The Valuation of Dependent Securities in a Diffusion Process

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ABSTRACT

A state vector, $x$, of market factors and time, $t$, are assumed to determine the values, $V^j(x,t)$, of a collection, $\mathcal{J}$, of assets. The state vector follows a continuous time diffusion process, asset trading markets are perfect, and riskless arbitrage opportunities are not allowed. Under these conditions there is a partial differential equation that must be satisfied by the value functions $V^j(x,t)$ for all assets $j \in \mathcal{J}$.

1. Introduction

Consider a market in which an $N+1$ element vector $x = (x_0, x_1, \ldots, x_N)$ and time $t$ determine the market value $V^j(x,t)$ for each asset $j$ in a collection $\mathcal{J}$. In this situation Garman, [3], derived a partial differential equation that must be satisfied by the value functions $V^j(x,t)$ for all $j \in \mathcal{J}$. Garman assumed the existence of what we call a valuation measure and used its attributed differentiability to deduce the partial differential equation. In contrast, we use a direct argument to prove and elaborate on Garman's result. Then we use the theorem to support the existence of a valuation measure. Since this paper is similar to [3], we refer the reader to [3] for embellishment and example.

The paper uses the continuous time methodology pioneered by Black and Scholes [1], and Merton [7]. Continuous time is extremely
useful in sorting out the risky and nonrisky aspects of any investment; 
the proof exploits this property.

The paper is organized as follows. In section 2, the assumptions 
are stated, along with the result. Section 3 discusses the theorem, and 
its proof is found in section 4.

Some notation might be new to readers: \( \Sigma_n \) means sum over all 
possible values of \( n = 0,1,2,\ldots,N \), \( \Sigma \) means sum over all indices 
\( j \in J \), in the index set \( J \).

2. Assumptions and Result

The theorem rests on several assumptions. The first concerns 
the stochastic evolution of the \( N+1 \) market factors:

Al: The market factors \( x_n \) for \( n = 0,1,\ldots,N \) obey the diffusion law:

\[
(1) \quad dx_n = \alpha_n dt + \sigma_n dZ_n
\]

where

\[
(ii) \quad E[dZ_n] = 0
\]

\[
(iii) \quad E[\sigma_n dZ_n \sigma_m dZ_m] = \sigma_{nm} dt + O(dt)
\]

and

\[
(iv) \quad \alpha_n, \sigma_n \text{ and } \sigma_{nm} \text{ are functions of } x,t.
\]

The differential operator \( \mathcal{L} \) applied to a function, say \( V(x,t) \), 
gives the result.

\[
(2.1) \quad V = V_t + \Sigma_n \alpha_n V_n + \frac{1}{2} \Sigma_n \Sigma_m \sigma_{nm} V_{nm},
\]
where \( V_n \) indicates partial differentiation with respect to \( x_n \) and the
partials are evaluated at \((x, t)\).

The second assumption describes more precisely the relation between
the market factors and the assets. There is a collection, indexed
by \( j \in J \), of assets. Each asset \( j \in J \) produces a flow of returns given
by the payoff function \( c^j(x, t) \). If the market is in state \( x \) at time
\( t \), then asset \( j \) yields a return \( c^j(x, t) dt \) in the time interval
\((t, t + dt)\). In addition, assets may have a terminal time \( \tau^j \) and a
known terminal value \( V^j(x, \tau^j) \) conditional on the state at time \( \tau^j \).
It is our objective to characterize the valuation function \( V^j(x, t) \) for
\( 0 \leq t < \tau^j \).

A2: The functions \( c^j, V^j, \sigma_n, \sigma_m, \sigma_{nm} \) are assumed to be sufficiently
regular to satisfy the requirements of Itô's lemma [6], pp. 309-310)
and to allow the expansion developed in section 3.

The third assumption is on market perfection:

A3: No transaction costs or taxes. Short sales are allowed with full
use of the proceeds. Investors are price takers, and assets are
divisible.

A4: Trading takes place continuously in time.

And, finally,

A5: There are no riskless arbitrage possibilities.

The paper's result is:

Theorem: (a) For each \((x, t)\) there exist a scalar \( r \) and \( N+1 \) element
vector \( \psi = (\psi_0, \psi_1, ..., \psi_N) \) such that:
(2.2) \[ c^j + \sum_n^j v^j = r v^j + \sum_n^j n^j_n n^j_n n^j_n \quad \text{for all assets } j \in \mathcal{J}. \]

(b) Let \( \lambda = (\lambda_0, \ldots, \lambda_N) \) be the unique vector with \( \lambda_n = 0 \) if \( \sigma_n = 0 \), and satisfying \( \psi_n \sigma_n = \sum_m \sigma_m \lambda_m \). Consider any portfolio, \( p \), that invests an amount \( y_j \) in each asset in a subset \( J \subseteq \mathcal{J} \). The value of this portfolio is \( v^p = \sum_j y_j v^j \). The expected instantaneous return on the portfolio is given by

(2.3) \[ r v^p + \sum_n^j v^j_n n^j_n n^j_n n^j_n = r v^p + \sum_n^j \lambda_n \left( \sum_m \sigma_m v^p_m \right) \]

where

(2.4) \[ v^p_n = \sum_j y_j v^j_n \quad \text{for } n = 0, 1, \ldots, N. \]

(c) There exists a risk-free portfolio of assets in \( \mathcal{J} \) if and only if there does not exist a vector \( w = (w_0, w_1, \ldots, w_N) \), such that:

\[ \sum_n^j w_n \sigma_n v^j_n = v^j \quad \text{for all } j \in \mathcal{J}. \]

(d) \( r \) and the products \( \psi_n \sigma_n \) are unique if and only if there does not exist a nonzero \( N+2 \) element vector \( (w_0 \sigma_0, w_1 \sigma_1, \ldots, w_N \sigma_N, \Theta) \) such that:

(2.5) \[ \sum_n^j w_n \sigma_n v^j_n = \Theta v^j \quad \text{for all } j \in \mathcal{J}. \]
A discussion, with examples, follows in section 3, and a proof of this result is included in section 4.

3. Discussion and Examples

The theorem was initially motivated by the author's attempt [4] to deal with capital asset pricing under conditions of inflation. Two markets are considered in [4]—a capital asset market made up of various equities and a market for discount bonds. The discount bond market contains bonds of maturity \( T > t \), and bond prices are assumed to depend entirely on time \( t \) and current rates of interest (real) and inflation. The investor wishes to take part in both markets. Equation (2.3) is used to characterize the investment strategy in the bond market. The investor need only select the amount invested \( (EY_jV^j) = V^P \), then the amount of factor sensitivity \( v^P_n = EY_jV^j_n \) for \( n = 0,1 \). It follows from (2.3) that the total expected return is then

\[
(3.1) \quad rV^P + \psi_0 \sigma_0 V^0 + \psi_1 \sigma_1 V^1
\]

The bond investment problem is reduced to the selection of three numbers: \( V^P, V^0, \) and \( V^1 \), subject to the constraint (3.1).

Notice that the second part of (2.3) is in the form of a capital asset pricing model. The term \( E_n \sigma_{nm} V^P \) represents the covariance of portfolio p's return with changes in market factor \( n \); this is a measure of type \( n \) risk in portfolio p. The parameters \( \lambda_n \) are the prices
for each type of risk. The terms \( \frac{\nu^p}{\nu^n} \) are much like the beta of a capital asset pricing model. They explain covariance through \( \sum_{m} \sigma_{mn} \nu^p \) and, with appropriate juggling of units, can be related to excess returns on portfolio that only has risk of type \( n \). In the best of all circumstances (the matrix \( B \) in the proof, section 4, has rank \( N+2 \)), there exists for each \( n = 0,1,\ldots,N \) a portfolio with values \( \nu^n = 1 \), and return

\[
d\nu^n = (r + \psi_n \sigma_n) dt + \sigma_n dZ_n.
\]

Portfolio \( n \) has only risk of type \( n \). Clearly \( \psi_n \sigma_n \) is the excess return one obtains for bearing a unit amount of risk type \( n \). For any portfolio \( p \), with value \( \nu^p \), the per dollar return is

\[
\frac{d\nu^p}{\nu^p} = (r + \sum_{n} \beta^p_n \psi_n \sigma_n) dt + \sum_{n} \beta^p_n \sigma_n dZ_n
\]

where

\[
\beta^p_n = \frac{\nu^p_n}{\nu^p}.
\]

If there exists a riskless asset or portfolio, then (2.3) indicates that \( r \) is the risk-free rate of return. Suppose, for example, that \( \nu^0 = \exp(\int_0^t X_0(t) dt) \); i.e., asset 0 accumulates value at the rate \( X_0 \). Since \( \nu^0_t = X_0 \nu^0 \) and \( \nu^0_n = 0 \) for all \( n \), equation (2.2) indicates \( X_0 = r \). We can continue this example with \( N = 0 \), and for each \( T > t \), let \( \nu^T \) be the value of a discount bond that matures at time \( T \). Set
\[ \alpha_0 = \alpha, \quad \psi_0 = \psi, \quad \text{and} \quad \sigma_0 = \sigma. \] Then (2.2) becomes:

\[ (3.4) \quad V_T^T + (\alpha - \psi \sigma) V_T^T + \frac{\sigma^2}{2} V_T^T = r V_T^T \quad \text{for all} \quad T > t, \]

which is Vasicek's, [9], equation (20) used to characterize the term structure.

Merton's formula, [8], p. 452, for the value of a derivative security is easily derived from (2.2). Let \( N = 1 \), and \( X = (r, S) \), where \( r \) is the certain risk-free interest rate and \( S \) is the value of the principal asset. The two assets are the principal asset with value \( S \), and a derivative asset with value \( F(r, S, t) \). Note that \( dr = 0 \), and we assume \( dS = \sigma S dt + \sigma S dX_1 \). In this case (2.2) reduces to

\[ (3.5) \quad (i) \quad c^S + (\alpha - \psi \sigma) = r S \]

\[ (ii) \quad c^F + (\alpha - \psi \sigma) F_S + F_t + \frac{\sigma^2}{2} F_{SS} = r F. \]

If we use (i) to eliminate \((\alpha - \psi \sigma)\) in (ii), we obtain Merton's formula

\[ (3.6) \quad r F = (r S - c^S) F_S + \frac{\sigma^2}{2} F_{SS} + F_t + c^F. \]

We saw above that the assumption of a risk-free asset implies \( r \) is its rate of return. Condition (2.4) of the theorem deals with the converse. Condition (2.4) holds if there is no risk-free asset. Note that (2.4) indicates the value of each asset depends only on its
sensitivity to changes in the risky, $\sigma_n > 0$, market factors. A riskless portfolio is constructed by choosing $y_j$ so that $\sigma_n > 0$ implies $\psi_n^p = \sum_j y_j \psi_n^j = 0$. From (2.4), we see such a portfolio must have zero value. The no riskless arbitrage then implies portfolio $p$ has an expected return equal to zero. If no riskfree portfolio with positive value can be constructed, then $r$ and $\psi$ in (2.2) are not unique. In fact, one can always select $r$ as zero. To see this, multiply (2.4) by $r$ and subtract from (2.2).

The uniqueness requirement (2.5) is stronger than the requirement (2.4). If $r$ and the $\psi_n^\circ_n$ are unique, then $r$ is a risk-free rate of return. The converse is not assured. If $r$ is a risk-free rate of return, then one must rule out solutions of (2.5) with $\Theta = 0$, in order to have uniqueness. A solution of (2.5) with $\Theta = 0$, implies that one or more of the market factors is redundant. If condition (2.5) is satisfied for all $(x,t)$ then the redundant market factors could be eliminated from the model.

We now show how the parameters derived in (2.2) relate to the market kernel that was assumed to exist in [3]. We use the term valuation measure rather than market kernel, since the measure is the analogue of valuation by state contingent claims in discrete time models.

Let $\mathcal{Y}$ be the set of all $N+1$ element vectors $(R^{N+1})$, and $\mathcal{B}$ the Borel subsets of $\mathcal{Y}$. For each $t, s > t$, and $x \in \mathcal{Y}$, there is a measure $Q(\cdot, s | x, t)$ defined on $\mathcal{B}$ such that for any $B \in \mathcal{B}$

$$
(3.7) \quad \text{Prob}\left[\{y(s) \in B | x, t\}\right] = Q(B, s | x, t) = \int_B dQ(y(s) | x, t).
$$
We wish to establish the existence of a valuation measure $K$ similar to $Q$, such that the value at time $t$, given state $x$ at time $t$, of a promise of $V(y,T)$ if state $y$ occurs at time $T > t$ is

$$(3.8) \quad \int_y V(y,T) dK(y,T|x,t).$$

In particular, the promise of §1 should event $B \in \mathcal{B}$ occur at time $T$ is worth

$$(3.9) \quad K(B,T|x,t) = \int_B dK(y,T|x,t).$$

The value of a flow of $c(y,s)$ in state $y$ at time $s$ is

$$(3.10) \quad \int_t^\infty \left[ \int_y c(y,s) dK(y,s|x,t) \right] ds.$$

When (3.8) and (3.10) are combined one obtains a formula for the value of assets in $\mathcal{F}$.

$$(3.11) \quad V^j(x,t) = \int_t^T \left[ \int_y c^j(y,t) dK(y,s|x,t) \right] ds + \int_y V^j(y,T_j) dK(y,T_j|x,t)$$

The valuation should assign value zero to any event that has probability zero; i.e., $Q(B,s|x,t) = 0$ implies $K(B,s|x,t) = 0$. This means the measure $K$ is absolutely continuous with respect to $Q$, and thus by the Radon-Nikodym theorem, [5] p. 128, there is a (unique almost
surely) finite valued measurable function \( G(y,s|x,t) \) such that

\[
K(B,s|x,t) = \int_B G(y,s|x,t)dQ(y,s|x,t).
\]

The valuation should also have a Markovian property. The stochastic process \( x(t) \) is Markovian: if \( t < \tau < T \) then for all \( B \in \mathcal{B} \).

\[
Q(B,T|\tau) = Q(B,T|x(t) \wedge (z,\tau)).
\]

In (3.13), we see the knowledge that the state was \( x \) at time \( t < \tau \), adds nothing about the probability of events at time \( T \) if we know that the state was \( x \) at a more recent time \( \tau \). By hypothesis, the values of all assets depend only on the state and time, thus the valuation measure must have the same Markovian property

\[
K(B,T|\tau) = K(B,T|x(t) \wedge (z,\tau)).
\]

From (3.14) we can rewrite (3.9) as

\[
v^j(x,t) = \int_t^{t+dt} \int_y c^j(y,s)K(y,s|x,t) \, dt + \int_y v^j(x+dx, t+dt)K(x+dx, t+dt|x,t)
\]

The first term in (3.15) is approximately \( c^j(x,t)dt \). The second term, using (3.12), is

\[
\int_y v^j(x+dx, t+dt)G(x+dx, t+dt|x,t)dQ(x+dx, t+dt|x,t).
\]

Now one follows the lead of Garman [3]. The second term is approximated by
(3.17) \( (i) \quad \mathbb{E}[(v^j + dv^j)(1 + dG)] \)

where

(ii) \( dG = (\mathcal{L} G) dt + \sum_m \sigma_m G_m dZ_m \)

and

(iii) \( G_m = \frac{\partial G(y,s|x,t)}{\partial y_m} \) evaluated at \( (y,s) = x,t \).

If we substitute (3.17 ii) in (3.16), take expectations, subtract \( v^j(x,t) \) from both sides, divide by \( dt \), and then let \( dt \) approach zero, we obtain

(3.18) \( c^j + \mathcal{L} v^j = (-\mathcal{L} G) v^j + \sum_n (-\varepsilon_m \sigma m G_m) v^j_n \)

A comparison with (2.2) quickly yields

(3.19) \( (i) \quad \psi_n \sigma_n = -\sum_m \sigma_m G_m \)

(ii) \( r = -\mathcal{L} G \).

We first note that \( \lambda_n = -G_n \) has already been introduced in (2.3) as the price of risk for market factor \( n \).

If one wishes to characterize the valuation measure, then (3.19) could be of use. If the purpose of the valuation measure is to derive (2.2), then it is not necessary; (2.2) can be derived directly and rigorously.
4. Proof of Main Result

The proof first derives a formula for the return on any asset or portfolio of assets. Then we establish the existence of a finite spanning set of assets--i.e., the returns combinations from portfolios of the spanning set are capable of replicating the returns from any portfolio. The nonexistence of riskless arbitrage is then used to establish the existence of $\psi$ and $r$ in (2.2).

The total gain for asset $j$, payoff plus capital appreciation, is:

\[(4.1) \quad c_j^j dt + dv_j^j = u_j^j dt + \sum_n \sigma_n v_n^j dZ_n,\]

where

\[(ii) \quad u_j^j = c_j^j + L v_j^j.\]

Equation (4.1) is obtained by applying Itô's lemma ([6], pp. 309-310) to get an expression for $dv_j^j$.

A portfolio is a finite collection $\mathcal{J}$ of assets and investment amounts $y_j v_j^j$ in each asset $j \in \mathcal{J}$.

Let $v_P^P = \sum_j y_j v_j^j$ be the value of the portfolio and $v_n^P = \sum_j y_j v_n^j$ the response of $v_P^P$ to a change in $x_n$. The stochastic change in $v_P^P$ is given by:

\[(4.2) \quad dv_P^P = \left(\sum_j y_j u_j^j\right) dt + \sum_n \sigma_n v_n^P dZ_n.\]
For each asset $j \in \mathcal{J}$, construct the $N+3$ element vector $a^j = (x_j^{0}, \sigma_j^{0}, \ldots, \sigma_j^{N}, V_j)$. The collection of vectors $a^j$, $j \in \mathcal{J}$ lies in Euclidean $N+3$ space and thus contains a maximal linearly independent subset of vectors called a basis. Let $H \subseteq \mathcal{J}$ index the vectors in the basis. $H$ contains $M \leq N+3$ elements and, for every $j \in \mathcal{J}$ there exists a vector $w^j$ with $M$ elements, such that:

$$a^j = \sum_{h \in H} w^j_h a^h.$$  

Equation (4.3) states that each asset $j \in \mathcal{J}$ can be considered as portfolio of assets in $H$. Thus, in answering any questions about portfolios drawn from $\mathcal{J}$, one can restrict attention to portfolios of the $M$ assets in $H$.

Let $A$ be the $M$ by $N+3$ matrix with rows $a^h$ for $h \in H$. Equation (4.3) in matrix form says that for each $j \in \mathcal{J}$ there exists a solution $w^j$ of the linear equations $a^j = w^j A$.

A riskless arbitrage portfolio exists if one can find an $M$ vector $y$, such that:

$$\sum_{h \in H} y^h a^h = 1$$

(i)  

$$\sum_{h \in H} \sigma_{n} w^h = 0 \quad n = 0,1,2,\ldots,N$$

(ii)  

$$\sum_{h \in H} y^h v^h = 0.$$  

(iii)  

Item (iii) says that the initial value of the portfolio is zero; item (ii) that the portfolio is riskless (see (4.2)); and item (i) that there is a positive expected return.
The arbitrage assumption implies $yA = b$ has no solution, where $b$ is an $N+3$ element vector with the first element equal to one and the remaining $N+2$ elements equal to zero. The fundamental theorem on the solvability of linear equations (Gale [2], p. 41), states that there exists a vector $u$ such that:

\[(4.5)\]  

\[Au = 0, \quad bu = 1.\]

Combining (4.5) with (4.3) yields:

\[(4.6)\]

\[a_j^T u = 0 \quad \text{for all} \quad j \in \mathcal{J}.\]

The first element of $u$ is 1, since $bu = 1$. Therefore, write $u$ as $u = (1, -\psi_0, -\psi_1, \ldots, -\psi_N, r)$. From the definition of $a_j^T$, (4.6) is:

\[(4.7)\]

\[u_j = rV_j + \lambda \sum_{n} \sigma_{jnm} V_{n}.\]

The definition of $u_j$ in (4.1) and (4.7) implies (2.2). If one multiplies (4.7) by $y_j^T$, sums for $j \in \mathcal{J}$, and uses $\sum_{j} y_j V_j = V^n$, then the first expression in (2.3) is immediate. To obtain the second expression in (2.3), merely use the definition of $\lambda^n$. Then:

\[\sum_{n} \psi \sigma_{n} V^n = \sum_{n} (\sum_{nm} \lambda^m V^n) = \sum_{n} \lambda^n (\sum_{nm} \sigma_{nm} V^n) = \sum_{n} \lambda^n (\sum_{m} \sigma_{mn} V^m)\]

In the last step, note that $\sigma_{nm} = \sigma_{mn}$, then reverse the role of the indices $n$ and $m$. 
Now we shall attempt to find a portfolio that invests $y_n v^h$ in asset $h$, $\sum_H y_n v^h = 1$, and $\sum_H y_n \sigma_H v^h = 0$ for $n = 0, 1, 2, \ldots, N$. From (2.3), this portfolio will have riskless return $\phi$.

Let $B$ be the $N$ by $N+2$ matrix obtained by dropping the first column of $A$ (containing the $\mu_j$). A risk-free portfolio can be constructed if and only if there exists a solution $y$ of $yB = d$, where $d$ is an $N+2$ element vector consisting of $N+1$ zeros and a one. If no risk-free portfolio exists, then (Gale [2], p. 41) there is a solution $\hat{u}$ to the equations $Bu = 0$, $d\hat{u} = 1$. The last element of $\hat{u}$ must be one, thus the $u_n$ in (2.4) can be taken as $-\hat{u}_n$.

To establish a uniqueness criterion, we examine the nonzero solutions $\tilde{u}$ of $Bu = 0$. Let such a solution be written as $\tilde{u} = (\tilde{u}_0, \ldots, \tilde{u}_N, \theta)$. Then:

\begin{equation}
0 = \sum_n \tilde{u}_n \sigma_n v^j + \theta v^j \quad \text{for all} \quad j \in J.
\end{equation}

Multiply (4.8) by a scalar $\rho$ and add to (4.7). That yields:

\begin{equation}
u_j = (r + \rho \theta)v^j + \sum_n (\psi_n \sigma_n + \rho \tilde{u}_n \sigma_n) v^j.
\end{equation}

From (4.9) we see that $r$ and $\psi_n \sigma_n$ are not unique if $(\tilde{u}_0 \sigma_0, \ldots, \tilde{u}_N \sigma_N, \theta)$ is nonzero. Nonuniqueness implies the existence of such a solution; thus equation (2.5) follows from (3.8) with $W_n = -\tilde{u}_n$. 
FOOTNOTE

1We are implicitly assuming that investments are restricted to a summable class. For example, if \( \mathcal{I} \) contains a continuum of assets, then investment policies must be integrable. The restriction is superfluous since the proof in section 4 indicates that no portfolio need contain more than \( N+2 \) assets.
REFERENCES


