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Inflation and Optimal Portfolio Choices

by

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INFLATION AND OPTIMAL PORTFOLIO CHOICES

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I owe a great debt to Richard Roll and his elegant method of deriving an analytical expression of the efficient frontier (Roll, (1976)). Part of this research was financed by the Fondation Jouy-Entreprises.
Abstract

This paper studies the influence of stochastic inflation on the composition of investors' optimal portfolios. Assuming that investors are real mean-variance optimizers, we are able to derive an analytical expression of the efficient set. Efficient portfolios for investors subject to different inflation rates are compared in some details. If consumption tastes vary, the market portfolio will not be efficient for most investors, neither will it be nominal mean-variance efficient. It is shown that even if the correlation between asset returns and inflation is small, the influence of inflation on optimal portfolio selection could be quite significant; an alternative a priori measure of the importance of stochastic inflation on market equilibrium is proposed.

The last section deals with the use of nominal return tests of asset pricing theory. It is suggested that these tests are poorly designed to test asset pricing theories and the influence of inflation on market equilibrium.
INFLATION AND OPTIMAL PORTFOLIO CHOICES

Capital market equilibrium has been extensively studied in the recent past, mostly in a mean-variance framework. In a perfect capital market with riskless assets and homogenous expectations among risk-averse investors, Sharpe and Lintner have shown that the efficient set of all investors could be described by only two portfolios (or mutual funds):

(1) the stock market portfolio
(2) the riskless asset.

More recently Black (1972) considered the case where no risk-free asset existed. He came to the conclusion that the stock-market portfolio was still efficient. The efficient set for all investors was then described by a combination of two portfolios:

(1) the stock market portfolio

(2) another efficient portfolio chosen to be the 0-beta minimum variance portfolio, or $R_Z$ portfolio.

Several authors have been tempted to link the nonexistence of a risk-free asset to inflation and interpret the Black results as dealing with the general case of capital market equilibrium under inflation.

It will be shown in this paper that the theoretical influence of inflation on mean-variance equilibrium is more perverse and will generally imply that the market portfolio be nonefficient.

With a few notable exceptions, CAPM specialists have focused on the descriptive implication of the theory (asset pricing) rather than on
the normative implications (composition of the efficient portfolios).

The usual asset pricing relation might be written as:

$$E(r_i) = E(r_Z) + \beta_i (E(r_m) - E(r_Z)) \text{ for all } i$$  \hspace{1cm} (1)

where $E(r_i)$ is the expected return on portfolio or stock $i$.

$Z$ is the 0-beta portfolio mentioned above which might be the riskless asset if it exists.

$$\beta_i = \frac{\text{Cov}(r_i, r_m)}{\text{Var}(r_m)}$$

As stressed by Roll (1976), this result is not interesting per se; this is a purely technical result due to the use of the mean-variance framework. Mathematically, equation (1) will always be valid for any mean-variance efficient portfolio $m$, independently of any assumption on investors' behavior or asset returns. The only economic conclusion of the CAPM is to state that the market portfolio should be mean-variance efficient and therefore among those portfolios $m$ for which equation (1) holds. Given this direct and technical link between portfolio efficiency and asset pricing (1), it seems preferable to focus on determination of optimal portfolios; pricing relations are then trivial derivations.

In this paper we will assume that assets have nominal payoffs and that investors care about real returns. It seems to be commonly accepted that real returns can be computed from nominal returns by simply subtracting the investor's inflation rate (Friend and Landskroner (1976); Roll (1973); Chen and Bonness (1975)). The characteristics of the multi-consumption utility functions which would lead to this behavior have not
been specifically studied, so we will simply assume from now on that
investors do measure real return by simply subtracting their relevant
inflation rate from nominal returns and are real mean-variance (MV)
optimizers.

The set of efficient portfolios and market equilibrium conditions
will be derived analytically when investors have diverse real wealth
measures. Then we will study what results should be expected from MV
analysis conducted on nominal returns in a world where investors are
real wealth optimizers. One of the conclusions will be that, if infla-
tion does affect asset returns, the market portfolio should not be nom-
inal efficient; therefore tests of the traditional CAPM or its zero-beta
version should not be expected to yield a positive and perfect answer.

Section I provides an analytical derivation of the portfolio
efficient set for each investor and how it will be viewed in the nominal
MV space. Section II considers the special case where there exist a
nominal risk-free asset and Section III investigates whether tests con-
ducted solely on the basis of nominal returns (Sharpe-Linter or zero-
beta version) are appropriate.

I. Portfolio Efficiency: The General Case.

1. The Efficient Frontier

The efficient portfolios of investor \( k \) will be the solutions
to the optimization problem, in matrix form:
Table 1

NOTATIONS

\[ X \text{ An } n \text{ column vector of investment proportions in each of the } \]
\[ n \text{ risky assets, } \| x_i \| \]
\[ V \text{ Is an } n \times n \text{ covariance matrix of nominal returns on these } \]
\[ \text{assets, } \| \text{cov}(\hat{\gamma}_i, \hat{\gamma}_j) \| \]
\[ C_k \text{ Is a } n \text{ column vector of covariances between each asset nom-} \]
\[ \text{inal return and the investor's inflation rate } \tau_k, \]
\[ \| \text{cov} (\hat{\gamma}_i, \tau_k) \| \]
\[ E \text{ Is a } n \text{ column vector of asset expected returns } \]
\[ u \text{ Is a } n \text{ vector where all elements are equal to one } - \]
\[ r, \tau \text{ Denote expected returns, i.e., } r = E(\hat{\gamma}) \]

\[
\min_{X} \left\{ \frac{1}{2} X'VX - X'C_k \right\}
\]

S.t. \[ X'E = \delta \]

\[ X'u = 1 \]  

with the notations given in Table 1.

Of course we will assume some correlation between inflation and nom-
inal asset returns (\( C_k \neq 0 \)), otherwise we are back to the standard problem
and this paper has no raison d'être.

Using Lagrange multipliers for the two constraints, first order optimal-
ity conditions are:

\[
VX = C_k + \lambda_1 E + \lambda_2 u
\]  

(3)
We will first treat the general case where no nominal nor real riskless asset exist and \( V \) is nonsingular. Then (3) can be solved for the optimal portfolios \( X \).

\[
X = V^{-1}C_k + \lambda_1 V^{-1}E + \lambda_2 V^{-1}u
\]  

(4)

To eliminate \( \lambda_1 \) and \( \lambda_2 \), we premultiply (4) by \((\text{Eu})'\) and use the two constraints:

\[
(\text{Eu})'X = (\text{Eu})' V^{-1}(\text{Eu}) \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} + (\text{Eu})' V^{-1}C_k
\]  

(5)

knowing that by constraint

\[
(\text{Eu})'X = (r_p, 1)'
\]

where \( r_p \) is the expected nominal return on the portfolio.

Let's call \( A \) the 2 x 2 matrix identical for each investor, \( A = (\text{Eu})' V^{-1}(\text{Eu}) \).

Then we can solve (5) for \( \lambda_1 \) and \( \lambda_2 \) and substitute in (4)

\[
\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = A^{-1} \begin{bmatrix} r_p - E' V^{-1}C_k \\ 1 - u' V^{-1}C_k \end{bmatrix}
\]

\[
X = V^{-1}(\text{Eu})A^{-1} \begin{bmatrix} r_p - E' V^{-1}C_k \\ 1 - u' V^{-1}C_k \end{bmatrix} + V^{-1}C_k
\]  

(6)

This is the analytical description of all the real mean-variance efficient portfolios for an investor with inflation rate \( \tau_k \). All interesting results will be derived from equation (6).
The efficient frontier will be the locus of portfolios $P$ described by (6). If we note $\sigma_p'$ and $r_p'$ the real standard deviation and expected return of such portfolios, it is demonstrated in the appendix that the analytical expression of the efficient frontier in the real mean-variance plane is:

$$\sigma_p'^2 = \sigma_k'^2 - \sigma_{\theta_k}^2 + \frac{1}{ad-b^2} \left[ d(r_p' + r_{p_k} - e_{\theta_k}r_{\theta_k})^2 - 2b(1-e_{\theta_k})(r_p' + r_{p_k} - e_{\theta_k}r_{\theta_k}) + a(1-e_{\theta_k})^2 \right]$$ (7)

where $\theta_k$ is the portfolio whose composition is:

$$\frac{1}{u'V^{-1}C} \text{ and } \sigma_{\theta_k}^2, r_{\theta_k} \text{ are its variance and return.}$$

$$a = E'V^{-1}E \quad d = u'V^{-1}u$$

$$b = u'V^{-1}E \quad e_{\theta_k} = u'V^{-1}C_{\theta_k}$$

While this expression (7) might look complicated, it only states that the real variance is a second degree polynomial of the real expected return of the portfolio. This is the expression of a parabola in the mean-variance (MV) plane and an hyperbola in the mean-standard deviation (M-SD) plane. This result is not new, since it is only a direct translation of the Merton (1972) and Roll (1976) findings using real returns terminology but other interesting conclusions will be drawn from this specific formulation.
The minimum real variance portfolio $\Theta^*_k$ is determined in Appendix A. Its composition:

$$V^{-1}C_k + \frac{1 - u'V^{-1}C_k}{u V^{-1}u} V^{-1}u$$

reflects the inflation hedging behavior of investors.

As seen in (6) the investor will hedge his total wealth against unexpected change in his purchasing power by selecting $V^{-1}C_k$. This decision is independent of risk aversion or expected returns. This is reflected in the real minimum variance portfolio $\Theta^*_k$. Intuitively, one might have expected this portfolio to be exactly equal or proportional to $V^{-1}C_k$; this will be true only if $V^{-1}C_k$ represents a portfolio $(u'V^{-1}C_k = e_k = 1)$. The reason is simple; by selecting $V^{-1}C_k$ the investor will be left with $1 - u'V^{-1}C_k$ to invest. This forced investment will increase the total risk of his portfolio. The optimal way to do it will be to invest this sum $1 - e_k$ in the stock portfolio with minimum nominal risk: $\frac{V^{-1}u}{u V^{-1}u}$. Hence the composition of $\Theta^*_k$.

A graphical representation of the efficient set in the real M-S.D. space is depicted on figure 1.
Analytical Expression: Nominal MV Space

Each investor might measure real returns differently because of his consumption tastes and therefore use a different "mean-variance" plane. Since nominal returns are the only directly-observable variables, it is interesting to compare the various sets of efficient portfolios in the nominal M-V plane. In other words, we assume that investors are real mean-variance optimizers and will observe how their efficient portfolios are represented in a nominal mean-variance framework.

The composition of their optimal portfolios $X$ is given in (6). These portfolios will have a nominal variance $\sigma^2_p = X' V X$. It is easily derived in the appendix A that the efficient set will be represented by:

$$\sigma^2_p = e_k' \sigma_{0k}^2 + \frac{1}{ad-b^2} \left[ dr_p^2 - 2br_p + a - e_k^2 (dr_{0k}^2 - 2b r_{0k} + a) \right] \quad (8)$$

The efficient set of investor $k$ will again appear as an hyperbola (parabola) in the mean-standard deviation (variance) plane. More importantly, since $a, b, \text{ and } d$ are common to all investors, curves for all investors
will have several common characteristics. Equation (8) might be written as:

\[ \sigma_p^2 = \frac{1}{ad-b^2} [dr_p^2 - 2br - a] + \lambda_k \]  

(9)

where \( \lambda_k \) is a constant. When this constant is zero, (9) describes the efficient set of a nominal mean-variance optimizer (no inflation):

\[ \sigma_p^2 = \frac{1}{ad-b^2} [dr_p^2 - 2br + a] \]  

(10)

This implies that all curves will be homothetic ("parallel") parabola in the nominal mean-variance plane. In the nominal mean-S.D. plane, the minimum variance point will have the same return, \( r = \frac{b}{d} \) and they will all have the same asymptotes. The real minimum-variance portfolio \( \delta^*_k \) described in appendix A will generally not appear at the minimum nominal variance point of curve \( T_k \); it will be somewhere on the curve but not with a nominal return \( \frac{b}{d} \).

![Figure 2](image-url)
It should be clear that the nominal efficient frontier $T_0$ is the envelope of all the curves. For investor $k$, his efficient curve $T_k$ represents a set of portfolios efficient in real terms, however, in the nominal plane, the nominal efficient frontier $T_0$ will dominate all other stock portfolios; so the curve $T_k$ will be strictly inside (or equal to) the curve $T_0$. This could be seen mathematically by noting that $A_k$ will never be negative. If $A_k$ was negative, it would mean that for portfolio $\theta_k$, we would have:

$$\sigma_{\theta k}^2 < \frac{1}{ad-b^2} [d\sigma_{\theta k}^2 - 2br_{\theta k} + a]$$

but this is impossible since the locus of the optimal nominal mean-variance portfolios is given by (10). For any portfolio, this inequality must hold:

$$\sigma_p > \frac{1}{ad-b^2} [d\sigma_p^2 - 2br_p + a]$$

One might wonder how a portfolio could be (in figure 1) to the left of $T_k$ and not be chosen by investor $k$. The obvious reason is that investors care about their real returns and covariances, not nominal ones. A portfolio which lies to the left of curve $T_k$ in the nominal plane will be strictly inside the efficient frontier of investor $k$ in his real mean-variance plane and as such unattractive. Since efficient curves $T_k$ are
either identical or have no common point (except asymptotically), ef-
ficient portfolios will generally be strictly different for investors with
different inflation rates; despite homogeneous nominal expectations,
there will be no overlap of portfolio efficient sets (even with varying
degrees of risk aversion).

2. Market Equilibrium and Separation Theorems

It will be useful to come back to equation (4):

\[ X = V^{-1}C_k + \lambda_2 V^{-1}E + \lambda_2 V^{-1}u \]  

since \( u'X = 1 \), it implies:

\[ 1 = u'V^{-1}C_k + \lambda_1 u'V^{-1}E + \lambda_2 u'V^{-1}u \]

therefore:

\[ \lambda_2 = \frac{1}{u'V^{-1}u} \left[ 1 - u'V^{-1}C_k \right] - \lambda_1 \frac{1}{u'V^{-1}u} u'V^{-1}E \]

and

\[ X = V^{-1}[C_k + \frac{1-e_k}{d} u] + \lambda_1 V^{-1}[E - \frac{b}{d} u] \] 

The first term on the right hand side is the minimum real variance port-
folio \( \theta_k^* \) introduced before the second term is common to all investors.
Let's note \( W \) the vector \( V^{-1}(E - \frac{b}{d} u) \), then:

\[ X = \theta_k^* + \lambda_1 W \]
This looks like a two funds separation theorem where one fund is common to all investors \((W)\). This would be slightly misleading since \(W\) does not represent a portfolio \((uX = 0)\) and would be only held at infinity \((\lambda_1 \rightarrow \infty)\). As mentioned above, no simple separation theorems can obtain since efficient portfolios will be strictly different among investors with different consumption tastes. A more common way to describe the set of efficient portfolios of investor \(k\) from a combination of two real (efficient) portfolios would be:

\[
X = (1 - \lambda)\theta_k^* + \lambda(W + \theta_k^*)
\]  

(13)

**Aggregation**

Aggregating the demand for all assets and adding it up to the supply (market portfolio \(M\)), we get:

\[
M = \theta_m + \lambda W
\]

(14)

Where \(\theta_m\) is a wealth weighted average of all \(\theta_k\), it can be written in exactly the same form as \(\theta_k\):

\[
\theta_m = \frac{1}{\lambda} \bar{\tau}_m + \frac{1 - \lambda}{\lambda} \frac{u}{\lambda}^{-1}
\]

(15)

where \(\bar{\tau}_m\) is the vector of covariances \(\text{cov}(\bar{\tau}_m, \bar{\tau}_m)\) with \(\bar{\tau}_m\) being the wealth weighted average of all \(\bar{\tau}_k\). In another fashion, (14) could be written as:

\[
M = (1 - \lambda)\theta_m^* + \lambda \bar{W} + \bar{\theta}_m^*
\]

(16)

These derivations lead to the following conclusions:

1. The market-portfolio will be efficient for the investor with inflation rate \(\bar{\tau}_m\). Such an "average" investor might not exist.
2. The market portfolio will be strictly inefficient for any other investor.

3. In particular, the market portfolio will always be strictly inside the nominal efficient frontier. It cannot be nominal efficient.

In the nominal mean-S.D. plane it will look somewhat like figure 3.

If everyone has the same consumption tastes and inflation rate, then the average inflation rate will also be that common one. The stock market portfolio will be efficient in real terms and a two fund separation theorem will apply with funds M and 0 (for example). However, M should still be inside the nominal optimal frontier T_0.

Now that the general case has been worked out mathematically, this paper will focus on the specific case where a nominal riskless asset exists.
II. **Equilibrium with a Nominal Riskless Asset**

1. **The Efficient Frontier**

Let's assume the existence of a nominal bond with return $r_0$. The demand vector $X$ for risky assets should now be complemented by the demand for this nominal riskless asset, $x_0$. With the previous notations, we will have:

$$u'X + x_0 = 1$$  \hspace{1cm} (17a)

$$VX = C_k + \lambda_1 E + \lambda_2 u$$  \hspace{1cm} (17b)

$$0 = \lambda_1 r_0 + \lambda_2$$  \hspace{1cm} (17c)

in (17b) we can replace $\lambda_2$ by $-\lambda_1 r_0$ and get:

$$VX = C_k + \lambda_1 (E - r_0 u)$$

The efficient portfolios are now described by (18):

$$X = V^{-1}C_k + \lambda_1 V^{-1}(E - r_0 u) = V^{-1}C_k + \lambda_1 V^{-1}E_0$$  \hspace{1cm} (18)

$$x_0 = 1 - u'V^{-1}C_k - \lambda_1 u'V^{-1}E - \lambda_1 r_0 u'Vu^{-1} = 1 - e_k - \lambda_1 (b-dr_0)$$

with $E_0 \equiv E - r_0 u$, the vector of expected excess returns $\rho_i \equiv r_i - r_0$.

In order to derive the equation of the efficient frontier, the route to be taken is less elegant than in the first part since the full matrix of nominal covariances would be singular if the nominal riskless asset was included. One has to separate as in (18) between the nominal risky and riskless assets. In appendix B, the various mathematical
derivations are performed. The main conclusions are as follows:

1. The efficient frontier is still an hyperbola in the real M-S.D. space. (Appendix B-1.)

Its equation is:

\[ \sigma_p^2 = \frac{\sigma_k^2 - \epsilon_k \rho \delta k}{\alpha} \left( \rho_p - \epsilon_k \delta k \right)^2 \]  

(19)

with \( \rho_p = r_p - r_o \) excess expected return

\[ \alpha = E_0 \mathcal{V}_0^{-1} = \mathcal{V}_o \frac{d \mathcal{V}_o^2}{\mathcal{V}_o^2} - 2br_o + a \]

2. The minimum-variance portfolio \( \Theta_k^* \) is composed of an investment \( \mathcal{V}_k \) in stocks and \( 1 - u \mathcal{V}_k^{-1} \) in the nominal riskless bond. (Appendix B-2.)

This portfolio has a return equal to \( \epsilon_k \Theta_k \) and standard deviation \( \epsilon_k \Theta_k \). The only difference with the previous case is that the riskless bond replaces the minimum nominal variance portfolio \( \mathcal{V}_k^{-1}u \). The same property applies in the no inflation case. (E.g., Roll (1976).)

3. The set of efficient portfolios is represented by an hyperbola in the nominal M-S.D. plane. For all investors, their hyperbola have the same asymptote (the nominal efficient line) and focus \( (r_o, \text{the nominal riskless asset}) \).

As shown in the appendix the hyperbola is given by:

\[ \sigma_p^2 = \epsilon_k \sigma_k^2 + \frac{\rho_p - \epsilon_k \rho \delta k}{\alpha} \]  

(20)

For an investor whose inflation rate is constant, it reduces to:

\[ \sigma_p = \frac{1}{\sqrt{\alpha}} \rho_p \]  

(21)
The asymptote of the hyperbolae. The only difference with the previous case is that the envelope of all the hyperbolae is now a straight line instead of another hyperbola.

![Diagram of hyperbolae and straight lines](image)

2. Market Equilibrium

The set of efficient portfolios may be described by a relation very similar to that of section 1-2.

\[ \theta_k^* + \lambda W \]  \hspace{1cm} (22)

where \( \theta_k^* \) and \( W \) are defined as:

\[
\begin{bmatrix}
X \\
x_0
\end{bmatrix} = \begin{bmatrix}
V^{-1}C_k \\
1 - e_k
\end{bmatrix} + \lambda \begin{bmatrix}
V^{-1}E_o \\
-1V^{-1}E_o
\end{bmatrix} \]  \hspace{1cm} (22a)

To clear the market, we must have:
\[ M = V^{-1}C_\omega + \lambda_m V^{-1}E_o \]

\[ 0 = 1 - e_\omega - \lambda_m u V^{-1}E_o \]

Eliminating \( \lambda_m \), we get:

\[ M = V^{-1}C_\omega + \frac{1 - e_\omega}{u V^{-1}E_o} V^{-1}E_o \]  \hspace{1cm} (23)

If \( \frac{V^{-1}E_o}{u V^{-1}E_o} \) is called portfolio \( M_o \), (21) is equivalent to

\[ M = V^{-1}C_\omega + (1 - e_\omega)M_o \]  \hspace{1cm} (24)

The conclusions derived in section I-2 apply similarly, and the situation is depicted in figure 4.

It should be stressed that, generally, everyone will be holding (short or long) some nominal risk-free bill \( (x_o \neq 0) \). This conclusion would be changed if there existed a real risk-free for every investor (a very unlikely feature!). Then the real efficient set of each investor would be made up of a combination of his real risk-free asset and a risky portfolio of stocks and other investors real risk-free bills and no one will wish to hold a nominal riskless bill. In the real M-S.D. plane the efficient frontier would now be a straight line, but it will still be represented by an hyperbola in the nominal M-S.D. plane and all the other conclusions will apply.

It should be noted from (22) that for each investor there exists only one value of \( \lambda \) which will make \( x_o \) to be zero. Therefore, there is only
one all stock portfolio which will be efficient for each investor. This all stock portfolio will turn out to be the market portfolio for the "average" consumer \( \bar{\tau} = \bar{\tau}_w \).

The interested reader will notice that even with this kind of diverse measure of wealth (real expectations), asset pricing equations are again simple by-products.

Equation (23) may be rewritten as:

\[
VM - C_\omega = \frac{1 - e_\omega}{u V^{-1} E_0}
\]  

(25)

Where the RHS is equal to a constant times the vector of (expected) excess returns. To eliminate this constant we left-multiply by \( M' \) and get

\[
M' VM - M' C_\omega = \frac{1 - e_\omega}{u V^{-1} E_0} \rho_m
\]  

(26)

Since \( VM - C_\omega = \| \text{cov}(\bar{\tau}_i, \bar{\tau}_m) - \text{cov}(\bar{\tau}_i, \bar{\tau}_w) \| = \| \text{cov}(\bar{\tau}_i, \bar{\tau}_m - \bar{\tau}_w) \| \) we can combine (25) and (26) to get a traditional form of the market pricing relation

\[
\rho_i = \frac{\text{cov}(\bar{\tau}_i, \bar{\tau}_m - \bar{\tau}_w)}{\text{cov}(\bar{\tau}_m, \bar{\tau}_m - \bar{\tau}_w)} \rho_m
\]  

(27)
III. Zero-betas and Tests of Asset Pricing

As stressed by Roll and others, all MV asset pricing tests are really tests of whether a specific portfolio (usually the market portfolio) is efficient or optimal. Most major tests of CAPM have been performed using nominal returns (e.g., Black, Jensen and Scholes (1972), Fama and McBeth (1973), etc.). This paper shows that the existence of stochastic inflation will modify the conclusion of the standard model as far as the composition of the optimal portfolios are concerned when measured in the minimal mean-variance space. Following the interesting theoretical work by Black (1972), some authors have thought that the inflation phenomenon could be accommodated by replacing the risk-free rate by a zero-beta portfolio. This is incorrect; the zero-beta test is testing whether the market portfolio is nominal mean-variance efficient (instead of tangent) but the theory shows that it should not be! To clarify the issue it might be useful to consider several theoretical views of the world and their testable implications.

From now on we will assume for simplification that investors have identical consumption tests (inflation ?) and investigate what should be the theoretical outcome of a nominal CAPM test, if investors are real MV optimizers and expected returns and covariances are known exactly.

1. Only Stocks

Let’s assume that the investment opportunity set is limited to nominal risky security. Then one might be tempted to perform a nominal
zero-beta test à la Black, Jensen and Scholes which would be a test of the market portfolio \( M \) efficiency. Unfortunately, we have shown in section II that \( M \) should not be nominal efficient. It would be an incorrectly specified test if uncertain inflation exist and the vector \( C \) is nonzero. Indeed, one might wonder how to interpret the results of such a "nominal" test of asset pricing theory.

The empirical results of Black, Jensen and Scholes or Fama and McBeth might be interpreted as a rejection of the role of inflation in asset pricing. However, because of the difficulties involved in identifying the true market portfolio, Roll (1976) and Fama (1976) have recently cast some serious doubts about the meaning of these tests. Here, for example, assume that we know the exact market portfolio \( M \) (stock, bonds, real estate—) which is ex ante real \( MV \) efficient; it is then most likely that there will be ex ante nominal efficient portfolios \( M_0 \) highly correlated with \( M \). Therefore, a good positive relation will be found between excess returns and nominal betas (using \( M \)). In other words, the ex ante and ex post \( R^2 \) of the cross sectional regression would be expected to be quite large.

However, as shown by Roll, observing only correlations would be very misleading. Even if the covariance of inflation with the market portfolio rates of return look very small compared to the market portfolio variance, it might make a big difference as far as the efficient portfolios composition is concerned and this is the only important conclusion. This is illustrated by a small example in section III-2 when a nominal risk-free asset exists.
It might be interesting to see what kind of covariance structure \( C \) (asset returns and inflation) would make the nominal and real efficient sets to coincide. From equation (9), it appears that inflation will not affect optimal portfolio selection if \( \Lambda \) is equal to zero. This implies that portfolio \( \Theta \) will be nominal efficient and so will the market portfolio. This means that there should exist two scalars \( \lambda_0 \) and \( \lambda_1 \) such that

\[
V^{-1}C = \lambda_0 V^{-1}u + \lambda_1 M
\]

or

\[
C = \lambda_0 u + \lambda_1 VM
\]

i.e.

\[
\text{cov}(\hat{r}_i, \hat{r}) = \lambda_0 + \lambda_1 \text{cov}(\hat{r}_i, \hat{r}_m) \text{ for all } i
\]

This says that covariances between asset returns and inflation should be equal to a constant plus the nominal \( \beta \) of the asset times a constant. This is an interesting result since it only involves observable nominal covariances and no expected returns. Relation (28) could be empirically investigated if the true market portfolio and inflation rate are known.

It is our contention that such a relation is unlikely when all types of assets are considered: stocks, bonds, real estate, commodity futures, etc. Stocks with similar betas would react differently to unanticipated inflation depending on the industry to which they belong. Moreover, this should vary greatly between different types of assets: bonds, for example, have a low beta but a seemingly larger negative reaction to inflation than stocks because of the interest rate mechanics.
Indeed, one might be tempted to conclude that inflation does not affect asset pricing if one conducts empirical tests on only one type of assets (e.g., stocks) since inflation is more likely to affect similarly assets of the same type than different types of assets (e.g., stocks vs real estate). This can be illustrated by considering two subsets of assets which are separately (but uniformly within each subset) influenced by inflation. Subsets I and II might be U.S. common stocks and all U.S. real estate investment opportunities. For each subset, relation (28) is supposed to be exactly verified but with two different values of \( \lambda_0 \) and \( \lambda_1 \). The ex ante nominal and real efficient set of each subset are identical. However, the efficient frontiers of the total set of assets will differ since relation (25) is not verified for the total set and there is no reason to expect the subset market portfolios \( M_1 \) and \( M_{II} \) to be efficient among each subset of assets. The situation is depicted in figure 5.

![Figure 5](image-url)
Any ex ante or ex post asset pricing tests performed on only one subset of assets will not detect any influence of inflation since the nominal and real efficient subsets coincide. However, it does show in the (theoretical) composition of the efficient subset portfolios since, for example, the stock market portfolio $M_i$ should not be efficient among stocks.

To summarize, zero-beta tests are poorly designed to accept or reject the influence of inflation and one should rather focus on testing optimal portfolio composition.

2. **Stocks and a Nominal Riskless Asset**

Let's now assume that the investment universe consists of stocks and a nominal riskless bill. This section will investigate the expected influence of inflation on traditional CAPM tests performed on nominal returns.

a. **Sharpe-Lintner Test**

Tests of the traditional Sharpe-Lintner relation:

$$\rho_i = \beta_i \rho_m \quad \text{for all } i$$

are only a test of whether the stock market portfolio is the tangent portfolio, i.e., the portfolio where the tangent to the (nominal) stock efficient frontier goes through the risk-free asset. However, if inflation is considered and the theory is correct, this tangent portfolio should not be the market portfolio.

As shown in appendix B, (b-3), the mathematics of MV algebra, implies that the nominal stock portfolio will have an equation
\[ \sigma_p^2 = \frac{1}{da-b^2} \left[ dr_p^2 - 2br_p + a \right] \]

and a tangent portfolio of composition:

\[ M_o = -\frac{1}{u'V^{-1}E_o} V^{-1}E_o \]

However, it has been shown in section II-2 that the market portfolio would verify

\[ M = V^{-1}C + \frac{(1-e)}{u'V^{-1}E_o} V^{-1}E_o = V^{-1}C + (1-e)M_o \]

These two portfolios, \( M \) and \( M_o \), are different. The situation is depicted on figure 6.
The two portfolios differ by:

\[ M - M_o = V^{-1}c - eM_o = \frac{1}{1-e} (V^{-1}c - eM) \]  

(29)

This difference has no reason to be negligible, but general statements on the signs of \( M - M_o \) are hard to make when \( V \) is a full matrix.

First let's study the case where the market portfolio will be identical to \( M_o \). This will happen if \( C = 0 \) which is of no interest to us. It will also happen if \( \frac{1}{u^{-1}V^{-1}c} = M = \frac{1}{u^{-1}V^{-1}E_o} \). The implication for the asset return-inflation covariance vector is that it should be proportion to the nominal \( \beta \) vector:

\[ C = e VM \]  

(30)

In this case any measure of inflation risk for individual assets is already contained in the betas and they can be used to summarize the total risk of a security in the asset pricing relations. Relation (30) is identical to (28) with \( \lambda_o = 0 \) since the risk-free asset replaces the minimum variance portfolio \( V^{-1}u \). It basically means that all assets are influenced similarly by inflation "through" their dependence on the market.\(^8\)

Again, this relation might be more easily subjected to empirical scrutiny since it does not involve expectations on future returns, but it does require a knowledge of the true market portfolio.

However, the two portfolios \( M \) and \( M_o \) will have a very different composition if assets react differently to inflation. Again, this would
be the case if some industries are positively affected by inflation and
other negatively. Besides, the market portfolio should include all
assets and there is extensive evidence that bond and stock nominal re-
turns are negatively affected by inflation (e.g., Lintner (1975)) while
nominal returns on other assets (real estate, commodities) have a posi-
tive correlation with unexpected inflation. In fact, there is no reason
to expect that \( M_o \) will have a positive investment in each asset.

Intuitively, one would expect \( M_o \) to be more invested than the
market portfolio in assets negatively correlated with inflation and less
invested in assets positively correlated with inflation. Assets are
priced according to their real risk; with the same nominal risk, assets
will have a lesser real risk the larger their correlation with unexpected
inflation and therefore the smaller their expected return in equilibrium.
This will make negatively correlated assets more attractive in the nominal
risk-return tradeoff. As mentioned previously, we are lacking the "qualita-
tive" mathematics theorems to sign the vector \( V^{-1}C \) and only a simple ex-
ample is provided below.

Let's consider a three asset world, with asset 1 uncorrelated with
inflation while asset 2 is positively correlated and asset 3 negatively:

\[
V = \begin{pmatrix} 100 & 0 & 0 \\ 0 & 10 & -5 \\ 0 & -5 & 10 \end{pmatrix}, \quad M = \begin{pmatrix} .8 \\ .1 \\ .1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 \\ 3 \\ -3 \end{pmatrix}
\]
Then $V^{-1} = \frac{1}{300} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 40 & 20 \\ 0 & 20 & 40 \end{pmatrix}$ and the reader can easily check that:

$$E_0 = \lambda \begin{pmatrix} 80 \\ -2.5 \\ 3.5 \end{pmatrix}, \quad M_0 = \begin{pmatrix} .8 \\ -.1 \\ .3 \end{pmatrix}, \quad \sigma_M^2 = 64.1, \quad \text{cov} (\hat{r}_t, \hat{r}_m) = 0,$$

$$\beta = \begin{pmatrix} 1.25 \\ .01 \\ .01 \end{pmatrix}$$

While assets 2 and 3 appear in equal proportions in the market portfolio, the optimal nominal portfolio shows a negative holding of asset 2 which is negatively correlated with inflation and a much larger proportion of the positively correlated asset. The asset uncorrelated with inflation appears in the same proportion in the two portfolios. Note that despite the very different normative implications, the correlation between the nominal and real optimal portfolio is equal to 0.99. Of course, less striking results would be obtained if a fuller covariance matrix $V$ was considered; the above example could be thought of in terms of respectively stocks, real estate and bonds.

Given the large correlation between $M$ and $M_0$, a Sharpe Lintner nominal test should give a positive answer although $M$ is not nominal efficient.
b. **Zero-beta Test**

Given poor results in tests of the Sharpe-Lintner model some empirical researchers might be tempted to forget about the existence of a nominal riskless asset and, again, perform a zero-beta test of whether the market portfolio is on the nominal MV stock efficient frontiers S. So our theoretical world is this time somewhat different from III-1, since we now compare nominal stock efficient portfolios to real efficient portfolios which might consist of stocks and the nominal risk-free bill. Looking at figure 7, it appears that the market portfolio will indeed be nominal stock efficient if, and only if, it lies at the intersection between the two hyperbolae S and T. We know that it should lie within or on S and that it is on T; the question is whether it is also on S. It can be proven that it will generally not be true. If it were, then any nominal stock efficient portfolio on S could be written as a linear combination of two other efficient portfolios, \( M_o \) and M:

\[
S = (1 - \lambda)M_o + \lambda(V^{-1}C + (1 - \epsilon)M_o)
\]

\[
S = (1 - \lambda\epsilon)M_o + \lambda\epsilon \frac{V^{-1}C}{uV^{-1}C} = (1 - \mu) \frac{V^{-1}E_o}{uV^{-1}E_o} + \mu \frac{V^{-1}C}{uV^{-1}C}
\]

which in turn means that the portfolio \( \frac{V^{-1}C}{uV^{-1}C} \) is also nominal efficient.
This implies that $\Theta$ can be written as a linear combination of two other nominal efficient portfolios and

$$v^{-1}C = \lambda_0 v^{-1}u + \lambda_1 M$$

We obtain the same relation (25) as Section III-1:

$$C = \lambda_0 u + \lambda_1 VM$$  \hspace{1cm} (25)

So, again, a necessary and sufficient condition for the market portfolio to be nominal stock efficient under inflation is that there exists two scalars $\lambda_0, \lambda_1$ so that relation (25) is verified.

Figure 7
Summary

This paper studied the influence of stochastic inflation on the composition of investors' optimal portfolios in a mean-variance framework. The analytical expression of the investor's efficient set has been derived assuming that he is a real mean-variance optimizer. Homogeneity of tastes was not assumed.

Since real efficient sets can be fully described by a linear combination of two portfolios, they can easily be represented in the nominal mean-variance space. All real efficient sets are depicted by homothetic parabolas (or hyperbolas in the mean-standard deviation space) whose envelop is the nominal efficient frontier.

Investors subject to different inflation rates (tastes) will never hold the same portfolios whatever their risk aversion is. Therefore, no simple separation theorems can be found. The market portfolio will not be efficient for all or most investors nor will it be nominal efficient.

Looking at the correlation between the market index return and inflation or at the vector of covariances of individual asset returns with inflation will not give meaningful indications on the importance of inflation for market equilibrium. Some other measures have been proposed in this paper. For example, inflation will not affect optimal choices if the vector of covariance of nominal asset returns with inflation is a linear function of the nominal covariance vector of assets returns with the market portfolio return. The larger the deviation from linearity the more important will inflation be. It appears that the issue is not whether asset
returns are strongly or weakly affected by inflation but rather whether
the influence of inflation is uniform across all assets. The casual
observation that nominal values of various investment vehicles react in
opposite directions to changes in the inflation rate would suggest that
stochastic inflation is a nonnegligible factor in asset pricing and op-
timal portfolio choices.

The last part of this paper dealt with the use of nominal return
tests of asset pricing theory. As emphasized by Roll, these CAPM tests
are only testing the nominal mean-variance efficiency of the specific
market index used. Since we concluded that the market portfolio should
not be nominal efficient, this is a misspecified test of asset pricing
theory under inflation. Furthermore, it seems that it would be misleading
to interpret the positive results of some of these tests as evidence
against the influence of inflation on market equilibrium.

To summarize, even though inflation might have a small variance
compared to stock prices, it could go a long way in affecting individuals' portfolio choices.
Footnotes

1A treasury bill is risk-free in nominal terms, but not in real terms. An individual cares about real returns, not nominal ones.

2Merton (1974); Roll (1976).

3If \( r_p \) is the return on portfolio \( p \) and \( \tau_k \) the investor's inflation rate, he will want to minimize the variance for a given level of return:

\[
\frac{\tau_p - \tau_k}{r_p - \tau_k} = \sum_{i=1}^{n} x_i \frac{\tau_i - \tau_k}{\tau_i - \tau_k}
\]

with

\[
E(\frac{\tau_p - \tau_k}{r_p - \tau_k}) = \frac{r_p - \tau_k}{r_p - \tau_k} = \frac{x_i \tau_i}{x_i} - \tau_k
\]

\[
\text{var}(\frac{\tau_p - \tau_k}{r_p - \tau_k}) = \sum x_i \sum_j \text{cov}(\frac{\tau_i}{r_p - \tau_k}, \frac{\tau_j}{r_p - \tau_k}) - 2 \sum x_i \text{cov}(\frac{\tau_i}{r_p - \tau_k}) + \text{var}(\frac{\tau_k}{r_p - \tau_k})
\]

The investor is maximizing over the vector \( X \) of investment proportions \( x_i \). This yields the matrix form (2), since \( \tau_k \) and \( \text{var}(\frac{\tau_k}{r_p - \tau_k}) \) are independent of \( X \).

4Lintner (1975) provided extensive evidence of negative correlation between inflation and nominal rates of return on common stocks.

5

\[
\frac{d\sigma_p}{dr_p} = 0 \rightarrow 2dr_p = 2b
\]

\[
\frac{\sigma_p}{r_p} \rightarrow \frac{d}{ad-b^2} \quad \text{when} \quad r_p \rightarrow \infty
\]
This will be true as soon as a single investor is subject to inflation (such that \( V^{-1}c_k \neq 0 \)), then \( V^{-1}c_\omega \neq 0 \).

The ex ante correlation between real and nominal efficient portfolios would be computed from equation (11).

For example, in a world where the market model is correct:

\[
\hat{r}_i = \beta_i \hat{r}_m + \hat{\varepsilon}_i
\]

It would require that the inflation rate be orthogonal to \( \hat{\varepsilon}_i \). All asset returns will be affected by the inflation rate through the market

\[
\hat{r}_m \quad \text{cov}(\hat{r}_i, \hat{r}) = \beta_i \text{cov}(\hat{r}_m, \hat{r})
\]

hence

\[
\text{cov}(\hat{r}_i, \hat{r}) = \text{cov}(\hat{r}_i, \hat{r}_m) \frac{\text{cov}(\hat{r}_m, \hat{r})}{\sigma_m^2} = e \text{cov}(\hat{r}_i, \hat{r}_m)
\]

since

\[
C = e \text{VM} \quad \text{implies} \quad M C = e \text{M VM}
\]

i.e.

\[
\text{cov}(\hat{r}_m, \hat{r}) = e \sigma_m^2
\]

One should notice that S and T might intersect at one point which represents two different portfolios, since S is built only with a subset of all assets. However, there is no problem with M which is by theory on T, if it turns out to be also on S then it will for sure be nominal stock efficient. Remember also that M is the only real efficient portfolio made up only with stocks.
Bibliography


Appendix A

Analytical Derivation of the Real Efficient Frontier

A-1. Real Efficient Frontier

The efficient frontier in real terms will be the locus of portfolios described by equation (6). The real variance will be given by:

$$\sigma_p^2 = \mathbf{x}^\prime \mathbf{v} \mathbf{x} - 2 \mathbf{x}^\prime \mathbf{c}_k + \sigma_{tk}^2$$

$$\sigma_p^2 = \sigma_{tk}^2 - \mathbf{c}_k^\prime \mathbf{v}^{-1} \mathbf{c}_k \left\{ \mathbf{r}_p - \mathbf{E}^\prime \mathbf{v}^{-1} \mathbf{c}_k \right\}^\prime \mathbf{A}^{-1} (\mathbf{Eu})^\prime \mathbf{v}^{-1} v (\mathbf{Eu}) \mathbf{A}^{-1} \left\{ \mathbf{r}_p - \mathbf{E}^\prime \mathbf{v}^{-1} \mathbf{c}_k \right\}$$

$$\mathbf{A} \equiv (\mathbf{Eu})^\prime \mathbf{v}^{-1} (\mathbf{Eu})$$

(Al) becomes:

$$\sigma_p^2 = \sigma_{tk}^2 - \mathbf{c}_k^\prime \mathbf{v}^{-1} \mathbf{c}_k + \left\{ \mathbf{r}_p^\prime \mathbf{E}^\prime \mathbf{v}^{-1} \mathbf{c}_k \right\}^\prime \mathbf{A}^{-1} \left\{ \mathbf{r}_p^\prime \mathbf{E}^\prime \mathbf{v}^{-1} \mathbf{c}_k \right\}$$

(A2)

Let's write:

$$\mathbf{A} = \begin{bmatrix} a & b \\ b & d \end{bmatrix} \text{ then } \mathbf{A}^{-1} = \frac{1}{ad-b^2} \begin{bmatrix} d & -b \\ -b & a \end{bmatrix}$$

with $a \equiv \mathbf{E}^\prime \mathbf{v}^{-1} \mathbf{E}$, $b \equiv u^\prime \mathbf{v}^{-1} \mathbf{E}$, $d \equiv u^\prime \mathbf{v}^{-1} \mathbf{E}$, and $c_k \equiv u^\prime \mathbf{v}^{-1} \mathbf{c}_k$

Let's first assume that the vector $\mathbf{v}^{-1} \mathbf{c}_k$ has the nice property that its elements sum to one so that it represents a portfolio $\mathbf{w}_k$ ($u^\prime \mathbf{v}^{-1} \mathbf{c}_k = 1$).
In this case, simple and intuitive results can be derived. The efficient frontier becomes:

\[ \sigma_p^2 = \sigma_{\theta k}^2 + \frac{1}{\sigma_{\theta k}^2} \left[ d(r_p - r_{\theta k})^2 \right] \]  

(A3)

where \( r_{\theta k} \) and \( \sigma_{\theta k}^2 \) are the expected return, \( E' \Sigma^{-1} C_k \), and variance, \( C_k \Sigma^{-1} \Sigma^{-1} C_k \), of portfolio \( \theta_k \).

A-2. Minimum Variance Portfolio

This portfolio \( \theta_k \) has several interesting characteristics. Its covariance of nominal return with any portfolio is equal to the covariance of nominal returns of that portfolio with the inflation rate \( \tau_k \):

\[ \text{cov}(r, r_{\theta_k}) = \Sigma V^{-1} C_k = \Sigma C_k = \text{cov}(r, \tau_k) \]  

(A4)

Its covariance of real return with any portfolio is constant and equal to \( \sigma_{\theta k}^2 \):

\[ \text{cov}(r, r_{\theta_k}) = \text{cov}(r - \tau_k, r_{\theta_k} - \tau_k) = \text{cov}(r, \tau_k) - \text{cov}(r_{\theta_k}, \tau_k) + \sigma_{\tau_k}^2 - \text{cov}(r, \tau_k) \]

but we know from (A4) that

\[ \text{cov}(r_{\theta_k}, \tau_k) = \text{cov}(r_{\theta_k}, \tau_k) = \sigma_{\theta_k}^2 \]

therefore:

\[ \text{cov}(r, r_{\theta_k}) = \sigma_{\tau_k}^2 - \sigma_{\theta_k}^2 \]  

(A5)

\[ \text{Note that } r_p - r_{\theta k} = r_p' - r_{\theta k}' + \tau_k - \tau_k = r_p' - r_{\theta k}' \]
It should be noted that this term is always positive since:

$$\sigma^2_{\theta k} = \text{cov}(\gamma_{\theta k}, \tilde{\tau}_k) = \rho \sigma_{\theta k} \sigma_{\tilde{\tau} k} - \sigma_{\theta k} \sigma_{\tilde{\tau} k}$$

implies $\sigma_{\theta k} \leq \sigma_{\tilde{\tau} k}$

As shown by Roll (1976), property (A5) demonstrates that $\theta_k$ is the real minimum variance portfolio. This could also be shown by computing the derivative of $\sigma^2_p$:

$$\frac{d\sigma^2_p}{dr_p} = 2\sigma_p \frac{d\sigma_p}{dr_p} = \frac{d}{dr_p} (r_p - r_p') = 0 \quad \text{for} \quad P = \theta_k$$

The portfolio $V^{-1}C_k$ is the stock portfolio used to hedge against unfavorable shifts in consumption prices.

Unfortunately, $u'V^{-1}C_k = e_k$ will generally differ from 1. $\theta_k$ will then be defined by $\frac{1}{u}V^{-1}C_k$. With $r_{\theta k} = \frac{1}{e_k}E'V^{-1}C_k$ and

$$\sigma^2_{\theta k} = \frac{1}{\sigma^2_k}C_k'V^{-1}C_k$$

(A3) is now replaced by (A3)':

$$\sigma^2_p = \sigma^2_{\tilde{\tau} k} - \epsilon^2_{\theta k} + \frac{1}{ad-b^2} [d(r_p' + r_k - e_k \tilde{\tau}_k)^2 - 2b(1-e_k)(r_k' + r_k - e_k \tilde{\tau}_k) + a(1-e_k)^2]$$

and (A4) by (A4)':

$$\text{cov}(\gamma_{\theta k}, \gamma_{\tilde{\tau} k}) = \frac{1}{\epsilon_k} \text{cov}(\gamma_k, \tilde{\tau}_k)$$

(A4)'
Portfolio $\theta_k$ is not any more the minimum-variance portfolio whose composition is given by:

$$\frac{d\sigma}{dr_p} = 0 \quad \rightarrow \quad 2d(r_p - e_k\theta_k) - 2b(1-e_k) = 0$$

$$r_p = e_k\theta_k + \frac{b(1-e_k)}{d} \quad (A6)$$

Replacing into (6), we get the exact composition of this real minimum variance portfolio:

$$\theta_k^* = v^{-1}c_k + \frac{1-u}{v^{-1}c_k v^{-1}u} = e_k\theta_k + \frac{1-e_k}{v^{-1}u} \quad (A7)$$

If $e_k = 1$ we do get $\theta_k^* = \theta_k'$. 
A-3. Nominal Efficient Frontier

The set of efficient portfolios $X$ is described in (6). In nominal terms, these portfolios will have a nominal return $r_p = E'X$ and a nominal variance $\sigma_p^2 = X'VX$. Therefore, in the nominal mean-variance plane the efficient frontier of investor $k$ will look like:

$$\sigma_p^2 = X'VX = \sigma_p'^2 - \sigma^2_{\tau_k} + 2X'C_k$$

$$= \sigma_p'^2 - \sigma^2_{\tau_k} + 2\left(\frac{r_p - E'V^{-1}C_k}{1 - u'V^{-1}C_k}\right)i^{-1}(E\nu)'V^{-1}C_k$$

$$= \sigma_p'^2 - \sigma^2_{\tau_k} + \frac{2}{ad-b^2}[d(r_p - e_k\theta_k)e_k\theta_k - be_k(r_p - e_k\theta_k) - (1-e_k)be_k\theta_k]$$

$$+ ae_k(1-e_k) + 2e_k\sigma^2_k$$

therefore

$$\sigma_p^2 = e_k^2\sigma^2_k + \frac{1}{ad-b^2}[dr_p^2 - 2br_p + a - e_k^2(dr_{\theta_k}^2 - 2br_{\theta_k} + a)]$$

(A8)
Appendix B

Equilibrium with a Nominal Riskless Asset

B-1. Efficient Frontiers

Let's note \( E_o = E - r_o u \)

then relations (15) might be written as:

\[ x = V^{-1}C_k + \lambda_1 V^{-1}E_o \]  \hspace{1cm} (15a)

\[ x_o = 1 - e_k - \lambda_1 u V^{-1}E_o \]  \hspace{1cm} (15b)

For any total portfolio \( P \), its nominal expected return is:

\[ r_P = E'X + r_o x_o = E'X + r_o (1 - u'X) = r_o + E_o'X \]

Let's note \( \rho_P \), the nominal excess expected return on \( P \) is equal to

\[ \rho_P = r_P - r_o = E_o'X \]  \hspace{1cm} (B1)

and \( \rho_P = r_P - \tau_k = (r_o - \tau_k) = \rho \)

In order to eliminate \( \lambda_1 \), from (15) we use the fact that \( \rho_P = E_o'X \):

\[ \rho_P = E_o'X = E_o'V^{-1}C_k + \lambda_1 E_o'V^{-1}E_o \]

but \( E_o'V^{-1}C_k = (E' - r_o u')V^{-1}C_k = e_k r_o k - e_k r_o = e_k \rho_k \)

then \( \lambda_1 = \frac{\rho_P - e_k \rho_k}{E_o'V^{-1}E_o} \)  \hspace{1cm} (B2)
replacing in (15), we get the composition of efficient portfolios as a
function of their expected return

\[ X = \mathbf{V}^{-1} \mathbf{C}_k + \frac{\rho_p - \mathbf{e}_k' \mathbf{\theta}_k}{\alpha} \mathbf{V}^{-1} \mathbf{E}_o \]  

(B3)

\[ x_o = 1 - \mathbf{e}_k' \mathbf{V}^{-1} \mathbf{C}_k - \frac{\rho_p - \mathbf{e}_k' \mathbf{\theta}_k}{\alpha} \mathbf{u}' \mathbf{V}^{-1} \mathbf{E}_o \]

where \( \alpha = \mathbf{E}_o' \mathbf{V}^{-1} \mathbf{E}_o = \mathbf{d}' \mathbf{r} \mathbf{d} - 2 \mathbf{b}' \mathbf{r} + \mathbf{a} \) is independent of \( k \).

Now the equation of the efficient frontier for investor \( k \)
measured in real returns can easily be derived:

\[ \sigma_p^2 = \sigma_{tk}^2 + X' \mathbf{V} \mathbf{X} - 2 \mathbf{X}' \mathbf{C}_k^* = \sigma_{tk}^2 - \mathbf{e}_k' \sigma_k^2 + \frac{(\rho_p - \mathbf{e}_k' \mathbf{\theta}_k)^2}{\alpha} \]  

(B4)

and its translation in the nominal plane:

\[ \sigma_p^2 = \mathbf{X}' \mathbf{V} \mathbf{X} = \mathbf{e}_k' \sigma_k^2 + \frac{\rho_p^2 - \mathbf{e}_k' \sigma_k^2}{\alpha} \]  

(B5)

**B-2. Minimum Variance Portfolio**

From (B4), it appears that the minimum real variance will be ob-
tained for \( \rho_p = \mathbf{e}_k' \mathbf{\theta}_k \). Replacing into (B3) we get the composition of the
minimum variance portfolio \( \mathbf{\theta}_k^* \):

\[ X = \mathbf{V}^{-1} \mathbf{C}_k \]

\[ x_o = 1 - \mathbf{u}' \mathbf{V}^{-1} \mathbf{C}_k \]  

(B6)

\[ \rho_{\theta k}^* = \mathbf{E}_o' \mathbf{V}^{-1} \mathbf{C}_k = \mathbf{e}_k' \mathbf{\theta}_k \]

\[ \sigma_{\theta k}^* = \mathbf{e}_k' \sigma_k^2 \]
This portfolio is similar to the previous one (A7), except that the riskless bond replaces the minimum nominal variance portfolio 
\[ \frac{1}{d} V^{-1} \nu \]. In real terms, the nominal riskless asset play the role of a pure inflation asset: to get a better understanding of the role played by the nominal riskless asset, it might be interesting to consider some special cases:

(a) \[ C_k = 0 \]

Then the riskless bond becomes the minimum variance portfolio since \( e_k = 0 \). The real variance of any portfolio \( P \) is given by:

\[
\sigma^2 (r_p - \nu) = \sigma_p^2 + \sigma_\tau^2
\]

The riskless asset will have the minimal real variance since its nominal variance is null.

(b) \[ u' V^{-1} C_k = e_k < 0 \]

Roughly speaking, nominal stock returns are negatively correlated with inflation which is "unpleasant" to consumer-investors. To minimize risk a risk-averted will "overinvest" in the riskless asset \((1 - e_k > 0)\), and sell short stocks.

(c) \[ u' V^{-1} C_k > 0 \]

Since nominal stock returns are positively correlated with inflation they become attractive as inflation hedges and partly replace riskless bills in the minimum real variance portfolio. The higher \( e_k \), the more attractive stocks are from a risk reduction viewpoint. It might be
worth borrowing \((1 - e_k < 0)\) to invest in stocks as an inflation hedge.

Note that when \(e_k = 1\), \(\theta^*_k\) is only made up of stock and its variance is equal to

\[
\sigma'^2 = \text{var}(r_{\theta_k} - \tau_k, r_{\theta_k} - \tau_k) = \sigma_{\theta_k}^2 - 2\text{cov}(r_{\theta_k}, \tau_k) + \sigma_{\tau k}^2
\]

but \(\text{cov}(r_{\theta_k}, \tau_k) = \sigma_{\theta k}^2\)

therefore

\[
\sigma'^2 = \sigma_{\tau k}^2 - \sigma_{\theta k}^2
\]

As we can see this will be inferior to the real variance of the riskless asset: \(\sigma_{\tau k}^2\).

B-3. The Nominal Stock Efficient Frontier

A graphical representation of the Sharpe-Lintner model is often given, where the market portfolio lies at the tangency point of the stock efficient frontier with the tangent drawn from the riskless asset point. In section III, we will need to compare the nominal stock efficient frontier (made up of only nominal risky assets) to the (total) real efficient frontier (made up of all assets including the nominal riskless bill). Therefore, it might be useful to recall the analytical expression of the nominal stock efficient frontier (called \(S\) hereafter) and the tangent portfolio as a function of covariances \(V\) and expected returns \(E\).
These expressions have already been derived by Merton (1972) and in a more compact manner by Roll (1976) whose method was used in this paper. They can be directly derived from the results of section I, assuming a constant inflation rate. From (6), the nominal stock efficient portfolios are given by:

\[ X = V^{-1}(Eu)A^{-1} \begin{pmatrix} r_p \\ 1 \end{pmatrix} \]  

(B7)

and the equivalent of relation (11) is:

\[ X = \frac{1}{d} \ V^{-1} u + \lambda \ (V^{-1}E - \frac{b}{d} \ V^{-1} u) \]

\[ X = \frac{V^{-1} u}{d} + \lambda \ b \left( \frac{V^{-1}E}{b} - \frac{V^{-1} u}{d} \right) \]

where \( d = u \ V^{-1} u \) and \( b = u \ V^{-1} E \)

calling \( \lambda_1 = \lambda b \), we get

\[ X = \frac{V^{-1} u}{d} \ (1 - \lambda_1) + \lambda \ \frac{V^{-1} E}{b} \]  

(B8)

The two portfolios \( \frac{V^{-1} E}{u \ V^{-1} E} \) and \( \frac{V^{-1} u}{u \ V^{-1} u} \) are efficient and the latter one is the minimum variance portfolio.

From (9), we see that the equation of the nominal stock efficient frontier \( S \) is:

\[ \sigma^2_{p} \ = \ \frac{1}{ad-b^2} \ (dr^2_{p} - 2br_{p} + a) \]  

(B9)
To get the composition of the tangent portfolio, it is sufficient to remember that it will be the only all stocks portfolio which will be efficient when a riskless asset exists for an investor with a constant inflation rate. In other words, we should consider equation (15) in section II, assume $C_k \equiv 0$ and $x_o = 0$ to find that the tangent portfolio is

$$M_o = \frac{1}{u} V^{-1} E_o$$

(B10)