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The proposition stated in the following quotation is shown to be false. The error stems from a mistaken application of the assumption of competitive markets.

Where there exist organized capital markets in which shares can be freely bought and sold and where these markets are perfect ... it is possible to develop an objective, operational decision criterion for management that (1) does not involve stockholder utility functions directly but (2) leads to precisely the same investment and operating decisions that each stockholder would make if he were running the firm himself.

- Eugene Fama and Merton Miller
The Theory of Finance, p. 69

I. INTRODUCTION

It is commonly believed that firms functioning in competitive product and financial markets should make decisions which maximize profits, or equivalently, the market value per share of the firm's securities. Decisions coincident with this goal are Pareto-efficient.
Moreover, if the firm has issued only one class of securities (i.e., equities), such decisions are also thought to be unanimously supported by the firm's shareholders.\footnote{Two initial circumstances are often separately treated. In the \textit{ex post} situation, production decisions are made only \textit{after} the firm's shareholders have traded to their equilibrium positions; in the \textit{ex ante} situation, production is simultaneous with exchange. The central purpose of this paper is to prove that the \textit{ex ante} justification of unanimity is false.} 

This departure from the conventional wisdom stems from circumspection about the meaning of competitive financial markets. I adopt the context of a static economy under certainty because this is sufficient to demonstrate the basic idea. In this context, competitive markets means that (1) consumers make consumption decisions as if they cannot influence the interest rate and (2) producers make production decisions as if they cannot influence the interest rate. A common justification, though not the only one, of this assumption is that, while each consumer and each producer in fact has some influence over the interest rate, every consumer and producer is sufficiently small relative to the rest of the economy that his decisions would be little changed if he were to consider the slight influence over the interest rate he actually exerts. I believe it is very important to realize that competitive financial markets does not mean that consumers and producers have no influence over the interest rate, but rather that they \textit{behave as if} this were true.
The correct justification of the competitive markets assumption is an argument of approximation: Even if consumers and producers were to take into account their slight influence over the interest rate, their choices and the resulting equilibrium interest rate would be approximately what they would have been under the assumption of competitive markets. In particular, the conclusions we would draw from competitive markets are little altered by this assumption as long as every participant is small relative to the rest of the economy.

The unanimity proposition of competitive markets, however, requires a third competitive assumption: (3) consumers, when assessing production decisions of producers, presume producers cannot influence the interest rate. I show below that approximation arguments, which may justify (1) and (2), fail to justify (3). That is, even though each firm in fact has a small influence over the interest rate, the difference between presuming even this arbitrarily small influence and zero influence will dramatically effect the ex ante unanimity conclusion. Competitive assumption (3) is, therefore, not justified and, therefore, it is unreasonable to presuppose unanimity in markets even where competitive assumptions (1) and (2) are viable.

To see intuitively why this might be true, suppose first all consumers are economically identical (i.e., same tastes and resources), then, since production decisions which maximize present value will be Pareto-efficient, they must also be unanimously preferred by all consumers in the economy. However, if consumers are different (for
example, with respect to time-preference), then some consumers will become net lenders and others net borrowers. The lenders will hope the interest rate will be high and the borrowers will hope the interest rate will be low. Moreover, firms, by their production decisions, effectively transfer existing aggregate resources into the future, thereby affecting the interest rate. More production, other things equal, implies a higher interest rate. Lenders, who consider these production effects on the interest rate, will prefer more production than borrowers. Consumers will, therefore, disagree over firm's production decisions.

However, suppose we accept the rationale for competitive assumption (2): Each firm only has an effect on the interest rate that we can make as small as we like by increasing the number of firms. Does this mean borrowers and lenders will only have a minor disagreement over the preferred level of production for each firm? Unfortunately not. While each firm is only a small fraction of a consumer's portfolio, each firm, through its impact on the interest rate, will be influencing the terms on which he can alter the composition of his portfolio. Therefore, from each consumer's point of view, the impact a given firm has on the interest rate (even though the impact is slight) may dominate a firm's impact on the value of its own shares (which comprise a small portion of the consumer's portfolio).
II. ANALYSIS

The imprecision of this intuitive argument seems to require a formal addendum. To focus on the essence of the problem, the example below will adopt many simplifications, none of which is critical to the conclusion. Let \( U_i(c_0^i, c_1^i) \) be a strictly increasing and strictly concave utility function over current and future consumption for consumer \( i = 1, 2, \ldots, I \). Let \( \bar{X}_0 > 0 \) be his endowment of current consumption and \( I^{-1} \) be the fraction of shares of each producer in the economy that he owns. Assume there are \( N \) identical producers with production technologies \( X_1 = f(X_0) \) where \( X_0 > 0 \) is current input and \( X_1 > 0 \) future output. \( f \) will be assumed to satisfy the usual Inada conditions so that \( 0 = f(0), \infty = f(\infty), f' > 0, f'' < 0, \infty = f'(0), 0 = f'(\infty) \). If \( \Delta_i > 0 (\Delta_i < 0) \) represents the amount consumer \( i \) lends (borrows) and \( r \) is the equilibrium rate of interest, then

\[
C_0^i = \bar{X}_0 - (N/I)X_0 - \Delta_i \quad \text{and} \quad C_1^i = (N/I)f(X_0) + \Delta_i (1+r)
\]

Suppose, on the demand side, there exists a composite consumer \(^2\) with tastes \( U(C_0, C_1) \) so that the equilibrium interest rate, given the level of aggregate production, is determined as if all consumers were identical to the composite consumer. The composite consumer then solves the following problem:

\[
\max_{\Delta} U[\bar{X}_0 - (N/I)X_0 - \Delta, (N/I)f(X_0) + \Delta (1+r)]
\]

Since by competitive assumption (1), \( \Delta \) is chosen as if \( \partial r / \partial \Delta = 0 \)
and by closure, \( \Delta = 0 \) for the composite consumer, at the optimum values of consumption

\[
U_0(C_0, C_1)/U_1(C_0, C_1) = 1 + r
\]

(1)

where \( C_0 = \bar{X}_0 - (N/I)X_0 \) and \( C_1 = (N/I)f(X_0) \).

\( U_0(C_0, C_1) \) denotes the first derivative of \( U \) with respect to \( C_0 \) and \( U_1(C_0, C_1) \) the first derivative of \( U \) with respect to \( C_1 \).

Turning to the supply side, let us compare the implications of production decisions made according to two criteria:

(1) competitive present value maximization, and

(2) consumer utility maximization.

Under criterion (1), each producer solves the following problem:

\[
\max_{X_0} - X_0 + \frac{X_1}{1 + r} \quad \text{s.t.} \quad X_1 = f(X_0)
\]

By competitive assumption (2), \( X_0 \) is chosen as if \( \partial r/\partial X_0 = 0 \). It follows that optimal production choices must satisfy

\[
f'(X_0) = 1 + r
\]

(2)

Equations (1) and (2) determine a unique competitive present value maximization equilibrium \( \{X_0, r\} \).

Under criteria (2), taking the production decisions of all producers but one (i.e., \( X_0^N \)) as given, each consumer solves the following problem:
\[
\max_{X_0^N, \Delta} U_i \left[ \bar{X}_0 - \frac{(N-1)X_0}{I} - \frac{X_0^N}{I} - \Delta, \frac{N-1}{I} f(X_0) + \frac{f(X_0^N)}{I} + \Delta(1+r) \right]
\]

where the interest rate now depends on consumer \(i\) since we assume he may not find competitive assumption (3) a useful approximation (i.e., \(\partial r/\partial X_0^N \neq 0\)). In this case, a given consumer can be roughly interpreted as first choosing the production level of firm \(N\); this determines the interest rate; and second, given \(X_0^N\) and \(r\), he chooses the amount he wishes to borrow and lend so as to maximize his utility. It follows that his optimal choices must satisfy

\[
f'(X_0^N) = 1 + r - \frac{\Delta}{1-1} \left( \frac{\partial r}{\partial X_0^N} \right)
\]

Equations (1) and (3) determine a unique consumer \(i\) utility maximization equilibrium \(\{X_0^N, \Delta, r\}_i\), given that \(X_0^1, \ldots, X_0^{N-1}\) have been determined in some prespecified manner.

Equation (3) reduces to equation (2) for firm \(N\) (i.e., the competitive present value maximizing solution) only if

1. \(\Delta_i = 0\) for all \(i\), the ex post case, or
2. \(\partial r/\partial X_0^N = 0\), competitive assumption (3).

Is it reasonable to assume \(\partial r/\partial X_0^N = 0\)? Since, regardless of what production decisions are made, the interest rate must satisfy equation (1), it is possible to calculate this derivative. Differentiating this equation...
\[
\frac{\partial r}{\partial x_0} = \frac{-1}{I} \left( \frac{U_0}{U_1} \right) \left[ \frac{U_{10}}{U_1} - \frac{U_{00}}{U_0} + \left( \frac{U_{10}}{U_0} - \frac{U_{11}}{U_1} \right) f'(x_0^N) \right]
\]

where \( U \) is evaluated at \( c_0 = \bar{x}_0 - \frac{(N-1)}{I} x_0 - \frac{x_0^N}{I} \) and

\[
c_1 = \frac{(N-1)}{I} f(x_0^N) + \frac{f(x_0^N)}{I}.
\]

Given a fixed ratio \( N/I \) of the number of firms to the number of consumers, \( \lim_{N \to \infty} \frac{\partial r}{\partial x_0^N} = 0 \). By keeping \( N/I \) fixed as the economy is enlarged, the per capita size of consumers and producers remains the same.

The fault of the standard ex ante analysis should now be apparent: The limiting argument on \( \partial r/\partial x_0^N \) was implicitly made prior to and not within equation (3). From this equation, there are two effects as the economy becomes large relative to a given consumer and producer: \( \partial r/\partial x_0^N \to 0 \) and also a consumers share in each producer, \( I^{-1} \to 0 \). Moreover, they both approach zero at the same rate. In particular, from equations (1) and (3)

\[
\lim_{N \to \infty} f'(x_0^N) = (1+r) \left\{ 1 - \Delta \left[ \frac{U_{10}}{U_1} - \frac{U_{00}}{U_0} + \left( \frac{U_{10}}{U_0} - \frac{U_{11}}{U_1} \right) f'(x_0^N) \right] \right\}
\]

where \( c_0 = \bar{x}_0 - (N/I)x_0 \) and \( c_1 = (N/I)f(x_0) \). Solving this for \( f'(x_0^N) \), in the limit,

\[
f'(x_0^N) = (1+r) \left\{ \frac{1 - \Delta(\frac{U_{10}}{U_1} - \frac{U_{00}}{U_0})}{1 + \Delta(1+r)(\frac{U_{10}}{U_1} - \frac{U_{11}}{U_1})} \right\}
\]

Since \( f'' < 0 \), \( \partial x_0^N/\partial \Delta > 0 \) so that net lenders prefer more production
than net borrowers, regardless of how the equilibrium interest rate is actually determined.

For example, suppose the composite consumer has an additive logarithmic utility function so that

$$U(C_0, C_1) = \ln C_0 + \rho \ln C_1$$

and \( f(X_0) = a \sqrt{X_0} \). Then, since from equation (1) \( 1 + r = C_1/(\rho C_0) \), the above equation simplifies to

$$X_0^N = \left( \frac{\alpha C_0}{2C_1} \right)^2 \left( \frac{C_0 + \Delta}{\rho C_0 - \Delta} \right)^2$$

where \( C_0 - \Delta \) may be interpreted as the current consumption for a consumer who lends \( \Delta \). Comparing this to the competitive present value maximizing decision, from equation (2),

$$X_0 = \left( \frac{\alpha \rho C_0}{2C_1} \right)^2$$

Therefore,

$$\frac{X_0^N}{X_0} = \left[ \frac{\rho C_0 + \Delta}{\rho (C_0 - \Delta)} \right]^2$$

If \( \rho = .8 \) and \( \Delta/C_0 = .2 \), then \( X_0^N/X_0 = 1.6 \); if \( \rho = .8 \) and \( \Delta/C_0 = -.2 \), then \( X_0^N/X_0 = .63 \). Unanimity is considerably wide of the mark.
III. CONCLUSION

Unanimity, even approximate unanimity, is not possible in an ex ante economy with a sufficiently diverse set of consumers. However, since for every borrower there a lender be, while competitive present value maximizing decisions are not unanimously acceptable, they are nonetheless Pareto-efficient. That is, provided $\Delta_1 > 0$ for some consumers so that $\Delta_i < 0$ for others, any sufficiently small movement away from the competitive present value maximizing amount will benefit one group only at the expense of the other. However, too extreme a change from competitive present value maximizing production would be disadvantageous for all consumers.

Unanimity, if it is to be supported, must rely on ex post arguments. If all consumers were identical, then competitive present value maximizing decisions would be unanimously accepted, and the greater the similarity among consumers, the closer their agreement would be. In essence, the great merit of competitive present value maximizing decisions is that, among other possibly Pareto-efficient decisions, they are the only decisions which are Pareto-efficient independent of the identities of consumers.
FOOTNOTES

1 For example, see Hirshleifer (1976, p. 422), Leland (1973), Radner (1974), and Samuelson (1970, p. 590).

2 For sufficient conditions for the existence of a composite consumer, see Rubinstein (1974).

3 The subsequent argument is even stronger if a consumer simultaneously considers his preferred production levels for all producers in the economy.
REFERENCES


