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WORKING PAPER NO. 50

A General Theory of Asset Valuation Under Diffusion State Processes

by

Mark B. Garman

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A GENERAL THEORY OF ASSET VALUATION
UNDER DIFFUSION STATE PROCESSES

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0. ABSTRACT

New foundations for contingent-claim valuation models are formulated in this paper. An absolute pricing theory is derived from extremely general partial equilibrium conditions without resort to the usual utility-maximization paradigm. These conditions provide for the absence of riskless arbitrage opportunities in continuous markets and deduce a fundamental principle, that of "intertemporal parity." The principle implies a central result: there is a single partial differential equation which must be satisfied by every existing marketable asset in a world governed by diffusion state processes. Our approach unifies many of the existing absolute valuation models with the modern relative pricing models, i.e., the Black-Scholes/Merton partial differential equation analysis.
1. INTRODUCTION

In this paper we formulate some new foundations for modern contingent-claim valuation models. The purpose of this introductory section is first, to develop a perspective on existing valuation models and second, to describe how the current paper fits into that perspective by summarizing the essential features of its content.

Generally speaking, modern contingent-claim valuation models may be divided into two classes. The first class includes what we might call the absolute valuation models, those that attempt to characterize asset prices (or alternatively, returns) directly in terms of underlying consumption units. The second class might be termed the relative valuation models, those that characterize asset prices solely in terms of other asset prices.

Almost all of the absolute valuation models developed in the modern finance literature have rested upon a foundation of expected utility maximization. Unfortunately, it has proved extremely difficult to derive from these models implications that are simultaneously interesting and general. In order to create interesting results, it typically has been necessary to make great concessions to generality, principally in one of two modes: (1) the imposition of restrictions on admissible beliefs, and (2) the imposition of restrictions on admissible utility functions ("tastes"). Among the works which might be included in the first category are those of Markowitz [1959], Sharpe [1964], Lintner [1965], Mossin [1966], Samuelson and Merton [1969], Fama [1970a], Jensen [1972], Black [1972], Merton [1973b], and Long [1974], to name but a few. The second
category would include at least the papers of Arrow [1964], Wilson [1968], Cass and Stiglitz [1970], Hakansson [1971], Kraus and Litzenberger [1975], and certainly several others. Yet, even after making the seemingly necessary restrictive concessions regarding beliefs and/or tastes, there have remained immense difficulties surrounding the meaningful integration of heterogeneous tastes and beliefs into a complete description of an economy, i.e., the "aggregation problem" (Rubinstein [1974], Brennan and Kraus [1975]). The general aggregation problem cannot now be said to be solved, and we are led to suspect that a general solution, if it exists at all, would be so cumbersome as to be analytically useless.

The second general class of models, the relative valuation models, attempt to avoid the difficulties of aggregation and the restrictive assumptions placed upon tastes or beliefs by avoiding expected utility maximization altogether. Instead, models in this class concentrate upon the "arbitrage" or "dominance" relationships which may be stated independently of tastes and/or beliefs. Thus generality is preserved, but not without considerable loss of power. The primary cost that historically has been paid is that such models have been able simply to compare the prices (alternatively, returns) of two or more assets. In the absence of belief assumptions, these comparisons have tended to be rather weak, resulting typically in price inequalities between very closely related assets (e.g., Stoll [1969], Merton [1973a], and in another context, by Modigliani and Miller [1958] and Stiglitz [1969]). When some belief assumptions are additionally included, stronger relative pricing statements can be made. In particular, a breakthrough was achieved in the Black-Scholes [1973] option pricing paper where it was shown that a
specific (geometric Brownian motion) restriction on beliefs leads to a powerful relative pricing theory: in effect, any two assets which both depend on the same diffusion random variable are "locally" (i.e., in an infinitesimal period of time) equivalent to each other, a fact which in turn must establish their relative prices. This theory was later developed and extended by Merton [1973a] and others, and will be termed herein the Black-Scholes/Merton (BS/M) approach. The essential limitation of the approach is that since it considers only a few assets in isolation from the remainder of the economy, it can provide at most the relative pricing relationships among those assets, whereas absolute pricing necessarily requires a more global equilibrium viewpoint.

The current paper shows that by applying arbitrage/dominance considerations as a partial equilibrium condition of an entire economy (i.e., across all existing securities simultaneously) it is possible to derive an absolute pricing theory. No expected utility maximization is required, and no aggregation problem need be faced to achieve this. In a sense, the present paper thus almost fits into a third class of valuation models (absolute pricing without utility maximization), within which Ross [1973] and Treynor [1961] have made some previous attempts in this general direction.

Our development of the paper follows this general outline. First, we consider all possible arbitrage portfolios that can be formed as linear combinations of existing assets. Because of new advances in the problem-attack technology for this area (Ross [1973, p. 15], Cox and Ross [1976, fn. 2], Garman [1976, Theorems 1, 2]) it is now possible to characterize all possible arbitrage possibilities simultaneously, whereas much previous literature has dealt with each arbitrage portfolio on an individual basis.
(e.g., Stoll [1969], Merton [1973a], Black and Scholes [1973]). An essential element of our approach is the Parkas-Minkowski Lemma, which applies in the following fashion: if we prohibit any linear combination of assets from producing an arbitrage profit, the Lemma then asserts that there must exist a function K representing a solution to a dual problem which turns out to be formulated in "valuation" space. It is precisely this solution function K, termed herein the "market kernel," which provides the basis for an absolute asset pricing framework. In other words, if arbitrage is prohibited as a partial equilibrium condition, K must exist and must represent an economy-wide characterization which is independent whether or not investor beliefs and tastes are heterogeneous, or indeed whether or not investors even possess von Neumann/Savage utility functions in the first place.

The second major step in the paper is the restriction of beliefs to permit only "local" Markovian changes in the random-valued state variables, i.e., diffusion processes. An arbitrary (diffusion) belief, q, is introduced in order to derive a fundamental partial differential equation which governs the absolute valuation of all assets appearing in the marketplace, at least from the point of view of the observer possessing that belief. Any combination of existing marketable assets may be given valuation formulæ in similar fashion. In particular, when a set of imperfectly related "primitive" assets are considered, a general form of the capital asset pricing model (in continuous time; cf. Merton [1973b]) is seen to result. Alternatively, when the assets taken in combination are perfect (local) substitutes for one another, the variables which depend upon the market kernel K will drop out and we are left with the usual
BS/M relative pricing differential equations, albeit in a more general form. Thus the current paper forms an integrative framework for previous absolute and relative pricing literature, and will develop several new findings as well.

One convenience of the diffusion assumption turns out to be that (given the diffusion belief q) exactly three quantities completely determine all asset prices. These quantities are seen to be identifiable via simple linear regression against the current riskless interest rate.

Besides yielding relative and absolute pricing information, the theory developed herein provides some interesting equilibrium implications. Primary among these is the result that as assets are introduced into the marketplace, increased belief consensus must take place among investors. This is because new assets provide new observables (their prices) which act simultaneously to constrain the admissible combinations of beliefs and market kernels. In the extreme case of the option pricing model, for example, it turns out that investors may disagree on some characteristics of the stock price, but must agree on its instantaneous variance. (Otherwise, one of our partial equilibrium assumptions will be violated.)

Section 2 of the paper establishes some preliminaries and introduces four very general assumptions as partial equilibrium characterizations. From these assumptions, the existence of the market kernel K is established via the Farkas-Minkowski Lemma. The customary weak definition of riskless arbitrage is shown to imply the usual "single price law of markets." (See, e.g., Rubinstein [1975].) A stronger definition of arbitrage (one which considers future markets) is shown to lead to the theoretically and pedagogically significant "intertemporal parity principle."
Section 3 adds two more assumptions restricting our consideration to diffusion processes and requiring the continuity of asset markets. A fundamental differential equation is thereupon derived from the intertemporal parity principle and shown to apply to the valuation of any marketable asset. No knowledge of Itô's lemma is required to follow the derivation, but readers who are familiar with this notation will find an elegant and satisfying description of the riskless interest rate and the fundamental differential equation in footnotes 12 and 15.

Section 4 particularizes our general theory by exploring economies which possess specific sets or subsets of assets available for exchange. (Since every marketable asset possesses its own version of the fundamental differential equation, introducing assets is our primary means of creating economic structure.) The models examined range from a single riskless asset to the capital asset pricing model, the option pricing model, corporate liabilities valuation, the term structure of interest rates, option pricing with random volatility, and other contexts.

Section 5 provides some interpretations and explores the equilibrium consequences of heterogeneous beliefs.

Section 6 gives a summary and suggests extensions.
2. ARBITRAGE, INTERTEMPORAL PARITY, AND ASSET PRICES

We shall first establish some preliminaries and notation. Our analysis will be set in continuous time and state spaces.

2.1 Preliminaries

An "asset" will be defined as constituting the ownership rights to a payoff stream, \( x_j(\theta_t, t) \), which will be interpreted as a function denoting the rate of payoff in the "state of the world" \( \theta_t \) at time \( t \) for the asset \( A_j \). In other words, if an investor holds asset \( A_j \) from time \( t \) to time \( t + \Delta t \), \( \Delta t \) sufficiently small, he will receive in that interval approximately \( x_j(\theta_t, t)\Delta t \) in underlying consumption units, the amount of which is also determined in part by the sample path of the stochastic process \( \tilde{\theta}_t \) in the interval \([t, t + \Delta t]\). (We shall use the tilde notation to denote stochastic processes and random variables, as distinct from ordinary functions and variables, and \( \tilde{\theta}_t \) may be considered to be vector-valued.)

Other than for discrete payoffs, we shall normally consider \( x_j(\theta_t, t) \) to be almost everywhere bounded in both variables. To allow also for the possibility of discrete payoffs, we shall in addition admit into the class of functions considered the popular mathematical artifice known as the Dirac delta function; for it does no great violence to the analysis and indeed provides a more intuitive approach than attempting to reinterpret our integrals in a more precise measure-theoretic (Stieltjes) sense. The Dirac delta function \( \delta_{s}(\tau) \) has the defining property that

\[
\int_{t_1}^{t_2} f(\tau) \delta_{s}(\tau) d\tau = \begin{cases} f(s) & \text{if } s \in [t_1, t_2] \\ 0 & \text{otherwise} \end{cases}
\]
for all functions \( f(\tau) \) continuous at \( s \). One of many possible functions that will serve in this regard is (loosely stated)

\[
\delta_s(\tau) = \lim_{n \to 0} \frac{1}{\sqrt{n}} e^{-(\tau-s)^2/n}.
\]

(The point of the loose notation is to provide for exchanging the order of the above limit operation with other limits, e.g., integration or differentiation.\(^1\))

Thus, for example, to describe the payoff of a pure-discount corporate bond with face value \( B \) maturing at time \( s \in [t_1, t_2] \), we would employ the notation

\[
\int_{t_1}^{t_2} B \delta_s(t) \, dt = B,
\]

the "rate" of payment at time \( \tau \) being \( B \delta_s(\tau) \).

One other function, the Heaviside step function, will later prove useful; we, therefore, define

\[
h_s(\tau) = \begin{cases} 
1 & \text{if } \tau \leq s \\
0 & \text{otherwise.}
\end{cases}
\]

Our assumptions are intended to be quite general, the first being the following:

1. Every marketable asset \( A_j \) possesses a valuation function (i.e., the values of a market price) \( V_j(\theta_\tau, \tau) \) and a payoff rate function \( x_j(\theta_\tau, \tau) \) which are uniquely determined by the state variable \( \theta_\tau \) and the time \( \tau \). \( \theta_\tau \) may be considered to be vector-valued.
This assumption presumes that we have fully identified all the state variables affecting the prices of some set of marketable assets. Moreover, the term "function" in Assumption (1) denotes (as usual) a deterministic mapping from states and dates to the real numbers. Thus, the only form of uncertainty permitted herein is uncertainty in the state variable \( \theta_t \). (If, for example, investors disagreed about the values of \( x_j(\theta_t, t) \) for given values of \( \theta_t \) and \( t \), we would simply expand the definition of \( \theta_t \) to cover all the possibilities, in the customary fashion.)

An asset will be said to be "marketable" if markets exist for that asset both in the present and in the future. If there are at most only present markets (but no subsequent ones), the asset will be termed "nonmarketable." Pursuant to this definition, an important distinction exists between the idea of just any "asset" and that of a "marketable asset." A Ph.D. degree is certainly an asset and may indeed provide its owner with a payoff stream. It is not a marketable asset, however, since it cannot be freely sold to the highest bidder. The crucial characteristic of a marketable asset is its future negotiability. This distinction seems lost, at least in its mathematical representation, in modern asset valuation theory. To remedy this, we introduce the set of induced payoff streams of a marketable asset, \( A_j \),

\[
(A_j) = \left\{ h_s(t) x_j(\theta_t, t) + \delta_g(t) V_j(\theta_t, t) \text{ for all future dates } s \text{ at which a market in } A_j \text{ exists} \right\}
\]

For contrast, we shall occasionally refer to \( x_j(\theta_t, t) \) as the natural payoff stream of asset \( A_j \).
The induced payoff stream of an asset provides explicit recognition of its marketable character: the owner may cut short the natural payoff stream $x_j(\theta_1, t)$ by simply selling the asset at a date when such is possible, i.e., when a market exists. Whenever he may choose to do so, he receives the truncated payoff function $b(\tau)x_j(\theta_1, \tau)$ plus the discrete payment of the proceeds of the sale, i.e., $\delta(\tau)v_j(\theta_1, \tau)$.

2.2 Arbitrage

We come now to the next assumptions made herein, which are:

(II) Asset markets are frictionless.

Here we assume the absence of transactions costs, that there are no restrictions on short sales, etc., and that markets are perfectly competitive and atomistic. (However, complete equilibrium is not required.)

(III) No riskless arbitrage is available.

Prohibiting riskless arbitrage amounts to asserting that if an asset or portfolio of assets has a positive (negative) price, it cannot have a nonpositive (nonnegative) payoff function over all dates and states. This concept can be expressed as a mathematical programming problem. Let $z_j$ be the amount of asset $A_j$ held in some portfolio which ranges over all assets available in the marketplace. Let $\theta_1$ be the state at the current date $t$ and $\theta_\tau$ be the state at some future date $\tau$. Then (III) includes at least the assertion that there does not exist a solution $z_j$ in "arbitrage space" to the problem.
\[(P1) \sum_{j} z_{j} V_{j}(\theta_{t}, t) > 0,\]

\[\sum_{j} z_{j} x_{j}(\theta_{\tau}, \tau) \leq 0 \quad \text{for all } \theta_{\tau} \text{ and all } \tau \geq t.\]

Thus, whatever else characterizes market equilibrium, it should be widely acceptable that at least prices are such as to preclude a solution $z_{j}$ to (P1). If there were a solution $z_{j}$ to (P1), traders would then execute riskless arbitrage by selling the corresponding portfolio and receiving the positive price (the first constraint) without incurring any future liabilities of payoff to buyers in any date or state (the second constraint).

We now make our fourth assumption:

\[(IV) \text{ The functions } V_{j}(\theta_{\tau}, \tau) \text{ and } x_{j}(\theta_{\tau}, \tau) \text{ are sufficiently regular.} \]

The meaning of this assumption will be explained shortly.

If we were dealing with a finite number of states and dates in problem (P1), we could immediately apply to (1)-(III) the Farkas-Minkowski Lemma of the Alternative (see, e.g., Mangasarian [1969]), which implies that/ to the problem (P1) if and only if there exists a solution $K(\theta_{t}, \tau, \theta_{\tau}, t)$ in the dual "valuation space" to the problem:

\[(DI) \quad V_{j}(\theta_{t}, t) = \int_{\tau \geq t} \int_{\theta_{t}} K(\theta_{t}, \tau, \theta_{\tau}, t) x_{j}(\theta_{\tau}, \tau) \, d\theta_{\tau} \, d\tau \]

\[K(\theta_{t}, \tau, \theta_{\tau}, t) \geq 0.\]

The first expression in (DI) states that there is a linear integral operator $\mathcal{K}$ which transforms every natural payoff stream into a current
price. Of note is the fact that $\tilde{K}$ does not depend on the asset considered.\(^3\)

The second expression insures that the kernel of this linear integral operator is nonnegative (positive payoff is a "good"). (D1) is a well-known result, having been derived in various fashions by Beja [1971, Eq. (2.1)], Rubinstein [1975, Eq. (1)], and Cox and Ross [1976, Eq. (3)]. It has been sometimes termed "the single-price law of markets."\(^4\)

The justification of (D1) follows immediately from (I)-(III) and the Farkas-Minkowski Lemma in the finite case; but unfortunately, there are deep mathematical questions in the continuous, unbounded case that affect the application of the Lemma. These questions involve issues of measurability, convergence, "duality gaps,"\(^5\) (and later) probability spaces, and series expansions, all of which assumption (IV) is intended to cover. While the weakest possible form of assumption (IV) is a question of some interest to mathematicians (see Hurwitz [1958] for the extension of the Farkas-Minkowski Lemma to linear topological spaces), finance theorists and practitioners will be quite safe in adopting the engineers' viewpoint by assuming that the real world will provide sufficient conditions to avoid any perverse cases: that assets will continue to have bounded prices and payoffs, individuals will continue to have finite lifetimes and endowments, etc. Thus, from (I)-(IV) we will assert the existence of a solution to (D1), i.e., the validity of the single-price law of markets.

The next step is to apply the same procedure to the induced payoff streams of a marketable asset. We do this by strengthening (P1) appropriately for the possibilities offered by future markets: arbitrage may occur if there exists a solution $z^*$ to
(P2) \[ \sum_j x_j V_j(\theta_{s}, t) > 0, \]

\[ \sum_j x_j \left( h_j(\tau)x_j(\theta_{s}, \tau) + \delta_j(\tau)V_j(\theta_{s}, \tau) \right) < 0 \]

for some \( s \), all \( \theta_{\tau} \), and all \( \tau \), such that \( s > \tau > t \).  

In other words, arbitrage is possible whenever one may sell for a positive current price a portfolio which involves no future payouts or for which future payouts may be avoided by "covering" one's position with a transaction at a price given by the valuation function at some future time.

The dual alternative of (P2) in valuation space is

(D2)  
\[ V_j(\theta_{t}, t) = \int_{\tau > t} \int_{\theta_{\tau}} K(\theta_{\tau}, \theta_{t}, \tau) \left( h_j(\tau)x_j(\theta_{\tau}, \tau) + \delta_j(\tau)V_j(\theta_{\tau}, \tau) \right) d\theta_{\tau} \, d\tau \]

for all \( s \) at which a market in asset \( A_j \) exists,  

and \( K(\theta_{\tau}, \theta_{t}, \tau) \geq 0 \).

Prohibiting arbitrage implies that a solution \( K \) of (D2) exists as a market-wide equilibrium characteristic.

2.3 Intertemporal parity

We shall term the concept behind (D2) the **intertemporal parity principle**: that the induced payoff streams of a marketable asset must all possess the same price, in the sense that a single integral operator transforms every such induced payoff into the same value. This principle provides the key to solving the general valuation problem in continuous markets.
To gain an intuitive grasp of the intertemporal parity principle consider the following example:

Suppose a claim A promises exactly the same payoff stream, including the dividends and liquidation value, as 100 shares of IBM common stock sold exactly one year hence. Suppose claim B will yield the same payoff stream as 100 IBM shares sold two years hence. These two claims then represent two distinct induced payoff streams of 100 IBM common shares. If IBM common stock is itself freely traded, then claims A and B must have the same current price. To see this, suppose that the two claims had different prices. Then an arbitrageur would sell the high-priced claim, covering the required pay-outs by buying 100 IBM shares and liquidating these when the time came. Conversely, he would buy the low-priced claim and hedge this by selling 100 IBM shares short, covering this eventually by the proceeds of his "long" claim. This yields the arbitrageur a current profit with no risk whatsoever. (Note that no future markets beyond the present are required for claims A and B in order to establish this arbitrage position, and so we may think of A and B as nonmarketable to avoid further concerns about their induced payoff streams.)

Another form of intuitive rationalization of intertemporal parity goes as follows: if one of the induced payoff streams is "preferred" to (has a higher implicit market price via K than) another, enough investors will rush to liquidate the marketable asset at that point in time which produces the "preferred" stream, and so the valuation function will adjust until parity is reached. Yet another argument is simplistic
and direct: that even though a marketable asset consists of a "bundle" of induced payoff streams, an asset may still possess only one price. In consequence, we may write \( V_j = R(A_j) \) as concise notation for the mathematical expression of the intertemporal parity principle (1).

It is worthwhile to note here that there are even more powerful arbitrage/valuation duality statements beyond (P2)/(D2) that may be inferred as following from assumption (III). In particular, (P2) says that there exists some future date \( s \) for which a transaction at \( s \) will produce arbitrage profits. The choice of \( s \) may therefore be viewed as a simple strategy for a future transaction. However, there are more complex strategies. For example, a strategy might depend on both a future state and date. Any strategy like "sell the option when the stock price goes below 7, but in any case, after 100 days after elapsed" may be similarly prohibited from generating arbitrage profits. The existential quantification of strategies in the arbitrage space leads always to their universal quantification in the dual valuation space. Thus if we extend the concept of induced payoff streams to include all such streams created by pursuing any complex strategy, the intertemporal parity principle is extended to an "interstrategic parity" principle and asserts that all of these induced payoff streams must again possess the same current price, regardless of what complex strategy is employed to create them.  

An even stronger concept of arbitrage may be formulated if we contemplate strategies based on an information set \( \Phi(t) \) available at time \( t \). Indeed, the corresponding parity principle then captures the very essence of the "efficient market hypothesis": that the current price "imparts" all available information in the sense that the linear
operator $\tilde{K}$ then depends on $\Phi(t)$ (stated in the dual valuation space) or, alternatively, that no strategy based upon $\Phi(t)$ will produce an arbitrage profit (stated in the primal arbitrage space). The Farkas-Minkowski Lemma assures us that these two statement forms are equivalent. The immediate goals of this paper, however, will be met by the alternative (P2)/(D2).

Assumptions (I)-(IV) imply the existence of a solution $K(\theta_t^r, \tau, \theta_t^d, t)$ to the dual alternative problem (D2) as a market-wide partial equilibrium characteristic, so we shall term $K(\theta_t^r, \tau, \theta_t^d, t)$ the market kernel. Notice that no assumptions regarding the homogeneity of tastes, beliefs, or other agent characteristics have been made, nor even that these other characteristics exist. Some investors might well have intransitive preferences, prefer inferior goods, smoke $100 bills, and engage in other strange economic behavior; provided their behavior does not lead to creation of arbitrage opportunities, our analysis still obtains.

Moreover, while existence of the market kernel is assured by (I)-(IV), uniqueness is not (i.e., markets may be "incomplete").

Useful economic properties of the market kernel $K$ include:

(i) $K(\theta_t^r, \tau, \theta_t^d, t) \geq 0$ (positive payoff is a "good") which has already appeared in (D2); (ii) $\lim_{\tau-t^+} K(\theta_t^r, \tau, \theta_t^d, t) \theta_t^r = 1$ (a current sure payoff of 1 has a price of 1), which follows from $x_{1} = 0, V_{1} = 1$, and $s \to t^+$ in (D2).

3. VALUATION UNDER DIFFUSION STATE PROCESSES

We now assume

(V) All observers agree that the stochastic process $\hat{\theta}_t$ is some diffusion process on the real line (but do not necessarily agree upon which diffusion process obtains). One (arbitrary) observer posits a transition
density \( q(\theta, \tau, \theta_t, t) \) {\text{put loosely}}, \( \text{Prob}\{\tilde{\theta}_t \in [\theta_t, \theta_t + d\theta_t] \} \) at \( \tau|\tilde{\theta}_t = \theta_t \) at \( t \) = \( q(\theta, \tau, \theta_t, t) \) \( d\theta_t \).

(Assumption (V) is relaxed in the appendix to include diffusion processes in \( \mathbb{R}^n \).)

The significance of the diffusion assumption is that the Lindeberg-type conditions, \(^{10}\) namely,

\[
\int_{|\theta - \theta_t| > \epsilon} q(\theta, \tau, \theta_t, t) \, d\theta = o(\tau - t),
\]

(2)

\[
\int_{|\theta - \theta_t| < \epsilon} (\theta - \theta_t) q(\theta, \tau, \theta_t, t) \, d\theta = o(\theta_t, t)(\tau - t) + o(\tau - t),
\]

(3)

and

\[
\int_{|\theta - \theta_t| < \epsilon} (\theta - \theta_t)^2 q(\theta, \tau, \theta_t, t) \, d\theta = \Omega(\theta_t, t)(\tau - t) + o(\tau - t)
\]

(4)

must hold for arbitrary \( \epsilon > 0 \), where \( o(\tau - t) \) represents "terms of order less than \( \tau - t \)," i.e., \( \lim_{\tau \to t} \frac{o(\tau - t)}{\tau - t} = 0 \). The functions \( \alpha \) and \( \Omega \) are termed the instantaneous drift (a vector in \( \mathbb{R}^n \) diffusions) and instantaneous variance (a covariance matrix in \( \mathbb{R}^n \) diffusions), respectively, of the diffusion process \( \tilde{\theta}_t \).

Define the (observer-specific) quotient kernel \( G(\theta_t, \tau, \theta_t, t) \) by

\[
G(\theta_t, \tau, \theta_t, t) = \begin{cases} \frac{K(\theta_t, \tau, \theta_t, t)}{q(\theta_t, \tau, \theta_t, t)}, & q(\theta_t, \tau, \theta_t, t) \neq 0; \\ 0, & \text{otherwise}; \end{cases}
\]
and so the intertemporal parity principle (1) now provides that

\[ V_j(\theta_t, t) = \int_{t}^{\infty} \int_{\theta_t}^{\theta_s} \sigma(\theta_s, \tau, \theta_t, t) \left( b_s(\tau)x_j(\theta_s, \tau) + \delta_\tau(\tau) V_j(\theta_s, \tau) \right) q(\theta_s, \tau, t) d\theta_s \ d\tau \]

(5)

for all \( s \geq t \) at which markets exist.\(^{11}\)

A market will be said to be continuous at time \( t \) if there exists an \( \varepsilon > 0 \) such that the market is in existence at every point in the time interval \([t, t + \varepsilon]\). Our final assumption is, then,

(VI) All asset markets are currently continuous.

(Assumptions (II) and (VI) together imply the more usual "continuous trading" assumption, but (VI) was not needed to derive the intertemporal parity principle (1).)

We now have a central consequence of assumptions (I)-(VI).

Theorem. (Fundamental Differential Equation) Under the assumptions (I)-(VI), the valuation function \( V_j(\theta_t, t) \) of every existing marketable asset \( A_j \) obeys the partial differential equation\(^{12}\)

\[
0 = x_j(\theta_t, t) + \frac{\partial V_j(\theta_t, t)}{\partial t} + \frac{1}{2} \frac{\partial^2 V_j(\theta_t, t)}{\partial \theta^2_t} + [a(\theta_t, t) + \Omega(\theta_t, t) \frac{\partial V_j(\theta_t, t)}{\partial \theta_t}] \frac{\partial V_j(\theta_t, t)}{\partial \theta_t} + [a(\theta_t, t)G_1(\theta_t, t) + G_2(\theta_t, t) + \frac{1}{2} G_{11}(\theta_t, t)] V_j(\theta_t, t),
\]

(6)
where

\[ G_1(\theta_t, t) = \frac{\partial G(\theta_t, \tau, \theta_t, t)}{\partial \theta_\tau} \bigg|_{\theta_\tau = \theta_t} \]

\[ G_2(\theta_t, t) = \frac{\partial G(\theta_t, \tau, \theta_t, t)}{\partial \tau} \bigg|_{\theta_\tau = \theta_t} \]

\[ G_{11}(\theta_t, t) = \frac{\partial^2 G(\theta_t, \tau, \theta_t, t)}{\partial \theta_\tau^2} \bigg|_{\theta_\tau = \theta_t} \]

**Proof.** Under assumptions (I)-(IV), we have shown that the intertemporal parity principle (1) obtains. When the diffusion process \( \tilde{\theta}_t \) is added (assumption (V)), (1) is re-stated in terms of the quotient kernel via (5). Given that our functions are sufficiently regular (required of assumption (IV)), all possess Taylor series expansions in time and the scalar state variable and so we may write (5) as

\[
V_j(\theta_t, t) = \int_{\theta_t}^{\theta_s} \left[ 1 + G_1(\theta_\tau - \theta_t) + G_2(\tau - t) + \frac{G_{11}}{2}(\theta_\tau - \theta_t)^2 + \cdots \right]
\times \left[ x_{j1}(\theta_\tau - \theta_t) + x_{j2}(\tau - t) + \frac{x_{j11}}{2}(\theta_\tau - \theta_t)^2 + \cdots \right]
\times q(\theta_\tau, \tau, \theta_t, t) \, d\theta_\tau \, d\tau
\]

\[
+ \int_{\theta_s}^\infty \left[ 1 + G_1(\theta_s - \theta_t) + G_2(s - t) + \frac{G_{11}}{2}(\theta_s - \theta_t)^2 + \cdots \right]
\times \left[ V_j + V_{j1}(\theta_s - \theta_t) + V_{j2}(s - t) + \frac{V_{j11}}{2}(\theta_s - \theta_t)^2 + \cdots \right]
\times q(\theta_s, s, \theta_t, t) \, d\theta_s
\]
(where \( G(\theta, t, \theta, t) = 1 \) clearly, by (5) and condition (ii) regarding the market kernel, and where the numeric subscripts denote partial derivatives with respect to the corresponding argument number) and thus it follows that

\[
V_j(\theta, t) = (s - t) \int_{|\theta - \theta_r| < \varepsilon} x_j q(\theta, s, \theta, t) \, d\theta_s + o(s - t) + V_j(\theta, t)
\]

\[
+ \int_{|\theta - \theta_r| < \varepsilon} [(V_j \gamma_1 + G_1 V_j)(\theta, \theta) + (V_j \gamma_2 + V_j \eta)(s - t)
\]

\[
+ \left( G_1 V_j \gamma_1 + \frac{G_1 V_j \gamma_2}{2} \right) + \frac{V_j \gamma_1}{2} (\theta - \theta_r)^2 q(\theta, s, \theta, t) \, d\theta_s + o(s - t).
\]

Collecting and canceling terms of the latter equation, we have

\[
0 = \int_{|\theta - \theta_r| < \varepsilon} [(V_j \gamma_1 + G_1 V_j)(\theta, \theta) + (V_j \gamma_2 + V_j \eta)(s - t)
\]

\[
+ \left( G_1 V_j \gamma_1 + \frac{G_1 V_j \gamma_2}{2} \right) + \frac{V_j \gamma_1}{2} (\theta - \theta_r)^2 q(\theta, s, \theta, t) \, d\theta_s + o(s - t).
\]

Dividing both sides of the last equation by \( s - t \), taking the limit \( s \to t \) (using assumption (VI) of continuous markets) and applying (3) and (4), we have the partial differential equation

\[
0 = (V_j \gamma_1 + G_1 V_j)\alpha + (V_j \gamma_2 + V_j \eta \gamma_2 + x_j) + (G_1 V_j \gamma_1 + \frac{G_1 V_j \gamma_2}{2} + \frac{V_j \gamma_1}{2} + \frac{V_j \gamma_2}{2})\alpha.
\]

The theorem follows by expanding the above terse notation and re-arranging terms. Q.E.D.
(An analogous theorem applies when the diffusion process is vector-valued; in this case we may give (2), (3), and (4) a matrix-vector interpretation to include the necessary cross-product terms of the state vector components in (6). For simplicity of exposition, this is deferred to the appendix.)

We shall next employ this fundamental asset equation in a collection of special cases. Structure may be introduced into the current context by two methods: (a) by assuming the existence of certain kinds of assets, or (b) by assuming specific features of the market kernel. We pursue the former course first.

1. STRUCTURAL MODELS

Since we have shown that every financial asset must obey the Fundamental Differential Equation, we may create economic structure by simply assuming the presence of certain assets, either singly or in groups. We shall depict the diversity of such structural models by means of three symbols: a square will represent a random variable or other nonmarketable value, a circle will represent a marketable asset, and an arrow from one asset or variable to another will mean that the former supplies the state variable for the valuation function of the latter.

In accordance with the considerations raised previously, we shall also provide separate terminology for two concepts: when an asset's valuation function is dependent only upon the price of another marketable asset, we shall term it a derivative asset; when an asset's valuation function is dependent upon a random variable which is not a marketable asset price (a primitive variable), we will call it a dependent asset. Otherwise, it will be termed a primitive asset. Primitive assets and
variables will have no arrows leading into them in our structural diagrams, while those that are dependent or derivative must have arrows leading to them.

Notice that our notation treats asset values and state variables interchangeably: it is entirely possible to build up structural models where an asset value becomes a state variable to another asset value, etc. It is therefore important that this recursive "building-block" procedure be valid, i.e., that various identifications of primitive, dependent, and derivative assets and variables not be inconsistent with one another. Fortuitously, since we are dealing only with diffusion processes and sufficiently regular valuation functions, our approach is internally consistent: if \( \theta_t \) follows a diffusion process with instantaneous drift \( \alpha(\theta_t, t) \) and instantaneous variance \( \Omega(\theta_t, t) \), then the valuation function \( V(\theta_t, t) \) will follow a diffusion process with corresponding parameters

\[
\left( \frac{\partial V}{\partial \theta_t} \right) \alpha(\theta_t, t) + \frac{\partial V}{\partial t} + \frac{\Omega(\theta_t, t)}{2} \left( \frac{\partial^2 V}{\partial \theta_t^2} \right)
\]

and

\[
\left( \frac{\partial V}{\partial \theta_t} \right) \frac{\partial^2 V}{\partial \theta_t^2}
\]

respectively.\(^{13}\) Thus our scheme is complete, since if all primitive variables and assets obey diffusion laws, they and all their dependent and derivative assets will obey diffusion laws and the marketable assets may be valued by our approach, forming in turn new state variables for their derivative assets, etc.

In the structural models that follow, three caveats should be noted. First, any scale parameters involving the "number of asset shares outstanding" are ignored, so the reader may treat the corresponding valuation
functions as applying to the entire amount of such assets. Second, the boundary conditions for the valuation differential equations are largely ignored. Third, all valuation differential equations containing terms in the kernel $G$ are unique only to the extent that the market kernel is unique. Nevertheless, this fact poses no empirical problems, as we shall see immediately in the first model.

4.1 The risk-free asset

A first question relates to the rate of return on a risk-free asset alone, presuming one exists. Our diagram shows the state variable $\theta_t$ affecting a single marketable asset $A_R$. We shall interpret its "risk-free" characteristic to mean that it has a valuation function $V_R(\theta_t,t) \equiv 1$ for all states $\theta_t$ and dates $t$.\textsuperscript{1}\textsuperscript{/} From the Fundamental Equation (6), solving for the payoff stream, we have then

\begin{equation}
    r(\theta_t,t) = -\{a(\theta_t,t)G_1(\theta_t,t) + G_2(\theta_t,t) + \frac{1}{2} \Omega(\theta_t,t)G_{11}(\theta_t,t)\},
\end{equation}

where $r(\theta_t,t)$ is the rate of return on the risk-free asset $A_R$.\textsuperscript{15}\textsuperscript{/} Interpreting quantities in the right-hand side of (7), $-G_2$ may be thought of intuitively as the "instantaneous market impatience" and $-G_1$ the "instantaneous market risk aversion," while $-G_{11}$ reflects the "increasing" or "decreasing" nature of "market risk aversion" $G_1$. However, these are just suggestive terms which are provided here only to enhance the reader's
intuitive grasp of the quantities $G_1$, $G_2$, and $G_{11}$; of course, nothing
derived to this point actually depends upon utility concepts, and such
a discussion is deferred to Section 5.

As an empirical matter, note that since our observer has posited a
diffusion density $q$ with its corresponding parameters $a$ and $\Omega$, he is now
in a position to measure $G_1$, $G_2$, and $G_{11}$. The left-hand side of (7) is
readily observed, and so such techniques as linear regression will allow
him to infer the latter quantities. In this fashion he may identify the
relevant features of the market kernel. This information will in turn
allow him to predict all other security prices in the economy by knowing
only the contractual characteristics (boundary conditions) of those securi-
ties, since all such securities must satisfy the fundamental equation (6)
which depends only upon the quantities $a$, $\Omega$, $G_1$, $G_2$, and $G_{11}$.

4.2 Primitive asset and risk-free asset

Next, we examine the valuation of that
primitive asset $A_p$ which is the state variable $\theta_t$
itslf, i.e., for which $V_p(\theta_t, t) = \theta_t$. In this
case the fundamental equation (6) for $A_p$ reduces to

\begin{equation}
0 = x_p(\theta_t, t) + a(\theta_t, t) + \Omega(\theta_t, t) G_1(\theta_t, t) + \theta_t a(\theta_t, t) G_1(\theta_t, t) \\
+ \theta_t G_2(\theta_t, t) + \theta_t \frac{\Omega(\theta_t, t)}{2} G_{11}(\theta_t, t)
\end{equation}

(8)

The certainty portion of (i.e., expected value of)
(where $x_p$ is the payout of the primitive asset) from whence the return
on the primitive asset is found by solving for $x_p + a$ and dividing through by $\theta_t$: 
\[ r_p(\theta_t, t) = \frac{x_p(\theta_t, t) + \alpha(\theta_t, t)}{\theta_t} \]

\[ = \left\{ \frac{\Omega(\theta_t, t)G_1(\theta_t, t)}{\theta_t} + \alpha(\theta_t, t)G_1(\theta_t, t) + G_2(\theta_t, t) + \frac{\Omega(\theta_t, t)}{2} - G_{11}(\theta_t, t) \right\}, \]

or, since the risk-free asset also exists, we may substitute from (7) to obtain

\[ r_p(\theta_t, t) = r(\theta_t, t) - \frac{\Omega(\theta_t, t)G_1(\theta_t, t)}{\theta_t}, \]

the last term of which is essentially the market-assessed "risk premium" associated with the primitive asset. (Typically, \( G_1 < 0 \).)

The basic methodological point arising here (with our consideration of a second asset) is that each marketable asset in a structural model has its own version of the Fundamental Differential Equation, and these equations all simultaneously govern.

### 4.3 Multiple primitive assets (The Capital Asset Pricing Model)

Consider now the case of \( n \) primitive assets with prices \( \theta_1, \theta_2, \ldots, \theta_n \). Drawing upon the appendix for the fundamental differential equation when a vector-valued state variable is present, we may rewrite (10) as

\[ r_i^j - r_j = \sum_{k=1}^{n} \Omega_{ik} \frac{\partial \theta_j}{\partial \theta_k} \]

\[ (11) \]
where the functional arguments and indices in $t$ have been suppressed, $r_j$ is the return of the asset $A_j$, and $\Omega_{jk}$ is the instantaneous covariance matrix of the $\theta_j$ variables.

Define the usual "market portfolio" as having price $E_M = \sum_{k=1}^{n} \theta_k + 1$ and return $r_M = (\sum_{k=1}^{n} r_k \theta_k + r) / \theta_M$. Then, trite calculations lead to

$$ (r_j - r) = \beta_j (r_M - r) \tag{12} $$

where

$$ \beta_j = \left( \sum_{k=1}^{n} \Omega_{jk} \frac{\partial \theta_j}{\partial \theta_k} \right) \left( \sum_{k=1}^{n} \sum_{l=1}^{n} \Omega_{lk} \frac{\partial \theta_l}{\partial \theta_k} \right)^{-1} \left( \frac{\partial \theta_j}{\partial \theta_M} \right) $$

a form which will be recognized by those familiar with the capital asset pricing model (CAPM). Note that the numerator of $\beta_j$ is much like the usual "ratio of covariance to variance" term appearing in the classical CAPM, except that now those terms involve $\partial \theta / \partial \theta_k$ also. (Note in particular that $r_j$, $r_M$, $r$, and $\beta_j$ would be constant in the case where lognormal diffusion is present and $\partial \theta / \partial \theta_k$ is inversely proportional to the $\theta_k$--implied, for example, by universal constant proportional risk aversion and universal lognormal beliefs amongst investors--cf. Merton's [1973b] assumption of a constant investment opportunity set.)

A similar derivation can be performed when the riskless asset is absent, using (9) without (7); cf. Black [1972].

Thus we find that the continuous-time CAPM is remarkably robust, at least in the very generalized form provided by (12), since the derivation above does not depend upon the usual homogeneous beliefs, time-additive
tastes, mean-variance tastes, constant opportunity sets, or even the existence of utility functions. Also, note that \( \beta \) is independent of price levels only in rare cases, an observation consistent with the arguments of Rosenberg and Ohlson [1976].

\[ \text{Dependent asset model} \]

The valuation of dependent assets is also governed by equation (6). When a risk-free asset is available, we may substitute (7) into (6) to yield for all dependent assets \( A_j \) the differential equation

\[
0 = x_j(\theta_i, t) + \frac{\partial V_j}{\partial t} + \frac{\Omega(\theta_i, t)}{2} \frac{\partial^2 V_j}{\partial \theta_i^2} + \left[ \alpha(\theta_i, t) + \Omega(\theta_i, t) \gamma_j(\theta_i, t) \right] \frac{\partial V_j}{\partial \theta_i} - r(\theta_i, t)V_j.
\]

(13)

The dependent asset valuation equation (13) is an absolute pricing result and depends on the stochastic drift \( \alpha \) and the market "risk-aversion" factor \( \gamma_j \). The dependent asset model applies in circumstances where the underlying state variable is not an asset, hence no "riskless hedge" may be formed against it. For example, consider a farmer with a marketable corn crop, the yield of which depends only on the stochastic process of rainfall. Since one cannot hold rainfall units in one's portfolio, it is generally impossible to eliminate the risk associated with rainfall in holding this asset, and thus market attitudes towards risk must enter the picture via \( \gamma_j \).
4.5 Derivative asset model (the Option Pricing Model)

In the case of derivative assets, the $C_1$ term of (13) drops out for the reason that a riskless hedge may be formed, and relative pricing independent of market risk structure results. When the derivative asset and its primitive asset are both extant in the marketplace, equations (8) and (13) both govern. Multiplying (8) by $\partial V / \partial \theta_t$ (a "hedge ratio") and subtracting the result from (13), we have for derivative assets $A_j$

$$0 = x_j(\theta_t,t) + \frac{\partial V}{\partial t} + \frac{\partial}{\partial \theta_t} \left[ \frac{\partial^2 V}{\partial \theta_t^2} + \frac{1}{2} \theta_t \left( \phi(\theta_t,t) - x_p(\theta_t,t) \right) \frac{\partial^2 V}{\partial \theta_t^2} - r(\theta_t,t)V_j \right].$$

(14)

which is identical to the Black-Scholes [1973, Eq. (7)] partial differential equation when $x_j(\theta_t,t) = x_p(\theta_t,t) = 0$ and completely identical to Merton's [1974, Eq. (7)] formula, except that we have allowed more general functions for the diffusion parameters and the interest rate. When these parameters are specialized to geometric Brownian motion and the boundary condition $V_j(\theta_T,t) = \max[0, \theta_T - E]$ is imposed (where $T =$ maturity date, $E =$ exercise price), the familiar "option pricing model" results.

4.6 Common-factor model

We now present some results concerning multiple assets that are dependent upon the same primitive variable. For instance, the prices of corn and wheat are both dependent upon the weather, but weather itself
is not a marketable asset. Yet, even though the formation of riskless hedges with the common state variable (weather) may be impossible, we would expect corn and wheat prices to be linked through this commonality. In fact, it usually turns out to be possible to form riskless hedges between the dependent assets themselves. Let $V_j(\theta_t, t)$ and $V_k(\theta_t, t)$ be two valuation functions which depend on the common primitive variable $\theta_t$. Then from the Fundamental Equation and the existence of a risk-free asset, we have the dependent asset equations

$$0 = x_j + \frac{\partial V_j}{\partial t} + \frac{\partial^2 V_j}{\partial \theta^2_t} + [\alpha + \Omega \delta] \frac{\partial V_j}{\partial \theta_t} - rV_j. \tag{15}$$

and

$$0 = x_k + \frac{\partial V_k}{\partial t} + \frac{\partial^2 V_k}{\partial \theta^2_t} + [\alpha + \Omega \delta] \frac{\partial V_k}{\partial \theta_t} - rV_k. \tag{16}$$

where we have now dropped all of the functional arguments in the interests of conserving space. We may eliminate the terms involving $\delta$ by multiplying (15) by $\partial V_k/\partial \theta_t$, (16) by $\partial V_j/\partial \theta_t$, and subtracting the two, yielding

$$0 = \left( x_j \frac{\partial V_k}{\partial \theta_t} - x_k \frac{\partial V_j}{\partial \theta_t} \right) + \frac{\partial V_j}{\partial t} + \frac{\partial^2 V_j}{\partial \theta^2_t} - rV_j + \frac{\partial V_k}{\partial t} + \frac{\partial^2 V_k}{\partial \theta^2_t} - rV_k. \tag{17}$$
Thus the two valuation functions are jointly determined through the coupling formula (17) which itself depends only on observable quantities. Formula (17) may be separated in a fashion that solves for a common function $\lambda(\Theta_t, t)$ such that

$$x_i + \frac{2 \lambda}{\lambda t} + \frac{\partial^2 \lambda}{\partial \Theta_t^2} - r \lambda = \lambda(\Theta_t, t), \quad i = j, k; \frac{\partial \lambda}{\partial \Theta_t} \neq 0, \quad (18)$$

and, indeed, (18) holds for all marketable assets $A_i$ that depend only on the common state variable $\Theta_t$. In other words, if a set of assets are all driven by a common variable, "riskless hedges" may be established to eliminate market risk aversion parameter $\Theta_t$. The exception to this general statement occurs when the denominator term $\frac{\partial \lambda}{\partial \Theta_t} = 0$ for some asset, in which case $\lambda(\Theta_t, t)$ might be undefined. Whenever $\frac{\partial \lambda}{\partial \Theta_t} \neq 0$, we may rewrite (18) as

$$0 = x_i + \frac{2 \lambda}{\lambda t} + \frac{\partial^2 \lambda}{\partial \Theta_t^2} - \frac{\partial \lambda}{\partial \Theta_t} - r \lambda \quad (19)$$

for some function $\lambda$. Of importance is the fact that $\lambda$ does not depend upon $\Theta$, and would take on the same value in every economy having securities with the same characteristics. With additional information we may identify $\lambda$ in some contexts, as shown in the following subsection.
4.7 A corporate liability valuation model

![Diagram]

This model is an extension of the common-factor model given above. The "value of the firm" is a state variable $V$ which is not itself traded, but which determines the values $V_i$ of each of a set of marketable corporate liabilities $A_i$, $i = 1, 2, \ldots, n$. The side constraint $\sum_{i=1}^{n} V_i = V$ is imposed, i.e., the value of the firm is defined as the sum of the values of its outstanding liabilities.

That $V$ is a state variable following some diffusion process independent of the form of the liabilities the firm issues may be viewed as an application of the Miller-Modigliani proposition. In this case, (19) must hold for all $i$, provided $\partial V_i/\partial q_i \neq 0$. Summing over $i = 1, 2, \ldots, n$ in (19) yields

$$0 = \sum_{i=1}^{n} q_i + \sum_{i=1}^{n} \frac{\partial V_i}{\partial t} + \frac{\partial}{2} \sum_{i=1}^{n} \frac{\partial^2 V_i}{\partial t^2} - \lambda \sum_{i=1}^{n} \frac{\partial V_i}{\partial V} - r \sum_{i=1}^{n} V_i$$

(20)
From the constraint \( \sum_{i=1}^{n} v_i = v \), we have \( \sum_{i=1}^{n} \frac{\partial v_i}{\partial v} = 1 \) and \( \sum_{i=1}^{n} \frac{\partial v_i}{\partial t} = \sum_{i=1}^{n} \frac{\partial^2 v_i}{\partial v^2} = 0 \).

Substituting these relationships into (19) gives

\[
\lambda(V,t) = \sum_{i=1}^{n} x_i - r(V,t)v. \tag{21}
\]

from whence the valuation function of each corporate liability must obey

the differential equation

\[
0 = x_1 + \frac{3v_1}{2} + \frac{n}{2} \frac{\partial^2 v}{\partial y^2} + [rV - \sum_{i=1}^{n} x_i \frac{\partial v_i}{\partial v} - rV]. \tag{22}
\]

The latter equation is identical to (14) and Merton's [1974, Eq. (7)] formula

for corporate liability valuation.

It is important to note the assumptions and limitations of this model. The derivation given above will falter whenever any of the following circumstances occur: (1) any corporate liability is nonmarketable (e.g., tax liabilities, some bank loans, bankruptcy costs), in which case the "value of the firm" variable becomes meaningless as being the sum of the market prices of liabilities issued (but, of course, we could respecify the model so that only the marketable portion of the firm's liabilities constituted the state variable); (2) only one corporate liability has a nonconstant valuation function (no "riskless hedge" is available, i.e., \( \lambda \) is undefined); (3) any other state variable besides \( V \) affects the values of corporate liabilities; or (4) interest rates are dependent on any other state variables besides \( V \).\(^{18}\)
4.8 Option Pricing Model with random volatility

When the derivative asset equation (14) is specialized to geometric Brownian motion, interest rates are assumed constant, and the boundary condition for an option's value at maturity is imposed, the option pricing model (Black and Scholes [1973]) results. Its solutions are characterized by certain parameters, these being (1) the riskless interest, (2) the time to option maturity, (3) the current stock price, and importantly, (4) the volatility, or instantaneous variance of stock price relatives. Of these parameters, the latter is probably the most difficult to evaluate. Moreover, recent evidence \(^{20}\) tends to show that the volatility is nonconstant and indeed may follow a stochastic process itself. In this structural model we therefore generalize the option pricing model to the case where the volatility parameter pursues an independent geometric Brownian motion. The model will serve to further illustrate the use of the vector version of the Fundamental Differential Equation given in the appendix. Employing the notation of the appendix, we have the differential equation

\[
0 = \frac{3V}{2t} + \frac{1}{2}(V^2)\Omega + V\nu[a + \Omega(V\nu')] - rV
\]  

(23)

(using the existence of the riskless asset and assuming the option payout is zero) where

\[V = \text{price of the option,}\]

\[\theta_1 = \text{stock price,}\]
$\theta = \text{a state variable affecting the volatility,}$

$\theta = (\theta_1, \theta_2),$

$\alpha = (\alpha_1, \alpha_2)$, the instantaneous drift vector of $\theta$,

$\Omega = \text{the instantaneous covariance matrix of } \theta.$

More specifically, we shall assume that

$$
\Omega = \begin{bmatrix}
\omega_1^2 & 0 \\
0 & \omega_2^2
\end{bmatrix}
$$

where $\omega_1$ and $\omega_2$ are constants, so that $\theta_1$ and $\theta_2$ are independent and $\theta_2$

follows its own geometric motion while affecting the volatility in the
usual (lognormal) multiplicative fashion. The volatility parameter for the

stock price is thus $\omega_1^2 \theta_2$. Expanding (23), we have

$$
0 = \frac{3V}{\delta t} + \frac{\omega_1^2}{2} \frac{\partial^2 V}{\partial \theta_1^2} + \frac{\omega_2^2}{2} \frac{\partial^2 V}{\partial \theta_2^2} + \left[ \alpha_1 + \omega_1 \theta_1^2 \frac{\partial}{\partial \theta_1} \frac{\partial}{\partial \theta_1} \frac{3V}{r(t)} \right] + \left[ \alpha_2 + \omega_2 \theta_2^2 \frac{\partial}{\partial \theta_2} \frac{\partial}{\partial \theta_2} \frac{3V}{r(t)} \right] - r(\theta, t)V
$$

(24)

for the option asset.

The Fundamental Differential Equation applied to $\theta_1$, on the other hand,
yields

$$
0 = \alpha_1 + \omega_1 \theta_1^2 \frac{\partial}{\partial \theta_1} - r(\theta, t)\theta_1,
$$

(25)

assuming that stock dividends are zero.
Clearly, no nonzero linear combination of (24) and (25) will eliminate all the terms in $3\theta /\theta_1$ and $3\theta /\theta_2$. However, we may multiply (25) by $\frac{3\theta}{\theta_1}$ and subtract from (24) yielding

$$0 = \frac{3\theta}{3t} + \frac{\omega_1}{2} \frac{3\theta^2}{\theta_1} \frac{3\theta}{\theta_1} + \frac{\omega_2}{2} \frac{3\theta^2}{\theta_2} \frac{3\theta}{\theta_2} + \theta_1 r(\theta, t) \frac{3\theta}{\theta_1} + [\alpha_2 + \omega_2^2 \frac{2\theta}{\theta_2}] \frac{3\theta}{\theta_2} - r(\theta, t)\theta$$

(26)

which eliminates "risk" arising from the stock price. Note that as long as $\theta_2$ represents a variable other than a marketable asset, risk associated with $\theta_2$, i.e., risk of volatility changes, cannot be diversified away: the solutions of (26) will depend upon $3\theta /3\theta_2$. (Convenient assumptions at this juncture might then be that $\alpha_2 = 0$ and that the market is neutral towards $\theta_2$, i.e., $3\theta /3\theta_2 = 0$.)

4.9 Risk-free term structure of interest rates

In this model, we assume as usual that the riskless asset $V_R = 1$ is present. Additionally, however, we shall suppose that other assets also exist which are "risk-free" in the sense that their market values do not depend upon $\theta$, i.e., $3V /3\theta \equiv 0$. From the Fundamental Differential Equation we then have

$$0 = x_j + \frac{3V}{3t} - rV_j$$

(27)

a first-order differential equation in time alone. The solution to (27) yields the usual formulae for a world of certainty, containing generalized
exponential terms in t, when appropriate boundary conditions are applied. While this model is relatively simple, it is instructive to compare it to that of the following section.

4.10 Default-free term structure of interest rates

The term "default-free" herein used refers to certainty regarding valuation at asset maturity (i.e., a boundary condition), but does not rule out state-dependent price fluctuations as do the "risk-free" assets of the preceding model. Following the lead of Vasicek [1976], we consider the model wherein the riskless "spot" rate r provides the primitive state variable to the default-free assets. It then follows immediately from the dependent asset formula (13) that

$$0 = x_j(r,t) + \frac{3V}{3t} + \frac{\Omega(r,t)}{2} \frac{3V}{3r} + [u(r,t) + \Omega(r,t)G_1(r,t)] \frac{3V}{3r} - rV_j,$$  (28)

which is equivalent to Vasicek's [1976, Eq. (20)] formula for the default-free bond price structure. Alternatively, $G_1$ may be eliminated by the same method as in the common-factor model of Section 4.6. From (19) we then have

$$0 = x_j(r,t) + \frac{3V}{3t} + \frac{\Omega(r,t)}{2} \frac{3V}{3r} - \lambda \frac{3V}{3r} - rV_j$$  (29)

for some function $\lambda$ independent of $G$. 
5. KERNELS, DENSITIES, AND EQUILIBRIUM: SOME INTERPRETATIONS

In this section we provide further economic interpretations of the quantities and derivations of Sections 1-4; no significant new mathematical facts are introduced, but some solution methodology is discussed.

5.1 The meaning of the kernels $K$ and $G$

Most of the finance literature dealing with absolute valuation is based upon the utility maximization paradigm, so it is natural for the reader to wonder what the kernels $K$ and $G$ "mean" in terms of that paradigm. If it happens that investors possess tastes and beliefs, then clearly these (along with endowments, information, market institutional features, etc.) must be impounded in the market kernel $K$. However, when tastes and beliefs are heterogeneous, it is not an easy task to explain the "meaning" of $K$. Indeed, to do so would be equivalent to solving the general aggregation problem, a feat not yet accomplished in the current literature. About the best we can do is to say that under heterogeneous tastes and beliefs, $K$ represents an implicit price at time $t$ in state $\theta_\tau$ attached to a unit payoff at date $\tau \geq t$ in state $\theta_\tau$. However, this really says no more than formula (1) states, and provides no additional insights. Yet, there are two important points that should be emphasized in this context. The first is that existence of the implicit prices $K$ is assured by the absence of investor arbitrage (via the Farkas-Minkowski Lemma) even with heterogeneity. The second is that the integral operator equation (1) involving $K$ is a linear transformation. Whatever nonlinear and otherwise bizarre individual tastes are impounded in the market kernel $K$, no nonlinear "marketwide tastes" actually appear in the operator $K$, given the absence of riskless arbitrage in the economy. Indeed, it seems clear that $K$ may be separated (perhaps
not uniquely) as \( K(\theta_1, r, \theta_1, t) = f(\theta_1, r, t)p(\theta_1, r, \theta_1, t) \), where \( p \) is some state transition density. This form makes it clear that we then might identify an aggregate "impatience" aspect \( f \) and an aggregate "beliefs" aspect \( p \), arbitrage-free but there can be no nonlinear aggregate "tastes" aspect in this/equilibrium setting.

When tastes, beliefs, endowments, information, and other such essential characteristics are homogeneous across investors, the interpretations of the kernels \( K \) and \( G \) are much more straightforward. \( K \), in this case, is essentially the marginal rate of substitution between consumption now (at date \( t \) in state \( \theta_1 \)) and future consumption at the later date \( r \) in state \( \theta_1 \), weighted by the (agreed upon) conditional probability of attaining state \( \theta_1 \) at \( r \). If our observer also agrees upon the conditional probability (his density \( q \)), then the quotient kernel \( G = K/q \) is just the marginal rate of substitution alone. The quantity \( G_{11} \) is then the current rate of change (with respect to states) of the substitution rate between alternative states currently. Alternatively, since the marginal substitution rate between the current date/state and itself is 1, we may normalize by this latter quantity and so \(-G_{11}\) is also the (universal) absolute risk aversion coefficient, hence our intuitive labels in Section 4.1. Likewise, \(-G_{22}\) is the (universal) impatience factor, a fact most easily perceived when the utility function is separable ("additive") in \( r \). (For example, if the utility function is of form \( e^{-\rho r}U(\theta_1, r) \), then \(-G_{22} = \rho \). \( G_{11} \) is less intuitive, but apparently accounts for increasing/decreasing risk aversion, and is a necessary consequence of the diffusion-process analysis. Most fortunately, the diffusion assumption also implies that we need not worry about any higher degree terms in \( G \): they cannot enter into price behavior.
5.2 Integral vs. differential equations

The above discussion leads naturally to consideration of the role of our integral equation (5). In other words, the solution of the integral equation (5) is the same as the solution of the Fundamental Differential Equation (6) and we know the latter is entirely governed by $G_1$, $G_2$, and $G_{11}$ (given q). This means that all quotient kernels having the same partial derivatives $G_1$, $G_2$, and $G_{11}$ will yield the same solutions to the integral equation (5). Hence, we may choose any such kernel and integrate to solve the problem at hand. This fact resolves a confusing issue that has dogged the Black-Scholes [1973] option pricing model since its inception. The issue is this: although the Black-Scholes valuation differential equation is derived from arbitrage considerations, its solution may be arrived at by considering universal risk neutrality discounted and integrating (computing the/expected value of) the boundary condition. Why should this be so? Clearly, the same trick does not work for all differential equations. As one possible answer, Cox and Ross [1975, 1976] have argued as follows: the BS/N differential equations are independent of investor preferences; hence any preference structure consistent with the attainment of equilibrium and the distributional assumptions will suffice. Therefore, we may employ risk-neutral preferences for valuation since they are computationally the simplest. This argument, while appealing, can now be seen to be somewhat inaccurate. Comparing the "option pricing" (derivative asset) differential equation (14) with the Fundamental Differential Equation (6), we see that only those universal tastes and beliefs such that the relation $r_{0_t} = \alpha + \Omega_q$ is satisfied will in fact work. It happens almost by accident that $G_1 = 0$ (implied by universal risk neutrality) and $\alpha = r_{0_t}$ (applicable when q is lognormal
and \( r \) constant) are convenient choices, since most other arbitrary preference structures will fail to integrate the boundary conditions correctly. (See formula (30) for a counterexample.) Finally, the relation \( r_0 = \alpha + \Omega_1 \) successfully answers the nontrivial question of why, given constant tastes and constant stock price variance, the option pricing formula is unaffected by changes in the stock drift term \( \alpha \): in an arbitrage-free setting, such changes must be exactly offset by the interest rate \( r \).

To solidify the rather general interpretations given above, it may be useful to examine a highly specific example. Let us consider again the corn-crop (dependent asset) example where we suppose that there is a state variable \( \theta \) ("weather") such that corn yield at harvest is given in consumption units by \( \max[0, \theta_T - W] \) where \( W \) is some critical value below which the crop fails and \( T \) is the harvest date. Suppose that \( \alpha(\theta, t) = \alpha_0 \theta \) and \( \Omega(\theta, t) = \Omega_0 \theta^2 \), where \( \alpha_0 \) and \( \Omega_0 \) are constants (geometric Brownian motion). Suppose further that the economy is characterized by \( G_1 \equiv \frac{\theta_T}{\theta_T} \), \( \theta_T \) a constant (implied by, for example, universal constant-proportional-risk-aversion (CPRA) tastes); \( G_2 = -\rho \); and \( G_{11} = \frac{\theta_T}{\theta_T} \). Then the quotient kernel \( G(\theta, t, \theta, t) = e^{\rho(t-t)}(\theta_T^{(t-t)})^{G_{11}} \) is a consistent choice. The riskless interest rate is found to be a constant \( r = \frac{\alpha_0}{\rho} + \frac{\Omega_0}{2} \frac{\theta_T}{\theta_T} \). Integrating the boundary condition \( \max[0, \theta_T - W] \) (since by the intertemporal parity principle we are free to choose any induced payoff stream and "wait until harvest" stream is the simplest choice) with respect to \( G \) and the geometric Brownian motion transition density, we find the valuation solution for the corn crop, \( V_c \), as

\[
V_c(\theta, t) = \theta e^{\int_0^t \left[ (\alpha_0 + \Omega_0 - r)(T-t) \right]} \left[ \frac{\ln(\theta_T/W) + (\alpha_0 + \Omega_0 - \Omega_0/2)(T-t)}{\sqrt{\Omega_0(T-t)}} \right]
\]

\[
- We^{-r(T-t)} \frac{\ln(\theta_T/W) + (\alpha_0 + \Omega_0 - \Omega_0/2)(T-t)}{\sqrt{\Omega_0(T-t)}}
\]

(30)
where \( \Phi(\cdot) \) is the cumulative normal distribution. Note that equation (30) has been constructed in a fashion that resembles the Black-Scholes solution, but that no riskless hedge argument was involved in its derivation; Rubinstein [1975] has employed a similar approach to examine option pricing in discrete time.

In more general terms, we see that there is a straightforward (if occasionally tedious) methodology for solving every differential equation developed in this paper: establish the relevant quotient kernel and density and integrate the boundary conditions via (1), numerically or otherwise. The intertemporal parity principle justifies this approach. (Alternatively, we may employ the classical tools of partial differential equation analysis. But the classical solution usually involves integration via the corresponding Green's function, which we already know to be just \( G_0 \).) The integral equation forms also possess the mild advantage that they can easily deal with discrete payoffs, such as occur with coupon bonds, whereas the differential equation forms will have singularities at those dates. 22 An additional advantage of employing integral representations over their differential counterparts is notational: the integral equation (1) is written the same whether \( \theta_t \) is a vector or scalar, whereas the differential equation (6) must be augmented with the cross-product terms arising from Itô's lemma, or else we must create additional notation as in the appendix.

5.3 Transition densities

In previous theory, the general approach to the valuation problem has been to impose boundary conditions on the valuation functions and solve their differential equations accordingly. The methodology proposed in the current paper suggests a direct solution of the integral form (1) as an alternative approach; hence it is necessary to determine the transition density \( q \) that governs in various contexts. Of course, the
parameters $a(\theta_t, t)$ and $\Omega(\theta_t, t)$ must be known from prior information. But it is also well known that the transition densities must satisfy the Fokker-Planck diffusion equation

$$
\frac{\partial q(\theta_{t+1}, \theta_t, t)}{\partial t} = \frac{1}{2} \frac{\partial^2}{\partial \theta^2} \left\{ a(\theta_{t+1}, t) q(\theta_{t+1}, \theta_t, t) \right\} - \frac{3}{2} \frac{\partial}{\partial \theta} \left\{ \Omega(\theta_{t+1}, t) q(\theta_{t+1}, \theta_t, t) \right\}.
$$

When the underlying state variable process is subject to absorbing and reflecting boundaries, formula (31) may be solved with the appropriate boundary conditions: $q(\theta_{t+1}, \theta_t, t) = 0$ along the absorbing boundaries and $\frac{1}{2} \frac{\partial}{\partial \theta} \left\{ a(\theta_{t+1}, t) q(\theta_{t+1}, \theta_t, t) \right\} - \Omega(\theta_{t+1}, t) q(\theta_{t+1}, \theta_t, t) = 0$ at the reflecting barriers. Absorbing boundaries arise for financial assets when bankruptcy, reorganization, bond calls, and other such events are considered. Reflecting barriers are found in regulated currency markets, commodity markets, and government intervention to save failing companies.

Solving for the transition densities $q$ by means of (31) and the boundary conditions thus provides a new set of possible applications: for example, valuation of options written on callable or convertible securities, valuation of options and corporate liabilities under threat of firm bankruptcy, etc. Almost all of these applications now remain unexplored.

5.4 Observers, observables, and general equilibrium

Beliefs enter into our analysis at two points. First, there are the multiple, usually heterogeneous, beliefs possessed amongst all investors, which are impounded in the market kernel $K$. Second, there is the belief $q$ posited by some observer of the asset markets. In this section we examine two interpretations of the observers' role.
One useful interpretation identifies $q(\theta_t, T, \theta_t, t)$ as a probability which represents the viewpoint and knowledge of an "external" observer of the asset markets. This interpretation is important because it gives empirical content to our theory, inasmuch as it permits the empirical scientist to identify certain quantities which are otherwise jointly unobservable with $q$ (primarily the quotient kernel $G$ and its derivatives). In other words, the external observer can independently study the movements of the relevant state variables and decide upon those functions $a(\theta_t, t)$ and $\Omega(\theta_t, t)$ that he feels are most descriptive of the diffusion process. Using these functions as a hypothesis, he is then in a position to identify a viable quotient kernel: the existence of asset prices provides the observables required to infer the values of $G_1$, $G_2$, and $G_{11}$. For example, existence of a risk-free asset alone is sufficient to simultaneously identify all of these quantities from formula (7), given $\alpha$ and $\Omega$, by regression techniques or similar means.

An alternative interpretation of the transition density $q$ is to suppose that it is a belief held internally by the $i$-th market agent. Since there is no a priori reason to suppose that all market agents will have the same beliefs, we are led notationally to provide the quantities with the subscript $i$: $q_i$, $G_i$, $a_i$, and $\Omega_i$. When the probabilities, as posited by the market agents themselves, are treated via the formalism of Sections 3 and 4, significant general equilibrium implications result. Intuitively speaking, the introduction of new marketable assets creates new observables (prices and payoffs) which will tend to cause the otherwise heterogeneous agent beliefs to converge.

For example, suppose that the riskless asset $A_r$ exists. Rewriting (7) for the beliefs of market agent $i$, we have
\[ r(\theta_t, t) = -\{\alpha_1 G_{11} + G_{12} + \frac{1}{2} G_{111}\}. \]  

(32)

The left-hand side of (32), \( r(\theta_t, t) \), is an observable and, moreover, does not depend upon \( i \), a fact which follows from the atomistic competition feature of assumption (II). Thus the right-hand side of (32) must also be invariant with respect to the market agent \( i \) selected. This means that not every collection of agent beliefs is compatible with assumptions (I)-(VI): when the riskless asset is present, only those beliefs which leave the right-hand side of (32) identical for each agent will work.

The introduction of additional assets will further serve to make agent beliefs converge. Consider next the primitive asset equation (8), rewritten as

\[ 0 = x_p + \alpha_i + \Omega_i G_{11} - r\theta_t. \]  

(33)

When this primitive asset is also present, beliefs will converge to the point that the quantity \( \alpha_i + \Omega_i G_{11} \) must be the same for each market agent, since \( x_p \) and \( r\theta_t \) are observable and independent of any agent.

Finally, when a derivative asset is also present, we may rewrite (14) as

\[ 0 = x_j + \frac{3V_j}{\delta_t} + \frac{1}{2} \frac{\partial^2 V_j}{\delta_t^2} + [r\theta_t - x_p] \frac{3V_j}{\delta_t} - rV_j \]  

(34)

The observables \( x_j, x_p, V_j, r, \) and \( \theta_t \) cannot depend on \( i \); this implies that \( \Omega_i \) must be independent of \( i \) also, that is, \( \Omega_i = \Omega \) for every agent \( i \). In other words, when the riskless asset, the primitive asset, and the derivative asset are all present in the economy, all agents' beliefs must converge to the point that each will hold the same estimate of the instantaneous variance \( \Omega(\theta_t, t) \). Otherwise, one of the assumptions (I)-(VI) has been violated, since we have shown that they together imply (34).
6. SUMMARY AND EXTENSIONS

The intertemporal parity principle, a differential equation which governs the valuation of all financial assets under the assumptions of no arbitrage, diffusion processes, and continuous, frictionless markets. As assets are introduced into the marketplace, structure arises through the simultaneous determination of asset values via the differential equations. We have discovered, for example, different valuation formulas for the cases of dependent vs. derivative assets, the former of which contains new terms not found in the BS/M equations. At the same time, we have provided a new and more general derivation of the BS/M equations themselves. The relationship between universal risk neutrality and the integral solution of the BS/M equations has also been explained, and completes the Cox-Ross [1975, 1976] argument. New insights and solution methods arising from the intertemporal parity principle have been set forth. Moreover, considerable integration over the existing absolute and relative pricing models has been achieved.

Yet, it is clear we have only scratched the surface of this very general topic. An obvious few among the several questions remaining are: How might $G_1$, $G_2$, and $G_{11}$ be best empirically estimated? What further identifications of the kernels [e.g., "return on the market portfolio"] vs. $(\partial / \partial_t)\theta$, Rubinstein, 1975] can be made? What about solutions for the effects of bankruptcy, reorganizations, and other "absorbing" or "reflecting" states upon the diffusion processes considered? Many such questions remain to be investigated.
Appendix

FUNDAMENTAL DIFFERENTIAL EQUATION FOR A VECTOR-VALUED STATE VARIABLE

Suppose the state variable $\theta_t$ is a vector in $\mathbb{R}^n$. In formulas (3) and (4), consider the instantaneous drift $a(\theta_t, t)$ a vector and $\Omega(\theta_t, t)$ an instantaneous covariance matrix. Let $\nabla^r$ be the gradient operator with respect to the components of $\theta_t$, and let $\nabla^2f$ be the corresponding Hessian matrix. Then the Taylor series expansion of the function $f(\theta_t, r)$ around $(\theta_t, t)$ is

$$f(\theta_r) = f(\theta_t) + \nabla^r f(\theta_t - \theta_r) + \frac{1}{2}(\theta_r - \theta_t)(\nabla^2 f(\theta_r - \theta_t)) + \cdots$$

Applying this to the proof of Fundamental Differential Equation, we arrive at the vector analogue of its result:

$$0 = x_j + \frac{3v_j}{2t} + \frac{1}{2}(v^2 v_j \Omega + \nabla v_j (a + \Omega \times (v v)^r)) + [(v v \cdot a) + \frac{6}{3t} + \frac{1}{2}(v^2 v \Omega \Omega)]v_j,$$

where a prime denotes the transpose, a dot stands for matrix multiplication, and the "box-multiply" symbol, $\Box$, is a dyadic matrix operator such that $A \Box B = \sum_{i,j} a_{ij} b_{ij}$, for matrices $A = [a_{ij}]$ and $B = [b_{ij}]$. 

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FOOTNOTES

1. It is interesting to note that while use of the Dirac delta notation has been made in connection with asset valuation equations, the solution formalism provided by the Dirac delta function method has apparently not been utilized. For example, Merton's [1974, p.468] formula for coupon bonds can apparently be solved by this interchange-of-limits technique.


3. This is true, however, only for a fixed set of assets. When a new asset is introduced, \( \mathbb{K} \) may change if the new asset is not already spanned by existing assets. (Therefore, the introduction of new assets provides structure to \( \mathbb{K} \), a fact we shall use later.) Once new assets have been introduced, the new set of assets is again fixed, and the new operator \( \mathbb{K} \) must again apply to all of them.

4. Rubinstein [1975].

5. For an illustration of this phenomenon, see, for example, Duffin and Karlovitz [1965].

6. Strictly speaking, the summation index \( J \) in both constraints ranges only over the set of assets marketable at time \( s \). Assumption (VI) will later dispense with this fine point.

7. Warning on notation: Although it could have been so expressed, (1) is not a stochastic integral. We use subscripts on the state variables only to distinguish them from each other, and not as being from index sets except when employed in conjunction with the "tilde" notation for stochastic processes.
8. This form of argument becomes important in considering the boundary conditions of the "decision-dependent" assets such as the American put option.

9. See Fama [1970b].

10. See Feller [1966, p. 321] for an exposition of these conditions. Basically, (2) is the defining characteristic of diffusion processes: in a short interval of time, the state changes only a small amount (with probability 1). Although they appear so, (3) and (4) are not really separate assumptions, since they can be derived from (2).

11. Note that (5) has not yet made any use of the diffusion assumption, and so applies equally well to jump processes, etc.

12. In terms of Itô derivatives, the Fundamental Differential Equation may be written simply as

\[ \frac{d(\gamma_j)}{\delta t} + x_j \frac{d}{d\delta t} = 0. \]

The rationale of this form becomes apparent when we consider the analogous formula for a world of certainty where the state variables move smoothly: any changes in the price (multiplied by the marginal substitution rate) must be exactly offset by immediate payoffs.


14. It is more common to define the risk-free asset in return space rather than in price space as we have done here. Besides possessing some subsequent mathematical advantages, the latter approach sensibly identifies risklessness with freedom from stochastic capital value changes. Definitions formulated in return space tend to confuse constancy in the interest rate with a risk-free nature of the asset. (Note that
stochastic changes in the interest rate represent no instantaneous risk; capital invested at this rate will instantaneously grow in a nonstochastic fashion, by Itô's lemma, regardless.)

15. In terms of Itô derivatives, the risk-free interest rate is simply expressed as

\[ r = \frac{\partial V}{\partial t} \]


17. Steve Ross suggested this model in a private communication.

18. Seemingly, the most reasonable way to approach the assertion that \( r \) depends on no other variable besides \( \Sigma \) is to additionally assume that \( r \) is constant. But, see footnote 21.

19. Fischer Black suggested this problem in a private communication.

20. Rosenberg [1972].

21. Note that the commonly employed assumption of constancy in the risk-free rate is really a severe restriction on admissible market kernels (i.e., tastes and beliefs) and diffusion processes. We see from (7), in fact, that it is generally **inconsistent** to assume simultaneously a constant interest rate, a given diffusion process, and arbitrary investor preferences. (Of course, inconsistent assumptions will then imply anything.) More than one paper has foundered upon these hidden shoals, which have been charted, in part, recently by Rosenberg and Ohlson [1976].

22. See footnote 1.

23. Of course, we cannot really say **which** of the assumptions is violated, since they are applied jointly.
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