Continuous-Time Finance Simplified

by

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Abstract. A simplified exposition of the valuation problem in analytic markets is developed. The paper attempts an intuitive explanation of how a single differential equation governs the price movements of all financial assets in a world that changes "smoothly". Two structural models, the Capital Asset Pricing Model and the Option Pricing Model are shown to be straightforward instances of the use of this "universal" differential equation.

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1. Introduction

This paper is aimed at developing some of the basic economic insights obtainable from the study of financial asset prices in analytic market environments. In casual terms, a market environment may be called "analytic" whenever relevant quantities like dividends, prices, consumption, investor tastes, and so forth change in a sufficiently smooth fashion. Much of the literature related to this paper goes by the more usual terminology "continuous-time models," but it must be said that this phrase is really inadequate to express all of the forms of continuity normally required of such models. Additionally, this title has become closely associated with diffusion (\$\text{It\~{a}}\$) processes, whereas we shall find that a considerable number of insights can be drawn out in the certainty cases as well. Indeed, many financial economists have permitted themselves to become somewhat intimidated by the apparent mathematical depth of "continuous-time finance," avoiding it altogether. We hope to show in this paper, at a tutorial level, that the study of analytic market environments can be made simple, natural, and understandable.

2. Foundations

In much of equilibrium-based finance, modeling activities can be classified as applying to one or both of two sub-problems: the aggregation problem and the valuation problem. The aggregation problem revolves about the question of how heterogeneous individual tastes, beliefs, and other characteristics are to be combined into meaningful quantities which summarize the market as a whole. The valuation question is that of how a market will set the prices of its financial assets.

The remarkable thing about arbitrage-free markets is that each of these two problems, aggregation and valuation, may be separated. This fact is not widely appreciated, and the previous approaches to the valuation problem have almost uniformly attempted to resolve the aggregation issues at the same time, usually leading to highly restrictive (and unnecessary) assumptions. The key to separating the two problems is the identification of a market-wide quantity which is sufficient to fully answer the valuation question, yet which is also guaranteed to exist in almost every economy regardless of the characteristics of its individual agents. The aggregation problem can then be left to focus on finding that quantity as a function of individual tastes, beliefs, etc., while the valuation problem can be confined to using the quantity.

It turns out that the appropriate summarizing quantity in analytic market environments is simply the current value of a sure $1 paid in a future date and state. Let us denote this quantity by $G(y,x,t)$. This function essentially represents an economy-wide marginal rate of substitution between values in the current state-of-the-world $x$ at current time $t$ and a future state $y$ occurring at future time $s$.

There are two fundamental questions to be answered regarding the summary measure $G$, which are: "For what economies does $G$ exist?" and "In what way is $G$ 'sufficient' for the valuation of financial assets?"

The first point, existence, is widely comprehended in the context of complete ("Arrow-Debreu") markets. It is less widely appreciated that existence is also assured in most incomplete markets. This fact is discussed in Beja [1], Garman [4], Ross [6,7], Rubinstein [8], and Cox and Ross [3], but does not seem to have gained wide recognition. Basically, all that is required for the existence of $G$ is that riskless arbitrage be unavailable in the economy.

The second point, i.e. precisely how $G$ figures in the general valuation question, can be best demonstrated by some simple illustration, as developed in the following section.

3. Certainty

Let us suppose that we inhabit a world of perfect certainty where everyone knows and agrees upon the future state-of-the-world sequence. Suppose further that the state-of-the-world variable changes in a "smooth" fashion. (Although we might treat vector-valued state variables, there is no loss of generality in confining ourselves to scalars for the moment.) We shall
employ notation as follows:

\[ x(t) \] - the value of the state variable at time \( t \)
\[ P_j(x(t), t) \] - the price of financial asset \( j \) at time \( t \) in state \( x(t) \)
\[ G(x(s), x(t), t) \] - the marginal rate of value substitution
\[ c_j(x(t), t) \] - the dividend payout rate of financial asset \( j \)

All of the above functions are assumed to be analytic (possess Taylor Series expansions) in their arguments.

In a world of certainty, how can it be possible for one financial asset to increase its dividend rate relative to another financial asset? The only possible answer is, of course, that the relative price changes must exactly offset the dividend rate changes. Let us consider a time interval of length \( \Delta t \) starting at time \( t \) and write out the necessary mathematical relationships. In the interval \( \Delta t \) the state will change by \( \Delta x \equiv x(t+\Delta t) - x(t) \), and so the "raw" price change will be \( P_j(x+\Delta x, t+\Delta t) - P_j(x, t) \), where we have further simplified notation by letting \( x \equiv x(t) \). However, we are comparing apples and oranges in taking this last difference, since the prices apply to different states and dates. This is the juncture where the economy's "marginal substitution rate" \( G \) comes in; we must make the two terms of this price difference comparable by means of multiplying each by the appropriate \( G \) term, and so the adjusted price difference is \( G(x+\Delta x, t+\Delta t; x, t) P_j(x+\Delta x, t+\Delta t) - G(x, t; x, t) P_j(x, t) \). Now the marginal rate of substitution between the current date/state and itself must be 1, so in fact the adjusted price difference is simply \( G(x+\Delta x, t+\Delta t; x, t) P_j(x+\Delta x, t+\Delta t) - P_j(x, t) \).

Next, consider the dividends of financial asset \( j \). At the beginning of the interval \( [t, t+\Delta t] \), the dividend rate is \( c_j(x(t)) \). At the end of the interval, the (value-adjusted) rate is \( G(x+\Delta x, t+\Delta t; x, t) c_j(x+\Delta x, t+\Delta t) \). So let us approximate the value of the dividend accumulation in the interval by taking the simple average of these two quantities, multiplied by the length of the interval:

\[
\Delta t \] \[ f(x, t, \Delta t) \equiv \frac{c_j(x(t)) + G(x+\Delta x, t+\Delta t; x, t) c_j(x+\Delta x, t+\Delta t)}{2} \Delta t
\]

The point of our analysis is that the two benefits of holding a financial asset, the adjusted price change and the adjusted dividend accumulation, must exactly offset one another for \( \Delta t \) sufficiently small, i.e.

\[
G(x+\Delta x, t+\Delta t; x, t) P_j(x+\Delta x, t+\Delta t) - P_j(x, t) + f(x, t, \Delta t) \approx 0 \quad (1)
\]

Suppose (1) were instead substantially different from zero. Then, given \( G \), all investors could possibly engage in pure arbitrage and all would become almost infinitely wealthy in a finite period of time, even from the most meager of initial stakes. We might then term such an opportunity an "intertemporal" arbitrage possibility.\(^1\) Let us divide (1) by \( \Delta t \) and take the limit as \( \Delta t \to 0 \). Then the approximation becomes exact and we have

\[
\lim_{\Delta t \to 0} \frac{G(x+\Delta x, t+\Delta t; x, t) P_j(x+\Delta x, t+\Delta t) - P_j(x, t)}{\Delta t} + c_j(x(t)) = 0 \quad (2)
\]

or put more simply,

\[
D(GP_j) + c_j = 0, \quad (3)
\]

where \( D \equiv d/dt \) represents the ordinary derivative operator. Note that according to (3).

\(^1\) To be more precise, such an actual arbitrage opportunity would be guaranteed to exist only in a complete market. When the market is incomplete, an intertemporal dominance situation would arise which may or may not allow investors to become infinitely wealthy (e.g. they may not be able to make short sales.) But since no investor would then want to hold a dominated security, rates of return (prices) would necessarily adjust to eliminate such dominance, i.e. make (1) zero. The most rigorous way of viewing this proposition is that a non-zero value of (1) would contradict the original definition and existence properties of \( G \).
dividends behave as if they were received at the beginning of the infinitesimal interval $dt$, while price changes act as if they were received at the end of the interval $dt$. (This observation will later take on great importance in the uncertainty case.)

By the product rule of derivatives, we have $D(fg) = fDg + gDf$ for ordinary functions $f$ and $g$, and so (3) becomes

$$DP_j + P_jDG + c_j = 0.$$  (4)

Thus we have shown that in our simple world of certainty, the prices and dividends of every financial asset $j$ must satisfy the differential equation (4). We can therefore provide this world with structure merely by specifying which types of financial assets inhabit it. For example, suppose there is one asset having price $P_j \equiv 1$ and dividend $r(t)$, and that $n$ arbitrary other assets are also extant. Equation (4) applies to each asset individually, thus we have the system of differential equations:

$$D1 + 1DG + r = 0$$  (5.0)
$$DP_1 + P_1DG + c_1 = 0$$  (5.1)
$$DP_2 + P_2DG + c_2 = 0$$  (5.2)
$$\ldots \ldots \ldots \ldots \ldots = 0 \ldots$$
$$DP_n + P_nDG + c_n = 0$$  (5.1)

From (5.0), it is clear that $r = -DG$, which is the market interest rate. Substituting (5.0) in the remainder of (5), we find that

$$DP_j - rP_j + c_j = 0, \quad j = 1, 2, \ldots, n.$$  (6)

An alternative way of writing (6) is

$$\frac{DP_j + c_j}{P_j} = r, \quad j = 1, 2, \ldots, n,$$

which points out the fact that the instantaneous rate of return on every asset must equal the instantaneous interest rate, since this is a world of certainty.

4. Uncertainty

While simplistic, our certainty world of the previous section has actually illustrated the entire methodology of our approach. The only change required to account for uncertainty is to reconsider the fundamental equation (3) with an appropriate new differential operator. Instead of $D \equiv \frac{d}{dt}$ we employ $L \equiv \frac{d}{dt} + a \frac{\partial}{\partial x} + b \frac{\partial^2}{2 \partial x^2}$ in its place. The operator $L$ is sometimes called the "differential generator" of a diffusion process, i.e. the stochastic process where the world changes randomly but continuously. The functions $a$ and $b$ are the instantaneous mean ("drift") and variance, respectively, of the diffusion process.

Thus equation (3) becomes instead

$$L(GP_j) + c_j = 0,$$  (7)

by essentially the same arguments regarding intertemporal arbitrage that we applied in the preceding section.

Now the concept that dividends are received at the beginning of the interval $dt$ and price changes are received at the end of the interval becomes critical. Dividends are certain while price changes involve risk. Only those assets whose prices are constant (nonstochastic) will be truly risk-free. (Even default-free bonds backed by the full faith and credit of the government will possess risk, since their prices may fluctuate stochastically in the short run.)

Unlike the differential operator $D$, $L$ does not obey the product rule of ordinary derivatives. From (7) we have the following analogue of equation (4):
\[ LP_j + P_j LG + b \frac{\partial G}{\partial x} \frac{\partial P_j}{\partial x} + c_j = 0. \] (8)

The new cross-product term appearing in (8), \( b \frac{\partial G}{\partial x} \frac{\partial P_j}{\partial x} \), amounts to a risk premium that could not have appeared in the certainty case. This risk premium reflects all the attributes that we would intuitively expect, namely proportionality to (i) \( b \) -- the volatility of the current state, (ii) \( \frac{\partial G}{\partial x} \) -- the market's attitude towards risk, and (iii) \( \frac{\partial P_j}{\partial x} \) -- the reactivity of the asset price to state changes.

We now turn to the consideration of specialized groups of assets. In particular, we will reformulate the two central paradigms of continuous-time finance, the option pricing model (OPM) and the capital asset pricing model (CAPM). The OPM is characterized by assuming the existence of three assets: one which is riskfree, one whose value is the state variable itself (the stock price) and one whose value depends upon the state variable (the option). Equation (8) applies to each, yielding the equation system (in the order just given):

\[ 0 \quad +LG \quad +b \frac{\partial G}{\partial x} 0 \quad +r = 0 \]
\[ a \quad +xLG \quad +b \frac{\partial G}{\partial x} 1 \quad +d = 0 \]
\[ LP \quad +PLG \quad +b \frac{\partial G}{\partial x} \frac{\partial P}{\partial x} +c = 0 \]

In (9), \( x \) is the stock price, \( d \) is the stock dividend, \( P \) is the option price and \( c \) its dividend. (We might assume \( d = c = 0 \), but it is just as simple to preserve generality here.) From (9.1) we have the riskless interest rate \( r = -LG \), so substituting this into (9.2) and (9.3) yields

\[ a \quad -rx \quad +b \frac{\partial G}{\partial x} \quad +d = 0 \] (10.2)
\[ LP \quad -rP \quad +b \frac{\partial G}{\partial x} \frac{\partial P}{\partial x} +c = 0 \] (10.3)

The important step in the OPM is substituting (10.2) into (10.3) to eliminate market risk attitudes inherent in the \( \frac{\partial G}{\partial x} \) term, making option pricing independent of investor tastes:

\[ LP - rP + (r - a) \frac{\partial P}{\partial x} + c = 0, \] (11)

which is the usual differential equation of the OPM. (Note that the terms in "\( a \)" cancel out in (11), so that option pricing is in fact independent of stock price drift.)

Next, consider an economy consisting of a riskless asset and \( n \) "primitive" assets, each of the prices of which constitutes a separate state variable \( x_j \). Since we are now working with a vector of state variables, we must consider the differential operator \( L \) for a vector-valued state description, namely

\[ L \equiv \frac{\partial}{\partial t} + \sum_{i=1}^n a_i \frac{\partial}{\partial x_i} + \sum_{j=1}^n \sum_{k=1}^n b_{ik} \frac{\partial^2}{\partial x_i \partial x_k}. \]

where \( a_i \) are the means and \( b_{ik} \) are the covariances of the asset prices \( x_i \). The analogue of (8) is then

\[ LP_j + P_j LG + \sum_{i=1}^n \sum_{k=1}^n b_{ik} \frac{\partial G}{\partial x_i} \frac{\partial P_j}{\partial x_k} + c_j = 0. \] (12)

For the economy just mentioned, (12) applies to yield

\[ 0 \quad +LG \quad +0 \quad +r = 0 \] (13.0)
\[ a_1 \quad +x_1 LG \quad +\sum_{i=1}^n b_{1i} \frac{\partial G}{\partial x_i} \quad +c_1 = 0 \] (13.1)
\[ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \]
\[ a_n \quad +x_n LG \quad +\sum_{i=1}^n b_{ni} \frac{\partial G}{\partial x_i} \quad +c_n = 0 \] (13.n)
Combining (13.0) with the remainder of (13) yields

\[ r_j = \frac{a_j + c_j}{x_j} = r - \frac{1}{x_j} \sum_{i=1}^{a} b_{ij} \frac{\partial G}{\partial x_i}, \quad j = 1, 2, \ldots, n \]  

(14)

where \( r_j \) is the return of asset \( j \). (The last term of (14) is a risk premium since normally \( \frac{\partial G}{\partial x_i} < 0 \).) Define the usual market portfolio (including the riskfree asset) as having price

\[ M = \sum_{j=1}^{a} x_j + 1 \]

and rate of return \( r_M = \left( \sum_{j=1}^{a} r_j x_j + 1 \right)/M \). Then trite calculations lead to

\[ (r_j - r) = \beta_j (r_M - r) \]  

(15)

where

\[ \beta_j = \frac{\sum_{i=1}^{a} b_{ij} \frac{\partial G}{\partial x_i} x_j}{\sum_{j=1}^{a} \sum_{i=1}^{a} b_{ij} \frac{\partial G}{\partial x_i}} M \]  

(16)

Note that when \( \frac{\partial G}{\partial x_i} \) is identical for each \( x_i \) (the market reacts symmetrically to a unit price change in any asset) these preference-dependent terms drop out, leaving the usual covariance/variance ratio of the discrete-time CAPM results, i.e., we have two-fund separation.

(The \( M/x_j \) term accounts for the fact that our covariances are in price space, not return space.)

5. Conclusions

The assumption of an analytic market provides a remarkable combination of power and simplicity. All that is required to solve the valuation question is the market characteristic \( G \), the (implicit) price of S1 in a later state and date. The existence of such a \( G \) can be established by the mere absence of riskless arbitrage possibilities. Prohibiting "inter-temporal" arbitrage for each financial asset gives rise to a differential equation (formula (7)) which must be universally satisfied by each asset. Prohibiting "intra-portfolio" arbitrage assures that all financial assets in an economy form a system of simultaneous differential equations.

In complete markets, \( G \) will be unique; in incomplete markets there may be many choices for \( G \). But the important thing is that we may employ the same \( G \) in all of our simultaneous equations in order to eliminate it (as in the OPM) or, where it cannot be eliminated, to simplify its occurrences to the point where empirical identification becomes possible (as in the CAPM). This methodology is direct, uncomplicated, and natural.

Our development here has been largely heuristic. We have not spelled out the exact nature of the concepts of analyticity, intertemporal arbitrage, etc. in a fully rigorous fashion. We have focused instead on the intuitive foundations of analytic markets. The interested reader who wishes a more detailed and rigorous exposition of the methods described here should see [4].
REFERENCES


