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ON THE ESTIMATION OF SECURITY PRICE VOLATILITIES FROM HISTORICAL DATA

by

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On the Estimation of Security Price Volatilities from Historical Data

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Abstract. Improved estimators of security price volatilities are formulated. These estimators employ data of the type commonly found in the financial pages of a newspaper: the high, low, opening, and closing prices and the transaction volume. The new estimators are seen to have relative efficiencies that are considerably higher than the usual estimation methods.

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1. Introduction

This paper examines the problem of estimating capital asset price volatility parameters from the most available forms of public data. While many varieties of such data are possible, we shall consider here only those which are truly universal in their accessibility to investors, namely data appearing in the financial pages of the newspaper. In particular, we shall consider volatility estimators which are based upon the historical opening, closing, high, and low prices and transaction volume. Alternative estimators of volatility may be constructed from such data as significant news events, "fundamental" information regarding a company's prospects, and other forms of publicly available data, but these will not be considered here.

Any parameter estimation procedure must begin with a maintained hypothesis regarding the structural model within which estimation is to be made. Our structural model is given exposition in section 2. Section 3 discusses the "classical" estimation approach which forms the basis of current practice. In section 4 we introduce some more efficient estimators based upon the high and low prices. "Best" analytic estimators which simultaneously use the high, low, opening, and closing prices are formulated in section 5. Section 6 considers the complications raised by trading volume and section 7 provides a summary.

2. The Structural Model

The maintained model employed herein assumes that security prices are governed by a diffusion process of the form

\[ P(t) = f(B(t)) \]  \hspace{1cm} (1)

where \( P \) is the security price, \( t \) is time, \( f \) is a monotonic time-independent transformation, \( B(t) \) is a diffusion process with the differential representation

\[ dB = \sigma \, dz \]  \hspace{1cm} (2)

where \( dz \) is the standard Gauss-Wiener process and \( \sigma \) is an unknown constant to be estimated. This formulation is sufficiently general to cover the usual hypothesis involving the geometric Brownian motion of stock prices, as well as some of the more recently proposed alternatives to the geometric hypothesis. Throughout the remainder of this paper, it shall always be understood that we are dealing with the transformed price series \( B = f^{-1}P \). Thus "price" would mean "logarithm of original price" and "volatility" would mean "variance of the logarithm of original prices," etc., in the case of the geometric Brownian motion hypothesis; the usage will be analogous for other hypotheses possessing other transformations.

Naturally, there are limitations to our maintained model. First, we are essentially considering each security in isolation, ignoring the covariation thought to exist amongst securities in various asset pricing models. Second, only one parameter is to be estimated; simultaneous estimation of other unknown parameters, e.g., the "drift," is not treated here. Third, the required form of \( f \) rules out a significant number of alternative diffusion processes, including many having arbitrary nonzero drift, even when this drift is known. Forth, dividends and other discrete capital payouts are neglected, since these violate the continuous nature of the assumed diffusion sample paths. Fortunately, almost all of the foregoing difficulties tend to vanish as we shorten the interval over which estimation is made.

Moreover, the current paper is not concerned with the question of whether the maintained model is the "correct" model of asset price movements. Such a study has been an ongoing subject with many authors over many years, and we certainly could not aspire to settle this complex issue here. Rather, our purposes are to develop the estimation consequences of the

---

1. Actually this assumption is somewhat stronger than required. It is sufficient that \( f \) and the operations \( \max \) and \( \min \) commute.
2. Cox and Ross [3].
3. See for example Sharpe [6].
model, given the data restrictions described earlier.

3. "Classical" Estimation

Under the maintained model, (transformed) price changes over any interval are normally distributed with mean zero and variance proportional to the length of the interval. Moreover, the prices will always exhibit continuous sample paths. Yet we will not assume that these paths may be everywhere observed. There are at least two factors that interfere with our abilities to continuously observe prices: the first is the fact that transactions often occur only at discrete points in time; the second is that stock exchanges are normally closed during certain periods of time. Our maintained model assumes that the continuous Brownian motion of (2) is followed during periods between transactions and during periods of exchange closure, even though prices cannot be observed in such intervals.

As a matter of choice, we shall concentrate herein on estimators of the variance parameter $\sigma^2$ of $B(t)$. Any such choice of estimation parameter will have disadvantages in some contexts. As Boyle and Ananthanarayan [2] have recently rediscovered, any estimation procedure for $\sigma^2$ will produce bias in the estimation of any nonlinear function of $\sigma^2$, their example being its use in the "option pricing formula." Since such bias typically vanishes with increasing sample size and is usually small relative to the other sources of error, we shall ignore this issue to concentrate upon the estimation of $\sigma^2$ alone.

Moreover, it is convenient to think of the interval $t \in [0, 1]$ as representing one trading day, since this will prove also to be a satisfactory paradigm for the problems of weekly and monthly data also. Our "day" will be divided into two portions, an initial period when the market is closed, followed immediately by a trading period. Figure 1 shows this diagrammati-

![Diagram](image)

**Figure 1: Price vs. time**


In Figure 1 trading is closed initially, starting with yesterday's closing with price $C_0$. The price sample path is then unobservable until trading opens, at time $f$ and price $O_1$. In the interval $[f,1]$ we shall assume (ignoring transaction volume for the moment) that the entire sample path is continuously monitored, having a high value $H_1$, a low value of $L_1$, and a closing value of $C_1$. (The effects of monitoring at discrete transactions will be considered later.) We adopt notation as follows:

\[ \sigma^2 = \text{unknown variance (volatility) of price change.} \]
\[ f = \text{fraction of the "day" that trading is closed.} \]
\[ C_0 = B(0), \text{ previous closing price.} \]
\[ O_1 = B(f), \text{ today's opening price.} \]
\[ H_1 = \max_{f \leq t \leq 1} B(t), \text{ today's high.} \]
\[ L_1 = \min_{f \leq t \leq 1} B(t), \text{ today's low.} \]
\[ C_1 = B(1), \text{ today's close.} \]
\[ f = \text{fraction of the interval } [0,1] \text{ that trading is closed.} \]
\[ u = H_1 - O_1, \text{ the normalized high.} \]
\[ d = L_1 - O_1, \text{ the normalized low.} \]
\[ c = C_1 - O_1, \text{ the normalized close.} \]
\[ n = \text{the number of transactions in } [f,1]. \]
\[ g(u,d,c; \sigma^2) = \text{the joint density of } (u,d,c) \text{ given } \sigma^2 \text{ and } f = 0. \]

The "classical" estimation procedure employs $\hat{\sigma}_2^2 \equiv (C_1 - C_0)^2$ as an unbiased estimator of $\sigma^2$. The advantages of the classical estimator $\hat{\sigma}_2^2$ are its simplicity of use and its freedom from obvious sources of error or bias. Closing prices are measured in a consistent fashion from period to period, and there is little question about the time interval being spanned by the estimator. Its principal disadvantage is the fact that it ignores other readily available information which may contribute to estimator efficiency. As we shall see, even minor additions to the utilized information can have remarkable impact.

Opening prices are not always published in the newspaper, but when they are, effective use may be may of them. In this case, form the estimator

\[ \hat{\sigma}_1^2 \equiv \frac{(O_1 - C_0)^2}{2f} + \frac{(C_0 - C_1)^2}{2(1-f)}, \quad 0 < f < 1. \] (3)

The classical estimator $\hat{\sigma}_2^2$ will provide the benchmark by which we shall judge all other estimators. Therefore define the relative efficiency of an arbitrary estimator $\hat{\gamma}$ by the ratio

\[ Eff(\hat{\gamma}) \equiv \frac{\text{var}(\hat{\sigma}_2^2)}{\text{var}(\hat{\gamma})}. \] (4)

Since $\text{var}(\hat{\sigma}_2^2) = 2$ and $\text{var}(\hat{\sigma}_1^2) = 1$, it follows that $Eff(\hat{\sigma}_1^2) = 2$, independent of $f$. Thus we see that simply by including the opening price in our estimation procedure, we may halve the variance of our volatility estimates, given known $f$. This point argues strongly for the inclusion of opening prices in a correspondingly useful data base. (However, the estimator $\hat{\sigma}_1^2$ is also more sensitive to the assumption of identical stochastic processes during the intervals when trading is open vs. when trading is closed. It remains an open empirical question as to whether this assumption is justified.)

The importance of high relative efficiency is obvious, inasmuch as estimates of improved confidence may be constructed from our data bases. Alternatively, investigators may adopt the tactic of purposely restricting data usage to combat unforeseen nonstationarities. For example, suppose a researcher possesses a data base spanning 10 months. If he discovers an estimator having a high relative efficiency, say 10, he might choose to reduce his estimator confidence regions by a factor of $\sqrt{10}$. Alternatively, he might decide to use only one month's data and retain the old confidence regions; his reason for doing this would be to use the most recent
month's data, since it has presumably more predictive content in the presence of unknown nonstationarities.

4. High/Low Estimators

High and low prices during the trading interval require continuous monitoring to establish their values. The opening and closing prices, on the other hand, are merely "snapshots" of the process. Intuition would then tell us that high/low prices contain more information regarding volatility than do open/close prices. This intuition is correct, as Parkinson [5] has recently shown. He assumes $f=0$ and constructs the estimator

$$\hat{\sigma}_2^2 = \frac{(H_1-L_1)^2}{4 \log 2} = \frac{(u-d)^2}{4 \log 2}. \quad (5)$$

Here, $Eff(\hat{\sigma}_2^2) \approx 5.2$. When the high, low, open, and close prices are simultaneously available, we can also form the composite estimator

$$\hat{\sigma}_2^2 = a\frac{(O_1-C_0)^2}{f} + (1-a)\frac{(u-d)^2}{(1-f)4 \log 2}, \quad 0 < f < 1. \quad (6)$$

which has minimum variance when $a = .17$, again independent of the value of $f$. In this case, $Eff(\hat{\sigma}_2^2) \approx 6.2$.

One criticism of the estimators which are based solely on the quantity $(u-d)$ is that they ignore the joint effects between the quantities $u, d, c$, which may be utilized to further increase efficiency. In the following section we shall characterize the "best" analytic estimators of $\sigma^2$.

5. "Best" Analytic Estimators

For our purposes herein, an estimator is "best" when it has minimum variance and is unbiased. We shall also impose the requirements that the estimators be analytic and scale-invariant, as explained later.

It suffices to suppose $f=0$, i.e., trading is open throughout the interval $[0,1]$. Then consider estimators of form $D(u,d,c)$, i.e., decision rules which are functions only of the quantities $u, d,$ and $c$. We restrict attention to these normalized values because the process $B(t)$ renews itself everywhere including $t=0$, and so only the increments from the level $O_1 (=C_0)$ are relevant. According to the Lemma established in Appendix A, any good estimator $D(u,d,c)$ should inherit the invariance properties of the joint density of $(u,d,c)$. Two such invariance properties may be quickly enunciated: For all $\sigma^2 > 0$ and all $d \leq c \leq u$, $d \leq 0 \leq u$, we have

$$g(u,d,c;\sigma^2) = g(-d,-u,-c;\sigma^2) \quad (7)$$

and

$$g(u,d,c;\sigma^2) = g(u-c,d-c,-c;\sigma^2). \quad (8)$$

The first condition represents price symmetry: For Brownian motion of form (2), $B(t)$ and $-B(t)$ have the same distribution. Whenever $B(t)$ generates the realization $(u,d,c)$, $-B(t)$ generates $(-d,-u,-c)$. The second condition represents time symmetry: $B(t)$ and $B(1-t) - B(1)$ have identical distributions. Whenever $B(t)$ produces $(u,d,c)$, $B(1-t) - B(1)$ yields $(u-c,d-c,-c)$. By the Lemma of Appendix A, then, any decision rule $\hat{\sigma}^2 \equiv D(u,d,c)$ may be replaced by an alternative decision rule which preserves the invariance properties (7) and (8) without increasing the expected loss associated with the estimator. Therefore we seek decision rules which satisfy

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7. Parkinson actually gives two estimators of volatility, the one described in formula (5) and another one which employs the square of the sum of the differences of high and low. Parkinson fails to warn the user that this latter estimator is biased, hence we do not consider it herein.
\[ D(u,d,c) = D(-d,-u,-c) \]  \hspace{1cm} (9) 

and

\[ D(u,d,c) = D(u-c,d-c,c-c) \]  \hspace{1cm} (10) 

Next, we observe that a scale invariance property holds in the parameter space: for any \( \lambda > 0 \),

\[ g(\lambda u, \lambda d, \lambda c; \lambda^2 \sigma^2) = g(u, d, c; \sigma^2) \]  \hspace{1cm} (11)

In consequence of (11), we restrict our attention now to decision rules for which

\[ D(\lambda u, \lambda d, \lambda c) = \lambda^2 D(u, d, c) \quad \lambda > 0. \]  \hspace{1cm} (12)

If we now adopt the regularity condition that the decision rules considered must be analytic in the neighborhood of the origin, condition (12) implies that the decision rule \( D(u, d, c) \) must be quadratic in its arguments. (Proof of this is given in Appendix B.) Thus we have

\[ D(u,d,c) = a_{101} u^2 + a_{010} d^2 + a_{002} u^2 + a_{110} ud + a_{101} uc + a_{011} dc. \]  \hspace{1cm} (13)

Scale invariance and analyticity have been combined to reduce the search for a method of estimating \( \sigma^2 \) from an infinite-dimensional problem to a six-dimensional affair. Applying the symmetry property (9) to (13), we have the implications \( a_{100} = a_{020} \) and \( a_{011} = a_{101} \). By virtue of property (10) we have the additional constraint \( 2a_{200} + a_{110} + 2a_{101} = 0 \), hence we have

\[ D(u,d,c) = a_{100} (u-d)^2 + a_{020} c^2 - 2(a_{020} + a_{101}) ud + a_{101} uc + a_{011} dc. \]  \hspace{1cm} (14)

Insisting that \( D(u,d,c) \) be unbiased, i.e., \( E[D(u,d,c)] = \sigma^2 \), leads to one further reduction. Since \( E[u^2] = E[d^2] = E[c^2] = E[uc(u+d)] = \sigma^2 \) and \( E[ud] = (1-2\log_e 2)\sigma^2 \), we may restrict attention further to the two-parameter family of decision rules \( D(\cdot) \) of the form

\[ D(u,d,c) = a_1 (u-d)^2 + a_2 \left( c(u+d) - 2ud \right) + \left( 1 - (a_1 + a_2) 4\log_e 2 + a_2 \right) c^2. \]  \hspace{1cm} (15)

To minimize this quantity, note that for any random variables \( X, Y, \) and \( Z \), the quantity \( V(a_1, a_2) \equiv E[(a_1 X + a_2 Y + Z)^2] \) is minimized by \( a_1 \) and \( a_2 \) which satisfy

\[ E[a_1 X + a_2 Y + Z|X] = E[a_1 X + a_2 Y + Z|Y] = 0. \]  \hspace{1cm} (16)

Solving the above for \( a_1 \) and \( a_2 \), we have

\[ a_1^* = \frac{E[XY]E[YZ] - E[Y]E[XZ]}{E[X]^2E[Y]^2 - (E[XY])^2} \]  \hspace{1cm} (17a) 

and

\[ a_2^* = \frac{E[XY]E[XY] - E[X]^2E[YZ]}{E[X]^2E[Y]^2 - (E[XY])^2} \]  \hspace{1cm} (17b) 

In our problem,

\[ X = (u-d)^2 - (4 \log_e 2) c^2 \]
\[ Y = c(u+d) - 2ud + (1 - 4 \log_e 2) c^2 \]
\[ Z = c^2 \]  \hspace{1cm} (18)

Analysis reveals the following forth moments:

\[ E[u^2c^2] = E[d^2c^2] = 2\sigma^4 \]
\[ E[u^3c] = E[d^3c] = 2.25\sigma^4 \]
\( E[u^3] = E[d^3] = 1.5\sigma^4 \)

Simulation was employed to evaluate these moments:

\[
E[ud^2c] = E[u^2dc] = -.18\sigma^4 \\
E[u^2d^2] \approx .222\sigma^4 \\
E[udc] \approx -.44\sigma^4 \\
E[ud^3] = E[u^3d] \approx -.425\sigma^4
\]

Substituting the above moments into (17a) and (17b) via (18), we find that \( a_1 \approx .2 \) and \( a_2 \approx .8 \), the approximation being subject to the limitations of our simulation studies. Employing these values in (15) yields the "best" unbiased analytic estimator

\[
\hat{\sigma}_4^2 \equiv .2(u-d)^2 + .8c(u+d) - 2ud - .97259c^2.
\]  

(19)

Simulation shows that \( \text{Eff}(\hat{\sigma}_4^2) \approx 7.6 \).

Now suppose that \( 0 < f < 1 \), i.e. trading is both open and closed in \([0,1]\). Then the opening price \( O_1 \) may differ from the previous closing price \( C_0 \), and so we may form the composite estimator

\[
\hat{\sigma}_3^2 \equiv (O_1-C_0)^2 f + (1-f) \frac{\hat{\sigma}_4^2}{(1-f)}
\]  

(20)

The variance of \( \hat{\sigma}_3^2 \) is minimized when \( a = .12 \), and in this case \( \text{Eff}(\hat{\sigma}_3^2) \approx 8.6 \). Thus the best analytic scale-invariant estimator obtainable possesses an efficiency factor which is almost nine times better than the classical estimator \( \hat{\sigma}_3^2 \), given only high, low, open, and close prices.

6. Volume Effects

The derivation of all of the high-low estimators of the previous sections depends critically upon the assumptions of continuously-monitored price paths (even though only a limited amount of data is extracted from these.) When the path can only be monitored at discrete transactions, the statistics which are based on the high and low will be biased downward in absolute magnitude. (Clearly, if we could observe the entire sample path, the high would have been higher and the low would have been lower.) Technically speaking, the knowledge that only a finite number of observations are available should lead us to commence a new search for the "best" estimator; however, we shall defer this task to a later paper. We instead confine our considerations here to determining the extent of the bias in using the estimators already described when only a finite set of observations is available. Simulation studies were employed to arrive at the following table:

<table>
<thead>
<tr>
<th>No. transactions</th>
<th>( \hat{\sigma}_3^2 )</th>
<th>( \hat{\sigma}_4^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>.56</td>
<td>.30</td>
</tr>
<tr>
<td>10</td>
<td>.63</td>
<td>.38</td>
</tr>
<tr>
<td>20</td>
<td>.79</td>
<td>.64</td>
</tr>
<tr>
<td>50</td>
<td>.83</td>
<td>.72</td>
</tr>
<tr>
<td>100</td>
<td>.85</td>
<td>.76</td>
</tr>
<tr>
<td>200</td>
<td>.90</td>
<td>.83</td>
</tr>
<tr>
<td>500</td>
<td>.93</td>
<td>.88</td>
</tr>
</tbody>
</table>

Expected values of volatility estimators \( \hat{\sigma}_3^2, \hat{\sigma}_4^2 \) are thus less than \( \sigma^2 \) whenever a finite number of transactions take place. As a practical procedure, one should then divide the estimators by the numbers given in the table for each transaction volume level.

As mentioned earlier, we are at this point faced with a number of alternative developments. It seems appropriate that further theoretical analysis should await empirical
investigations to better determine the right paths.

7. Conclusions

We have examined a number of estimators of price volatility. Efficiency factors which are almost nine times better than the classical estimators have been demonstrated. These same estimators are also subject to more sources of predictable bias, one of the most evident of which is finite transaction volume. Unpredicted sources of bias await further empirical work.

Appendix A: Estimator Invariance Properties

Lemma. Let \( \Theta \) be a parameter space. Let \( X = (X_1, X_2, \ldots, X_n) \) be a vector of (not necessarily independent) observations whose joint density \( f_\theta(X) \) depends on an unknown parameter \( \theta \in \Theta \), to be estimated. Let \( T: \mathbb{R}^n \rightarrow \mathbb{R}^n \) be a fixed measure-preserving transformation. Suppose that for all \( \theta \in \Theta \) and all \( X \) in the support of \( f_\theta \),

\[
    f_\theta(TX) = f_\theta(X). \tag{A1}
\]

Let \( D(X) \) be any decision rule which estimates \( \theta \). Let \( L(\theta, D(X)) \) be any loss function such that \( L(\theta, \cdot) \) is a convex function for each fixed \( \theta \in \Theta \). Defining \( T^j \equiv T^{j-1} T \) where \( T^0 \) is the identity operator, let \( A_k \) be an averaging operator which maps decision rules into decision rules according to the prescription

\[
    A_k(D(X)) = \frac{1}{k} \sum_{j=1}^{k} D(T^{j-1}X). \tag{A2}
\]

Then for all \( \theta \),

\[
    E_\theta L(\theta, A_k(D(X))) \leq E_\theta L(\theta, D(X)). \tag{A3}
\]

Comment. In particular, if \( L(\theta, \cdot) \) is strictly convex, then every non-invariant rule is subject to improvement. (A rule is invariant with respect to \( \theta_0 \) iff \( \text{Prob}(D(X) = D(TX)) = 1 \). In many cases an invariant rule can be constructed from \( D(\cdot) \). If, for example, \( T^\ast X \equiv X \), then

\[
    A_k(D(TX)) = \frac{1}{k} \sum_{j=1}^{k} D(T^jX) = \frac{1}{k} \sum_{j=1}^{k-1} D(T^jX) + \frac{D(X)}{k} - A_k(D(X)). \tag{A4}
\]

Secondly, if \( T \) is measure-preserving then, according to the Mean Ergodic Theorem, there exists a measurable function \( D^\ast(\cdot) \) such that for almost all \( X \),

\[
    \lim_{k \to \infty} A_k(D(X)) = D^\ast(X), \tag{A5}
\]

\[
    D^\ast(TX) = D^\ast(X), \tag{A6}
\]

and

\[
    E_\theta L(\theta, D^\ast(X)) \leq E_\theta L(\theta, D(X)). \tag{A7}
\]

Thus, in the interest of minimizing expected loss, attention may often be restricted to those rules for which \( D(X) = D(TX) \).

Proof. By convexity,

\[
    L(\theta, A_k(D(X))) \leq \frac{1}{k} \sum_{j=1}^{k} L(\theta, D(T^{j-1}X)).
\]
Taking expectations,

\[ E_\theta L(\theta, A_k(D(X))) \leq \frac{1}{k} \sum_{j=1}^{k} E_\theta L(\theta, D(T^{j-1}X)). \]

It suffices to verify that

\[ E_\theta L(\theta, D(T^jX)) = E_\theta L(\theta, D(X)) \]

for all \( j \geq 1 \). Computing,

\[ E_\theta L(\theta, D(T^jX)) = \int_{\mathbb{R}^n} L(\theta, D(T^jX)) f_\theta(X) \, dX. \]

\[ = \int_{\mathbb{R}^n} L(\theta, D(T^{j-1}X)) f_\theta(T^jX) \, dX, \]

the latter by (A1). By change of variable, this equals

\[ = \int_{\mathbb{R}^n} L(\theta, D(Y)) f_\theta(Y) J_{T^{-j}} \, dY \]

\[ = E_\theta L(\theta, D(X)), \]

where the Jacobian \( J_{T^{-j}} = 1^{-j} = 1 \), since \( T \) is measure-preserving.

Appendix B: Analytic Estimators are Quadratic

Lemma. Estimators \( D(u, d, c) \) of \( \sigma^2 \) which are analytic in the neighborhood of the origin are quadratic in form.

Proof. If \( D(u, d, c) \) is analytic in the neighborhood of the origin, we may write its Taylor series expansion as

\[ D(u, d, c) = \sum_{i, j, k \geq 0} a_{ijk} u^i d^j c^k. \] \hspace{1cm} (B1)

Next define

\[ F_{u, d, c}(\lambda) \equiv \lambda^2 \sum_{i, j, k \geq 0} a_{ijk} u^i d^j c^k - \sum_{i, j, k \geq 0} a_{ijk} \lambda^{i+j+k} u^i d^j c^k. \] \hspace{1cm} (B2)

From the scale invariance property (12), \( F_{u, d, c}(\lambda) \) must be identically zero. It may also be extended to an analytic function in some open neighborhood of the origin. Thus by uniqueness, all coefficients of powers of \( \lambda \) in (B2) must be identically zero. It follows that \( a_{ijk} u^i d^j c^k = 0 \) for \( i+j+k \neq 2 \), i.e., \( D \) is quadratic.
REFERENCES


