A CHARACTERIZATION OF OPTIMAL MULTIPERIOD PORTFOLIO POLICIES*

by

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I. INTRODUCTION

In the single-period portfolio problem, the optimal investment policy is completely sensitive to the utility function being used; the set of policies that are inadmissible or dominated across all utility functions is relatively small. The same observation holds in the multi-period case when the number of periods is not large. But as the number of periods does become large, the set of investment policies that are optimal for current investment tends to shrink drastically, at least in the simple reinvestment case without transaction costs. Many strikingly different investors will, in essence, invest the same way when the horizon is distant and will only begin to part company as their horizons near [Ross (1974), Hakansson (1974)].

It is tempting to conjecture that all long-run investment policies to which risk-averse investors with monotone increasing utility functions will flock, under a favorable return structure, insure growth of capital with a very high probability. Such a conjecture is false; many investors will, even in this case, converge on investment policies which almost surely risk run in the long run, in effect ignoring feasible policies which almost surely lead to capital growth. One purpose of this note is to set down, in simple terms, some of the relationships between the behavior of capital itself over time, the behavior of the expected utility of that same capital over time, and certain limits in the simple multi-period reinvestment model. A second purpose is to summarize the important role played by the isoelastic family of utility functions.
in this model, with particular attention given to the family member
known variously as the "growth-optimal" policy, as the geometric mean
policy, and as the maximization of expected logarithm policy.¹

II. PRELIMINARIES

In the interest of simplicity, and following Merton and Samuelson (1974),
Goldman (1974), and others, I limit the discussion, in this and the next sec-
tion, to two assets with stationary and (intertemporally) independent returns
and to constant (proportion) policies. Let \( w_t \) denote wealth after \( t \) periods
(where \( w_o \) is initial wealth), \( r_1 \) the rate return from the safe asset, and
\( r_2 \) the rate of return from the risky asset, where

\[
(1) \quad r_2 \geq -1, \quad 0 < r_1 < \mathbb{E}[r_2] < M, \quad \text{some } M.
\]

If \( v \) is the proportion (possibly greater than 1) of wealth \( w_{t-1} \)
invested in the risky asset at the beginning of period \( t \), we obtain

\[
(2) \quad w_t(v; w_o) = w_o \prod_{j=1}^{t} R_j(v) = w_o \left[ \exp \left( \sum_{j=1}^{t} \ln R_j(v) / t \right) \right]^t = w_o (1 + g_t)^t,
\]

where

\[
R_j(v) = R(v) \equiv (r_2 - r_1)v + 1 + r_1,
\]

provided

\[
(3) \quad \Pr\{R_j(v) \geq 0\} = 1, \quad j = 1, \ldots, t;
\]

here, \( R_t(v) \) is what is commonly known as the wealth relative in period
\( t \) and \( g_t \) is the average compound growth rate of capital over the first
\( t \) periods.
Consider the class of isoelastic utility of wealth functions

\[ u(w_t) = \frac{1}{\gamma} w_t^\gamma, \quad \gamma < 1, \]

where \( \gamma = 0 \) represents \( \ln w_t \). Under our assumptions the optimal policy for each member of this class is stationary [Mossin (1968)] and calls for investing proportion \( v_\gamma > 0 \) in the risky asset each period, where \( v_\gamma \) is the solution to

\[ k(\gamma) \equiv \gamma \max_v E[\frac{1}{\gamma} R(v)^\gamma]; \]

as long as \( v_\gamma \) is an interior solution with respect to constraint (3), we note that \( v_\gamma \) is increasing in \( \gamma \), that

\[ \lim_{\gamma \to -\infty} v_\gamma = 0, \]

and that

\[ k(\gamma) > 0, \quad k(0) = 1, \quad k'(\gamma) > 0 \]

[see Hakansson (1974)]. For any \( r_1 \) and \( r_2 \) satisfying (1), it follows from (6) and the continuity of \( v_\gamma \) in \( \gamma \) that there exists a \( \gamma \) such that

\[ \Pr[R(v_\gamma) \geq 1] = 1, \quad \gamma \leq \gamma \].

By the strict concavity of \( E[\frac{1}{\gamma} R(v)^\gamma] \) in \( v \), (5), and (7) we obtain

\[ k(\gamma, \gamma) \equiv \gamma E[\frac{1}{\gamma} R(v_\gamma)^\gamma] > k(\gamma), \quad \gamma < \gamma < 0, \]
and (if (3) is not binding at $v_0$),

$$
\frac{dE[\ln R(v_\gamma)]}{d\gamma} > 0, \quad \gamma < 0.
$$

III. A BOUNDED UTILITY DISCRIMINATION THEOREM

We are now in a position to juxtapose the behavior of capital and the behavior of expected utility over time.

Choose numbers $\gamma_1$ and $\gamma_2$ such that

$$
\gamma_1 < \gamma_2 \leq 0, \quad \gamma_2 \leq \hat{\gamma}, \quad k(\gamma_1, \gamma_2) < 1,
$$

(property (7) makes this possible), and consider the utility function

$$
u(w) = \frac{1}{\gamma_1} w^{\gamma_1}, \quad w \geq \delta > 0.
$$

The function (12) is clearly bounded both below and above and can be used to evaluate

$$E[u(w_t(v; w_0))] \quad \text{for} \quad v \leq \nu_\hat{\gamma}, \quad w_0 \geq \delta,$$

and all $t$ in view of (2) and (8); the optimal policy $v^*$ satisfies

$$v^* = \nu_\gamma < \nu_\hat{\gamma},$$

and gives

$$E[\frac{1}{\gamma_1} w_t(v^*; w_0)^{\gamma_1}] = \frac{1}{\gamma_1} w_0^{\gamma_1} k(\gamma_1)^t \to 0$$

by (7) and (11). Applying policy $v_\gamma$, $\gamma_1 < \gamma < \gamma_2$, the utility function (12) gives, using (11),

$$E[\frac{1}{\gamma_1} w_t(v_\gamma; w_0)^{\gamma_1}] = \frac{1}{\gamma_1} w_0^{\gamma_1} k(\gamma_1, \gamma)^t \to 0.$$
Thus, policies \( v^* \) and \( v_\gamma \), as noted by Samuelson (1971) and others, cannot be distinguished by reference to expected utility in the limit. But by (7) and (9), \( k(\gamma_1)/k(\gamma_1, \gamma) = \lambda_1 < 1 \) so that in addition to (13) and (14) we also obtain

\[ \frac{E[u(w_t(v^*; v_o))] - E[u(w_t(v_\gamma; v_o))]}{E[u(w_t(v_\gamma; v_o))]} = \lambda_1 \to 0 , \]

or equivalently, defining the sequence \( b_1, b_2, \ldots \) by

\[ E[u(w_t(v^*; v_o))] = E[u(w_t(v_\gamma; v_o b_t))], \quad t = 1, 2, \ldots , \]

we obtain the Goldman (1974) observation

\[ b_t = \left( \frac{\lambda_1}{\lambda_2} \right)^t \to \infty . \]

Both (15) and (17) are incontrovertible testimony to the superiority of \( v^* \) over \( v_\gamma \) with respect to the (bounded) utility function (12). More importantly, either the behavior of the ratios (15) or the sequence of bribes given by (16) can be used to distinguish between policies whenever the limits (13) and (14) fail to do so.

Let us now characterize the long-run effects on capital resulting from policies \( v^* \) and \( v_\gamma \). In view of (10), \( E[\ln R(v^*)] < E[\ln R(v_\gamma)] \). Thus, by the law of large numbers, (2) gives (since \( \exp(x) \) is monotone increasing) for some numbers \( a \) and \( T(\varepsilon) \)

\[ Pr\{w_t(v^*; v_o) < v_o^t < w_t(v_\gamma; v_o)\} \geq 1 - \varepsilon , \quad t \geq T(\varepsilon) , \]

for every \( \varepsilon > 0 \); in this case \( a > 1 + r_1 > 1 \) \cite{Hakansson and Miller (1975)}.

Thus, even though, in the long run, policy \( v_\gamma \) will almost surely produce a (compound) return above rate \( a - 1 > 0 \) and policy \( v^* \) will almost surely yield a return below it, some rational investors will prefer \( v^* \) to \( v_\gamma \).
The preceding may be summarized as

The Rational Investor's Reassurance Theorem. For every \((1 >) \varepsilon > 0\), there exist bounded utility of wealth functions \(u(w)\) (which are monotonically increasing and possess decreasing absolute risk-aversion), investment policies \(z_1, \ldots, z_n\), and numbers \(1 < a_1 < a_2 \cdots < a_{n-1}\) and \(T(\varepsilon)\) such that, for any initial wealth \(w_0 \geq \delta > 0\) and any return distributions satisfying (1),

\[
E[u(w_t(z_1))] > E[u(w_t(z_2))] > \cdots > E(u(w_t(z_n))) , \quad t = 1, 2, \ldots ,
\]

with the limits

\[
\lim_{t \to \infty} E[u(w_t(z_i))] = \lim_{t \to \infty} E[u(w_t(z_i))] , \quad i = 2, \ldots, n ,
\]

and (if zero is used as the upper bound on \(u(w)\))

\[
\lim_{t \to \infty} \frac{E[u(w_t(z_i))]}{E[u(w_t(z_{i+1}))]} = 0 , \quad i = 1, \ldots, n-1 ,
\]

while, at the same time,

\[
Pr\{w_t(z_1) < w_0 a_t < w_t(z_2) < w_0 a_t < \cdots < w_t(z_{n-1}) < w_0 a_t < w_t(z_n) \geq 1-\varepsilon \}, \quad t \geq T(\varepsilon),
\]

Paraphrased, the theorem says the following: there exist bounded utility of wealth functions reflecting a preference for more to less and (decreasing) risk aversion which rank wealth distribution 1 ahead of wealth distribution 2, which in turn is ranked ahead of wealth distribution 3, etc., despite the fact that wealth distribution 1 lies almost entirely to the left of wealth distribution
2, which in turn lies almost entirely to the left of distribution 3, etc. What makes the difference, even though the utility function is bounded, is the fact that the lower tail of distribution 1 is shorter and (imperceptibly) thinner than the (bounded) left tail of distribution 2, which in turn is shorter and (imperceptibly) thinner than the (bounded) left tail of distribution 3, etc. Very small adverse changes in the lower tail overpower the value of almost surely ending up with a higher compound return.

It should be noted that no reference has been made so far to the "growth-optimal" investment policy (denoted \( v_0 \) by reference to (5) since it maximizes \( E[\ln R(v)] \), and hence \( E[\ln(w_{t+1})] \), in each period \( t \)). If the return structure is such that \( \hat{\gamma} \geq 0 \) in (8), then this policy may be associated with policy \( z_n \) in the theorem. We now turn to a classification scheme which summarizes the crucial role served by the isoelastic investment family and various branches thereof, including the "growth-optimal" twig, under pure reinvestment in a linear returns technology without transaction costs.

IV. "GROWTH-OPTIMAL" VERSUS OTHER ISOELASTIC VERSUS ALL OTHER INVESTMENT POLICIES: A CLASSIFICATION

Let \( u_h(w_n) \), \( w_n \geq 0 \), be the utility of terminal wealth at some horizon \( h \), where \( u'_h > 0 \), \( u''_h < 0 \), and let \( v^*_n(w_{h-n}) \) be the optimal fraction of \( v_{h-n} \) to invest in the risky asset with \( n \) periods to go. Then for the class (4)

\[
v^*_n(w_{h-n}) = v_\gamma, \quad n = 1, 2, \ldots,
\]

where, as noted, \( v_\gamma \) is increasing in \( \gamma \) until constraint (3) is binding.

Modifying assumption (1) and expressions (2)-(3) to permit many risky assets and nonstationary returns [see; e.g., Hakansson (1974, p. 203)],
the select nature of the isoclastic family of investment policies in the "pure" multiperiod portfolio model is evidenced by the following properties in relation to all other policies based on monotonically increasing and strictly concave (terminal) utility functions:

A1. When the feasible wealth relatives are exactly lognormal, the set of optimal policies for the isoclastic family (4) spans the set of policies which are optimal for the class of all risk-averse investors in any given period.

A2. The preceding is also true when there is one risky asset (with arbitrary return distribution) and one riskless asset.

A3. Only the class (4) yields myopic optimal policies, i.e., the optimal policy for the current period is independent of the return structure beyond the current period (\( v_n^* (w_{h-n}) = v_\gamma \), \( n = 1, 2, \ldots \), in the two-asset stationary returns case) [Mossin (1968)].

A4. Convergence of \( v_n^* (w_{h-n}) \) to a "simple" policy has been demonstrated for a large class\(^3\) of terminal utility functions \( u_h (w_h) \) [Hakansson (1974); see also Leland (1972) and Ross (1974)]. Significantly, convergence has always been to an isoclastic policy,\(^4\) (i.e., of the form

\[
(19) \quad v_n^* (w_{h-n}) + v_\gamma ,
\]

in the case of two assets and stationary returns).\(^5\)

For investors who lack explicit utility of (terminal) wealth functions and therefore use other criteria, such as the mean-variance criterion, we can state
A5. To obtain a "simple" approximation to the policies which yield mean-variance efficient portfolios with respect to average compound return over \( h \) periods, i.e. the mean and variance of
\[
\left[ \frac{1}{h} \sum_{t=1}^{h} R_t(v) \right]
\]
the class (4) for \( \gamma \leq 1/h \) is sufficient [when there are only two assets, one risky and one riskless, and returns are stationary, this class yields the set of policies which is exactly efficient] [Hakansson (1971b); Hakansson and Miller (1975)].

The preceding reminds us that the simplicity of the class (4) (spelled out in A3) does not detract from its richness (as evidenced by A1, A2, A5) and its strong influence over the (investment) behavior of many other families (A4). (Furthermore, as we have seen, members of the class can be chosen to prove the theorem in Section III.)

The fundamental equality of the "growth-optimal" policy \( v_0 \) as a member of its immediate family (4) is evidenced by the following:

B1. Expression (5) yields a different optimal policy for each \( \gamma \), i.e.,
\[ v_{\gamma_1} \neq v_{\gamma_2} \text{ for } \gamma_1 \neq \gamma_2 \]
whenever constraint (3) is not binding.

B2. In (19), the class of terminal utility functions whose optimal policies converge to \( v_{\gamma} \), \( \gamma = 0 \), is no "larger" than the class whose policies converge to \( v_{\gamma} \) for any other \( \gamma \).
The "equality" of the "growth-optimal" policy as a member of the subset of the isoelastic family (4) which insures "growth of capital" (in the sense that for any \( \varepsilon > 0 \) we can find numbers \( T(\varepsilon) \) and \( a > 1 \) such that

\[
\Pr \{ w_t(z) > w_o a^t \} \geq 1 - \varepsilon, \quad t \geq T(\varepsilon)
\]

shows up in the following property:

Cl. Suppose the return structure is as in (1) or its generalization and let \( z_\gamma \) denote the optimal policy (sequence) when \( u_h(w_h) = (1/\gamma)w_h^\gamma \). Then (20) holds for all \( z_\gamma \) such that \( \gamma \leq \gamma_{\bar{\gamma}} \), where \( \gamma_{\bar{\gamma}} > 0 \); in fact, we can (in the stationary case) set \( a = 1 + r_1 \) in (20) for all \( z_\gamma \) such that \( \gamma \leq \gamma_{(r_1)} \), where \( 0 < \gamma_{(r_1)} < \gamma_{\bar{\gamma}} \). Thus, all isoelastic terminal utility of wealth functions with nonpositive exponents, and at least some with positive exponents, have the property that they cause growth of capital at a rate which almost surely exceeds the return on the risk-free asset. On the other hand, isoelastic terminal utility of wealth functions with exponents close to 1 may, under certain return structures and in the long run, almost surely lead to ruin. 6

The distinctive features of the "growth-optimal" policy (i.e., the policy based on \( u(w) = \ln w \)), by itself, are the following:

Dl. Let \( z_0 \) be the "growth-optimal" policy and \( z \) be any other investment policy which does not converge to \( z_0 \). Then for any \( \varepsilon > 0 \) there exist numbers \( a_1 < a_2 \) (where \( a_2 > 1 \) and \( a_1 \) is possibly less than 1) and \( T(\varepsilon) \) such that

\[
\Pr \{ w_t(z) < w_o a_1^t < w_o a_2^t < w_t(z_0) \} \geq 1 - \varepsilon, \quad t \geq T(\varepsilon).
\]
D2. Only the logarithmic investment policy remains myopic under serially dependent returns [Hakansson (1971a)].

Property D1 in effect says that one can virtually drive a wedge of arbitrary width between the capital distribution resulting from a geometric mean maximizing policy and the distribution produced by any other (significantly different) policy. In view of this, the appeal of the so-called "growth-optimal" investment policy to those who are somewhat unsure about their preferences is understandable: under the usual portfolio model assumptions it promises (and delivers) almost surely more wealth in the long run than (asymptotically) different policies. At the same time, the investor who knows his preferences, and is certain that they are not "close" to logarithmic, should have no difficulty discarding the "growth-optimal" policy as inferior for long-run investment purposes (see theorem in section III). Again, the reason is simply that in the long run "small" changes in the tails of the wealth distribution begin to overwhelm "large" changes elsewhere in the distribution when integrated with any given utility function. Contemplation of the integral in question renders this fact rather apparent for unbounded utilities, much less so for bounded utility functions (but recall the theorem). The resulting nonoptimality of the "growth-optimal" investment policy for certain nonlogarithmic preferences was routinely noted in Hakansson (1966, p. 69) and Samuelson (1969, p. 246) and demonstrated by explicit counterexamples in Samuelson (1971), Merton and Samuelson (1974), and Goldman (1974).

I stress again that the preceding properties do hold in the case of many risky assets and under "bounded" nonstationarity of returns (in the latter case the optimal policy will of course be nonstationary). However, the
extent to which the conditions that give rise to property A4 must be strengthened when preferences based on consumption streams, as opposed to terminal utility of wealth in a pure reinvestment model, are introduced is as yet unclear. In any case, it is evident that the isoelastic family (4) occupies a rather special position in multiperiod portfolio theory sans transaction costs.
1 For discussions of the geometric mean policy, see, for example, Williams (1936), Kelly (1956), Latané (1959), and Breiman (1960).

2 Warning: Exact lognormality does not obtain from a mixture of a riskless asset and a lognormally distributed risky asset, or mixtures of lognormal assets (except under continuous rebalancing under geometric Brownian motion). Thus, $A_1$ is notable for its existence rather than for its usefulness.

3 For an example in which convergence does not take place, see Hakansson (1974, pp. 219-20).

4 In view of (6), the riskless policy may be viewed as a member of class (4).

5 (19) does not necessarily imply that $v_n^*$ can validly be substituted for $v_n^*(w_{n-1})$ for small $n$, i.e., near the horizon [Ross (1974, p. 184), Hakansson (1974, p. 202)].

6 Consider for a moment the utility function $u(w) = w^{1/2}$, one of the most frequently cited examples of "substantial" risk aversion since Bernoulli's time. Even this venerable function may, however, lead to (almost sure) ruin in the multiperiod case: suppose, for example, that the riskfree asset yields 2 percent per period and that there is only one risky asset, which gives either a loss of 8.2 percent, with probability .9, or a gain of 206 percent, with probability .1. The optimal policy then calls for investing fraction 1.5792 in the risky asset (by borrowing fraction .5792 of current wealth to complete the financing) in each period. But the average compound growth rate $x_2$ in (2) will now tend to $-0.00756$, or $-3/4$ percent. Thus, expected utility $+\infty$ as capital itself almost surely vanishes.

7 One might argue that the proof of the pudding is in the eating and that the test-driving of cars has been known to change preferences. But once the ignorance has been dispelled, the point still remains.

8 As an illustration of this point and of the theorem, suppose there are two investment opportunities, one yielding 5 percent per period, the other a 110 percent gain or a 47.5 percent loss with equal probabilities, the returns being independent from period to period. Consider the capital distributions obtained from repeated reinvestment over $N$ periods in such a way that
(i) the assets at the beginning of each period are divided equally between the two opportunities (distribution $A_N$), and (ii) a quarter of the assets are placed in the risky asset, the other three quarters in the riskless one at the beginning of each period (distribution $B_N$). Then $A_N$ clearly has longer tails than $B_N$; furthermore, most of the probability mass of $A_N$ is "near" $w_01.05^N$ and most of the mass of $B_N$ is "near" $w_01.11^N$ for large $N$, where $w_0$ is the initial asset level. For any $N$, a logarithmic investor will always prefer $B_N$ to $A_N$ while the converse is true, for example, for an investor whose utility may be measured by the square root of wealth.

Less obvious, implicit counterexamples are provided by (for example) Mossin (1968), Marton (1969), and Hakansson (1971a).
REFERENCES


