riskfree Rate

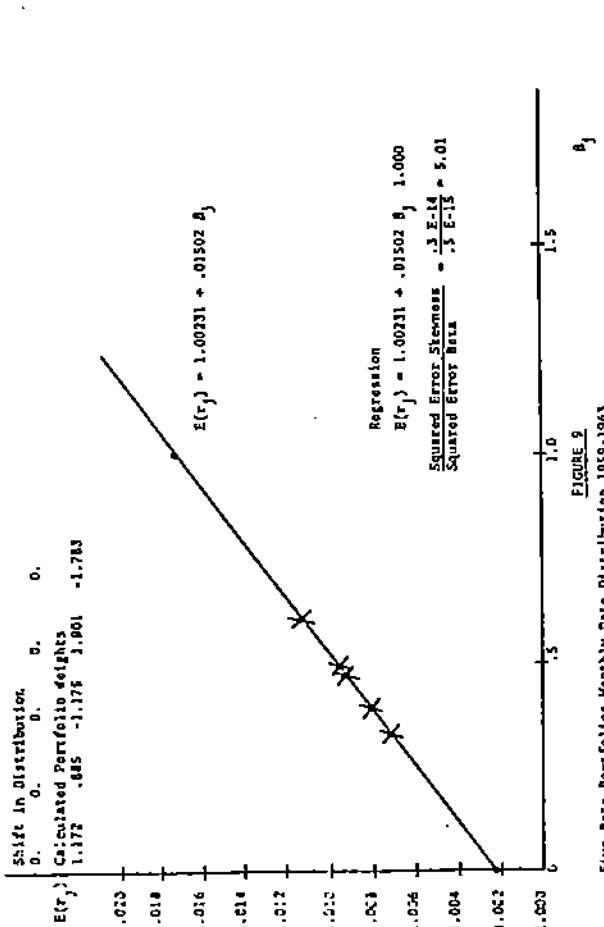


FIGURE 11
Five Beta Portfolios Monthly Data Distribution 1959-1963
1959 Riskfree Rate

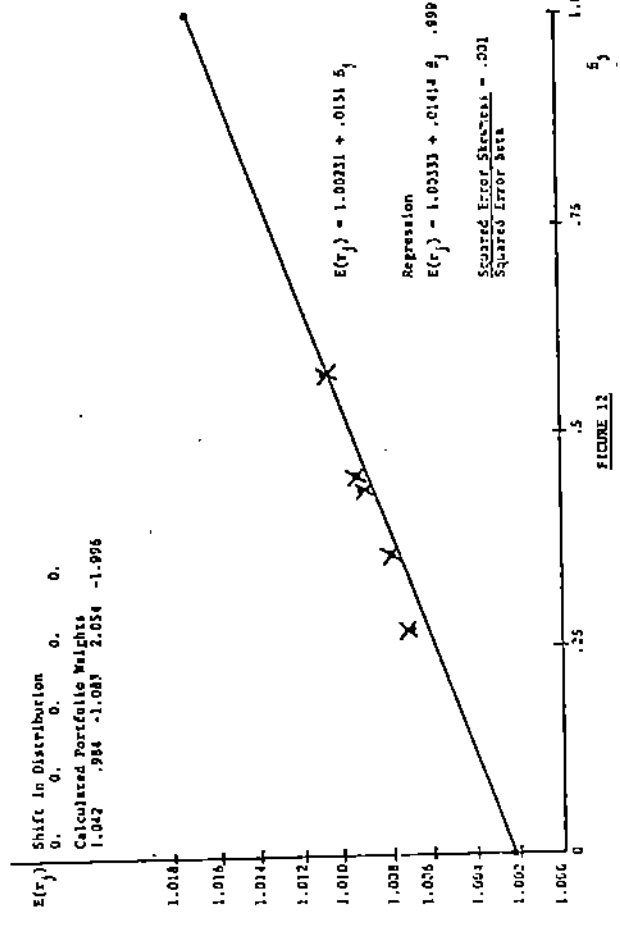


FIGURE 12
Five Beta Portfolios Monthly Data Distribution 1959-1963
1959 Riskfree Rate

where negative weights were calculated using a raw return distribution, were resolved with the means of the distribution shifted so as to obtain positive portfolio weights. (Actually the returns in each state of nature were shifted by a stated amount. This method of shifting the return distribution has the effect of shifting the means while leaving the covariance structure unaffected.)

Observations. With respect to the expected return-beta relationship the following points stand out. For quadratic economies the plot is exactly linear whether or not the calculated portfolio weights are all positive. For the power utility LRT economies the scatter of points appears to be fairly linear and generally becomes more so when the portfolio weights are positive. But consistent with the empirical evidence the slope of the regression line is on average too flat and the intercept greater than the riskless return while on occasion just the opposite result occurs.¹⁵ There is only one instance of a negatively sloping relationship. This result occurred when the market portfolio was calculated from actual market value weights and incidentally provided the only instance where the slope of the regression equation was not statistically significant.

Turning to the predictions of expected returns by the beta and skewness models note first that the skewness predictions were perfect for the bond stock portfolios. However, for the five stock portfolio cases the predictions of the skewness model become very erratic. In some instances the predictions are accurate but on other occasions they are

extremely inaccurate compared to the beta model. While the complete results are not reported in the paper, the skewness model predicted expected returns better than the beta model in only eight of twenty-two cases examined and in only seven of seventeen cases for the power utility models.¹⁶

C. Conclusions

The expected return-beta plots. Overall, the calculated expected return-beta plots for the power utility LRT economies display the same type of patterns as the empirical studies found--on average low (high) β securities are plotting above (below) the security market line predicted by the (MV)CAPM. Or alternatively stated, on average the slope of a regression line fitted through the plots was flatter than predicted by the (MV)CAPM. Therefore, it appears that a power utility LRT CAPM may offer a more consistent explanation of capital asset pricing than does the explanation offered by the (MV)CAPM.¹⁷

A three-parameter model as the determinant of the structure of security prices. From (11) it is apparent that for nonspecial cases we would have to know all the joint moments of the individual security returns with the market return as well as the prices of the joint moments in order to accurately predict expected returns. From an abstract theoretical point of view this is a most important result, but from an operational point of view, we desire simpler valuation equations. As the simplest (MV)CAPM valuation equation has been found to be somewhat deficient in explaining expected returns, the question becomes: Can we more accurately predict (explain) expected returns with a three- rather than a two-parameter model?

We found that for an economy where there are only two risky securities the three-parameter model predicts (explains) expected returns exactly, whether the two-parameter model predicts them very well or very poorly. However, this turns out to be a special case. The reason becomes apparent when we substitute into equation (11)

$$E(r_2) = r + \lambda_1 \beta_2 + \lambda_2 \theta_2,$$

$$E(r_3) = r + \lambda_1 \beta_3 + \lambda_2 \theta_3.$$

Here we have a system of two equations in two unknowns λ_1 and λ_2 which has a unique solution because the two underlying risky securities were linearly independent. The result clearly generalizes so that any time we have an n risky security world, an $n + 1$ -parameter model can explain the structure of security prices (assuming, of course, that $n + 1$ parameters exist).

Moving away from these special cases, we found the results turned mixed--sometimes the three-parameter model predicted (explained) expected returns appreciably better than the two-parameter model, but in the majority of cases examined the reverse held. This result might seem surprising at first in that (i) we might expect that adding a third parameter to a two-parameter valuation equation would lead either to an equally as good or an improved explanation of expected returns,¹⁸ (ii) *ceteris paribus* a preference for positive skewness can be shown for power tastes, and (iii) Kraus and Litzenberger have presented empirical evidence that is consistent with

a three-parameter valuation model as a positive theory of security pricing. However, one way to see how the explanation of expected returns can actually become worse with a three-parameter model is to note that when we add the coskewness parameter we have not simply added an extra term to the two-parameter model but have also changed the price of beta as is evidenced by a comparison of the two valuation equations (1) and (11). A second explanation is that while more than beta is needed to explain expected returns on average the addition of a systematic skewness parameter more than offsets the effect of the higher order of moments. Finally, a third explanation may be tied to the fact that a three-parameter model based on a restriction of investor tastes is theoretically correct only if all investors have LRT cubic utility.¹⁹ Thus equation (1) holds exactly if $\gamma = 2$ but in moving to (11) to approximate any of the power LRT economies, where $\gamma < 1$, we have in fact exactly described an LRT economy where $\gamma = 3$.

In sum then within a single period framework the power utility LRT CAPMs produce an essentially linear expected return-beta plot, where low beta securities, on average, earn a higher return than predicted by the (MV)CAPM. As the power utility LRT CAPMs, or normal goods models, are more consistent with the empirical evidence it is suggested that they may offer a better explanation of security pricing than the explanation offered by the (MV)CAPM. Given that the explanation of security pricing offered by the (MV)CAPM is too simple for power LRT economies it was to be hoped that a skewness model, the next simplest equation, would offer a better approximation of the structure of security prices. However, in the majority of the examples examined we found that the beta model proved to be superior.

FOOTNOTES

¹See Friend and Blume (1970); Black, Jensen, and Scholes in Jensen, ed. (1972); Blume and Friend (1973); and Fama and MacBeth (1973).

²Limited liability effectively rules out the possibility that security returns follow a joint multivariate normal probability distribution-- the alternate assumption used to justify the (MV)CAPM.

³Equation (2) is a special case of equation (4), where $\gamma = 2$, and equation (3) reduces to $\log(w + a_i)$ when $\gamma = 0$. An interesting interpretation of a_i is that it may reflect a subsistence level of income (wealth). A large negative a_i would indicate that individual i is unwilling, be it for physiological or psychological reasons, to allow his date 1 wealth to drop below some subsistence level.

⁴It is emphasized that we are working with a discrete-time model. If we assume homogeneous beliefs, continuous-time decision making by investors, and a stochastic process of the Itô variety, an (MV)CAPM also obtains. See, for example, Merton (1973).

The SML would also hold in discrete time if investors share homogeneous, normal probability assessments and have utility functions (at the minimum) defined on the whole line. However, we have noted that limited liability would rule out the possibility of normal probability assessments. In addition, the empirical evidence indicates that security returns can be more accurately described by a lognormal distribution. See, for example, Fama (1965), or Rosenberg and Marathe (1974). In light of this evidence it is mildly surprising to see the stable Paretian or normal probability defense of the (MV)CAPM.

⁵As already mentioned, in this case the power utility functions should be of primary interest as they imply that risky assets are normal goods. Parenthetically, we note that, although a formal discussion of multiperiod models is beyond the scope of this paper, an examination of these models reinforces the idea that power tastes are of primary interest. The LRT CAPMs are consistent with individual discrete-time multiperiod decision making only if the class of LRT utility functions is restricted to the power functions in (3) where $a_i \leq 0$ for all i . This is equivalent to saying that to derive an LRT CAPM we must have an economy populated by individuals who all exhibit either constant ($a_i = 0$) or decreasing ($a_i < 0$) proportional risk aversion where the coefficients of absolute and

relative risk aversion are defined to be respectively $R_a = \frac{-u''(w)}{u'(w)}$ and $R_p = \frac{-wu''(w)}{u'(w)}$. See, for example, Hakansson (1974) or Grauer (1975).

⁶The v_j are independent of individual information and hence not superscripted. To see this solve the $I \times J-1$ first-order conditions to the individual decision problems and note that $\{v_j^i\}$ $i = 1, \dots, I$; $j = 1, \dots, J$, are invariant across individuals. (This in fact is one statement of the separation theorem.) Furthermore, Hakansson (1970) has shown that under a plausible set of assumptions there is a unique optimal solution to (6).

⁷From the market clearing conditions it can be easily shown that this set of ratios corresponds to the relative values of the risky securities.

⁸In general, equation (11) is an approximation of the "true" multi-parameter valuation equation (10). However, the three-parameter valuation equation (11) will be an "exact" valuation equation in an economy populated by individuals with linear risk tolerance cubic utility functions. Likewise the general equations reduce to (1) if investors have quadratic utility (or make normal probability assessments). For details see Rubinstein (1973).

⁹The return distribution and the solution values $\{v_2^*, v_3^*\}$ were taken from an example presented in Hakansson (1971).

¹⁰In any state the return on the market portfolio is simply a weighted average (where $\left\{ \frac{v_2}{\sum_{j=2} v_j}, \frac{v_3}{\sum_{j=2} v_j} \right\}$ are the weights) of the returns on the two risky securities in that state.

¹¹The plot will be similar for the six other power utility economies.

¹²Actually they have to in this two risky security case as the market portfolio is a weighted average of the two risky securities.

¹³A number of algorithms were employed in solving and cross-checking the examples. However, the majority of computations were performed using the University of California's (Berkeley) version of Best's Feasible Conjugate Direction Algorithm. See Best (1975).

¹⁴A much fuller description of the data and computational procedure is to be found in Grauer (1975).

¹⁵This last finding is not inconsistent with the empirical evidence. See, for example, the 1932-1933 plot recorded in Black, Jensen, and Scholes (1972), p. 101.

¹⁶The paper was revised when it was discovered that a computer error resulted in the five stock market portfolio being ranked by $\frac{\text{cov}(r_j, |r_M|)}{\sigma^2(|r_M|)}$ instead of $\beta \equiv \frac{\text{cov}(r_j, r_M)}{\sigma^2(r_M)}$. While the plots were similar

in both cases the expected return predictions were somewhat different. In particular before the revision the skewness model predicted expected returns better than the beta model in only four of fifteen cases examined and in only three of twelve cases for the power utility models. In addition the original figure 8, based on 1959 market values, exhibited a squared error skewness to squared error beta ratio of 827.68.

¹⁷However, we should note that while the scatter of points for the power utility economies is not the linear relationship predicted by the (MV)CAPM it may be sufficiently close to make it difficult to tell the various models apart empirically. In fact, Roll (1973) attempted to discriminate between the (MV)CAPM and a "growth optimal" model (a special case of equation (3) where $\gamma = 0$ and $a_i = 0$ for all i) and based on his results concluded that the two models are empirically identical.

¹⁸A note of caution regarding methodology may be in order. Our procedure was to explain (or predict) expected returns employing the two valuation equations (1) and (11) in hypothetical (but hopefully realistic) economies where individuals made their investment decisions in strict accordance with theory. We did not employ regression analysis to address the question of whether the three-parameter model appreciably better explained expected returns. Had we done so we could not help but have done at least as well in explaining expected returns with the three-parameter model.

¹⁹See footnote 8.

REFERENCES

- Best, Michael J. "A Feasible Conjugate Direction Method to Solve Linearly Constrained Optimization Problems." *Journal of Optimization Theory and Applications* 16 (July 1975): 25-38.
- Black, Fischer; Jensen, Michael C.; and Scholes, Myron. "The Capital Asset Pricing Model: Some Empirical Tests." In *Studies in the Theory of Capital Markets*. Edited by M. Jensen. New York: Praeger, 1972.
- Blume, Marshall E., and Friend, Irwin. "A New Look at the Capital Asset Pricing Model." *Journal of Finance* 28 (March 1973): 19-34.
- Chen, Andrew, and Boness, James. "Effects of Uncertain Inflation on the Investment and Financing Decisions of a Firm." Presented at the Annual Meetings of the American Finance Association, December 1974.
- Fama, Eugene F., and MacBeth, James D. "Risk, Return, and Equilibrium: Empirical Tests." *Journal of Political Economy* 81 (May 1973): 607-636.
- Fisher, Lawrence, and Lorie, James H. "Rates of Return on Investments in Common Stock: The Year-by-Year Record, 1926-65." *Journal of Business* 41 (July 1968): 291-316.
- Fisher, Lawrence, and Weil, Roman L. "Coping with the Risk of Interest Rate Fluctuations: Returns to Bondholders from Naive and Optimal Strategies." *Journal of Business* 44 (October 1971): 408-431.
- Friend, Irwin, and Blume, Marshall. "Measurement of Portfolio Performance Under Uncertainty." *American Economic Review* 60 (September 1970): 561-575.
- Grauer, Robert R. "Risk Aversion and the Structure of Asset Prices: An Examination of Linear Risk Tolerance Capital Asset Pricing Models." Ph.D. dissertation, University of California, Berkeley, May 1975.
- Hakansson, Nils. "Optimal Investment and Consumption Strategies Under Risk for a Class of Utility Functions." *Econometrica*, September 1970, pp. 587-607.
- _____. "Multiperiod Mean-Variance Analysis: Toward a General Theory of Portfolio Choice." *Journal of Finance* 26 (September 1971): 857-884.
- _____. "The Capital Asset Pricing Model: Some Open and Closed Ends." Finance Working Paper No. 22, Institute of Business and Economic

Research, University of California, Berkeley, June 1974. Forthcoming in *Risk and Return in Finance*. Edited by Irwin Friend and James Bicksler. Cambridge, Mass.: Ballinger.

Kraus, Alan, and Litzenberger, Robert. "Skewness Preference and the Valuation of Risky Assets." Research Paper No. 130, Graduate School of Business, Stanford University, December 1972.

Merton, Robert. "An Intertemporal Capital Asset Pricing Model." *Econometrica* 41 (September 1973): 867-887.

Roll, Richard. "Evidence on the Growth Optimum Model." *Journal of Finance* 28 (June 1973): 551-566.

Rosenberg, Barr, and Marathe, Vinay. "Testing the Capital Asset Pricing Model: A Robust, Powerful and Easily Generalized Procedure." Presented at the Stanford-University of California Finance Colloquium, December 1974.

Rubinstein, Mark. "The Fundamental Theorem of Parameter-Preference Security Valuation." *Journal of Financial and Quantitative Analysis* 8 (January 1973): 61-69.

_____. "An Aggregation Theorem for Securities Markets." *Journal of Financial Economics*, September 1974, pp. 225-244.

Sharpe, William F. *Portfolio Analysis and Capital Markets*. New York: McGraw-Hill Book Co., 1970.