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INSTITUTIONAL INVESTMENT WITH MULTIPLE PORTFOLIO MANAGERS

by

Barr Rosenberg

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INSTITUTIONAL INVESTMENT WITH MULTIPLE PORTFOLIO MANAGERS

Barr Rosenberg*

A corporate pension fund is generally apportioned among several portfolio managers by the corporate sponsor. Such diversification across carefully chosen managers is believed to promise improved performance. This paper is concerned with the coordination of the decentralized decisions of the managers. A minimal set of parameters is found, which should be determined by the sponsor and communicated to the manager, so as to induce management that aggregates to an optimal investment policy for the total portfolio. The policy is optimal with respect to the sponsor's risk preference in a mean-variance context.

The essential control parameters are as follows: first, the sponsor's assessment of the information content of the managers' appraisal processes, captured in "covenant information ratios"; second, "dependence-adjustment factors" that correct for information redundancies between managers; third, the sponsor's "risk acceptance" for systematic and residual risks; fourth, the investment proportions entrusted to the managers.

The analysis of optimal decentralization policy provides a standard of comparison for institutional investment. From this perspective, several features of current practice appear to be counterproductive. A series of simple remedies is suggested: the control framework just discussed allows managers to determine appropriate aggressiveness in various aspects of portfolio investment; "chits" for market return and asset return traded among managers reduce losses from the costs of off-setting transactions and ameliorate inefficiencies arising from legal restrictions on portfolio holdings; distinction between the "equal-risk" and "reward" principles for apportioning funds among managers is necessary to achieve management fees that reflect managers' contributions to the sponsor's utility.

*Associate Professor, Schools of Business Administration, University of California, Berkeley. Michel Houglet's contributions to this study were indispensable, as were Ellen Mc Gibbon's efforts in preparing the manuscript. Stimulating suggestions from Douglas Love, Harold Arbit, and Peter Dietz are gratefully acknowledged.
1. Introduction and Summary

Employee benefit funds (pension and profit-sharing plans) are usually apportioned among more than one manager. Large funds, with $100 million or more in assets, rarely have fewer than four managers, and "groups" of fifteen or more managers do occur.

Diversification appears to be the primary motive underlying multiple management. The corporate sponsor of the plan attempts to select above-average managers who will return an appraisal premium relative to the market portfolio. Since the assessment of any single manager is subject to error, the sponsor diversifies across the apparently better managers to earn the average reward accruing to his (the sponsor's) selection process. A second justification for multiple management is to obtain a representative selection of "management styles" or "habitats." Certain managers emphasize subsets of the universe of assets (bonds, growth stocks, higher-yielding stocks, special situations). A collection of such specialists is required for an exhaustive investment policy.

Each manager charges a management fee, which increases with the amount under management. The unit fee generally declines as the amount managed grows. This is strictly an investment management fee, earmarked to support the process whereby the manager appraises assets, determines appropriate investments, and implements these through trading activity. Transaction costs, comprising commission fees to brokers and spreads paid to dealers, are additional and are deducted directly from the corpus of the portfolio.

1.1. Passive and Active Management

Both management fee and transaction costs would be much lower for a passive investment strategy, in which a diversified "informationless" portfolio were bought and held. The basic Capital Asset Pricing Model (CAPM) singles out the market portfolio as the efficient passive investment vehicle, in the absence of superior forecasting ability.¹

¹Generalizations of the CAPM, reflecting realistic considerations such as differential taxation, restricted borrowing, uncertain inflation,
Active management entails differential holdings, relative to the efficient passive portfolio, in order to profit from forecasts of abnormal asset returns. Inevitably, an incremental fee must be charged to support the appraisal and investment process. The justification for the sponsor to pay the fee is a belief that active management will achieve superior returns relative to the passive portfolio.

The sponsor, to justify active management, must believe that his selection process, superior to that of other sponsors, identifies an above-average group of managers, enough superior in expected gross return to offset the three forms of costs of active management: (a) the additional management fee paid to the active managers, (b) the transaction costs of active investment, and (c) the disutility of the unsystematic risk caused by the active holdings. In this paper, the validity of the sponsor's selection process is not questioned. The sponsor's belief in positive appraisal premia is accepted, and the question of appropriate portfolio management is considered.

1.2 Basic Precepts

The institutional investment arrangement of multiple portfolio managers is too often taken for granted. To understand it better, it is useful to consider three kinds of modifications that could easily be made if they were desirable. The first is to transfer funds from the direct responsibility of individual managers to a single master account controlled

and exchange-rate uncertainty, admit of several different "mutual funds," with the efficient passive portfolio for each individual investor being a weighted combination of these, with weights depending on the investor's particular circumstances. Special considerations on the liability side of the investor, such as the duration and degree of inflation adjustment of the liabilities of a pension fund or insurance company, or the atypical exposure to wage inflation of an endowed university, when incorporated into the theoretical context, lead to still greater distinctions between the optimal "passive" portfolios of investors. The fact remains that, for each investor, in general, there is an efficient passive portfolio, which, in the absence of superior returns generated by appraisal, would be optimal for that investor.
by the sponsor, so that the managers become advisors who send recommended transactions to the central agency. The second is concerned with changes in investment proportions entrusted to different managers—with appropriate fee adjustments so that management fees are unaffected—so as to change the relative sizes of constituent portfolios. The third concerns changes in the "aggressiveness" of constituent portfolios. Aggressiveness, as defined in [1], refers to the extent to which individual holdings deviate from market proportions. Greater aggressiveness can be understood as greater concentration. For any set of security appraisals, there is a family of optimal portfolios, ranging from the conservative, passive portfolios through a range of increasingly aggressive postures to the portfolio that is 100% invested in the security (or securities) with the highest appraisal premium.

By considering the changes in the aggregate portfolio that result from modifications of the three kinds discussed above, nine fundamental precepts concerning multiple portfolio management can be deduced. These will be useful background for the systematic analysis of decentralized portfolio management which follows.

A. *Multiple management can at best match optimal practice in a centralized portfolio with multiple advisors.* This precept can be demonstrated by considering the following arrangement. Instead of producing a managed portfolio, each "manager" transmits a model portfolio to the central transaction agency. The model portfolio is managed as if it is a real portfolio, with recommended purchases and sales. At the central agency, these model portfolios are somehow coordinated, and a single decision as to transactions of the aggregate portfolio is made. This centralized procedure can exactly replicate multiple management, insofar as recommended purchases and sales are concerned, by maintaining distinct pools, one for each manager, in which the model transactions are reproduced. Since this option is available to the centralized agency, optimal centralization must be able to match—and possibly better—the performance of the decentralized arrangement. However, this argument applies only to transaction recommendations and not to the trading process itself.
The possibility of centralization opens several questions for consideration: What is the optimal way of reflecting the managers' recommendations in the aggregate portfolio? Can a decentralized arrangement, such as the multiple-manager process, accomplish this optimal procedure? The multiple-manager arrangement has obvious advantages with regard to the sharing of responsibility, the motivation of managers, and the preservation of their independence. However, it is important to consider whether these advantages are not offset by disadvantages of the multiple-manager process, which preclude an optimal aggregate policy.

B. Whether there is one or many managers, the return on the aggregate portfolio is what matters to the sponsor. This precept is self-evident and increasingly so as one reflects upon it. The sponsor is concerned with the value of the entire pool. Whether the same return is obtained from one manager only, or from a group of managers, or from high returns on some managers that are offset by low returns on other managers, is not material to the purposes of the fund.

C. Aggregated normal holdings should coincide with the sponsor's optimal passive portfolio. Each managed portfolio, although continually changing due to active management, exhibits a normal posture due to managerial style. The normal posture is results when expected portfolio holdings are averaged over time. For example, the normal emphasis of a growth stock manager is toward growth-oriented companies, with average beta greater than one, probably shaded toward small companies when compared with the S&P 500. The sponsor must allocate funds among managers so that the aggregate of constituent portfolios (weighted by investment portions) coincides with the desired passive portfolio. The beta must be correct, and there must be no bias in the direction of a particular group of companies. A permanent bias in favor of some group of companies should be based on the belief that some aspect of Capital Asset Pricing Theory promises compensatory reward for the added residual risk. Ordinarily, active management should cause variations in holdings around the passive portfolios. Permanent bias is not the proper result of active management, unless a
manager asserts that there is a perpetual market inefficiency favoring some kind of company.

D. Aggressiveness should vary inversely with the manager's investment proportion. This precept follows from precept B. To demonstrate it, suppose that the initial situation is optimal and that, suddenly, a manager is given twice as many funds to manage as he had previously. The information in the hands of the manager, on the basis of which portfolio decisions are made, is identical to what it was before. Since the previous situation was optimal, it follows that the same contribution to the aggregate portfolio should now be made as was made previously. As there is twice as large a pool, the same contribution can be accomplished by an investment strategy with exactly one-half as much aggressiveness. The result of this is that twice as much money is committed to holdings which deviate one-half times as much from the market. The dollar value of resulting active positions is identical to the previous set. Since the previous situation was optimal, so is the modified situation. Thus, as the investment proportion entrusted to a manager is increased, his aggressiveness should decrease. The reverse argument can also be carried through. Some subtleties relating to restrictions on investment holdings are discussed in detail in section 3.4, below.

Another way of explaining this precept is to consider the following hypothetical situation. Suppose that the strategy offered by a money manager is presently optimal. Suddenly, the money manager breaks up his operation into four independent managers, each concentrating on one portion of the market (large companies, small companies, high-growth companies, and high-yield companies). Each one of these four new entities is given a portion of the manager's previous pool. (These portions might be equal to the proportions of the market capitalization in the four sectors.) The sponsor now deals with each of the four managers as a separate entity. Suppose, for the sake of argument, that one of these entities has 25% of the previous pool. Since the previous situation was optimal, this new entity must produce within his quarter of the previous pool the same holding which were previously present in his sector. To do this, the active holdings, as a proportion of 25% of the pool, must be four times greater.
than the active holdings in the original pool. In this way, active holdings four times as great multiplied by an equity pool which is one-fourth as large will produce the same dollar values of active positions as before.

E. Investment aggressiveness should vary in response to the sponsor's assessment of the managers' ability. It is common practice for a manager to act with comparable aggressiveness regardless of the degree of belief that the sponsor places in his security appraisal process. From the sponsor's viewpoint, this is counterproductive. A sponsor who is convinced that the manager has no information does not want the manager to take any risks and desires a passive market portfolio. The sponsor who believes that the manager has a superior appraisal process which offers the potential for significant abnormal returns must encourage the manager to undertake risks in order to profit from this superior appraisal. The precept that aggressiveness should vary with information was developed in detail in [1].

F. Never pay an active management fee for passive management, and use passive management as a means to reduce unsystematic risk. This precept is best understood by considering a hypothetical situation. Suppose that a manager presently delivers an unaggressive portfolio that approximates an index fund. That is, none of the active holdings, either positive or negative, are very large. In this case, the sponsor can effect a reduction in management fee as follows: Demand that half of the portfolio be managed as an index fund, with the lower management fee charged for index funds, and that the other half of the portfolio be managed at the previous active management fee, but managed with twice the aggressiveness. The end result is that the fee, instead of being charged 100% at the active level, is now charged 50% at the active level and 50% at the lower passive level. The portfolio, on the other hand, is identical to the previous portfolio, since one-half of the pool is invested with twice the active holdings. Thus, the sponsor has reproduced the same portfolio position with a reduction in fee.

With this situation in mind, consider another case where the sponsor decides to reduce residual risk by some means other than diverting
funds to passive investment. This implies that he demands a revised active portfolio with smaller aggressiveness than before, but without a reduction in fee. This is clearly undesirable, from the sponsor's viewpoint, since as an alternative, he could have retained the initial aggressiveness but achieved the risk reduction by reducing the actively managed portion of the portfolio. The reduced actively managed portion would reduce fees and lead to the same result as reduced aggressiveness which did not reduce fees.

G. Multiple active managers are not a means to reduce risk, but rather a means to obtain superior reward. Consider a hypothetical situation where the sponsor begins with one manager (or J managers). Assume that this group is balanced, in the sense that no sector of the market is disproportionately represented. Or, alternatively, if some sector is emphasized, assume that this stems from deliberate active investment, not from accidental bias in the group.

The sponsor then takes some funds away from the original group and entrusts them to an additional active manager, manager #2 (or J+1). This must be justified by the belief that the new manager offers a potential for superior reward, since otherwise a passive portfolio would serve the purpose at lower fee. He may not be superior to the other managers in the pool, but he must be regarded as (1) superior to the average manager and (2) providing some information in addition to that provided by the previous managers, which offers a potential for superior return. Otherwise, the sponsor would be better off keeping to the original pool of managers or reducing risk through passive investment. When these requirements are met, it is easy to prove that the revised portfolio should have a higher level of unsystematic risk than the original portfolio! (This conclusion follows from Corollary 9.1.)

Thus, if the sponsor retains his previous assessment of the abilities of an existing group of managers, and adds another manager to the pool, the unsystematic risk of the aggregate portfolio should increase. The multiple-manager procedure is not suitably applied as a means of reducing unsystematic risk, but rather as a means of obtaining opportunities for unsystematic reward, opportunities that are sufficiently improved to warrant
taking higher unsystematic risk in their pursuit. The sponsor who thinks
he has found only one superior manager will operate the aggregate port-
folio at a lower level of unsystematic risk than will the sponsor who be-
lieves that he has found four or five managers of the same degree of superiority.

H. The optimal aggressiveness for each manager is influenced by
the correlation between his information process and that of other managers.
Correlation of information refers to the fact that two managers may use
similar information sources and may apply similar analytical procedures,
with the result that they tend to arrive at similar recommendations. This
can be thought of as a redundancy between the information offered by the
two managers. The greater the redundancy, the greater the intermanager
correlation. To understand that investment aggressiveness must be adjusted
for correlations, a simple and extreme example is useful. Suppose that a
money-management firm suddenly clones a subsidiary, with a different name,
which uses exactly the same analytical process and produces identical port-
folios. The subsidiary obtains additional funds from the sponsor
on its own account, so that the sponsor now has two ostensibly different
portfolios managed identically. Clearly, precept D, relating to aggres-
siveness and investment proportions, applies in this case. Since a larger
amount of funds have been entrusted to one management, aggressiveness
must be reduced to obtain the same contribution to the aggregate portfolio.
Consequently, the aggressiveness of each of the two nominally separate
portfolios must be reduced, and some adjustment must be made for the re-
dundancy between their recommendations.

The same problem arises, although to a lesser degree, in the sit-
uation where there is partial but not complete redundancy between two man-
gers. In that case, the aggressiveness of each must be reduced to avoid
overresponding to the common information

I. The dependency adjustment applies to information, in general,
and not to individual positions. The adjustment for dependancy, which was
introduced as precept H, serves to avoid overresponding to a common source
of information. It is a natural mistake to apply this concept on an item-
by-item basis, and to conclude that when two managers recommend the same
stock, the sponsor should reduce the holding to correct for dependency. To understand the problem with the approach, visualize two different security analysis departments. Each department must scrutinize a variety of sources and apply the analytical methods it has developed to come up with a set of recommendations. If both operations are useful ingredients in the aggregate portfolio, then there must be something that is both valuable in each one of these analytical processes and also different between processes. Each process typically adds an atom of useful information as a result of a special and superior approach to data. When the recommendations from the two processes are in the same direction, then two different atoms of information imply the same conclusion. The sponsor should place greater credence in consistent recommendations than he would in either recommendation separately. When two recommendations match, a larger active position is appropriate, and when two recommendations differ—implying that the distinct atoms of information are offsetting—it is appropriate to take a neutral stance.

If there is some redundancy between the two processes, this should be taken into account by an estimate of the degree of overlap which applies for the typical stock. The sponsor should not necessarily conclude that a matching of recommendations implies a greater-than-usual degree of overlap. This would serve to devalue the two processes inappropriately, since the sponsor is seeking precisely that situation, where the two processes legitimately come up with the same recommendation, indicating a superior opportunity.

1.3 Optimal Decentralization

The balance of this paper derives and explains appropriate decentralization procedures. As the precepts have made clear, one necessary action of the sponsor is to determine appropriate aggressiveness for the managers. It is shown in the appendix that this can be done optimally, so that the end result of the managers' autonomous purchase and sale recommendations is the same aggregate optimal portfolio which the sponsor would prepare if he were coordinating all decisions in-house. Thus, optimal decentralization of purchase and sale recommendations is possible.
Two inefficiencies arise from the nominally independent status of the constituent portfolios: first, unnecessary transaction costs whenever a purchase recommendation of one manager and a sale recommendation of another are offsetting; second, reductions in achievable reward/risk ratios due to restrictions on holdings in constituent portfolios.

Section 2 of the paper is concerned with decentralized management of beta. This topic is taken first, not because it is more important, but because it is easier to present. After the discussion of beta is completed, the more complicated case of decentralized control of stock selection is taken up in section 3. The appendix derives the results that are used in the text.

The conclusions may be summarized as follows. The first responsibility of the sponsor is to determine his own attitudes toward systematic risk and, based upon long-term forecasts of the normal expected excess return and variance of return on the market portfolio, to determine the normal beta that is optimal for the aggregate portfolio. Next, normal betas must be assigned to the constituent portfolios, so that the aggregate result is the desired aggregate beta.

The sponsor next formulates his attitude toward residual risk, as opposed to systematic risk: in theory, the sponsor should be equally averse to these or less averse to residual risk, but current practice appears to involve much higher aversion to residual risk.

Next the sponsor must assess the superiority of managers, as measured by the "covenant information ratio." This is equivalent to assessing the alpha or appraisal premium that can be earned at some fixed level of residual risk, for the information ratio is equal to the alpha divided by the residual standard deviation. The sponsor must also establish redundancy adjustments, called dependence-adjustment factors, for the managers. These account for intermanager correlations in information. Since correlations are different for market forecasts than for forecasts of specific returns on common stocks and different, also, for forecasts of common factors, it is necessary to analyze each of these three components of return separately. Thus, covenant information ratios and dependence-adjustment factors are prepared for each manager for each of these three classes of information.
Although the implied demands on the sponsor may appear formidable, in fact, the information that is called for is really not much more difficult to prepare than that which has traditionally been the province of the sponsor. All that is involved is a more quantitatively explicit decision as to the superiority of managers. It is important to recall that sponsors have always evaluated the abilities of their managers in the process of determining who should be entrusted with investment responsibility.

The paper also considers the question of appropriate investment portions to assign to the managers. Portions may be assigned either as a method of reward to the more valuable managers, or as a means to minimize the inefficiencies caused by investment restrictions on constituent portfolios. These two goals are similar but not identical and lead to different allocations of funds. An important issue in this regard is the effectiveness with which the sponsor copes with the problems of offsetting purchase and sale recommendations and with the restrictions against short sales. Two simple accounting devices, called the "market return chit" and "asset return chit" are suggested, which have the potential of removing the losses from offsetting transactions, and which are an ideal solution to the problem of restrictions on portfolio holdings.

The procedures suggested in the paper do call for some institutional changes. The manager must be responsive to the small number of guiding parameters that the sponsor communicates to him, and he must be prepared to offer portfolio management strategies at differing levels of aggressiveness. The sponsor must explicitly acknowledge responsibility for assessing the appraisal ability of managers and dependence adjustments among managers, and must acknowledge that it is his responsibility to determine appropriate aggressiveness. On the other subject of efficient portfolio interactions, the sponsor, or an agent such as the master trustee, must implement an environment which facilitates intermanager transactions and opens up opportunities for negative nominal positions in constituent portfolios. Finally, all parties must recognize the intrinsic risk of investment and the necessity for an optimal risk/reward trade-off. The relationship between sponsor and money managers is shown schematically in figure 1.
FIGURE 1

OPTIMAL DECENTRALIZED INVESTMENT THROUGH MULTIPLE PORTFOLIO MANAGERS

The Relationship Between Sponsor (S) and Manager (M)

In response to $E_M$, $\sigma_M^2$

S sets normal beta $\beta_E$ for aggregate equity portfolio.

This determines $\lambda_AS$.

S sets $\lambda_{AU} = \lambda_AS/\kappa$

For each of three components of returns (market, common factors, specific), S sets dependence-adjustment factor $b_i$ for M's forecasts, and S and M agree on covenant information ratio $z_i$

$M$ sets market alpha

$M$ sets factor alphas

$M$ sets specific alphas

$S$ sets $M$'s portion, $W_i$

$W_i$

$S$ sets $M$'s normal beta, $\beta_i$

$\beta_i$

Portfolio beta

Active holdings

$E_M$ normal long-term expected excess equity market return

$\sigma_M^2$ normal long-term variance of equity market return

$\beta_E$ normal beta for aggregate equity portfolio

$\lambda_AS$ systematic risk acceptance for aggregate portfolio

$\kappa$ relative aversion to unsystematic and systematic risk

$\lambda_{AU}$ unsystematic risk acceptance for aggregate portfolio

$\beta_i$ normal beta assigned to manager $i$

$W_i$ portion of equity portfolio allocated to manager $i$

$z_i$ covenant information ratio for manager $i$

$b_i$ dependence-adjustment factor for manager $i$
2. Management of Beta

The equity portfolio beta determines exposure to systematic or market return. From expected market return arises expected reward, and from the variance of market return arises systematic risk. Portfolio beta should be set so as to optimize the trade-off between systematic reward and systematic risk. The long-term or normal forecasts of expected market return and market variance provide a usual environment, and short-term forecasts of extraordinary market return (and possibly of abnormal market variance) produce changes from the normal environment. The sponsor determines the optimal risk/reward trade-off in view of the usual environment, and from this sets the "normal beta" for the equity portfolio. The deviations of short-term forecasts are exploited by active changes in beta relative to this normal value. These active deviations are expected to average to zero over the long run.

Suppose, first, that the market forecasts of the various managers are brought to the sponsor, and that the sponsor sets the beta for the aggregate portfolio autonomously. Thus, the managers are serving as advisors, and portfolio decision making is in the hands of the sponsor. The sponsor's problem can be divided into two parts: first, how to combine the forecasts of the separate managers into a single best forecast; second, what portfolio beta to establish so as to optimize the risk/reward trade-off in view of the forecast.

2.1. The Optimal Combined Forecast

To prepare an optimal combined forecast, the sponsor must ascertain three kinds of data: (1) the managers' forecasts; (2) the quality or information content of the managers' forecasts; (3) the degree of interdependence among the different managers' forecasts. Each of these kinds of information will be considered in turn.

Each manager presumably begins with a forecast of extraordinary market return. This forecast will average to zero over time, sometimes being optimistic, sometimes pessimistic. Extraordinary return is defined as the difference between the manager's best forecast of market return and the
usual or long-term forecast accepted by the sponsor. Each manager must convert this forecast into a "market alpha." The conversion process involves scaling the basic forecast to correct for erroneous information content. The result, as explained in [1] (sections A7 through A9), is endowed with the important property that the information content (or covariance with the future market return) is the same as the variance of forecast. To carry out this scaling, the manager must determine how much valid information is contained in the forecast. Of course, this is a difficult process which must be largely judgmental, but it is inescapable. Without ascertaining how much information is present, the manager cannot determine how aggressively to react in response to the forecast. Ideally, this measure of information content should be accepted by both manager and sponsor. In this case, it may be termed a "covenant information ratio." Agreement concerning the covenant information ratio assures the sponsor that the forecast has been scaled consistently with his assessment of the manager's abilities. The covenant information ratio also states the information content of the forecast. In the formal analysis of this problem, the covenant information ratio can be replaced by the "covenant correlation coefficient," which is almost identical. This correlation coefficient is the correlation between the market forecast and the market return.\(^1\) Thus, the forecast by the manager is expressed as a market alpha, \(\alpha_i\), for manager \(i\). The information content of the forecast is expressed by the correlation coefficient \(\rho_i\).

If there were only one manager, the \(\alpha\) and \(\rho\) for that manager would be sufficient. When there are two or more managers, it is also necessary to respond to redundancy between their forecasts. This redundancy would arise when different managers were using the same sources or the same mode of analysis and tended to reach similar conclusions concerning market

\(^1\)The covenant information ratio is proportional to \(\rho^2/(1-\rho^2)\). (See (2.2) in the appendix.) Since \(\rho^2\) is small, the information ratio is close to the correlation coefficient.
overpricing or underpricing. The degree of redundancy is measured by the correlation coefficient between forecasts. For any pair of managers, say manager \( i \) and manager \( j \), the correlation coefficient is \( \pi_{ij} \). This can be estimated either by inspection of the managers' judgmental processes or by monitoring the correlation between their forecasts over time. Hopefully, the correlation will usually be low. If it is positive and near to 1, then two managers are almost duplicating one another in terms of the information which they are providing. If it is negative, then it must be the case that one manager is systematically misevaluating information which is correctly processed by the other.

Based on the quality of the forecasts \( (\rho_i, i=1,\ldots,J) \) and the intermanager correlations \( (\pi_{ij}, i=1,\ldots,J, j=1,\ldots,J) \), the sponsor applies a simple formula to compute "dependence-adjustment factors" for the different managers. These adjust for the redundancy problem. If all managers are truly independent, without redundancy, then all dependence-adjustment factors are unity. The greater the redundancy between one manager's information and another's, the smaller the dependence-adjustment factor will be. These factors, \( b_i, i=1,\ldots,J \), are crucial in combining the managers' forecasts. The formula is given in the appendix (Theorem 1, Equation 2). The best combined forecast is the weighted sum of the individual forecasts, each weighted by the dependence-adjustment factor [Appendix: Theorem 1, Equation 1]. In the special case, where all managers are independent, the best forecast is the sum of their forecasts [Appendix: Theorem 1, Equation 3].

Notice that the best forecast of extraordinary return is akin to a sum, not to an average. This is because the individual forecasts have already been converted to alphas (by scaling adjustments) and are analogous to units of valid information. Thus, the best forecast is accomplished by cumulating these units. The dependence-adjustment factors correct for redundancies between these units of information, so that, in general, the best forecast is the dependence-adjusted weighted sum.

The combined forecast is better than any of the constituent forecasts. Its coefficient of determination \( (\rho_c^2) \) is the dependence-adjusted
weighted sum of the coefficients of determination of the individual forecasts [Appendix: Theorem 1, Equation 4]. Thus, the sponsor, by combining information from many above-average managers correctly, is able to invest with better information than any single manager can do on his own.

2.2. Optimal Investment in the Aggregate Portfolio

The sponsor can express the optimal beta for the portfolio at any time as the sum of the normal beta and a differential beta. The former optimizes the risk/reward trade-off, given the normal forecast [Appendix: Theorem 2, Equation 1]. An interesting fact is that when managers are above average in their ability to forecast market returns, then the remaining variance of the market, after deducting that part explained by their forecasts, is smaller than the variance faced in the absence of market forecasting. The proportional reduction is the coefficient of determination ($\rho^2_C$) for the combined forecast. As a consequence of this reduction, the sponsor faces less market risk, and will therefore, on average, maintain a higher systematic risk coefficient or beta [Appendix: Theorem 2, Equation B1]. The ability to do this increases the utility of the sponsor [Appendix: Theorem 2; Equations 3 and 4 applied to a comparison of Equations A2 and B2].

When the combined forecast predicts abnormal market return, the sponsor benefits by setting a differential beta that optimizes the risk/reward trade-off promised by the forecast. The adjustment to beta is proportional to the abnormal forecast. Thus, a positive abnormal return forecast leads to an increased beta, a negative forecast to a decreased beta [Appendix: Theorem 2, Equation C1]. The manipulation of differential beta to exploit market appraisals adds to sponsor utility [Appendix: Theorem 2, Equations 3 and 4 applied to Equation C2].

The formula for optimal beta relies on two assumptions which are not exactly correct. The first is that there are no transaction costs. The second is that the changes in beta are obtained by mixing the risk-free asset with the market portfolio and therefore do not entail imperfect diversification. Neither of these assumptions is completely satisfied in the real world.
Modification of beta optimization to accommodate transaction costs is not too difficult. In general, a transaction should only be undertaken when the anticipated profit (the net expected benefit from exploiting the forecast, less round-trip transaction costs) is favorable. Moreover, the portfolio beta is only moved to the degree that the marginal benefit (the increased utility from higher expected reward, adjusted for the changed disutility of residual risk) is greater than the incremental transaction costs of the move. When transaction costs are recognized, transactions occur less frequently, since the benefit is sometimes too small to overcome the transaction-costs hurdle, and adjustments are less extreme, because the adjustment stops where increased benefit does not balance incremental transaction costs.

The assumption that beta adjustments are achieved through a mixture of perfectly diversified equities and risk-free assets is not unrealistic for most sponsors. The sponsor generally has a pool of riskless assets or very low-beta bonds, which can be drawn on at will. Thus, an increased beta can be accomplished by transfers into equities, and a decreased beta by transfers out of equities. Equity portfolio betas between 0 and 1 can be achieved by a mixture of cash and equity within the equity portfolios. Imperfect diversification is only necessitated when an increase in portfolio beta must be obtained via an equity portfolio beta greater than 1. Since attainment of a high beta by borrowing and leverage is usually unfeasible, it is necessary to shade the portfolio toward high-beta assets. As demonstrated in Rosenberg and Rudd [2], the added unsystematic risk due to imperfect diversification begins to increase sharply beyond a portfolio beta of 1.3, so that in considering betas higher than this, the disutility of necessary unsystematic risk must be considered.

2.3 Decentralized Management

The centralized scheme considered thus far is rarely encountered. The usual case is for each portfolio manager to manipulate his portfolio beta independently of the others. Is there a decentralization procedure
which will allow the managers to act autonomously, each manipulating his
own beta without attention to events elsewhere in the portfolio, so that
the end result is an aggregate optimal portfolio strategy? In other words,
is there a decision procedure which allows individual managers to make
their own decisions and which yet results in an optimal combined outcome?

Two levels of complexity are important in answering this question.
First, consider the idealized case excluding transaction costs and assum-
ing that beta is manipulated through (possibly levered) mixtures of the
market portfolio and the risk-free asset. Here the answer is affirmative:
optimal decentralization can be accomplished quite easily. At the second
level of complexity, one must also minimize transaction costs between
managers and eliminate unnecessary unsystematic risk due to equity
portfolio betas greater than 1 in some managers' portfolios. To
cope with these considerations in a multiply managed portfolio, certain
intermanager transfer arrangements are necessary. These arrangements are
analogous to those offered by a market inventory fund, but with some en-
hancements.

The idealized case, ignoring transaction costs and imperfect divers-
sification, will be discussed first. The sponsor's first difficulty is to
establish normal portfolio betas for the separate managers. The
weighted average of these betas, when weighted by investment portions, must
add up to the desired normal beta for the aggregate equity portfolio [Ap-
pendix: Equation 2.17]. One simple way of accomplishing this is to give
the same normal beta, equal to the aggregate value, to all managers. How-
ever, some managers may prefer to operate with a different normal beta,
due to specialization in a universe of stocks, such as high-yielding stocks
or growth stocks or smaller companies, having a nonunit beta. The sponsor
can allow for this by setting the normal beta for each manager to be the
typical value for the manager's universe. As long as the universes aggre-
gate to the market portfolio, the normal betas will also correctly aggre-
gate to unity. The sponsor can further adjust some or all to effect his
desired normal beta.
The manager and sponsor have agreed concerning the covenant information ratio (or correlation coefficient, $\rho_j$). With this guide, the manager converts his market forecast to a market "alpha." Next, the manager multiplies this by the dependence-adjustment factor, $b_j$, which the sponsor has communicated to him. The dependence-adjusted alpha, $b_j\alpha_j$, is the expected extraordinary market return which the manager responds to in setting his portfolio beta.

The sponsor also communicates to the manager the manager's investment portion in the equity pool, $W_j$; the normal forecast of market excess reward, $E$; and the normal beta for the aggregate equity portfolio, $\beta_E$. The manager adjusts for investment portion and normal beta to compute his differential beta:

$$\beta_j = \beta_E \left( \frac{1}{W_j} \right) \left( \frac{b_j\alpha_j}{E} \right).$$

When this is added to his normal beta, $\beta_j$, the appropriate portfolio beta is found [Appendix: Theorem 3, Equation 2]:

$$\beta_j = \beta_E \frac{1}{W_j} \left( \frac{b_j\alpha_j}{E} \right).$$

To summarize, the sponsor communicates six parameters to the manager:
1. the covenant correlation coefficient, $\rho_j$, which expresses the degree of belief in the manager's appraisal ability;
2. the dependence-adjustment factor, $b_j$, which expresses the degree of redundancy between this and other managers;
3. the manager's investment portion in the total portfolio, $W_j$;
4. the sponsor's normal expectation for excess market return, $E$;
5. the normal beta for the aggregate equity portfolio, $\beta_E$, arrived at in response to the normal market forecast; and
6. the normal beta $\beta_j$ assigned to the manager. With the exception of the investment portion, none of these parameters is likely to change often. Indeed, the parameters might ordinarily be fixed for several years, subject to annual review. The manager, in the day-to-day environment in which market forecasts are made
and evaluated, implements market-timing decisions in response to these parameters.

2.4 The Role of the Market Return Chit (MRC) in Decentralized Management

The difficulties with transaction costs and imperfectly diversified high-beta portfolios can now be considered. The transaction-cost problem arises when one manager reduces his market forecast at the same time that another manager increases his. In the absence of some arrangement, the result would be for the two managers to incur transaction costs in offsetting trades. However, the sponsor, or his agent, being aware of these offsetting transactions, can effect both at zero transaction costs. This can either be done by a physical transfer of assets from one portfolio to the other or, much more elegantly, by the transfer of a "market return chit" (MRC).

An MRC is an imaginary device which should perhaps be given reality. Essentially, it takes the form of a bet between one manager and another as to the market return. It has no effect on the holdings within each manager’s equity portfolio. It also has no effect whatsoever on the aggregate portfolio. What it does do is to transfer, within the portfolio-monitoring process of the sponsor, certain returns from one manager to another. For example, suppose "A" manages $400 million and "B" manages $300 million. If manager A wishes to reduce his portfolio beta by .3, and manager B wishes to increase his by .4, these offsetting changes are accomplished by an MRC. Manager A issues an MRC to manager B in the amount of $120 million, .3 times the dollar value of his portfolio. This MRC causes the excess return on a $120 million investment in the market portfolio to be deducted from manager A's return and incremented to manager B's return, within the portfolio-monitoring system maintained by the sponsor. Thus, as far as manager A is concerned, he has successfully reduced his portfolio beta by .3. Similarly, manager B has successfully increased his by .4. This has been accomplished with no transactions whatsoever and without imperfect diversification.
By contrast, an additional beta adjustment of .1 in manager A's portfolio, not offset by B, should have a net effect on the sponsor's portfolio. This can only be accomplished by a transaction with the outside world. The MRC wipes out the necessity for offsetting intermanager transactions, but should not impact the net effect of managers' actions. Interestingly, the MRC also allows each manager the potential of carrying his portfolio beta well beyond the range which is obtainable by ordinary means. Recall that when the portfolio beta rises much beyond 1.3, substantial unsystematic risk due to imperfect diversification must be undertaken. The MRC allows the manager to achieve this beta without imperfect diversification, as long as other managers are pessimistic with regard to the market and will issue him an MRC.

The MRC thus eliminates unnecessary transaction costs and also expands the ability of individual managers to respond to abnormal market forecasts. For example, suppose that the manager of 20% of the portfolio, with unusual ability in market forecasting, formulates a positive forecast that warrants an increase in his beta from 1.0 to 2.0. This increase is virtually impossible within the confines of the manager's own portfolio, because of the great residual risk of all portfolios having that beta. However, the manager can offer to purchase .2 MRCs from the other managers. Other managers (or the sponsor's agent) can issue these to the manager. If other managers have offsetting forecasts and are prepared to short the market, the MRC can be delivered with no modification in their portfolios. Otherwise, the managers can increase the betas of their portfolios to 1.2, passing the .2 increment on to the market forecaster and "charging" him appropriate compensation for the transaction costs and reduced diversification incurred thereby. The end result is that the aggregate sponsor portfolio achieves a beta of 1.2 through each portion adjusting from a beta of 1.0 to 1.2. This should be contrasted with the existing situation, where 80% of the portfolio remains at 1.0, while the other 20% must go to 2.0. It is easily proved that when the information of the manager warrants an increased aggregate beta to 1.2, the solution achieved by the trading of MRCs is preferable, because it is achieved at lower residual risk. The same conclusion applies to all adjustments in beta.
An increase from 1.0 to 2.0 may seem unreasonably large, but in the context of a multiply managed portfolio, quite modest forecasts of abnormal market return can justify substantial beta shifts. Consider the following example: Normal equity portfolio beta equals 1; five managers, each managing 20%; no dependence adjustment necessary; normal expected excess market return equals 6%. Assume, finally, that transaction costs are negligible. Then, a manager with a market alpha of +1.2% should increase his portfolio beta to two times his normal portfolio beta. The reader may find this inconceivable, but a check of Appendix Theorem 3, Equation 2, will show that it is correct. The surprising size results from two factors: first, the diversification effect of multiple managers, which reduces the risk exposure from response to any one manager's forecasts; second, the fact that what is being discussed here is not a market forecast, but a scaled market forecast—market alpha—which is a unit of valid information. Therefore, if the manager does his scaling process correctly, a market alpha of 3% implies that the expected excess return on the market in the holding period will be 9% (6% market excess return plus 3% market alpha).

2.5 Appropriate Managerial Rewards

Increased management fees, in addition to the basic fee for passive management, should be paid to active managers in proportion to their contribution to the well-being of the sponsor. The increased well-being from active management can be measured by the increase in expected utility due to the improved investment opportunities provided by the managers. To determine appropriate management fees, it is necessary to identify the utility contribution of each manager.

Consider three different approaches to defining this contribution: first, the contribution which could be produced by the manager acting alone; second, the dependence-adjusted contribution to total utility; third, the increment to client utility which occurs when this manager is joined to the others in the group. Independence among managers leads to the same conclusion in all three cases.
The definition of contribution becomes ambiguous once intermanager dependency is encountered. Each of the three approaches leads to a different conclusion [Appendix: Theorem 4, Equations 3 to 5]. It is therefore necessary to consider which is the most appropriate rule to use in rewarding managers. The issue can be clarified by considering an example in which two managers provide some common information and, in addition, each manager provides an increment of unique information. Under the first approach, each manager would be rewarded for the common information, since each would enhance sponsor utility by using that information if he were acting on his own. Under the second approach, each manager is rewarded for one-half of the common information. Under the third approach, neither manager is rewarded for the common information, since when either manager is viewed as an increment, with respect to the other, the common information is not added.

In the simple case just considered, the second approach, with dependence-adjusted reward, is clearly the most appealing. An argument for a mixed approach could be made if the managers were hired sequentially. Then the first manager might be rewarded for the entire utility contribution when acting alone (the first approach), and the second manager could be rewarded for the incremental contribution (the third approach). This mixed procedure does not treat managers symmetrically, so it can perhaps be rejected on grounds of equity. The dependence-adjusted reward approach is therefore suggested.

Having established the managers' contributions to sponsor utility, some compensation procedure must be adopted. Compensation can be varied between managers by means of differing unit management fees, by means of differing amounts of funds entrusted to the managers, or by a combination of these two. This issue will be discussed further in section 3.8 below.
3. Management of Unsystematic Return

To provide a realistic context for the discussion of unsystematic risk and return, we will refer to an illustrative pension fund. This is a sizable fund of about $500 million, divided into ten components managed by seven different managers. Table 1 shows the investment portions entrusted to the managers.

<table>
<thead>
<tr>
<th>A</th>
<th>A2</th>
<th>A3</th>
<th>B</th>
<th>B2</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>I</th>
</tr>
</thead>
<tbody>
<tr>
<td>.128</td>
<td>.017</td>
<td>.012</td>
<td>.292</td>
<td>.029</td>
<td>.153</td>
<td>.061</td>
<td>.042</td>
<td>.045</td>
<td>.231</td>
</tr>
</tbody>
</table>

Manager A has almost 15 percent of the equity pool divided among a major account, "A," and two special equity accounts, "A2" and "A3." Manager B has about 32 percent of the total, with 3 percent in special equity account "B2." Active manager C has 15 percent of the account, and active managers D, E, and F have smaller proportions. Finally, account I, with 23 percent of the total, is a well-diversified index fund, closely approximating the S&P 500 index.

3.1. Risk and Reward in the Aggregate Portfolio

This portfolio is representative of the exceedingly well-diversified, multiply managed portfolios discussed in [1]. As shown in table 2, almost 99 percent of the portfolio variance of return arises from systematic risk. (All aspects of variance are predicted by a model of investment risk [3, 4].) This corresponds to a residual standard deviation of 2.85 percent per annum. As explained in [1], the reward/risk ratio for active investments should not be greater than the normal reward/risk ratio for systematic return. Applying this principle in the present case with reasonable assumptions concerning expected excess return and variance of return on the market, the required appraisal premia, or required alphas, to justify the
small amounts of exposure to residual risk are at most as great as the figures given in panel 2 of the table.

TABLE 2
INVESTMENT RISK IN THE AGGREGATE ILLUSTRATIVE PORTFOLIO

<table>
<thead>
<tr>
<th></th>
<th>Annual Variance ($^{2}$)</th>
<th>Annual Expected (Excess) Return (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Systematic return</td>
<td>473.6</td>
<td>6.300 (normal forecast)</td>
</tr>
<tr>
<td>Residual return</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Specific</td>
<td>1.6</td>
<td>.021 (required)</td>
</tr>
<tr>
<td>XMC</td>
<td>5.7</td>
<td>.089 (required)</td>
</tr>
<tr>
<td>Total</td>
<td>8.3</td>
<td>.110 (required)</td>
</tr>
<tr>
<td>Grand total</td>
<td>481.9</td>
<td>6.410 (inferred)</td>
</tr>
</tbody>
</table>

Source: Fundamental Risk Measurement Service. Forecasts of systematic variance and return are derived from predicted beta = 1.05; annual mean and variance of S&P 500 excess return equal 6.0 and 429.5, respectively.

Note that all figures in this and subsequent tables may be read as annualized returns for simplicity, but actually refer to properties of the logarithm of return (multiplied by 100 to approximate a percentage return).

In table 2, residual risk is broken down into two components: risk arising from specific returns and risk arising from extra-market covariance (XMC). This decomposition is a natural one in modeling investment risk, and in analyzing portfolio investments. Specific returns are those which are specific to individual companies and independent across companies. Extra-market covariance arises from the tendencies of stocks to exhibit similar returns, net of their common dependence on the market, due to common factors in investment returns which influence them. In the present portfolio, only 1.6 units of variance occur because of exposure to the specific returns of individual companies, and only 5.7 units of residual variance occur because of exposure to common factors impacting security returns.

The required appraisal premia for these two components of investment risk can be approximated by applying the reward/variance ratio for systematic
return to these variances. The results are 2.1 basis points per annum for specific risk, and 8.9 basis points per annum for extra-market covariance. The required appraisal premium for total residual risk is 11 basis points per annum. In [1], it was suggested that the active management fee should not exceed one-half of the required appraisal premium for unsystematic risk (plus the required abnormal expected return from market timing, if any). In this case, the argument suggests that the management fee devoted to security analysis and sectoral investments should not exceed one-half of 11 basis points per annum, or 5.5 basis points per annum. Since more than 75 percent of this portfolio is actively managed, it is probable that the active management fee does exceed this amount. This suggests an internal inconsistency in the present portfolio, probably arising from excess conservatism with regard to residual risk, as discussed in [1].

3.2. Residual Variances and Required Alphas in the Constituent Portfolios

Table 3 reports residual variances in the ten constituent portfolios.

<table>
<thead>
<tr>
<th>Model</th>
<th>A</th>
<th>A2</th>
<th>A3</th>
<th>B</th>
<th>B2</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>I</th>
</tr>
</thead>
<tbody>
<tr>
<td>Specific variance</td>
<td>6.4</td>
<td>12.0</td>
<td>14.8</td>
<td>7.8</td>
<td>42.8</td>
<td>9.3</td>
<td>17.5</td>
<td>14.2</td>
<td>6.3</td>
<td>0.1</td>
</tr>
<tr>
<td>Extra-market covariance</td>
<td>2.5</td>
<td>65.8</td>
<td>142.9</td>
<td>1.4</td>
<td>183.6</td>
<td>39.3</td>
<td>61.3</td>
<td>8.0</td>
<td>8.6</td>
<td>0.0</td>
</tr>
<tr>
<td>Total</td>
<td>8.9</td>
<td>77.8</td>
<td>157.6</td>
<td>9.2</td>
<td>226.4</td>
<td>48.6</td>
<td>78.8</td>
<td>22.1</td>
<td>14.8</td>
<td>0.1</td>
</tr>
<tr>
<td>Simulation total</td>
<td>7.1</td>
<td>51.8</td>
<td>134.4</td>
<td>8.0</td>
<td>190.7</td>
<td>35.2</td>
<td>40.0</td>
<td>15.3</td>
<td>13.5</td>
<td>0.0</td>
</tr>
</tbody>
</table>

*In this and later tables, figures may not add correctly because of rounding.

Specific variance and extra-market covariance for each portfolio, as predicted by the model, are totaled to obtain the model's residual risk
prediction. The last row of the table provides a retrospective simulation, in which the current portfolio weights are used to compute the portfolio returns that would have been experienced in each of the last sixty months, and the residual variance in that period is obtained by regression on a similarly reconstructed market portfolio.

First, the coincidence between the model prediction and the historical simulation is heartening. Notice that the model predictions are somewhat larger than the simulation in all cases, but that the relative ordering of the portfolios is extremely close. Next, notice the substantial differences in relative emphasis between portfolios. Portfolio B and, to a lesser degree, portfolios A and E emphasize stock selection, with larger specific variance than extra-market covariance. On the other hand, the special equity portfolios, A2, A3, and B2, and portfolios C and D strongly emphasize extra-market covariance as an element of risk. These portfolios have well-diversified holdings but show pronounced differential exposure to common factors, relative to the S&P 500. This differential exposure to common factors leads to extra-market covariance exposure.

There is an enormous range of investment risks between portfolios. Portfolios A, B, and C, which are roughly equal portions of the total, range in variance from 8.9 to 48.6. Portfolios D, E, and F, which are roughly equal portions of the total range from 14.8 to 78.8. The special equity portfolios range from 77.8 to 226.4.

The contribution of each portfolio to aggregate residual risk is found by adjusting its residual variance for the investment portion. Once this contribution is obtained, the approach already used for the aggregate portfolio provides a required appraisal premium for each constituent portfolio. When no dependence adjustment is necessary, the required appraisal premia are as indicated in the first row of table 4. These are minuscule premia, varying from 0 basis points for the index fund to a maximum of 10 basis points for portfolio C. However, as will be seen below, there are pronounced intermanager correlations which require dependence adjustments. When these adjustments are made, the corrected required appraisal premia are as given in the second row of table 4. Now the appraisal
premia for the special equity funds are more substantial, ranging as high as 45 basis points for fund B2. Note the great differences in the magnitude of dependence adjustments: for portfolio E, the adjustment is by a factor of 10, while for portfolio C, it is by a factor of only 2.4, and for portfolio B, it is by a factor of only 2. These adjustments reflect differing degrees of correlation between these portfolios and the aggregate.

**TABLE 4**

**REQUIRED APPRAISAL PREMIA EXPRESSED AS BASIS POINTS OF ANNUAL RETURN FOR CONSTITUENT PORTFOLIOS**

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>A2</th>
<th>A3</th>
<th>B</th>
<th>B2</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>I</th>
</tr>
</thead>
<tbody>
<tr>
<td>Assuming independence</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>8</td>
<td>10</td>
<td>7</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>Dependence-adjusted</td>
<td>6</td>
<td>25</td>
<td>39</td>
<td>6</td>
<td>45</td>
<td>24</td>
<td>30</td>
<td>10</td>
<td>9</td>
<td>0</td>
</tr>
</tbody>
</table>

3.3. Investment Strategy and Exposure to Extra-Market Covariance

The model of extra-market covariance includes 45 factors: 39 industry groups and 6 continuums characterizing individual companies. Two of these continuums are "immaturity and smallness," which relates to company size, and "growth orientation," which is inversely related to dividend yield. A portfolio becomes exposed to extra-market covariance through positions with respect to these factors that differ from the market portfolio (in this analysis, the S&P 500).

Differentiated positions may come from two distinct causes. First, the "style" of the manager may lead him to invest in an atypical "habitat," which is highly differentiated from the market in some characteristics. For example, high-yielding portfolios are low on the growth-orientation axis, and growth stock managers are high on the growth-orientation axis. Many managers, emphasizing large companies, used to be low on the immaturity and smallness axis, although this tendency has been weaker since the collapse of the "two-tiered market." Special equity funds are generally very high on the immaturity and smallness axis.
A second source of exposure comes from deliberate active positioning of the portfolio. Here, the manager, based upon forecasts of abnormal return for some sectors or some kinds of companies, takes an investment posture which is exposed to extra-market covariance in order to profit from the forecast. In this case, the characteristics of the portfolio will change over time.

When the exposure to extra-market covariance arises from manager's style, without significant change over time, then such exposure is the responsibility of the sponsor. If the sponsor chooses a diversified group of specialists, the risk exposures of different managers will wash one another out, and the end result will be little or no difference from the market portfolio. On the other hand, if the sponsor emphasizes one managerial style more than in proportion to the capitalization weight of that kind of company, then the aggregate portfolio will be permanently biased in that direction. This permanent exposure to residual variance would not ordinarily be adjusted by the managers, and hence can really not be viewed as their responsibility. It is the sponsor who must have predicted long-run abnormal returns arising from the emphasized sector, which are sufficient to meet the alpha requirement from that exposure (precept C).

On the other hand, transitory risk exposures due to active management are clearly the responsibility of the managers. It is their forecasts leading to this exposure which must provide the justifying appraisal premium.

The extra-market-covariance model allows correlations among residual returns due to XMC to be computed for all managers. These measure the degree to which differing managers have coincident exposure to extra-market covariance. Where the correlation is positive, exposures of different managers reinforce one another in contributing to residual risk in the aggregate portfolio. Table 5 lists the correlations among the portfolios. Correlations with the index fund are relatively uninteresting, since the exposure of the index fund to XMC is minute. Excepting that case, notice that every single correlation is positive. Thus, no portfolio within the system is in an off-setting position relative to other portfolios. Correlations
among the three special equity funds range from .82 to .9. The correlation between managers C and D is .85, and both managers are also highly correlated with the special equity funds. Portfolio B is the least correlated with the others.

TABLE 5

INTERPORTFOLIO CORRELATION OF RESIDUAL RETURNS
ARISING FROM COMMON FACTORS (XMC)

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>A2</th>
<th>A3</th>
<th>B</th>
<th>B2</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>I</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1.0</td>
<td>.26</td>
<td>.28</td>
<td>.25</td>
<td>.26</td>
<td>.30</td>
<td>.28</td>
<td>.27</td>
<td>.42</td>
<td>.05</td>
</tr>
<tr>
<td>A2</td>
<td>.26</td>
<td>1.0</td>
<td>.90</td>
<td>.21</td>
<td>.82</td>
<td>.64</td>
<td>.64</td>
<td>.48</td>
<td>.43</td>
<td>.43</td>
</tr>
<tr>
<td>A3</td>
<td>.28</td>
<td>.90</td>
<td>1.0</td>
<td>.20</td>
<td>.89</td>
<td>.76</td>
<td>.76</td>
<td>.52</td>
<td>.55</td>
<td>.07</td>
</tr>
<tr>
<td>B</td>
<td>.25</td>
<td>.21</td>
<td>.20</td>
<td>1.0</td>
<td>.18</td>
<td>.20</td>
<td>.19</td>
<td>.13</td>
<td>.21</td>
<td>.07</td>
</tr>
<tr>
<td>B2</td>
<td>.26</td>
<td>.82</td>
<td>.89</td>
<td>.18</td>
<td>1.0</td>
<td>.76</td>
<td>.76</td>
<td>.49</td>
<td>.51</td>
<td>.07</td>
</tr>
<tr>
<td>C</td>
<td>.30</td>
<td>.64</td>
<td>.76</td>
<td>.20</td>
<td>.76</td>
<td>1.0</td>
<td>.85</td>
<td>.55</td>
<td>.63</td>
<td>.02</td>
</tr>
<tr>
<td>D</td>
<td>.28</td>
<td>.64</td>
<td>.76</td>
<td>.19</td>
<td>.76</td>
<td>.85</td>
<td>1.0</td>
<td>.54</td>
<td>.60</td>
<td>-.02</td>
</tr>
<tr>
<td>E</td>
<td>.27</td>
<td>.48</td>
<td>.52</td>
<td>.13</td>
<td>.49</td>
<td>.55</td>
<td>.54</td>
<td>1.0</td>
<td>.38</td>
<td>.05</td>
</tr>
<tr>
<td>F</td>
<td>.42</td>
<td>.43</td>
<td>.55</td>
<td>.21</td>
<td>.51</td>
<td>.63</td>
<td>.60</td>
<td>.38</td>
<td>1.0</td>
<td>-.09</td>
</tr>
<tr>
<td>I</td>
<td>.05</td>
<td>.13</td>
<td>.07</td>
<td>.07</td>
<td>.07</td>
<td>.02</td>
<td>-.02</td>
<td>.05</td>
<td>-.09</td>
<td>1.0</td>
</tr>
</tbody>
</table>

Aggregates: .48 .74 .83 .52 .82 .89 .86 .60 .68 .08

These correlations are based on the positions of each portfolio with regard to the 45 common factors. Inspection of those positions shows that the predominant characteristics of these portfolios are biases toward high growth orientation and smaller companies. There are smaller positions with regard to some industries. The bias toward higher growth orientation is present in almost all portfolios and is the principal source of substantial intermanager correlations. It is not clear whether this position arises from a permanent style bias in the aggregate or from coincidental optimistic forecasts for high-growth companies. If the latter is the case, it is remarkable and interesting that all managers' forecasts coincide. In
the former situation, most of the appraisal premia previously identified for the active policies must be provided by the sponsor's stylistic emphasis rather than by the managers' forecasts.

In table 6, the required appraisal premia for extra-market covariance exposure are given. Row 1 contains the minute values which would apply if there were no correlations among managers. Row 2 gives the much larger values which reflect the high intermanager correlations. The very large increase due to the dependence adjustment reflects the pronounced communality in strategy across the portfolios.

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>A2</th>
<th>A3</th>
<th>B</th>
<th>B2</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>Assuming independence</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>7</td>
<td>8</td>
<td>5</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Dependence-adjusted</td>
<td>4</td>
<td>23</td>
<td>37</td>
<td>3</td>
<td>42</td>
<td>21</td>
<td>26</td>
<td>8</td>
<td>8</td>
</tr>
</tbody>
</table>

It is useful to recall that the aggressiveness of an active investment can be measured by the risk that is assumed in consequence of it. In the present portfolio, by far the most aggressive position is a rather simple orientation toward high-growth companies, which could easily be constructed by an unskilled investment organization using readily obtainable summary data on corporations (such as the COMPUSTAT tapes). The industry investments, which are often thought of as a major ingredient in investment strategy, are timid by comparison. There is little reflection in the aggregate portfolio of industry forecasting or of active positioning with regard to the continua characterizing stocks. Two factors interact to produce these relatively passive positions. First, industry investments in the constituent portfolios closely approximate market proportions, as the managers show little aggressiveness in this area. Second, the managers do appear to be operating on relatively independent information
in determining their industry investments, so that large and small positions tend to average out, resulting in an aggregate portfolio which is still nearer to the market. These two factors also operate in minimizing the portfolio's exposure to specific risk, as discussed in the following section.

3.4. Exposure to Specific Returns

Specific risk occurs when the portfolio holding in an asset differs from the market proportion (adjusted for portfolio beta where necessary, as explained in [Appendix: Lemma 2, Equation 4]). When the active holding is positive, the portfolio will benefit from a positive specific return and suffer from a negative specific return. The situation is reversed when the active holding is negative. Since short-selling is not allowed in this portfolio, a substantial negative active holding is only possible for a stock that is a significant portion of the market.

Table 7 gives the correlations between the portfolio residual returns arising from the specific returns of individual common stocks. The correlations between managers are generally small, with the exception of the correlation of .4 between managers C and D. There is slightly higher correlation among the three special equity portfolios, .39, .42, and .52, respectively; this correlation arises not so much from the stocks which they hold as from the negative active holdings with respect to large companies in the S&P 500. The general picture contrasts sharply with the case of extra-market covariance. Here, the managers do seem to be operating with relatively independent information, and there is no strong inter-manager similarity.

Correlations between constituent portfolios and the aggregate, given in the bottom row, are somewhat higher. This is not surprising, since each portfolio is perfectly correlated with its own contribution to the aggregate, and therefore more correlated to the aggregate than to the other portfolios.

Inspection of the holdings in individual assets bears out the prediction of the model. In Table 8, this pattern is summarized by reporting individual portfolio holdings in ten common stocks. First, the five largest capitalization companies in the S&P 500 are listed. Since these are
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<td>.30</td>
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<tr>
<td>B2</td>
<td>.01</td>
<td>.39</td>
<td>.42</td>
<td>-.03</td>
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<td>.30</td>
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<td>.40</td>
<td>.17</td>
<td>.17</td>
<td>.06</td>
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<tr>
<td>D</td>
<td>.08</td>
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<td>F</td>
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<tr>
<td>I</td>
<td>.11</td>
<td>.09</td>
<td>.18</td>
<td>.07</td>
<td>.05</td>
<td>.06</td>
<td>.06</td>
<td>.07</td>
<td>.04</td>
<td>1.0</td>
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| Aggregate | .40| .31| .31| .68| .31| .56| .49| .29| .33| .19|

Large proportions of the market portfolio, it would be possible to have substantial active holdings which were either positive or negative. However, the individual portfolio holdings are such that only in the case of AT&T does a substantial active holding result. Here, the market proportion is 6.2% and the aggregate portfolio investment is 3.0%, resulting in an active holding of approximately -3.2%. This active holding is by far the largest in the universe of 950 stocks which are included in the aggregate portfolio! For a contrast, consider the case of IBM. Managers A and B are optimistic and hold 9.6% and 8.2% of their portfolios, respectively, in IBM. The special equity funds, of course, hold no IBM. Managers D and F are also optimistic but hold only slightly more than the market proportion. The index fund holds exactly the market proportion. Managers C and E are slightly pessimistic and hold slightly less than the market proportion. The end result is an aggregate holding of 6.8%, corresponding to an active holding of .4%. The cases are similar for EXXON, GM, and GE.
For these three stocks, also, some managers are optimistic, some pessimistic, and the end result is an aggregate holding that is little different from the market proportion.

TABLE 8
REPRESENTATIVE HOLDINGS (EXPRESSED AS PERCENTAGES)

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<tbody>
<tr>
<td><strong>A. Five Largest Capitalization Companies</strong></td>
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<tr>
<td>IBM</td>
<td>6.4</td>
<td>6.8</td>
<td>9.6</td>
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<td>0.0</td>
<td>8.2</td>
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<td>5.6</td>
<td>6.5</td>
<td>5.4</td>
<td>7.4</td>
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<td>AT&amp;T</td>
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<td>0.0</td>
<td>0.0</td>
<td>4.6</td>
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<td>0.0</td>
<td>0.0</td>
<td>2.8</td>
<td>6.2</td>
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<tr>
<td>EXXON</td>
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<td>2.9</td>
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<td>0.0</td>
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<td>0.0</td>
<td>2.2</td>
<td>3.3</td>
<td>3.6</td>
<td>3.7</td>
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<tr>
<td>GM</td>
<td>3.2</td>
<td>2.5</td>
<td>4.6</td>
<td>0.0</td>
<td>0.0</td>
<td>3.8</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>2.7</td>
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<tr>
<td>GE</td>
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<td>2.1</td>
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<td>0.0</td>
<td>4.2</td>
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<td>0.0</td>
<td>3.2</td>
<td>0.0</td>
<td>2.0</td>
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<tr>
<td><strong>B. Five Greatest Active Holdings</strong></td>
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<tr>
<td>Atlantic</td>
<td>1.0</td>
<td>1.8</td>
<td>1.7</td>
<td>0.0</td>
<td>0.0</td>
<td>2.9</td>
<td>0.0</td>
<td>1.4</td>
<td>3.3</td>
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<tr>
<td>Clark</td>
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<td>1.1</td>
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<td>0.0</td>
<td>0.0</td>
<td>2.7</td>
<td>0.0</td>
<td>1.0</td>
<td>0.0</td>
<td>0.0</td>
<td>1.9</td>
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<tr>
<td>General Tel.</td>
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<td>1.5</td>
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<td>0.0</td>
<td>0.0</td>
<td>3.8</td>
<td>0.0</td>
<td>1.3</td>
<td>0.0</td>
<td>0.0</td>
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<tr>
<td>&amp; Electro.</td>
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<tr>
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<td>1.0</td>
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<td>3.2</td>
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<td>0.0</td>
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<tr>
<td>&amp; Rubber</td>
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<tr>
<td>Intn'l. Tel.</td>
<td>0.5</td>
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<td>3.9</td>
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<td>0.0</td>
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<td>0.0</td>
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<tr>
<td>&amp; Tele.</td>
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</table>

The second panel in the table shows the five greatest active holdings in the aggregate portfolio. These are the cases where managers' optimism was reinforcing and resulted in an aggregate portfolio concentration. However, the word "concentration" is hardly apt, since in no case is the aggregate more than 1% greater than the S&P 500 market proportion. The concentrations are produced by manager B and, secondarily, by manager C. This is not surprising, since these are the two active managers with the largest investment proportions, and both are also more active in stock selection than manager A.
Inspection of the holdings for all 950 assets bears out the pattern seen here. In no case is the active holding of an individual manager large enough to cause a useful position in the aggregate portfolio. The word "useful" is employed deliberately. The point being made is that if the managers do have the ability to identify overvalued and undervalued stocks, then the holdings which they are constructing are not adequate to exploit this ability. The active holdings of the individual portfolios, already timid, are averaged out in the aggregation process, so that the active holdings in the aggregate portfolio are minimal. This point can be underscored by noting that the existing residual standard deviation from specific risk (1.25% per annum, from table 2) would be quite acceptable in an index fund, although it is substantially larger than that maintained in the carefully diversified index funds that are widely marketed.

The required appraisal premia to justify the specific risk undertaken by the constituent portfolios appear in table 9. These are so small

**TABLE 9**

**REQUIRED APPRAISAL PREMIA (BASIS POINTS PER ANNUM)**

FROM FORECASTS OF SPECIFIC RETURNS (SECURITY ANALYSIS)

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<th>D</th>
<th>E</th>
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<tbody>
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<td>Assuming independence</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Dependence-adjusted</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
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</table>

that, when expressed in basis points, they sometimes round out to zero. Row 1 of the table again gives required alphas under the assumption that the managers act from independent information. The second row gives required alphas adjusted for the apparent correlation between managers' information. Here, the adjustment has little effect, since the managers are so nearly independent. The only important effect applies to the special equity funds and to manager D, since these four constituents do have positive intercorrelations. Allowing for the dependence adjustment, the highest required appraisal premium is 4 basis points per annum for manager
D. Again, under the rule that no management fee should be greater than one-half the appraisal premium, this implies that no active manager should charge more than 2 basis points per annum to support the security analysis effort which results in stock selection.

This conclusion at first seems surprising but, upon reflection, the historical explanation emerges. Active managers have tended to increase the diversification of their portfolios in the last decade. The methods employed to diversify have generally been too crude to provide protection against extra-market-covariance exposure but have worked quite well to reduce exposure to specific risk. Even the simplest rule of thumb, which is to put a lot of names in the portfolio, will reduce specific risk. At the same time that exposure to specific returns has declined in the constituent portfolios, sponsors have sought diversification through larger numbers of managers. Consequently, the aggregate portfolio is excessively insulated against the specific returns which security analysis identifies as opportunities for profit.

3.5. Failings of the Present Approach to Decentralized Control

The active holdings in the portfolio show several signs of poor coordination. Apparently risks are now being taken in a manner that imperfectly reflects the intentions of sponsor and manager alike. For example, the analysis of specific returns (more commonly called "security analysis") is a major function in almost all active money-management processes, and a large portion of the budget is devoted to this. Yet security analysis is little reflected in the active holdings, for the minute deviations from index proportions would only be optimal if the appraisal premium to be earned thereby were 2 basis points (table 2), implying a management fee to support security analysis on the order of 1 basis point.

As another illustration, it is not easy to imagine an investment-strategy process that, after evaluation of expected returns to all common factors in the economy, decided upon nearly perfect industry diversification at the same time that a relatively aggressive position were taken with regard to growth stocks. Of course, this pattern is the possible outcome of an analytical process, but we may be forgiven for doubting that it is the
anticipated outcome of the analysis undertaken by the managers. The aggressive growth-orientation strategy is possibly the accidental consequence of the sponsor's choice of managerial style, rather than of the deliberate outcome of the managers' analysis. Finally, the portfolio, as a whole, falls into the logical fallacy pointed out in [1]. If the existing investments were optimal, from the sponsor's assessment of appraisal ability, then the benefit from active management must be smaller than the active management fee, so that it would be better to terminate active management entirely. Thus, the portfolio should be modified, either by increased aggressiveness or by the termination of active management.

This pattern is readily explained. For one thing, it is natural for the sponsor to fall into an exaggerated aversion toward residual risk, for the reasons discussed in [1]. Perceiving this, the managers naturally reduce exposure to residual risk by shrinking their active holdings. Second, very few managers have understood the diversification effect in the multiple manager portfolio. This effect requires that portfolio aggressiveness be matched to the investment portion. Thus, a manager facing different sponsors, for whom he manages different portions of the total, must be prepared to offer portfolios with differing aggressiveness to the different sponsors. Very few managers have been so prepared, with the result that the sponsor experiences less and less aggressiveness in the aggregate portfolio as investment portions of individual managers drop.

Third, very little cognizance has been taken of the manner in which active holdings reinforce themselves across constituent portfolios. In the present case, the growth-orientation position of the various portfolios is mutually reinforcing and leads to substantial risk exposure in the aggregate. On the other hand, the industry holdings and individual stock holdings tend to average out across the portfolios, with little reinforcement, so that the end result approximates indexation. This averaging out of active holdings is a desirable feature, for it suggests that the managers are acting on independent information. If this is truly a superior group of managers, then those cases where their decisions align and a concentration results, are cases where the sponsor can be relatively confident of earning
a superior reward. However, with the minimal level of activity in the constituent portfolios, the aggregate portfolio shows so little concentration that there is negligible opportunity to profit from these superior forecasts.

These problems are not difficult to remedy. What is required is the same kind of decentralized control which was already developed for management of beta. The process is somewhat more complex, since a distinction must here be drawn between specific risk and extra-market covariance. However, the process is also somewhat easier to carry out, because the data on many asset holdings allow computation of the intermanager correlations which can be the basis for accurate dependence adjustments. These matters are taken up in the remainder of the paper. Before going on to that subject, however, it is important to consider the one possible weakness in the argument presented thus far.

This weakness concerns the relationship between the investment risk predicted by the model, on the one hand, and the required appraisal premia, on the other. The model is not perfect, and model errors might distort the risk predictions. However, it is safe to say that these errors could never change the conclusions with regard to industry risk or specific risk enough to substantially change the required appraisal premia for these aspects of investment. So, a weakness, if it occurs, must arise in the derivation of the required appraisal premia from the model risk predictions.

Within this derivation, it is assumed that the investor is free to take on whatever holdings he wishes, including short-sales and unlimited concentrations. The required appraisal premia are deduced under this assumption of free holdings. When the size of holdings is constrained by legal restrictions, rather than by risk considerations, it is possible for the appraisal premium to exceed the required premium which we have computed. The manner in which this happens is best explained through an illustration.

Suppose that a manager's information process is not able to discriminate between fine gradations of alpha. In fact, the only outcome is a judgment of either "good" or "bad." The former stocks are viewed as desirable, the latter as undesirable. Then the resulting portfolio cannot be
better, from the manager's point of view, than an optimally diversified list of the "good" stocks. This follows because the manager does not have the opportunity to short-sell the inferior assets and increase the investment in superior assets. Thus, he can do no better than to invest 100% in the good assets and, if this is to be done, portfolio optimization consists in producing minimum residual risk. The resultant portfolio, optimal in view of short-selling restrictions, is not affected by the magnitude of the alpha, so long as this is large enough to offset transaction costs and the required appraisal premium for residual risk. The alpha might be 1% or 10%, but the portfolio would be the same, and so the required alpha would not change.

However, this extreme situation only arises in the absence of gradations within judgmental alphas. When good stocks are further subdivided into higher and lower alphas, the portfolio alpha can always be increased by larger concentrations in the better assets. The smaller the number of "best" assets in the appraised list, the greater is the scope for increasing aggressiveness. As a matter of realism, it is hard to imagine an information process that is able to identify good and bad stocks successfully and yet is unable to distinguish between good and better stocks. Thus, the extreme case cited in the previous paragraph does not seem to be probable.

Returning to the present case, by inspecting the holdings of the constituent portfolios, it becomes clear that the major active portfolios are not constrained. Funds are readily available for increased concentrations in assets whose judgmental appraisals are greater than required appraisals. This can be seen from table 8, panel A. For example, manager A holds 2.5% of his portfolio in EXXON, substantially less than the market proportion of 3.7%. Therefore, manager A has taken a negative active holding in EXXON and must view it as an inferior asset. If there were substantial unexploited opportunities for concentrations in desirable assets, the position in EXXON could easily be driven to zero, and this 2.5% of the portfolio invested in those superior stocks. Manager B is in a similar position, with a 3% holding in EXXON and, also, a 4.6% holding in AT&T. Thus, 7.6% of manager B's portfolio is available for further concentrations. Manager C has
a 5.6% holding in IBM and a 2.2% holding in EXXON, both corresponding to negative active positions. Manager C has 7.8% of the portfolio invested in stocks that were regarded as overvalued. These funds would be available for concentrated investments if such opportunities existed. Manager D has a 3.3% position in EXXON; manager E a 5.4% position in IBM; manager F has a 2.8% position in AT&T and a 2.7% position in GM. So, every one of the active managers is clearly restrained from added concentration—-not by unavailable funds, but rather by risk aversion. In this case, we can rely on the required appraisal premia to be accurate. Consequently, the potential weakness in the argument does not apply.

3.6. Decentralized Control of Active Holdings

The decentralized-control problem for individual assets is very similar in nature to the control problem for market forecasting. However, there are some added complexities due to the fact that forecasts are made for N different assets. A fully general approach to the problem would need to take account of all possible dependencies among these N forecasts, as well as between the N forecasts and the actual returns. The solution for this general approach is given in [Appendix: Theorem 6]. The approach does require far more information than the sponsor could reasonably provide, since it would be necessary to analyze intermanager dependencies relating to each individual asset and also between assets.

Two possible simplifications can be used to make the problem manageable, each with a good deal to recommend it. The first simplifying assumption is that the managers are independent of one another. This may be quite a reasonable assumption concerning forecasts of specific returns. Table 7 showed that intermanager correlations in this context are quite small, and such correlations as there are may have resulted from the subordination of security analysis to style. (By this is meant that individual stocks are chosen, not so much because of an optimistic forecast for that individual asset, as because of an optimistic forecast for a wide
category of assets within which that individual case falls.) On the other hand, the assumption of independence does not seem to be a good one with respect to common factors. For one thing, intermanager correlations in this area are frequently large, so the assumption seems to be contradicted in practice. Furthermore, since there are relatively few common factors to be forecasted, it is quite possible that different managers would develop overlapping information sources or analytical methods which might lead to correlations.

When the independence assumption is made for a component of residual return, such as specific returns, optimal combination of the managers' alphas is straightforward [Appendix: Theorem 8]. The best combined appraisal, which should ideally be used by the sponsor, is simply the sum of the appraisals of the different managers. The information content of the combined appraisal is the sum of the information contents of the constituents.

The second appealing simplifying assumption is the Constant Proportional Explanation (CPE) assumption. This states that each manager can predict the same proportion of future variance of return for every asset. Thus, a manager's abilities are comparable across all assets in his universe and do not vary substantially from one asset to another. An argument justifying this assumption might be that the manager applies a standardized process to investment analysis, which offers the same potential in all applications. Of course, once the process is applied, there will be some individual assets that stand out as superior to others. But this does not contradict the assumption, which only requires that the intrinsic ability of the process is the same in all cases, not that the process always comes up with the same answer.

If the CPE assumption applies for each manager taken in isolation, then it is reasonable to assume also that, for each pair of managers, there will be some unique correlation coefficient that summarizes the dependence between all pairs of forecasts. Again, this sort of assumption is justified on the basis of two standardized processes; one for each manager, which tend to have similar degrees of redundancy in all cases.
Under the CPE assumption, the information content of each manager's set of \( N \) forecasts is summarized by a single information ratio or correlation coefficient. Also, the dependence between any pair of managers' forecasts is summarized by a single correlation coefficient. The situation therefore becomes formally identical to the market-forecasting problem, insofar as the description of information content is concerned. The solution for the optimal combined forecast follows easily. As in the market-forecasting case, the best combined appraisal is the dependence-adjusted sum of the appraisals of the individual managers [Appendix: Theorem 7]. The information content of the combined forecast is the dependence-adjusted sum of the information contents of the constituents.

The two major components of residual returns—specific returns and common factors—can be treated separately. Either the independence assumption or the CPE assumption can be applied to specific returns, resulting in a set of dependence adjustments for these forecasts. Similarly, the forecasts of common factors can be subjected to a CPE assumption, having different correlation coefficients than assumed for specific returns.

Once the problem of combining the different appraisals is resolved, in principle the sponsor could manage the portfolio in-house, determining active holdings, with the managers serving as advisors. Again, the important question arises as to whether the managers, acting autonomously in what is now the usual arrangement, can produce an optimal active policy for the aggregate portfolio. Although the distressing properties of the illustrative case study might suggest otherwise, optimal decentralized control can be accomplished. One caveat is important here: As in the case of market forecasting, the sponsor must be protected from off-setting transactions and from worsened performance due to restrictions on individual portfolio holdings.

The decentralized-control procedure is developed in [Appendix: Theorems 9, 10, and 11] for residual returns as a whole, and for residual returns divided into distinct components (specific returns and common factors) in [Appendix: Theorem 15, Equations 1 through 18]. The procedure is quite analogous to that for market forecasting. For each component, the sponsor establishes covenant information ratios for each manager and
dependence adjustments for each manager. The other parameters that are required were already discussed in the context of market timing: the investment portion, the normal portfolio beta, and the normal mean and variance of the market portfolio. The sponsor may also wish to apply a different risk acceptance to residual risk than that which is applied to systematic risk. This parameter (\( \kappa \) in the appendix) would probably be set greater than 1 in the present institutional environment, although arguments of capital market theory strongly indicate that the parameter should be equal to or less than 1.

The manager’s optimization problem is quite straightforward, once he has received the necessary guidance from the sponsor. He applies the risk-acceptance parameter communicated by the sponsor, with adjustment for the investment portion which is entrusted to him. He uses, as security appraisals, his internally generated alphas, multiplied by the appropriate dependence-adjustment factors. The problem of constructing a portfolio that is optimal with regard to this risk/reward tradeoff is easily solved with presently available technology.

3.7. Establishing the Control Parameters

The sponsor’s problem of determining control parameters to communicate to the managers is considered in greater detail in this section. The first and crucial parameter is the Covenant Information Ratio. Under the GPE assumption, establishing this ratio is tantamount to establishing the correlation coefficient between the managers’ appraisals and actual returns. Conceptually, this parameter is nothing more than a judgment of the superiority of the manager. It has always been the case that the sponsor seeking active management has been forced to search for superior managers and to assess their abilities once they were found. Thus, this problem is no more than an old problem in new clothing. There can be no easy solutions, since if there were, all sponsors could apply it equally. Since the average sponsor cannot come up with a superior manager, it follows that any solution that can be applied by all is devoid of content. The sponsor must do his best to ascertain the intrinsic potential of the manager’s appraisal process and must, in the final analysis, come down to a single quantified measure of that potential.
The second category of information parameters which must be established is the set of dependence-adjustment factors. Here, the situation is much more sanguine. The reason is that the intermanager correlation coefficients that are required to set dependence adjustments can be estimated from recognized portfolio properties.

To make the problem manageable, some simplifying assumption, such as CPE, must be made. Under that assumption, the dependence-adjustment factors can be estimated from the correlations between residual returns. This approach was followed in the case study, where the dependence-adjusted required appraisal premia were estimated in each instance from the correlation matrix of residual returns. The reasoning that underlies this approach is developed in the appendix. Theorem 13 deals with the problem when there is a single component of residual return, and Theorem 15, Equations 19 through 33, generalize the solution to the case where there are two components.

It is interesting to notice that the correlations can be established from portfolios, without knowing the individual appraisal premia. Thus, the sponsor can establish the necessary correlation coefficients from portfolio asset lists without scrutinizing the internal processes of the managers. This approach relies on the fact that any portfolio which is optimized with respect to the risk/reward trade-off discloses the underlying appraisal premia that led to its construction.

With a model of investment risk, the necessary residual correlations can be computed from a single snapshot of portfolio asset lists at a point in time. No history of returns is required. This snapshot approach was used in the case study. Of course, more accurate estimates can be obtained from a historical series of snapshots. Such a history allows correlations to be estimated at many different times, and the average may be more representative than any single value. Moreover, a history of portfolio asset lists allows the sponsor to study the coherence between changes in appraisals by different managers, one more source of information about correlations.

In summary, the question of decentralized control does not raise any significant new problems for the sponsor. Instead, it requires
quantification of the solutions to traditional problems, and communica-
tion of these solutions to the managers in such a fashion that optimal
decentralized control can be achieved.

3.8. Apportioning the Portfolio; Use of the Asset Return Chit (ARC)

Two principles can be followed in dividing the portfolio among man-
agers. Both principles have some precedent in current practice. The
first may be called the "equal-risk" principle. This requires that money
be apportioned among the managers so that each will be taking roughly the
same amount of residual risk from specific returns. If this principle is
followed, then the sponsor will expect to see the same residual standard
deviation from specific return in all constituent portfolios, when the
optimal aggregate policy is achieved.

The second principle which is often used is the "reward" principle.
Here, funds are apportioned among the managers to reward their contribu-
tions to the sponsor's well-being. The greater the contribution to the
sponsor's utility, the greater is the proportion of funds awarded to the
manager. When the sponsor requires that each manager be paid the same
fee schedule, this is the only way of increasing the management fee in
reflection of a greater contribution.

It is interesting to consider whether funds can be apportioned so
as to satisfy both of these principles at once. The answer is negative.
When the optimal decentralized policy is followed, both these principles
do lead to larger portions for the more valuable contributors. However,
the rate at which the investment portion increases differs from one prin-
ciple to the other. The investment portion for the superior manager is
larger under the reward principle than under the equal-risk principle
[Appendix: Theorem 14]. If the reward principle is followed exactly,
then, with investment proportions being proportional to contributions to
utility, the sponsor will find that the superior managers are operating
at lower levels of residual risk than the less superior managers. If the
equal-risk principle is followed, then the superior managers must be paid
higher unit fees to bring their total fee in line with their proportional
contribution to utility.
Since unit fees can be changed rather easily, the reward principle can be sacrificed with no inequity to the managers. The equal-risk principle, on the other hand, is potentially important because it can protect active managers from an uneven burden of investment restrictions. Recall that the optimal aggressiveness of the manager increases as the investment portion decreases. If a manager is assigned an investment portion which requires that his optimal aggressiveness be higher than some other manager, he will be forced to take larger active positions, both positive and negative. Negative active holdings may drive him against the short-selling constraint, so that he is unable to provide as much expected reward to the sponsor as he potentially can. Where maximum holdings in the amount of 5% of the portfolio have been established, the same problem can be encountered with positive active holdings. Thus, fairness to all managers may suggest that the equal-risk principle is a reasonable one.

If the sponsor employed the managers as advisors and established aggregate positions in response to the combined appraisals, the restrictions on constituent portfolios would not be relevant. Since these restrictions can only harm the aggregate portfolio, it follows that one of the weaknesses of decentralized management is the existence of these restrictions. Fortunately, a simple device allows these restrictions to be relaxed to the same extent that they can be relaxed by centralized management. This device is the "Asset Return Chit" or ARC. The ARC is issued with regard to an individual asset and is otherwise identical to an MRC. It is a promise by one manager to deliver the return on an asset to another manager and is quite similar to a short sale. When two managers enter into an ARC transaction, the real holdings in their portfolios are unchanged. However, in the sponsor's monitoring system, the credit for return on this asset is shifted from one portfolio to another.

This very simple paper transaction accomplishes two important goals in one step. On the one hand, it makes it possible for any manager to sell short a stock to the extent that it is held as a positive holding elsewhere in the portfolio. This allows the sponsor to benefit from the manager's negative appraisals of stocks to an extent which is not now
possible. At present there is an undesirable asymmetry caused by the short-selling constraint. For example, if a stock is 1% of the market, a manager can take a 4% position in the stock, corresponding to a 3% active holding, but the smallest position he can hold is 0%, corresponding to only a -1% active holding. Thus, the manager really cannot respond to negative information actively. The ARC allows the manager to sell short up to the entire portfolio holding. If he issues an ARC to another manager, then the other manager is faced either with a position of holding more of the stock or, alternatively, with the necessity of selling out his own holding so as to restore his initial situation. Therefore, one manager, by offering ARCs to other managers, can induce them to sell their holdings of the stock. The result is that the sponsor finds his entire holding of the stock eliminated, not just that part held by the manager with the negative appraisal. Of course, the entire holding would only be eliminated when the manager had extreme confidence in his negative appraisal. But in this case, it is entirely appropriate that a device such as the ARC be used to clear out the position.

The second advantage of the ARC is that it can be used to remove offsetting transactions. When one manager wishes to sell stock and the other wishes to buy it, this can be effected by an ARC without any net outside transactions. This is the same as an automated transfer between the accounts of the different managers. The result is an economy in commissions and dealer spreads paid to outsiders.

Incidentally, notice that the ARC can be gracefully accomplished within the framework of a master trust. Since all common stocks are held in a single pool, the bookkeeping transaction analogous to an ARC can be effected as an accounting entry. In fact, if confidentiality between managers is an important issue, the ARC can be accomplished without either manager being able to distinguish between this and a real transaction. Thus, manager A sells and manager B buys, and neither knows whether this is a real transaction which affects the aggregate portfolio or a wash transaction which is offset by another manager in the pool. A function very similar to this is now performed in some market inventory funds, but
none have yet extended the concept to permit short-selling by managers. This is a small extension, and the argument for it appears to be compelling. Also, notice that the ARC does not imply that the active manager's judgment be overridden, as is often the case in existing market inventory funds. Rather, it implements the manager's judgment.

REFERENCES


APPENDIX

This appendix rigorously derives the relationships underlying the statements in the text. It is a continuation of the analysis in the appendix of "Security Appraisal and Unsystematic Risk in Institutional Investment" (Rosenberg, 1976), which dealt with the aggregate portfolio. The notation and terminology of that appendix are retained here. Additional terminology is collected in the glossary below, which is an addendum to the glossary in the earlier paper.

TABLE A.1
GLOSSARY OF ADDITIONAL TERMS AND NOTATION
FOR THE MULTIPLE-MANAGER PROBLEM
(Other Notation Is in ([1], Table A.2)

1. Mnemonic subscripts
A the aggregate portfolio
c the combined forecast

2. Constructs defined for each manager i
b_i dependence adjustment factor; b is vector for J managers
W_i investment portion in the aggregate equity portfolio
q_i variance explained by alpha; q is vector for J managers
\rho_i correlation between alpha and actual return; \rho is vector for J managers
OmegaA residual covariance with the aggregate portfolio
k_i N-element vector of covariances k_{in} between residual returns of
asset n and residual returns of portfolio

3. Intermanager correlations and covariances
Omega matrix of Omega_{ij}, residual covariance between managers i,j
P matrix of P_{ij}, residual correlation between managers i,j
C matrix of C_{ij}, covariance parameters between alphas of
managers i,j
A_{ij} N \times N covariance matrix between asset alphas of managers i,j

4. Miscellany
\Delta J \times J matrix with \rho_i along the diagonal, zero elsewhere
\Delta, \Delta large matrices composed of submatrices \Delta_{ij}
K, L measures of covariance for different components of residual
return, adjusted for overlap
y^s, y^r, y^s measures of importance of components of residual return,
and overlap between components
The two major sections of the appendix consider the management of beta (section A.2) and active investments (section A.3). Section A.1 provides some preliminary lemmas.

A.1. Lemmas

A.1.1. Lemma on Forecasting

The first lemma concerns optimal forecasting. The formulas will be used to meld the managers' forecasts of returns into best combined forecasts.

Let \( x \) be a variable (or vector of variables) which is to be employed in constructing a forecast of the vector \( y \). Let \( E(x) = \bar{x} \) and let \( E(y) = \bar{y} \). Let \( C = \text{COV}(x, y) \); \( D = \text{VAR}(x) \); \( \Sigma = \text{VAR}(y) \). Thus, \( x \) and \( y \) are jointly distributed random variables. Then the optimal forecast and its properties are given in the following lemma.

**Lemma 1: Optimal Forecasts**

The optimal forecast for \( y \), denoted by \( \hat{y} \), is

\[
\hat{y} = \bar{y} + C'D^{-1}(x - \bar{x})
\]

The variance of \( \hat{y} \), denoted by \( \tilde{\Sigma} \), equals its covariance with \( y \):  

\[
\tilde{\Sigma} = \text{VAR}(\hat{y}) = \text{COV}(\hat{y}, y) = C'D^{-1}C.
\]

Variance of \( y \) equals the sum of the mean square error matrix for \( \hat{y} \), denoted by \( \tilde{\Sigma} \), and the matrix \( \tilde{\Sigma} \), which is therefore the explained variance.

\[
\tilde{\Sigma} = \Sigma + \tilde{\Sigma}
\]

\( \Sigma \) is the minimum mean square error linear unbiased prediction in the following sense: For all other linear functions of \( y \), written as \( Ky + k \) for arbitrary matrix \( K \) and vector \( k \), which are unbiased, in that \( E(Ky + k) = E(y) \), the mean square error matrix exceeds that of \( \hat{y} \) by a positive semidefinite matrix \( \Omega \) (which depends on \( \bar{y} \) and \( \bar{x} \))

\[
E((Ky + k - y)(Ky + k - y)') = \Sigma + \Omega
\]

\( \hat{y} \) is the posterior mean for \( y \) in the case of a normal distribution: If...
$x$ and $y$ are jointly multivariate normal, then $E(y | x) = \alpha$, $\text{VAR}(y | x) = \Sigma$, and $\alpha$ is the minimum mean square error prediction among all possible predictions (not just linear ones).

**A.1.2. Reward and Risk in a Multiply Managed Portfolio**

Let $h_n, n = 1, \ldots, N$ denote investment proportions in a portfolio. Let $h_{MN}$ denote the capitalization proportion in the market portfolio. Let $h_{jn}$ denote investment proportions in manager $j$'s portfolio, and let $h_{An}$ denote proportions in the aggregate portfolio. Hereafter, all constructs defined for portfolio $j$ will also apply to the aggregate portfolio (with subscript A). Let $\delta_{jn} = h_{jn} - \beta_j h_{MN}$ denote the active holding of stock $n$ in portfolio $j$, where $\beta_j$ is the portfolio beta.

Let $\mu_M$ be the expected excess return on the market. Let $\mu_{jS}$ be expected portfolio excess systematic return—that is, portfolio expected return, in excess of the riskless rate, due to exposure to the market return. Let $\alpha_n, n = 1, \ldots, N$ denote forecast unsystematic returns (risk-adjusted abnormal returns) for the $n$ assets. Let $\mu_{jU}$ denote the expected unsystematic return (risk-adjusted abnormal return) for portfolio $j$.

For each manager $j$, let $w_j$ denote the portion of the aggregate portfolio entrusted to that manager.

For each of the constructs applying to the individual securities, such as $\alpha_n, \beta_n, h_n, \delta_n$, let an unsubscripted symbol with tilde beneath denote the n-element column vector made up of values for all securities (e.g., $\tilde{\alpha}, \tilde{\beta}, \tilde{h}, \tilde{\delta}$).

Then the following formulas, relative to investment holdings and expected reward, can easily be derived.

**Lemma 2: Investment Holdings and Expected Reward**

**Aggregation of Holdings:**

(L2.1) \[ h_{An} = \sum_{j=1}^{J} w_j h_{jn}, \quad n = 1, \ldots, N. \]
Portfolio Betas, Aggregation of Beta:

(L2.2) \( \beta_j = \sum_{n=1}^{N} h_{jn} \beta_n \); \( \beta_A = \sum_{n=1}^{N} h_{An} \beta_n \); \( \beta_A = \sum_{j=1}^{J} W_j \beta_j \).

Expected Excess Systematic Return; Aggregation Thereof:

(L2.3) \( \mu_{JS} = \beta_j \mu_M \); \( \mu_{AS} = \beta_A \mu_M \); \( \mu_{AS} = \sum_{j=1}^{J} W_j \mu_j \).

Active Portfolio Holdings:

(L2.4) \( \delta_{jn} = h_{jn} - \beta_j h_{Mn} \); \( n = 1, \ldots, N \)

or, as a matrix equation:

(L2.4A) \( \delta_j = h_j - \beta_j h_M = (I - h_j h_j^T) h_j \).

Aggregation of Active Holdings:

(L2.5) \( \delta_A = h_A - \beta_A h_M \); \( \delta_A = \sum_{j=1}^{J} W_j \delta_j \).

Expected Unsystematic Return:

(L2.6) \( \mu_{jU} = h_j \alpha = \sum_{n=1}^{N} h_{jn} \alpha_n \); \( \mu_A = h_A \alpha = \sum_{n=1}^{N} h_{An} \alpha_n \).

or, in terms of active holdings:

(L2.6A) \( \mu_{jU} = \delta_j \alpha = \sum_{n=1}^{N} \delta_{jn} \alpha_n \); \( \mu_A = \delta_A \alpha = \sum_{n=1}^{N} \delta_{An} \alpha_n \).

Aggregation of Expected Unsystematic Return:

(L2.7) \( \mu_{AU} = \sum_{j=1}^{J} W_j \mu_{jU} \).

Proof: The formulas in the lemma follow directly from the properties of the constructs and may be easily verified. The only assumption underlying the results which is not self-evident is the "market neutrality" condition.
that the market exhibit zero unsystematic return, or

$$\sum_{n=1}^{N} h_{Mn} \alpha_n = h'_{M} \alpha = 0.$$  

This condition is required for the step from (L2.6) to (L2.6A), which is accomplished by substituting (L2.4A):

$$\delta'_{j} \alpha = h'_{j} (I - \beta h') \alpha = h'_{j} \alpha - h'_{j} \beta h'_{M} \alpha = h'_{j} \alpha - \beta_{j} 0 = h'_{j} \alpha.$$  

The next lemma concerns investment risk. Let $$\sigma_{M}^{2}$$ denote the variance of the market return. Let $$R_{mn}$$ denote the (unsystematic) residual covariance between assets $$m$$ and $$n$$, and let $$R$$ denote the $$(N \times N)$$ matrix of residual variances and covariances. For each portfolio $$j$$ (or the aggregate portfolio $$A$$), let $$\sigma_{JS}^{2}$$ denote systematic variance, and $$\omega_{j}^{2}$$ denote residual variance. For each pair of portfolios $$i$$ and $$j$$, let $$\Omega_{ij}$$ denote the residual covariance between them. (Hence, $$\Omega_{ii} = \omega_{i}^{2}$$). Finally, let $$\Omega_{jA}$$ denote the residual covariance between portfolio $$i$$ and the aggregate portfolio.

Let $$r$$ denote an excess return, and let $$u$$ denote a residual (risk-adjusted) excess return, $$u = r - \beta_{M} r_{M}$$. Residual returns on individual assets will be written as $$u_{m}$$ or $$u_{n}$$, for assets "m" or "n"; residual returns on portfolios as $$u_{j}$$, $$u_{j}$$, or $$u_{A}$$, for portfolios "i," "j," or the aggregate portfolio. Some basic formulas concerning systematic and residual portfolio risk are then easily derived.

**Lemma 3: Investment Holdings and Risk**

**Systematic Risk:**

$$\sigma_{JS}^{2} = \beta_{j}^{2} \sigma_{M}^{2}; \quad \sigma_{AS}^{2} = \beta_{A}^{2} \sigma_{M}^{2}.$$  

**Residual Risk for Constituent Portfolios:**

$$\omega_{j}^{2} = \delta_{jR_{S}}^{2} = \sum_{m=1}^{N} \sum_{n=1}^{N} \delta_{j} \delta_{n} \delta_{jm} \delta_{mn}.$$
Residual Covariance Between Portfolios:

\[(L3.3)\]
\[\Omega_{ij} = \delta_i^\prime R \delta_j = \sum_{m=1}^N \sum_{n=1}^N \delta_{im} \delta_{jn} \rho_{mn}.\]

Aggregate Residual Variance:

\[(L3.4)\]
\[\omega_A^2 = \delta_{-A}^\prime R \delta_{-A} = \sum_{m=1}^N \sum_{n=1}^N \delta_{Am} \delta_{An} \rho_{mn}.\]

Aggregation of Residual Variance:

\[(L3.5)\]
\[\omega_a^2 = \frac{J}{\sum_{i=1}^J \sum_{j=1}^J W_i W_j \Omega_{ij}}.\]

Residual Covariance Between Aggregate and Constituent Portfolios:

\[(L3.6)\]
\[\Omega_{iA} = \sum_{i=1}^J W_i \Omega_{ij}.\]

Aggregate Variance As the Weighted Sum of Covariances with Constituents:

\[(L3.7)\]
\[\omega_A^2 = \sum_{j=1}^J W_j \omega_{jA}^2.\]

Proof: (L3.1) follows because

\[(1.3)\]
\[\sigma_{jS}^2 \equiv \text{VAR}(\beta_j r_M) = \beta_j^2 \text{VAR}(r_M) = \beta_j^2 \sigma_M^2.\]

(L3.2) is the special case of (L3.3) for \(i = j\). The general formula (L3.3) is verified by rewriting portfolio residual return as the weighted sum of asset returns:

\[(L3.4)\]
\[\Omega_{ij} \equiv \text{COV}(u_i, u_j) = \text{COV} \left( \sum_{m=1}^N \delta_{im} u_m, \sum_{n=1}^N \delta_{jn} u_n \right) = \sum_{m=1}^N \sum_{n=1}^N \delta_{im} \delta_{jn} \text{COV}(u_m, u_n).\]
(L3.4) is derived similarly. To derive (L3.5), note that the aggregate portfolio residual return is the weighted sum of the constituent returns:

\[(1.5)\quad \omega_A^2 \equiv \text{VAR}(u_A) = \text{VAR}\left(\sum_{j=1}^{J} W_j u_j\right) = \sum_{i=1}^{J} \sum_{j=1}^{J} W_i W_j \text{COV}(u_i, u_j)\]
\[= \sum_{i=1}^{J} \sum_{j=1}^{J} W_i W_j \Omega_{ij}.\]

By the same approach, (L3.6) is obtained, since

\[(1.6)\quad \Omega_{ij} \equiv \text{COV}(u_i, u_j) = \text{COV}\left(\sum_{i=1}^{J} W_i u_i, u_j\right) = \sum_{i=1}^{J} W_i \text{COV}(u_i, u_j) = \sum_{i=1}^{J} W_i \Omega_{ij}.\]

Finally, (L3.7) is shown to be equivalent to (L3.5) by substitution of (L3.6):

\[(1.7)\quad \sum_{j=1}^{J} W_j \Omega_{jA} = \sum_{j=1}^{J} W_j \left(\sum_{i=1}^{J} W_i \Omega_{ij}\right) = \sum_{i=1}^{J} \sum_{j=1}^{J} W_i W_j \Omega_{ij}.\]
A.2. Management of Beta

This section investigates systematic risk control in a multiply managed portfolio. The analysis is of some interest in its own right and is also preliminary to the analysis of unsystematic returns in the next section. The essential steps are developed in the first four subsections: A.2.1. The Optimal Combined Forecast, A.2.2. Optimal Aggregate Investment, A.2.3. Decentralization Procedure, and A.2.4. Utility Contributions of the Managers. A.2.5. contains a numerical example. Finally, A.2.6. evaluates the simplifying approximation that market forecasts negligibly reduce market uncertainty.

A.2.1. The Optimal Combined Forecast

Let \( r_M \) be the market portfolio return in excess of the risk-free rate over some time interval, and let \( E_M \) and \( \sigma_M^2 \) be the mean and variance of excess market return, in the absence of market forecasting. Thus, \( u_M = r_M - E_M \) is the unexpected market return, with mean zero and variance \( \sigma_M^2 \). Each manager \( j, j=1,...,J \), forecasts \( u_M \). The forecast is converted by the manager to a minimum mean square error forecast \( \alpha_{Mj} \), by the Bayesian adjustment procedure discussed in Rosenberg ([1], A.7 through A.9). Since this section concerns the market portfolio return exclusively, the subscript "M" will be suppressed without confusion. When the time interval is viewed as a future occurrence, as yet undetermined, the forecasts and actual return become jointly distributed random variables. Since \( u \) has expected value zero, and the forecasts are equal to \( u \) on average, the forecasts also have expected value zero.

Because of the Bayesian adjustment, \( \text{VAR}(\alpha_j) = \text{COV}(\alpha_j, u) \). Let \( q_j \) denote this explained variance, the basic measure of forecast quality. The \( J \) managers' forecasts are prepared independently. However, because the forecasting process may overlap in terms of method or information, correlations between the forecasts are possible. For each pair of managers \( i \) and \( j \), let \( c_{ij} = \text{COV}(\alpha_i, \alpha_j) \) be the variance between forecasts.
Let $\mathbf{q}$ be the column vector of forecasts. The variances and covariances of the forecasts are arrayed in the $J \times J$ "variance matrix," $\mathbf{C}$, and the covariances between forecasts and market return in the column vector, $\mathbf{q}$, as follows:

$$
\mathbf{C} = \text{VAR}(\mathbf{\alpha}) = \begin{pmatrix}
    c_{11} & c_{12} & \cdots & c_{1J} \\
    c_{21} & c_{22} & \cdots & c_{2J} \\
    \vdots & \vdots & \ddots & \vdots \\
    c_{J1} & c_{J2} & \cdots & c_{JJ}
\end{pmatrix};
\mathbf{q} = \text{COV}(\mathbf{\alpha}, \mathbf{u}) = \begin{pmatrix}
    q_1 \\
    q_2 \\
    \vdots \\
    q_J
\end{pmatrix};
\text{VAR}(\mathbf{u}) = \sigma^2.
$$

(2.1)

The mean and variance of the joint probability distribution determines the characteristics of the optimal combined forecast $\mathbf{\alpha}_c$, based upon the $J$ forecasts. The meaning of the solution is more transparent when expressed in terms of correlations rather than covariances. Let $\pi_{ij} = \text{corr}(\alpha_i, \alpha_j)$ and let $\rho_i = \text{corr}(\alpha_i, u)$. Let $\mathbf{\Pi}$ be the correlation matrix of forecasts, and $\mathbf{\rho}$ the column vector of forecast correlations with the market. (Also, let $\pi_{ij}$ denote element $(i,j)$ in $\mathbf{\Pi}^{-1}$.) For each manager, the ratio $\frac{\rho_i}{\sqrt{\rho_j^2/(1-\rho_j^2)}}$ is a measure of goodness of forecast. There is no perfect analog to the information ratio in market forecasting, but the most appropriate definition, because of (T4.1), below, is probably:

$$
z = \frac{\rho_i}{\sqrt{(1-\rho_j^2)}} \frac{1 + \frac{E^2}{\sigma^2}}{
$$

(2.2)

With these definitions, the optimal combined forecast is easily found.

**Theorem 1. Optimal Combined Prediction**

The minimum mean square error combined prediction is:

$$
T1.1 \quad \mathbf{\alpha}_c = b_1\mathbf{\alpha}_1 + b_2\mathbf{\alpha}_2 + \cdots + b_J\mathbf{\alpha}_J,
$$

where the dependency adjustment factors (DAFs) are:
\[ b_j = \frac{1}{\rho_j j} (\rho_1 \rho_{j1} + \rho_2 \rho_{j2} + \ldots + \rho_d \rho_{jd}) , \quad j = 1, \ldots, r . \]

In the special case, where all correlations between managers are zero,

\[ \alpha_c = \alpha_1 + \ldots + \alpha_j , \quad \text{since} \quad b_j = 1 \quad \text{for all} \quad j . \]

The coefficient of determination of the prediction is:

\[ \rho_c^2 = b_1 \rho_1^2 + b_2 \rho_2^2 + \ldots + b_d \rho_d^2 . \]

The remaining market variance not explained by the prediction is:

\[ M = \text{VAR}(u - \alpha_c) = (1 - \rho_c^2) \sigma^2 . \]

**Proof:** The means, variances, and covariances of the forecasts and the market return were given in (2.1). From Lemma 1, the minimum mean-square-error linear unbiased predictor of \( u \) is:

\[ \mathbf{E}(u) + \text{COV}(u, \mathbb{Q})(\text{VAR}(\mathbb{Q}))^{-1}(\mathbb{Q} - \mathbb{E}[\mathbb{Q}]) = 0 + q^t \mathbb{Q}^{-1}(\mathbb{Q} - \mathbb{E}) = q^t \mathbb{Q}^{-1} \mathbb{Q} = \mathbb{Q}^t \mathbb{Q} = \mathbb{Q}^t \mathbb{Q} . \]

where \( b = \mathbb{Q}^{-1} \mathbb{Q} . \)

The Lemma further states that:

\[ q_c = \text{COV}(\alpha_c, u) = \text{VAR}(\alpha_c) = q^t \mathbb{Q}^{-1} \mathbb{Q} = q^t \mathbb{Q} , \]

and that

\[ \text{VAR}(u) = \text{VAR}(\alpha_c) + \text{VAR}(u - \alpha_c) \Rightarrow \text{VAR}(u - \alpha_c) = \sigma^2 - q_c . \]

Equation (2.3) is statement (T1.1). Equations (2.4), (2.5), and (2.6) are the analogs of (T1.2), (T1.4), and (T1.5), but expressed in terms of variances rather than correlations. To complete the proof, all that is necessary is to convert from covariances to correlations.

First, the correlations between any alpha and the market return is expressed in terms of covariances:

\[ \rho_j = \text{CORR}(\alpha_j, u) = \frac{\text{COV}(\alpha_j, u)}{\sqrt{\text{VAR}(\alpha_j)} \sqrt{\text{VAR}(u)}} = \frac{q_j}{\sqrt{q_j \sigma^2}} = \frac{\sqrt{q_j}}{\sigma} . \]
Therefore, the standard deviation of each forecast is:

\[ SD(\alpha_j) = \sqrt{q_j} = \rho_j \sigma. \]

Next, since each intermanager correlation is obtained by dividing covariance by the product of the standard deviations, \( \pi_{ij} = c_{ij} / (\sigma_i \sigma_j) = c_{ij} / (\sigma^2 \rho_i \rho_j) \). Therefore, the covariance matrix is

\[ \Pi = \frac{1}{\sigma^2} \Delta^{-1} \Pi \Delta^{-1}, \]

where \( \Delta = \text{diag}(\rho_j) \) is the diagonal matrix with forecast correlations in the diagonal entries and zeros elsewhere. Therefore,

\[ \zeta = \sigma^2 \Delta \Pi \Delta. \]

Also, \( q \) may be written from (2.7) as:

\[ q = \sigma^2 \begin{pmatrix} \rho_1^2 \\ \vdots \\ \rho_j^2 \end{pmatrix} = \sigma^2 \Delta \rho. \]

When (2.9) and (2.10) are substituted into (2.4), (Tl.2) results:

\[ b = c^{-1} q = \frac{1}{\sigma^2} \Delta^{-1} \Pi^{-1} \Delta^{-1} (\sigma^2 \Delta \rho) = \Delta^{-1} \Pi^{-1} \rho. \]

In the special case where all correlations between managers are zero, \( \Pi \) is the identity matrix. Therefore, \( \Pi^{-1} \) is also the identity matrix and (Tl.3) follows:

\[ b = \Delta^{-1} \Pi^{-1} \rho = \Delta^{-1} I \rho = \Delta^{-1} \rho = \begin{pmatrix} \rho_1 / \rho_1 \\ \vdots \\ \rho_j / \rho_j \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}. \]
Thus, $b$ is a vector of units, and $\alpha_c$ becomes the sum of the $\alpha_j$.

When (2.10) is substituted into (2.5), (T1.4) results:

$$\rho^2_c = \frac{q_c}{\sigma^2} = \frac{1}{\sigma^2} q'b = \left(\rho^2_1, \ldots, \rho^2_J\right)' = \sum_{j=1}^J b_j \rho^2_j.$$  

(T1.5) follows from (2.6), since $q_c = \rho^2_c \sigma^2$. Notice, also, that from (2.10) and (2.11),

$$\rho^2_c = \frac{q_c}{\sigma^2} = q'b(\Delta \rho)' \Delta^{-1} \Gamma^{-1} q = q' \Gamma^{-1} q.$$  

---

A.2.2. Optimal Aggregate Investment

The next topic to consider is the appropriate portfolio response to the combined forecast. Suppose that the sponsor controls all equity investments, with the managers acting in the role of advisors, so that the combined forecast is an input to the sponsor’s investment decision. The sponsor’s attitude toward systematic risk is characterized by the risk-acceptance parameter (RAP) $\lambda_{ES}$ for systematic risk in the equity portfolio as a whole. The simplest assumption, as developed in ([1], Theorem 4) is that the investor possesses a linear mean/variance (LMV) utility function.

$$U = \mu_{ES} - \frac{1}{2 \lambda_{ES}} \sigma^2_{ES}. $$

Alternatively, the utility function may be more general, but preferences in the locality of the optimal portfolio are approximated by a function of this form ([1], Theorem 6). The characterization of the optimal portfolio is valid under either assumption, but the statements concerning the utility function (or certainty equivalent excess return (CEER)) apply only in the simpler case where the utility function is linear mean/variance.

The following theorem exhibits the optimal portfolio beta in four contexts: first, in the absence of market forecasting, where the risk/reward opportunities are determined by the mean $E$ and variance $\sigma^2$ of
the market return; second, in the case where the availability of the market forecasting process had reduced uncertainty, so that market variance is smaller than \( \sigma^2 \), but where the abnormal forecast \( \alpha \) is zero, so that \( E \) is unchanged; third, in the case where a particular nonzero value of \( \alpha \) occurs; and fourth, for the average over many time periods, taking into account the effect of a time-varying \( \alpha \). The Theorem gives the optimal beta in each case, under the assumption that transaction costs are zero and there are no constraints on holdings. Also, the expected systematic return, \( \mu_{AS} \); the variance of systematic return, \( \sigma_{AS}^2 \); and the expected utility under a quadratic utility function, \( U \), are stated in terms of the Sharpe ratio of portfolio systematic return, defined as \( z^2 = (E(r))^2 / \text{VAR}(r) \).

**Theorem 2. Optimum Beta in the Aggregate Equity Portfolio**

**Case A: No Market Forecasting**

*Optimum beta*

\[
(\text{T2.A1}) \quad \beta_A = \lambda_{ES} \frac{E}{\sigma^2}.
\]

*Sharpe Ratio of Excess Systematic Return*

\[
(\text{T2.A2}) \quad z^2 = \frac{E^2}{\sigma^2}.
\]

**Case B: Uncertainty Reduction Due to the Forecasting Process, But \( \alpha_c = 0 \)**

\[
(\text{T2.B1}) \quad \beta_A = N\beta_E = \lambda_{ES} E / M, \text{ where } M = (1 - \sigma^2_c \sigma^2).
\]

\[
(\text{T2.B2}) \quad z^2 = \frac{E^2}{M}.
\]

**Case C: Market Forecasting, with a Specific Forecast \( \alpha_c \)**

\[
(\text{T2.C1}) \quad \beta_A = N\beta_E + \lambda_{ES} \alpha_c / M = N\beta_E \left( \frac{\alpha_c + E}{E} \right).
\]

\[
(\text{T2.C2}) \quad z^2 = (E + \alpha_c)^2 / M.
\]
Case D: Expectation from the Market Forecasting Process

\begin{align*}
& (T2.D1) \quad \bar{\beta}_A = N\beta_E^*. \\
& (T2.D2) \quad \bar{z}^2 = (E^2 + \rho_c^2\sigma_c^2)/M. \end{align*}

In every case, the mean and variance of portfolio return, expressed in terms of the Sharpe Ratio are:

\begin{align*}
& (T.2.3) \quad \mu_{AS} = \lambda_E\bar{z}^2, \quad \sigma_{AS}^2 = \lambda_E^2\bar{z}^2. \\
& \text{In each case, the expected utility or CEER under LV utility is:} \\
& (T.2.4) \quad U_{AS} = \frac{1}{2} \lambda_E^2\bar{z}^2. \end{align*}

Proof: In cases A, B, and C, the solution is obtained by substituting the mean and variance of excess market return into ([1], Theorem 5). For case D, the expected value of the case C solution is computed:

\begin{align*}
& (2.15) \quad E[\beta_A] = E \left[ N\beta_E \left( \frac{\alpha + E}{E} \right) \right] = N\beta_E \left( 1 + \frac{E(\alpha_c)}{E} \right) = N\beta_E \\
& (2.16) \quad E[z^2] = E \left[ \frac{(E + \alpha_c)^2}{M} \right] = \frac{E^2 + \text{VAR}(\alpha_c)}{M} = \frac{E^2 + \rho_c^2\sigma_c^2}{(1 - \rho_c^2)\sigma_c^2}. \end{align*}

Comparing cases A and B, the effect of the uncertainty reduction from the forecasting process is to increase the normal beta and the Sharpe Ratio by the factor $1/(1 - \rho_c^2) = \sigma_c^2/M$, the ratio of total market variance to variance unpredicted by the forecast process. In case C, any particular $\alpha_c$ is found to adjust beta by the factor $(1 + \alpha_c/E)$, and to increase the Sharpe Ratio by the square of this factor. Finally, case D confirms that the average optimum beta to be expected over time equals the "normal" $N\beta$ found in case B and shows that the expected contribution of the market forecasts is to increase the case B Sharpe Ratio by the factor $(1 + \rho_c^2\sigma_c^2/E^2)$.
A.2.3. Decentralization Procedure

Suppose, next, that the funds are entrusted to the managers in portions \( W_j, j=1,...,J \). Of course, \( \sum_j W_j = 1 \). The problem is now to guide the managers so that, even though all managers are acting independently and are unaware of one another's actions, the independent actions of the managers aggregate to the optimal strategy for the portfolio as a whole. This is the problem of optimal decentralized control.

The sponsor has determined the properties of the individual forecasting processes, as summarized by the forecast correlations \( \rho \) and intermanager correlations \( \Pi \). From these, he has computed DAFs \( b_j \), \( j=1,...,J \). From his \( \lambda_{ES} \), he has determined the normal beta \( \mathbf{N\beta}_E \).

Based upon this information, the sponsor must communicate instructions to the managers, so that each manager, when optimizing his own portfolio in light of his own \( \alpha \), generates an optimal contribution to aggregate strategy.

The first key to decentralization is to assign normal beta \( \mathbf{N\beta}_j \), \( j=1,...,J \) to the managers. One possibility would be to assign the same normal beta, \( \mathbf{N\beta}_E \), to all managers. Alternatively, normal betas may differ, as long as they aggregate to the correct normal beta, i.e.,

\[
\sum_j W_j \mathbf{N\beta}_j = \mathbf{N\beta}_E .
\]

Each manager is informed of three items which relate specifically to his own case—\( \mathbf{N\beta}_j, W_j \), and \( b_j \)—and three items which relate to the sponsor's aggregate context—\( E, M \), and \( \mathbf{N\beta}_E \). Recall that \( M = (1-\rho^2_C)\sigma^2 \) is the market variance not explained by the forecasting process, \( E \) is the expected excess return, and \( \mathbf{N\beta}_E \) is the portfolio beta chosen in response to these.

Using \( \mathbf{N\beta}_E, E, \) and \( M \), each manager can compute \( \lambda_{ES} \) by (T2.B1):

\[
\lambda_{ES} = \mathbf{N\beta}_E M / E .
\]
Alternatively, if the sponsor informs the managers explicitly of $\lambda_{ES}$, then $N^E_j$ need not be communicated to them.

Each manager then computes the RAP for systematic risk within his portfolio according to the following principle: Let $\lambda_{ES}$ be the RAP with respect to risk and reward contributed by the manager to the aggregate portfolio. Then the RAP which that manager should apply within his own portfolio is:

$$\lambda_{JS} = \frac{\lambda_{ES}}{W_j}. \tag{2.19}$$

This is the adjustment to the RAP for the diversification effect of a multiple-manager portfolio and may be thought of as the RAP corrected for investment portion. The correctness of this adjustment is confirmed in the following theorem.

**Theorem 3. Optimal Decentralized Control of Beta**

Suppose that each manager sets his beta at

$$\beta_j = NB_j + \lambda_{JS} \frac{B_j}{W_j} \alpha_j. \tag{T3.1}$$

The adjustment is optimal for an autonomous portfolio, where $\lambda_{JS}$ is the RAP for systematic risk; the dependence-adjusted alpha, $b_j \alpha_j$, is the forecast of return; and $W$ is the unexplained variance of market return.

An equivalent formula is:

$$\beta_j = NB_j + \lambda_{ES} \frac{B_j}{W_j} \alpha_j \tag{T3.2}$$

Then the managers' independent actions result in the optimal aggregate strategy.

**Proof:** When $\lambda_{JS}$ as RAP and $b_j \alpha_j$ as dependence-adjusted alpha are substituted into (T2.C1), the result is (T3.1). This confirms that (T3.1) would be the optimal strategy for this RAP and forecast. Substitution for
\( \lambda_{js} \) from (2.18) and (2.19) yields (T3.2). When (T3.1) is aggregated across portfolios, the effect of these strategies is to produce portfolio beta:

\[
\beta_A = \sum_j W_j N\beta_j + \sum_j W_j \lambda_{js} b_j \alpha_j / N .
\]

Substituting \( \sum_j W_j N\beta_j = N\beta_E \) and \( \lambda_{js} = \lambda_{ES} / W_j \), (2.20) is seen to be identical to the optimal beta in (T2.3), which completes the proof.

---

A.2.4. Utility Contributions of the Managers

Turning next to the question of compensation for the managers, Theorem 4 first states the overall contribution to the portfolio Sharpe Ratio and CEER from the forecasting process. Then, three possible formulas for recompensing the managers are considered: in proportion to the ability of the manager to contribute on his own; in proportion to his dependence-adjusted contribution; and in proportion to his incremental contribution relative to the others. When the managers are independent, these three formulas coincide. Otherwise, there are differences and a choice among the formulas must be made.

**Theorem 4. Contributions to Utility**

The increment in the portfolio Sharpe Ratio from market forecasting is:

\[
(T4.1) \quad \Delta \tilde{z}^2 = \frac{\rho^2}{(1 - \rho^2)} \left( \frac{E^2 + \sigma^2}{\sigma^2} \right) .
\]

With a linear mean/variance utility function, the expected increase in CEER is:

\[
(T4.2) \quad \Delta U = \frac{1}{2} \lambda_{ES} \Delta \tilde{z}^2 .
\]

Thus, portfolio utility is a monotonic increasing function of \( \rho^2 \), and
managers should be rewarded for their contributions to this ratio. When reward is proportional to the ability of the managers, acting on their own, the fee formula is:

\[(T4.3)\]

\[f_j \propto \rho_j^2 \quad j = 1, \ldots, J\]

The dependence-adjusted contributions of the managers yield the fee formula:

\[(T4.4)\]

\[f_j \propto b_j \rho_j^2 \quad j = 1, \ldots, J\]

Finally, if each manager is rewarded in proportion to his incremental contribution, relative to the collective contributions of other managers, the fee formula is:

\[(T4.5)\]

\[f_j \propto \frac{b_j^2 \rho_j^2}{\sum_{i=1}^{J} b_i^2 \rho_i^2} = b_j \rho_j \left( \frac{\sum_{i=1}^{J} \rho_i}{\sum_{i=1}^{J} \rho_i} \right).
\]

When there are no correlations among managers, (T4.4) and (T4.5) are identical to (T4.3).

Proof: The expected Sharpe Ratio inclusive of market forecasting is, from (T2.62),

\[(2.21)\]

\[\bar{\varepsilon}^2 = \frac{E^2 + \rho^2 \sigma^2}{(1-\rho^2)\sigma^2} = \frac{E^2}{(1-\rho^2)} + \frac{\rho^2 E^2 + \rho^2 \sigma^2}{(1-\rho^2)\sigma^2} = \frac{E^2}{\sigma^2} + \frac{\rho^2}{(1-\rho^2)\sigma^2} \left( \frac{E^2 + \sigma^2}{\sigma^2} \right).
\]

Since \(E^2/\sigma^2\) is the Sharpe Ratio attainable without market forecasting, from (T2.62), this verifies (T4.1). (T4.2) then follows from (T2.4).

(T2.3) and (T2.4) follow by the definition of the manager's coefficient of determination when acting alone \((\rho_j^2)\), and by definition of the dependence-adjusted contribution (T1.4). The remark at the end of the theorem, that (T2.4) and (T2.5) reduce to (T2.3) when all managers are independent, follows because in the case of independence, \(b_j = 1\) for all \(j\).

It remains to derive the incremental reward formula (T2.5). This is a relatively complex derivation that can be skipped without loss of
continuity. For convenience, the result will be derived for the first manager, without loss of generality. To exhibit the incremental contribution of the manager, let the correlations be partitioned into those relating to the first manager and to the other managers (subscript "0"):

\[
\begin{pmatrix}
\eta_{11} & \eta_{10} \\
\eta_{01} & \eta_{00}
\end{pmatrix}
\begin{pmatrix}
\rho_1 \\
\rho_0
\end{pmatrix}
\]

When information from the other managers only is available, from (2.13),

\[
\rho_c^2 = \rho_0^{-1} \Pi_{00} \rho_0.
\]

When manager 1 is added, the result is:

\[
\rho_c^2 = \rho^\prime_0 \Pi^{-1} \rho = (\begin{pmatrix}
\rho_1 \\
\rho_0
\end{pmatrix})^\prime
\begin{pmatrix}
\eta_{11} & \eta_{10} \\
\eta_{01} & \eta_{00}
\end{pmatrix}^{-1}
\begin{pmatrix}
\rho_1 \\
\rho_0
\end{pmatrix}.
\]

To compare this with (2.23), we use the familiar formula for the inverse of a partitioned matrix (Theil [5], p. 18) in the form:

\[
\begin{pmatrix}
\eta_{11} & \eta_{10} \\
\eta_{01} & \eta_{00}
\end{pmatrix}^{-1}
= \begin{pmatrix}
\eta_{11} & \eta_{10} \\
\eta_{01} & \eta_{00}
\end{pmatrix}^{-1}
= \begin{pmatrix}
0 & 0 \\
0 & 1
\end{pmatrix}
+ \frac{1}{\eta_{11}}
\begin{pmatrix}
\eta_{11} & \eta_{10} \\
\eta_{01} & \eta_{00}
\end{pmatrix}
\begin{pmatrix}
\eta_{11} & \eta_{10} \\
\eta_{01} & \eta_{00}
\end{pmatrix}^{-1}
\begin{pmatrix}
\eta_{11} & \eta_{10} \\
\eta_{01} & \eta_{00}
\end{pmatrix},
\]

where the superscripts indicate elements in the inverse matrix. When (2.25) is substituted into (2.24), some matrix algebra and the definition of \( b_1 \) (T1.2) yield:

\[
\rho_c^2 = \rho_0^{-1} \Pi_{00} \rho_0 + \frac{1}{\eta_{11}} \left( \begin{pmatrix}
\rho_1 \\
\rho_0
\end{pmatrix}^\prime
\begin{pmatrix}
\eta_{11} & \eta_{10} \\
\eta_{01} & \eta_{00}
\end{pmatrix} \begin{pmatrix}
\rho_1 \\
\rho_0
\end{pmatrix} \right)
= \rho_0^{-1} \Pi_{00} \rho_0 + \frac{b_1^2}{\eta_{11}}.
\]

Comparing (2.26) with (2.23), (T4.5) is confirmed.
A.2.5. An Illustrative Example

Let there be two managers. Manager #1 is given one-third of the pool \( (W_1 = 1/3) \), and manager #2 is given two-thirds \( (W_2 = 2/3) \). Assume \( E = 6\% \), \( \sigma^2 = 400(\%)^2 \). Each manager explains 3(\%)^2 market variance, resulting in \( \rho_1^2 = \rho_2^2 = \frac{3}{400} \). The sponsor’s \( \lambda_{ES} \) is 66-2/3\%. To show the effect of intermanager correlation, three cases are considered: Case A, no correlation \( (\tau_{12} = 0) \); case B, substantial positive correlation \( (\tau_{12} = 2/3) \); and case C, substantial negative correlation \( (\tau_{12} = -2/3) \). Both managers are assigned \( N_{\theta_1} = N_{\theta_2} = N_{\theta_E} \). The various constructs for the three cases are arrayed in three columns, as follows:

Correlation Matrix of Predictions, \( \Pi \):

\[
\Pi = \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 2/3 \\
2/3 & 1
\end{pmatrix}
\begin{pmatrix}
1 & -2/3 \\
-2/3 & 1
\end{pmatrix}
\]

Correlation Vector of Predictions with Market Return, \( \rho \):

\[
\rho = \begin{pmatrix}
\frac{3}{\sqrt{1200}} \\
\frac{3}{\sqrt{1200}} \\
\frac{3}{\sqrt{1200}}
\end{pmatrix}
\begin{pmatrix}
\frac{3}{\sqrt{1200}} \\
\frac{3}{\sqrt{1200}} \\
\frac{3}{\sqrt{1200}}
\end{pmatrix}
\begin{pmatrix}
\frac{3}{\sqrt{1200}} \\
\frac{3}{\sqrt{1200}} \\
\frac{3}{\sqrt{1200}}
\end{pmatrix}
\]

Coefficient of Determination for Each Manager Singly:

\[
\rho_1^2 = \rho_2^2 = \frac{3}{400}
\]

Inverse of Correlation Matrix:

\[
\Pi^{-1} = \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & -2/3 \\
-2/3 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 2/3 \\
2/3 & 1
\end{pmatrix}
\]

Dependence Adjustment Factors (same for both managers):

\[
b_1 = b_2 = 1
\]

\[
3/5
\]

3
Coefficient of Determination for Combined Prediction:

\[
\rho_c^2 = \sum b_i^2 \rho_i^2 = \frac{6}{400} \quad \frac{3.6}{400} \quad \frac{18}{400}
\]

Unexplained Variance of Market Return:

\[M = \sigma^2 (1 - \rho_c^2) = 394 \quad 396.4 \quad 382\]

Dependence-Adjusted \( \rho^2 \) for a Manager (same for both):

\[
b_{1i}^2 \rho_{1i}^2 = \frac{3}{400} \quad \frac{1.8}{400} \quad \frac{9}{400}
\]

Increment to \( \rho^2 \) Added by the Second Manager (same for either):

\[
b_{2i}^2 \rho_{2i}^2 = \frac{3}{400} \quad \frac{0.6}{400} \quad \frac{15}{400}
\]

Expected Total Contribution to Utility (CEER) with LMV Utility:

\[
\frac{1}{2\lambda} \left( 1 + \frac{E^2}{\sigma^2} \right) \frac{\rho_c^2}{(1 - \rho_c^2)} = .55 \quad .33 \quad 1.71
\]

Portfolio Beta, in Absence of Market Forecasting:

\[\lambda_{ES} = \frac{E}{\sigma^2} = 1 \quad 1 \quad 1\]

Optimal Portfolio Beta, Reflecting Explained Variance by Forecasting Process, When Combined Market Alpha Happens to Be Zero (\( \alpha_c = 0 \)):

\[\lambda_{ES}^* = \lambda_{ES} \frac{E}{M} = 1.015 \quad 1.009 \quad 1.046\]

RAP for Manager #1 in Decentralized Optimization:

\[\lambda_{1S} = \lambda_{ES} / \omega_1 = 200 \quad 200 \quad 200\]

RAP for Manager #2 in Decentralized Optimization:

\[\lambda_{2S} = \lambda_{ES} / \omega_2 = 100 \quad 100 \quad 100\]
The remaining computations refer to a hypothetical realization, where $\alpha_1 = +2\%$, $\alpha_2 = +2\%$.

**Table: Best Combined Alpha:**

<table>
<thead>
<tr>
<th>Case A</th>
<th>Case B</th>
<th>Case C</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_c = b_1\alpha_1 + b_2\alpha_2$ =</td>
<td>$4%$</td>
<td>$2.4%$</td>
</tr>
</tbody>
</table>

**Optimal Aggregate Portfolio Beta, with LMV Utility:**

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_E + \lambda_{SE} \alpha_c /M = $</td>
<td>1.691</td>
<td>1.413</td>
</tr>
<tr>
<td><strong>Optimal Beta for Portfolio #1, with LMV Utility:</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$N_{SE} + \lambda_{S1} b_{1}\alpha_1 / M = $</td>
<td>2.030</td>
<td>1.615</td>
</tr>
<tr>
<td><strong>Optimal Beta for Portfolio #2, with LMV Utility:</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$N_{SE} + \lambda_{S2} b_{2}\alpha_2 / M = $</td>
<td>1.520</td>
<td>1.312</td>
</tr>
</tbody>
</table>

An illustrative example of this sort generally helps to clarify the nature of the model. Moreover, it often helps to highlight instances where one's intuition may lead one astray. There are two such instances in the present application.

First, we are accustomed to thinking of the "best estimate" obtained from a number of sources as being an average of the separate sources. In the case of market alphas from different sources, however, the best combined market alpha is nearer to a sum than an average. Indeed, when the two managers are independent sources (case A), the best prediction is the sum of their predictions $\alpha_c = \alpha_1 + \alpha_2 = 2\% + 2\% = 4\%$. When the two managers are highly correlated, in case B, the best prediction is more conservative, but is still greater than either of the separate predictions $\alpha_c = b_1\alpha_1 + b_2\alpha_2 = (3/5) 2\% + (3/5) 2\% = 2.4\%$; and in case C, where the two managers' information is negatively correlated, the best prediction is a startling $\alpha_c = (3) 2\% + (3) 2\% = 12\%$. 

Speaking heuristically, the reason for this is that each "market alpha" has already been scaled down to reflect the imperfection of the information contained in the forecast. After being scaled down, the alpha is to be treated as valid information, for which no further adjustment is necessary. When two independent predictions are obtained, each being a valid bit of information, then the best prediction is the sum of the two. When two positively correlated items are obtained, then this is equivalent to having part of the valid information in one duplicated in the other. This redundant information should not be double-counted; the prediction becomes further conservative because it is not possible, from the reported predictions, to determine what portion of each is being double-counted and what portion is in fact from distinct sources. The resulting prediction is less than the sum of the separate predictions, but greater than their average.

Finally, when the alphas are negatively correlated, it is implied that each must contain a greater degree of valid information, partially offset by common information that is being misused by one or the other and that therefore causes the negative correlation. Hence, when we have access to both predictions, a sum of the two will tend to cancel out the misused information, leaving the great bulk of the valid information, so the best prediction is a multiple of sum and is much superior to either prediction taken singly.

The second topic on which intuition tends to be misleading is the usefulness of the market forecasting process and the strength of the response in optimal beta. Despite small coefficients of determination (.015, .009, and .045 in the three cases), the added information makes a substantial expected contribution to utility (with certainty equivalent excess return of .55, .33, and 1.71 percent per annum, respectively). The forecasts of excess market return (4%, 2.4%, and 12%) lead to substantial increases in beta (1.69, 1.41, and 3.14). At first sight, from the perspective of common practice in institutional portfolios, these are shockingly large shifts.

However, recall that transaction costs are not charged, that high beta may be obtained by leveraging with borrowing at the risk-free rate,
and that the 12% is a true expectation of extraordinary market return, an "alpha," not a raw forecast before it is scaled to convert it to valid information. In the case of the 12% forecast, mean excess return for the market is 18% (12% + 6%), in addition to the risk-free return, and the standard deviation of excess return is about 19.5% (slightly reduced from 20% by the information in the forecasting process). In practice, the increase in beta could not ordinarily be obtained by borrowing, but might be attainable, at least in part by a combination of two strategies: first, an increase in the beta of the equity portfolio, by purchase of higher-beta stocks; second, transfer of investments from cash and bonds into equity.

A.2.6. The Approximation That Explained Variance Negligibly Reduces Market Variance

The sponsor should utilize the unexplained variance \( M = \sigma^2 (1 - \rho_c^2) \) as the measure of systematic risk, not the total variance \( \sigma^2 \). It is common practice--and convenient--to ignore this reduction, assuming \( M = \sigma^2 \). This concluding subsection considers the impact of that approximation. The reader can skip this subsection and go to section 3 without loss of continuity.

When the approximation is used, market variance is set to an excessively high value, portfolio beta is consequently set too low, and, as a result, the portfolio utility is lower than it otherwise would be. This is a real cost to the sponsor. Further, since portfolio properties are computed using exaggerated market variance, the portfolio variance prediction is upward biased, and the resulting computation of utility is downward biased. These are errors which do not affect portfolio properties but rather cause confusion between expected and real properties. The following theorem displays these effects.

**Theorem 5.** The Approximation \( M = \sigma^2 \)

When \( M \) is set to \( \sigma^2 \), the resulting "approximately optimal beta," denoted by \( \beta^* \), is related to the optimal beta, \( \beta^0 \), by:
\[ (T5.1) \quad \hat{\beta} = \lambda_{ES} \frac{(E+\alpha_c)}{\sigma^2} = \beta^0 \left( \frac{M}{\sigma^2} \right) = \beta^0 (1 - \rho_c^2). \]

The resulting mean and variance of portfolio systematic return are:

\[ (T5.2) \quad \mu^*_AS = \lambda_{ES} \frac{(E+\alpha_c)^2}{\sigma^2} = \mu_{AS} (1 - \rho_c^2); \quad \sigma^2_{AS} = \lambda_{ES} \frac{(E+\alpha_c)^2}{\sigma^2} (1 - \rho_c^2)^2 = \sigma^2_{AS} (1 - \rho_c^2)^2. \]

The CEER of the portfolio, denoted by \( U_{ES}^4 \), is:

\[ (T5.3) \quad U_{AS}^4 = U_{AS} (1 - \rho_c^4). \]

Moreover, if characteristics of this portfolio are computed under the erroneous assumption that \( M = \sigma^2 \), the computed values, indicated by a super tilde, are related to the correct values, by:

\[ (T5.4) \quad \tilde{\beta} = \hat{\beta} \]

\[ (T5.5) \quad \tilde{\mu}_{AS} = \mu^*_AS; \quad \tilde{\sigma}^2_{AS} = \frac{1}{(1 - \rho_c^2)^2} \sigma^2_{AS} \]

\[ (T5.6) \quad \tilde{U}_{AS} = \frac{1}{1 + \rho_c^2} U^*_AS. \]

**Proof:** (T5.1) is obtained by substitution of the assumed market mean and variance into (T2.1). (T5.2) then follows directly from (T2.3). Substitution of these formulas into the utility function yields (T2.3):

\[ U^*_AS = \mu^*_AS - \frac{1}{2\lambda_{ES}} \sigma^2_{AS} \]

\[ = \lambda_{ES} \frac{(E+\alpha_c)^2}{\sigma^2} - \frac{1}{2\lambda_{ES}} \left( \frac{\lambda_{ES} (E+\alpha_c)^2 (1 - \rho_c^2)}{\sigma^2} \right) = \lambda_{ES} \frac{(E+\alpha_c)^2}{\sigma^2} \left( 1 - \frac{(1 - \rho_c^2)^2}{2} \right), \]

\[ (2.27) \quad = \frac{\lambda_{ES} (E+\alpha_c)^2}{2\sigma^2} (1 + \rho_c^2) = \left( \frac{\lambda_{ES} (E+\alpha_c)^2}{2\sigma^2 (1 - \rho_c^2)} \right) (1 + \rho_c^2) (1 - \rho_c^2) = U_{AS} (1 - \rho_c^2)^4. \]
(T5.4) and (T5.5) follow because the approximation for $\sigma^2$ does not affect computation of $\beta$ or $\mu_{AS}$, and increases $\sigma^2_{AS}$ by the factor $(b/1-\rho_c^2)$. (T5.6) then follows, since:

$$
\tilde{u}_{AS} - u^*_{AS} = -\frac{1}{2\lambda_{ES}}(\sigma^2_{AS} - \sigma^2_{AS}^*) = -\frac{1}{2\lambda_{ES}} \frac{(E+\alpha_c)}{\sigma^2} (1 - (1-\rho_c^2)) = -\rho_c^2 \frac{u^*_{AS}/(1+\rho_c^2)}
$$

(2.28) $\implies \tilde{u}_{AS} = u^*_{AS}(1 - \rho_c^2/(1+\rho_c^2)) = u^*_{AS}/(1+\rho_c^2)$.

Table A-2 shows the effects of the approximation for a range of values of $\rho_c$. It is clear that, for $\rho_c < .2$, the effect of the approximation is negligible. (For $\rho_c = .2$, the CEER decreases by only .16 of 1 percent of the optimal value, and the percentage computational error is 4%.)

**TABLE A-2**

PERCENTAGE CHANGES IN PORTFOLIO CHARACTERISTICS AND PERCENTAGE COMPUTATIONAL ERRORS DUE TO THE APPROXIMATION

<table>
<thead>
<tr>
<th>Value of $\rho_c$</th>
<th>Percentage Changes</th>
<th>CEER</th>
<th>Percentage Computational Errors</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\beta_{AS}$, $\mu_{AS}$, and $\sigma^2_{AS}$</td>
<td>CEER</td>
<td>$\sigma^2_{AS}$</td>
</tr>
<tr>
<td>.1</td>
<td>-1.</td>
<td>-.01</td>
<td>+1.</td>
</tr>
<tr>
<td>.2</td>
<td>-4.</td>
<td>-.16</td>
<td>+4.</td>
</tr>
<tr>
<td>.5</td>
<td>-25.</td>
<td>-6.25</td>
<td>+33.</td>
</tr>
<tr>
<td>.7</td>
<td>-49.</td>
<td>-24.01</td>
<td>+98.</td>
</tr>
</tbody>
</table>

However, for larger values of $\rho_c$ the losses become sizable, and as $\rho_c$ approaches .5, it is clearly necessary to carry forward the true mean square forecast error rather than the market variance. Forecasting processes with
\[ \rho_c \geq 0.3 \] are probably rare; hence, the approximation is generally defensible.

### A.3. Forecasts of Unsystematic Returns and Optimal Active Holdings

This section considers the forecasting of unsystematic returns and the associated problem of optimal active investments. The problem is very similar to that considered in section 2. A vector \( \alpha \) forecast for the vector of unsystematic returns on securities, \( \mathbf{u} \), replaces the forecast \( \alpha_M \) for the unexpected market return \( u_M \). The problem is now in \( N \) dimensions rather than one, but the formalism of matrix algebra allows the solution to be presented in a way that closely parallels section 2.

The subsections deal with the following topics: A.3.1., The Optimal Combination of Managers' Appraisals; A.3.2., The Special Case of Constant Proportional Explanation of Variance; A.3.3., The Special Case of Nil Intermanager Correlation; A.3.4., Optimal Aggregate Investments; A.3.5., Decentralization Procedure; A.3.6., Utility Contributions of the Managers; A.3.7., Portfolio Covariances of Unsystematic Return; A.3.8, Apportioning Funds and Management Fees; and A.3.9., The Extension to Distinguish Specific Returns from Common Factors of Return.

#### A.3.1. The Optimal Combination of Managers' Appraisals

Suppose that there are \( N \) assets for which abnormal returns are forecast. The forecast for each asset might arise from a predicted event, specific to that company, in which case it would be a forecast for specific return. Or it might arise from a forecast of an economic event to which that company, among others, is exposed, in which case it would be due to a common factor in security returns. Each of the \( J \) managers prepares one forecast for every one of the \( N \) assets. These forecasts are then transformed by the manager into a set of judgmental alphas, as explained in ([1], sections A.7 and A.8). Let \( \alpha_j \) be the vector of judgmental alphas for manager \( j \).

Each manager's appraisal vector has two important properties. The first is "market neutrality," a term due to Treynor-Black [6], which
requires that the market portfolio exhibits zero abnormal return, relative to itself. In mathematical terms, this is the condition that:

\[ \sum_{n=1}^{N} h_{mn} \alpha_n = h_{m}^\prime \alpha_j = 0 \quad j = 1, \ldots, J. \]

The second important property is that forecast variance equals covariance with actual future returns.

The judgmental alpha vector is characterized by a variance matrix \( \Sigma_j \). Each entry in that matrix—for example, the entry in row \( m \) and column \( n \)—is equal to the covariance between the judgmental alpha for item \( m \) and the actual return for item \( n \); it is also equal to the covariance between the judgmental alpha for item \( m \) and the judgmental alpha for item \( n \).

If there were a single manager, this covariance matrix would be the only essential measure of information content for the forecasts. When there are multiple managers, another essential property is the covariance between the information content of any one manager's forecasts and that of other managers. For each pair of managers, \( i \) and \( j \), these covariances can be arranged in a square matrix, \( \Sigma_{ij} \), such that the entry in the \( m \)th row, \( n \)th column is the covariance between the judgmental alpha of manager \( i \) for asset \( m \) and the judgmental alpha of manager \( j \) for item \( n \).

\[ \text{In the notation of ([1], A.7 and A.8), where the matrix "C" had a different meaning, the raw forecasts of the \( J \) managers, \( \hat{u}_j \), \( j = 1, \ldots, J \), would have first and second moments \( E(\hat{u}_j) = 0; \quad \text{VAR}(\hat{u}_j) = \Sigma_j; \quad \text{COV}(\hat{u}_j, \hat{u}_j) = \Sigma_{jj} \); \quad i = 1, \ldots, J; \quad j = 1, \ldots, J. \) Then each manager would transform his forecast vector by the matrix \( C_{ij} \), resulting in \( \alpha_j = C_{ij} \hat{u}_j \); \( \Sigma_j = C_{ij} \Sigma_{ij} \); \( \Sigma_{ij} = C_{ij} \Sigma_{ij} \).} \]
Suppose that the sponsor has in hand the alphas of the \( J \) managers and knows the information properties of those forecasts, as given by the covariance matrices. His problem is to construct the optimum combination of these forecasts, that which provides a best prediction for the returns on the assets. In this section, the combined forecast is exhibited for the general case, where the covariance matrices have no simplifying properties. This is easily done through the formalism of matrix algebra. Then, in the next subsections, simplifying assumptions turn the general solution into an easily comprehended form.

**Theorem 6. Best Combined Alphas; General Solution**

The joint distribution of predictions and returns has mean and variance:

\[
E \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_J \\ u \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}
\]

_(T6.1)_

\[
VAR \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_J \\ u \end{pmatrix} = \begin{pmatrix} A_1: A_{12} : \ldots : A_{1J} : A_1 \\ A_{21}: A_2 : \ldots : A_{2J} : A_2 \\ \vdots \\ A_{J1}: A_{J2} : \ldots : A_{J} : A_J \\ A_1: A_2 : \ldots : A_J : R \end{pmatrix} \equiv \begin{pmatrix} \zeta \mid \zeta' \end{pmatrix},
\]

The right-hand expression for the variance matrix is a partition, where \( A \) denotes the square matrix made up of the \( J \times J \) array of variance and covariance matrices of the forecasts, and \( \zeta \) is the column of \( J \) variance
matrices. Then the minimum mean square error prediction for returns based upon all information, say, $\mathbf{z}$, its variance matrix $\mathbf{A}$, and its mean square error matrix $\mathbf{M}$ are:

**Best Combined Forecast:**

\[
\mathbf{z} = \mathbf{L} \cdot \mathbf{L}^{-1} \mathbf{z}
\]

**Variance of Forecast Equals Explained Variance:**

\[
\mathbf{A} = \mathbf{L} \cdot \mathbf{L}^{-1} \mathbf{L}
\]

**Mean Square Error Or Unexplained Variance:**

\[
\mathbf{M} = \mathbf{R} - \mathbf{A}
\]

If all forecasts and returns are jointly normally distributed, $\mathbf{z}$ is the conditional mean and $\mathbf{M}$ the conditional variance for $\mathbf{u}$, conditional upon the values of $\mathbf{z}_1, \ldots, \mathbf{z}_J$.

**Proof:** The mean and variance of the joint distribution of predictions and returns follow from the previously mentioned properties of the alphas. The best prediction is the straightforward application of Lemma 1.

**A.3.2. The Best Combined Prediction When Constant Proportions of Variance Are Explained**

Suppose that each manager is unable to predict greater proportions of variance of return for some items than for others, so that a constant proportion of variance is explained. Then it is reasonable to assume the same constant proportional explanation of covariance between returns for different items. In this case, the covariance matrix $\mathbf{A}_j$ for each manager simplifies to $\mathbf{A}_j = \rho_j^2 \mathbf{R}$, where $\rho_j^2$ is the proportion of variance explained.
by that manager. It is also reasonable to assume that the covariance between the information of different managers will be a constant proportion of variance, so that for each \( i \) and \( j \), \( A_{ij} = c_{ij} R \). This simplified context is "Constant Proportional Explanation" (CPE).

The information structure is captured by the covariance matrix \( \Sigma \), the diagonal entries of which are \( \rho_j^2 \), \( j = 1, \ldots, J \).

Let \( q = (\rho_1^2; \ldots; \rho_J^2)' \) denote the column vector of squared correlation coefficients.

Under the CPE assumption, for every possible pair of assets \( m \) and \( n \) (possibly equal to \( m \)):

- \( \rho_i^2 \) is the proportion of unsystematic return variance for asset \( n \) explained by the judgmental alpha of manager \( i \).
- \( \rho_i \) is the correlation between forecast and actual return.
- \( \eta_{ij} = \frac{c_{ij}}{\rho_i \rho_j} \) is the correlation between the \( i \)th manager's alpha for asset \( n \), and the \( j \)th manager's alpha for asset \( m \).

Define \( \tilde{\Sigma} \), as in A.2, as the matrix of intermanager correlations. A number of constructs have now been defined, which are formally identical to those in A.2. Of course, these relate to the forecasting process for unsystematic returns, and correlations for this process are probably different than for the market forecast. It would be possible to underscore this point by assigning a distinguishing subscript, but to simplify the notation, none will be used. No confusion will result as long as it is remembered that all properties are newly defined in this section.

The matrices of all covariances can be written, using the Kronecker product of matrices,\(^1\)

\[
\mathcal{A} = \mathcal{C} \otimes \mathcal{R} \quad \mathcal{Q} = q \otimes \tilde{\Sigma}.
\]

\(^1\) See most texts on matrix algebra or econometrics (e.g., Goldberger [7], p. 15).
It is now possible to derive the optimal prediction vector $\alpha_c$ in precisely the same format as Theorem 1, above.

**Theorem 7. Best Combined Prediction of Unsystematic Returns, in the CPE Case**

**Optimal Combined Prediction:**

\[
\alpha_c = \sum_{j=1}^{J} b_j \tilde{\alpha}_j,
\]

where the DAFs are:

\[
b_j = \frac{\sum_{i=1}^{J} \pi_{ij} \rho_i}{\rho_j}, \quad j = 1, \ldots, J.
\]

**Explained Variance of Combined Prediction:**

\[
\Lambda_c = \text{VAR}(\alpha_c) = \rho_c^2 \tilde{\sigma}_c^2, \text{ where } \rho_c^2 = \sum_{j=1}^{J} b_j \rho_j^2.
\]

**Mean Square Forecast Error:**

\[
\tilde{M} = \text{MSE}(\alpha_c) = \text{VAR}(\tilde{u} - \alpha_c) = (1 - \rho_c^2) \tilde{\sigma}_c.
\]

**Correlation of Combined Prediction with Unsystematic Returns:**

\[
\text{CORR}(\alpha_{cm}, \tilde{u}_n) = \rho_c \text{ CORR}(u_m, \tilde{u}_n) \text{ for all } m, n.
\]

**Proof:** By the formula for the inverse of a Kronecker product,

\[
\tilde{\alpha}^{-1} = (\tilde{c} \otimes \tilde{R})^{-1} = \tilde{c}^{-1} \otimes \tilde{R}^{-1}. \text{ When this formula is substituted into (T6.2) and (T6.3),}
\]

\[
\alpha_c = (q' \otimes \tilde{R})(\tilde{c}^{-1} \otimes \tilde{R}^{-1})(\tilde{\alpha}_1 \ldots \tilde{\alpha}_J) = (b_1' \otimes \tilde{I})(\tilde{\alpha}_1 \ldots \tilde{\alpha}_j) = \sum_{j=1}^{J} b_j \tilde{\alpha}_j
\]

\[
\tilde{M} = \tilde{R} - (q' \otimes \tilde{R})(\tilde{c}^{-1} \otimes \tilde{R}^{-1})(q \otimes \tilde{R}) = \tilde{R} - (q' \tilde{c}^{-1} q) \tilde{R}.
\]
With these equalities, resulting from the simple structure of $\mathbf{A}^{-1}$, the proofs of (T7.1) to (T7.4) follow by the same reasoning as the corresponding statements in Theorem 1. Then, for any assets $m$ and $n$,

$$\text{CORR}(\alpha^c_{cm}, \alpha^c_{cn}) = \rho^c_{mm} \cdot \sqrt{\rho^c_{nn} \cdot \rho^c_{mm}} = \rho^c_{mm} \cdot \sqrt{\rho^c_{nn} \cdot \rho^c_{mm}} = \rho^c \cdot \text{CORR}(\alpha^c_m, \alpha^c_n),$$

which gives (T7.5).

---

### A.3.3. The Special Case of Nil Intermanager Correlation

Suppose that the $J$ managers determine unsystematic return forecasts from independent information, or in distinct ways, or over mutually exclusive subsets of the universe of assets. Then it may be a reasonable approximation to assume that the alphas reflect uncorrelated information. Mathematically, this amounts to the assumption that $\alpha^c_{ij} = 0$ for all $i \neq j$.

Then the optimal combined prediction simplifies greatly, as stated below.

**Theorem 8. Optimal Combined Prediction When Alphas Are Uncorrelated**

**Optimal Combined Prediction is the Sum of the Alphas:**

(T8.1)

$$\alpha^c = \alpha^c_1 + \ldots + \alpha^c_J.$$

**Variance of Combined Prediction Is Sum of Variances:**

(T8.2)

$$A^c = A^c_1 + \ldots + A^c_J.$$

**Unsystematic Variance Decomposes to Mean Square Prediction Error Plus Explained Variance:**

(T8.3)

$$\sigma^2 = \mu^2 + A^c.$$

**Proof:** Since the off-diagonal matrices $A^c_{ij}$ equal zero,

$$A^{-1} = (\text{diag}(A^-))^{-1} = \text{diag}(A^-).$$

When this expression is substituted into (T8.2) to (T8.4), $\mathbf{A}^{-1} \mathbf{A}$ and $\mathbf{A}^c$ are found to cancel out, and the simplification results.
A.3.4. Optimal Aggregate Investments

To obtain the optimal aggregate active holdings, the forecast \( \bar{\alpha} \) and unexplained variance \( \bar{\nu} \) need only be substituted into \( (1, \text{Theorem 5}) \). As explained in \( (1, \text{A.4}) \), the matrices \( \bar{\nu} \) and \( \bar{\nu}' \) are singular and possess no inverses. Since natural approximations to these matrices do possess inverses, the usual notation \( (\bar{\nu}^{-1}, \bar{\nu}'^{-1}) \) will be used, except where the singularity is important, in which case exact generalized inverses \( (\bar{\nu}^+, \bar{\nu}'^+) \) will be employed.

**Theorem 9. Optimal Active Holdings**

In the absence of alphas for unsystematic returns, optimal active holdings are zero. In the case of a general multiple-manager appraisal process, the optimal portfolio active holdings are:

\[
\delta_A = \frac{\lambda}{\kappa} \bar{\nu}^{-1} \bar{\alpha}_c = \frac{\lambda}{\kappa} \bar{\nu}^{-1} \left( \begin{array}{c} \alpha_1 \\ \vdots \\ \alpha_J \end{array} \right),
\]

resulting in portfolio unsystematic appraisal premium and unsystematic variance

\[
\alpha_A = \frac{\lambda}{\kappa} \bar{\nu}^{-1} \bar{\alpha}_c ; \quad \omega_A^2 = \left( \frac{\lambda}{\kappa} \right)^2 \bar{\nu}^{-1} \bar{\nu}'^{-1} \left( \begin{array}{c} \alpha_1 \\ \vdots \\ \alpha_J \end{array} \right),
\]

and CEER of unsystematic return, with linear MV utility,

\[
U_{AU} = \frac{1}{2} \alpha_A,
\]

where the Sharpe Ratio of unsystematic return is:

\[
\omega_A^2 = \left( \begin{array}{c} \alpha_1 \\ \vdots \\ \alpha_J \end{array} \right) \bar{\nu}^{-1} \bar{\alpha}_c = \left( \begin{array}{c} \alpha_1 \\ \vdots \\ \alpha_J \end{array} \right) \bar{\nu}^{-1} \bar{\nu}'^{-1} \left( \begin{array}{c} \alpha_1 \\ \vdots \\ \alpha_J \end{array} \right).
\]

The complex matrix expressions in formulas 1 and 4 simplify greatly in the two special cases. In the Constant Proportional Explanation Case:

\[
\delta_A = \frac{\lambda}{\kappa} \left( 1 - \sum_{j=1}^{J} b_j \alpha_{j+} \right) \bar{\nu}^{-1} \left( \sum_{j=1}^{J} b_j \alpha_{j+} \right)
\]

\[
\omega_A^2 = \left( \sum_{j=1}^{J} b_j \alpha_{j+} \right) \bar{\nu}^{-1} \left( \sum_{j=1}^{J} b_j \alpha_{j+} \right) \left( 1 - \sum_{j=1}^{J} b_j \alpha_{j+} \right),
\]
and in the zero correlation case:

\[ (T9.1B) \quad \delta_A = \frac{\lambda}{k} \left( R - \sum_{j=1}^{J} \alpha_j \right)^{-1} \left( \sum_{j=1}^{J} \alpha_j \right) \]

\[ (T9.4B) \quad \varepsilon_{AU}^2 = \left( \sum_{j=1}^{J} \alpha_j \right)' \left( R - \sum_{j=1}^{J} A_{-j} \right)^{-1} \left( \sum_{j=1}^{J} \alpha_j \right) . \]

Finally, suppose that the approximation \( M = \tilde{R} \) is made, so that the reduction in unsystematic variance due to the explained variance of the predictions is ignored. For the case of zero correlation, define \( b_1 = b_2 \ldots = b_J = 1 \), as is appropriate when forecasts are uncorrelated. Then in both cases (CPE and the Zero Correlation), the formulas simplify to:

\[ (T9.1C) \quad \delta_A = \left( \frac{\lambda}{k} \right) R^{-1} \left( \sum_{j=1}^{J} b_j \alpha_j \right) \]

\[ (T9.4C) \quad \varepsilon_{AU}^2 = \left( \sum_{j=1}^{J} b_j \alpha_j \right)' \left( R^{-1} \right) \left( \sum_{j=1}^{J} b_j \alpha_j \right) . \]

**Proof:** Expressions (T9.1) to (T9.4) come from substituting the formulas for \( \alpha_c \) and \( M \) from Theorem 6 in ([1], Theorem 5). The formulas in cases A and B result by substitution of the simplified \( \alpha_c \) and \( M \) from Theorems 7 and 8.

The approximation \( M = \tilde{R} \) implies \( \sum b_j \rho_j = 0 \) in case A, and \( \sum A_{-j} = 0 \) in case B. These substitutions result in case C.

The approximate formulas in case C are convenient, since portfolio properties depend only on the weighted sum, \( \sum b_j \alpha_j \). The approximation \( M = \tilde{R} \) removes the adjustment to the variance matrix. Fortunately, that approximation is an accurate one in this case and can be used with confidence. As shown in ([1]: Appendix, Section 9), it is probable that \( \rho_{c}^2 \leq 0.03 \) or \( \rho_{c} \leq 0.175 \). This follows because the opportunity to diversify across assets
allows highly significant performance for small $\rho_c^2$. For $N$ assets, the contribution to utility is $N$ times as great for a given $\rho_c$ as would be obtained from the same $\rho_c$ for market return. Knowledge of historical performance of managed portfolios, which allows us to place a bound on realistic performance, hence establishes a low bound on explained unsystematic variance.

Consequently, the approximation $\bar{N} = \bar{R}$ will be employed hereafter, thereby simplifying the formulas. The reader can carry through the exact formulas on his own, following the same structure as in section A.2, where exact formulas were maintained until A.2.6, which presented the approximation. In dealing with market forecasting, it was appropriate to retain the exact formulas, since an upper bound on $\rho$ cannot be asserted with certainty.

It is convenient at this point to insert a corollary concerning the effect of an added manager on portfolio properties.

**Corollary 9.1. Effect of Adding Another Superior Manager**

Suppose that an initial group of managers is supplemented by an additional manager. If the addition results in increased utility to the sponsor, then the new optimal portfolio possesses higher residual variance.

**Proof:** The most direct proof comes from comparison of T9.3 and T9.2. Since utility is proportional to the optimal appraisal premium $U_{AUP} = \alpha_A$, and optimal residual variance is proportional to the appraisal premium $\bar{U}_A^2 = \bar{\alpha}_A$, it follows that optimal variance is proportional to utility. Hence, utility increases only when optimal residual risk rises.

**A3.5. Decentralized Control of Differential Holdings**

The outcome of A3.4 is that, in two special cases, CPE and zero correlation, the optimal active holdings depend only on the weighted sum of the alpha vectors $\sum_{j=1}^{J} b_j \alpha_j$, weighted by the "dependence adjustment factors" $b_j$. This optimal portfolio could be achieved if the information of the distinct managers was brought to the sponsor, centrally combined by him, and the resulting portfolio was constructed by him.
The more realistic case is one where the individual managers each maintain a separate part of the equity pool under their management. Managers optimize their own pools of funds, acting under instructions from the sponsor. Each manager has access only to the $\alpha_j$ generated by himself, not those of other managers. A crucial question, therefore, is whether the same optimal strategy could be obtained through decentralized management. Is it possible for the sum total of the activities of the $J$ managers to produce the optimal combined policy? The following theorem states the straightforward procedure that accomplishes this goal.

**Theorem 10. Optimal Decentralized Control of Active Holdings**

Let each manager be informed of his proportional weight in the total equity portfolio, $W_j$, the sponsor's RAP for unsystematic risk, $(\lambda/\kappa)$, and the DAF, $b_j$, for the manager's appraisals. Each manager is instructed to adjust his appraisal premia by the multiplicative factor $b_j$, resulting in the "dependence adjusted alpha" $\alpha^*_j = b_j \alpha_j$. He is further instructed to apply a risk acceptance parameter divided by the investment proportion, $\lambda_j^* = (\lambda/(W_j \kappa))$. Each manager optimizes his active holdings separately, using the adjusted RAP with respect to his alpha. The consequence for each manager is:

\[ \delta_j = \left( \frac{\lambda}{W_j \kappa} \right)^{1/2} b_j \alpha_j \quad j = 1, \ldots, J \]

with the result that the aggregate portfolio is optimal

\[ \delta_A = \sum_{j=1}^J W_j \delta_j = \left( \frac{\lambda}{\kappa} \right)^{1/2} \left( \sum_{j=1}^J b_j \alpha_j \right) \]

**Proof:** From ([1], Theorem 5.3), (T10.1) is the optimal active holding under the stated conditions, $\delta_j = \lambda_j^* b_j^{-1} \alpha_j^*$. The overall active holding, $\delta_A$, which equals the investment-weighted sum of the $\delta_j$ by (L2.5) is the expression in (T10.2), equal to the approximate optimal investment in (T10.1C). The exact optimum could be attained by communicating $\delta_A$. 
rather than \( R \), to the individual managers, but the refinement is not necessary.

A manager's active holdings are changed, relative to a portfolio under his single management, in two ways: scaling upward by the multiplicative factor \( (1/w_j) \), and scaling by the DAF, \( b_j \). The decentralization principle is identical to that for market forecasting, and the discussion concluding section A.2.3 applies equally here.

**Theorem II: Required Appraisal Premia for Aggregate and Constituent Portfolios**

For the MV optimal aggregate portfolio,

\[
\alpha_A = \left( \frac{\lambda}{\kappa} \right) \omega_A^2.
\]

The formula gives the "required alpha" or the appraisal premium required for optimality of a given unsystematic variance. Also, let \( \tilde{k}_A \) denote the vector of unsystematic covariance between aggregate portfolio return and individual asset returns:

\[
\tilde{k}_A = R \tilde{\omega}_A.
\]

Then, for every asset \( n \), the optimality condition for required alpha is:

\[
\alpha_{on} = \left( \frac{\kappa}{\lambda} \right) k_{on} \quad n = 1, \ldots, N.
\]

Next consider an individual manager's portfolio. Let \( \tilde{k}_j = R \tilde{\omega}_j \) be the unsystematic covariance vector for that portfolio. Then, the optimality conditions for portfolio \( j \) is:

\[
\alpha_j = \left( \frac{\lambda}{\kappa} \right) \left( \frac{w_j}{b_j} \right) \omega_j^2.
\]

Required alphas for individual holdings in portfolio \( j \) are:

\[
\alpha_{jn} = \left( \frac{\kappa}{\lambda} \right) \left( \frac{w_j}{b_j} \right) k_{jn}, \quad n = 1, \ldots, N.
\]
Proof: \[ \text{COV}(u^*_A, u^*_n) = \sum_{m=1}^{N} A_m \text{COV}(u^*_m, u^*_n), \] which is the \(n\)th entry in the column vector \( k = R \delta_A \). This proves (T11.2).

From ([1], Theorem 5.2), the MV optimal portfolio is characterized by the vector equation:

\[ \begin{pmatrix} \kappa \\ \lambda \end{pmatrix} \tilde{\delta}_A = \alpha_c. \]

This proves (T11.3). Premultiplying by \( \delta'_A \),

\[ \begin{pmatrix} \kappa \\ \lambda \end{pmatrix} \delta'_A R \delta_A = \delta'_A \alpha. \]

Since \( \omega^2 = \delta'_A R \delta_A \) and \( \alpha = \delta'_A \alpha \), this verifies (T11.1), which could also have been taken directly from ([1], Theorem 5.6).

From (T10.1), the MV optimal component portfolio for manager \( j \) is

\[ \delta_j = \frac{\lambda b_j}{\kappa W_j} R^{-1} b_j \alpha_j. \]

Premultiplying by \( R \),

\[ k_j = R \delta = \frac{\lambda b_j}{\kappa W_j} \alpha_j, \]

which is (T11.5). (T11.4) is obtained by premultiplying by \( \delta'_A \).

\[ \text{A.3.6. Utility Contributions of Managers' Security Appraisals} \]

From Lemma 2, the security appraisal process results in an increment to the portfolio's Sharpe Ratio, which is a general measure of portfolio goodness and which, in the case of linear MV utility, results in a specific increase in the CEER of the portfolio. What is important here is to compute the contributions of the individual managers to the aggregate portfolio Sharpe Ratio. The exact contributions, at any point in time, depend on the particular alpha which has been obtained at that time. The expected value of this contribution, which can be expected to apply on average, can be computed from the information matrices.

For example, in the case of zero correlation between managers, it is easy to show that the expected aggregate Sharpe Ratio from unsystematic
return is the sum of the expected Sharpe Ratios of the individual managers. The expected Sharpe Ratio for each manager depends, in turn, on his information matrix $A_j$, in a relatively complex fashion.\footnote{Treynor-Black (6) and Ferguson (8) have worked this out in detail for the case of independent specific returns. In general, with a full covariance model, the expectation is $E(z_{jU}^2) = \text{TRACE}(R_j^\dagger A_j)$.} The CPE assumption, in contrast, results in greatly simplified formulas which can be naturally implemented by the sponsor.

\textbf{Theorem 18. Expected Utility Contributions of Security Appraisals:}
\textbf{The CPE Case}

Assume that the CPE assumption holds, and that $R$ may be approximated by $\tilde R$. Then the expected portfolio Sharpe Ratio from unsystematic return is:

\begin{equation}
E(z_{AU}^2) = (N-1) \rho_a^2 = (N-1) \sum_{j=1}^J b_j \rho_j^2.
\end{equation}

With an EMV utility function, the expected incremental CEER due to unsystematic return is:

\begin{equation}
E(U_{AU}) = \frac{1}{2} \left( \frac{\lambda}{\kappa} \right) (N-1) \sum_{j=1}^J b_j \rho_j^2.
\end{equation}

Proof: The Sharpe Ratio for unsystematic return in the portfolio is $z_{AU}^2 = \alpha_c^\dagger R_c^\dagger \alpha_c$, from ([1], T5.4), where $R_c$ has been substituted as an approximation for $M$. The expectation is evaluated by a matrix manipulation, familiar to statisticians, using the trace (sum of diagonal entries) of a matrix:

\begin{equation}
E[z_{AU}^2] = E[\alpha_c^\dagger R_c^\dagger \alpha_c] = E(\text{TRACE}(R_c^\dagger \alpha_c \alpha_c)) = \text{TRACE}(R_c^\dagger E(\alpha_c \alpha_c'))
= \text{TRACE}\left( R_c^\dagger \left( \sum_{i=1}^J \sum_{j=1}^J b_{ij} \alpha_{ij} \right) \left( \sum_{i=1}^J \sum_{j=1}^J b_{ij} \alpha_{ij}' \right) \right) = \text{TRACE}\left( R_c^\dagger \sum_{i=1}^J \sum_{j=1}^J b_{ij} b_{ij} A_{ij} \right)
\end{equation}
\[ = \text{TRACE} \left( \sum_{i=1}^{J} \sum_{j=1}^{J} b_i b_j c_{ij} R \right) = \left( \sum_{i=1}^{J} \sum_{j=1}^{J} b_i b_j c_{ij} \right) \text{TRACE}(R) \]

\( (3.6) = (N-1) \frac{J}{J} \sum_{i=1}^{J} \sum_{j=1}^{J} b_i b_j c_{ij} = (N-1)b'Cb' \)

\( \text{TRACE}(R^\dagger R) = N - 1 \) because \( \tilde{R} \) is the idempotent projection matrix onto the \( (N-1) \) dimensional space of all assets which are available for appraisal premia. (One dimension, that of the market, has a zero alpha, by definition, and is therefore unavailable.) But \( b'Cb = b'C(C^{-1}q) = b'q = \frac{J}{J} \sum_{j=1}^{J} b_j^2 = \rho_c^2 \). This completes the derivation of (T12.1). (T12.2), CEER for an LNV utility function follows from ([1], T5.7).

Thus, in the CPE case, the Sharpe Ratio equals the dependence-adjusted sum of the coefficients of determination of the individual managers. It is natural to consider each manager's contribution as the dependence-adjusted term \( b_j \rho_j^2 \). However, the entire discussion concerning the three alternative measures of contribution (single, dependence-adjusted, and incremental) presented in section A.2.4 applies equally here. If, for some reason, incremental contributions are to be used, these can be computed as functions of the correlation matrix \( \rho \) and the vector \( q \) of coefficients of determination, just as in section A.2.4.

A.3.7. Useful Properties of Portfolio Covariances of Unsystematic Return

The sponsor's assessment of the correlation among managers' appraisal processes will necessarily rely heavily on the degree to which the actual appraisal premia generated by those processes tend to covary. Observation of the pattern of correlation over time is one major source of such information—in the case of market forecasts, the only source. But in the case of security appraisals, the correlation across \( N \) different assets is available at each moment in time, and suffices for a current point estimate of the correlation between appraisals. Furthermore, if the CPE
assumption applies, and if the portfolios are optimally constructed, there is no need to inspect the alphas themselves. The covariance between unsystematic returns provides a sufficient statistic. This section develops the relevant properties of the variances and covariances of unsystematic returns.

**Theorem 13. Properties of Unsystematic Covariances of Returns**

For each component portfolio, $j$, the variance of unsystematic returns is:

(T13.1) \[ \omega_j^2 = \delta_j^R \delta_j, \]

and for each pair of component portfolios, $i$ and $j$, the covariance of unsystematic returns is:

(T13.2) \[ \Omega_{ij} = \delta_i^R \delta_j. \]

Let $\Omega$ denote the matrix of unsystematic variances and covariances of the component portfolios, and let $\tilde{\Omega}$ denote the corresponding correlation matrix. Let $\Omega_{iA}$ denote the unsystematic covariance between portfolio $i$ and the aggregate portfolio.

Suppose, first, that the active holdings in a manager's portfolio are optimized with respect to his alphas, but may or may not reflect the correct RAP or DAF for the aggregate. (This might even be a portfolio of that manager maintained for some other sponsor.) Then,

(T13.3) \[ \tilde{\Omega} \text{ is the natural estimator for } \Omega, \]

which is consistent as the number $N$ of assets increases asymptotically.

Next, add the further assumption that the active holdings are optimal with respect to the RAP and DAF of the sponsor. Then the expected values of portfolio covariances reflect the underlying information covariances:

(T13.4) \[ E[\Omega_{ij}] = (N - 1) \left( \frac{\lambda}{k} \right)^2 \left( \frac{b_{i}}{\tilde{W}_i} \right) \left( \frac{b_{j}}{\tilde{W}_j} \right) \omega_{ij}. \]
\[(T13.5) \quad E[\Omega_{iA}] = (N-1) \left( \frac{\lambda}{\kappa} \right)^{2} \frac{b_2 \varrho_i^2}{\bar{W}_i}. \]

From (T13.4),

\[(T13.6) \quad c_{ij} = \frac{E[\Omega_{ij}]}{(N-1) \left( \frac{\lambda}{\kappa} \right) \left( \frac{b_2}{\bar{W}_i} \right) \left( \frac{b_2}{\bar{W}_j} \right)} \]

and, in particular,

\[(T13.7) \quad \rho_i^2 = \frac{E[\omega_i^2]}{(N-1) \left( \frac{\lambda}{\kappa} \right) \left( \frac{b_2}{\bar{W}_i} \right)} \quad i = 1, \ldots, J. \]

Also, from (T13.5),

\[(T13.8) \quad b_i \varrho_i^2 = \frac{E[\Omega_{iA}]}{(N-1) \left( \frac{\lambda}{\kappa} \right) \left( \frac{1}{\bar{W}_i} \right)} \quad i = 1, \ldots, J. \]

\[(T13.9) \quad \bar{W}_i E[\omega_i^2] \quad i = 1, \ldots, J \]

\[(T13.10) \quad E[\alpha_i] = \left( \frac{\kappa}{\lambda} \right) E[\Omega_{iA}] \quad i = 1, \ldots, J. \]

\textbf{Proof:} For each manager \(j\), any MV optimal portfolio based upon \(\alpha_j\) will have active holdings \(\delta_j = S_j R^T \alpha_j\), where the parameter \(S_j\) is a "scale parameter" reflecting the RAP, DAF, and investment portion in that instance. Consider the unsystematic covariance between any two managers' portfolios, \(\Omega_{ij} = \delta_i^T R \delta_j\). Its expectations, in view of the underlying information processes, is:

\[E(\Omega_{ij}) = E(\delta_i^T R \delta_j) = E(S_i \alpha_i^T R R^T \alpha_j S_j)\]

\[= S_i S_j E(\alpha_i^T R \alpha_j) = S_i S_j E(\text{TRACE}(R \alpha_i \alpha_j^T)) \]
by the CPE assumption. Therefore, as in equation (3.6),

\[(3.7) \quad E[\Omega_{ij}] = S_i S_j c_{ij} \text{TRACE}(R_i^c c_{ij} R_j)\]

The same derivation, applied for \(i = j\), results in \(E(\omega_j^2) = S_j^2 \rho_j^2 (N-1)\).

The \(\rho_j^2\) and the \(c_{ij}\) are the entries in the information matrix \(\Sigma\). Thus, the expected value for each entry \((i,j)\) in \(\Omega\) is \((N-1)S_i S_j\) times that entry in \(\Sigma\).

The natural estimate of a correlation is the sample covariance divided by the sample standard deviations: \(\frac{\Omega_{ij}}{\omega_i \omega_j} \equiv p_{ij}\). As \(N\) increases, the probability limit of this estimator is the ratio of expectations,

\[(3.9) \quad \text{plim}(p_{ij}) = \frac{E(\Omega_{ij})}{E(\omega_i)E(\omega_j)} = \frac{(N-1)S_i S_j c_{ij}}{\sqrt{(N-1)S_i^2 \rho_i^2 (N-1)S_j^2 \rho_j^2} = \frac{c_{ij}}{\rho_i \rho_j} = \tau_{ij}.}\]

Thus, \(p_{ij}\) is a consistent estimator of \(\tau_{ij}\). It has the sampling properties of correlation in an \((N-1)\) observation sample.

Next, suppose that portfolios have been constructed so that the scale parameters \(S_j\) are the appropriate parameters for the sponsor's aggregate, which, from Theorem 10, are:

\[(3.10) \quad S_j = \frac{\lambda b_j}{\kappa W_j}, \quad j = 1, \ldots, J.\]

Therefore, substitution of (3.10) into (3.8) yields:

\[(3.11) \quad E(\Omega_{ij}) = (N-1)S_i S_j c_{ij} = (N-1)\left(\frac{\lambda}{\kappa}\right)^2 \frac{b_i}{W_i} \frac{b_j}{W_j} c_{ij} \quad i = 1, \ldots, J\]

This proves (T13.4). (T13.6) then follows directly, and (T13.7) is the special case where \(i = j\).

Next, consider the unsystematic covariance between the \(i\)th manager's portfolio and the aggregate portfolio:
\[ \Omega_{iA} = \sum_{j=1}^{J} W_j \Omega_{ij} \] 

The expected value of this covariance is:
\[ E[\Omega_{iA}] = \sum_{j=1}^{J} W_j (N-1) \left( \frac{\lambda}{\kappa} \right)^2 \left( \frac{b_i}{W_i} \right) \left( \frac{b_j}{W_j} \right) c_{ij} \]
\[ = (N-1) \left( \frac{\lambda}{\kappa} \right)^2 \left( \frac{b_i}{W_i} \right) \sum_{j=1}^{J} b_j c_{ij} \]

But \( \sum_{j=1}^{J} b_j c_{ij} \) is the \( i \)th entry in the matrix product \( Cb \). Substituting \( b = C^{-1}q \), the \( i \)th entry in \( C C^{-1}q \) is just the \( i \)th entry in \( q \), or \( \rho_i^2 \).

This substitution results in (T13.5):
\[ E[\Omega_{iA}] = (N-1) \left( \frac{\lambda}{\kappa} \right)^2 \frac{b_i \rho_i^2}{W_i} \]

(T13.8) then follows directly, and (T13.9) follows from the ratio of (T13.8) and (T13.7):
\[ b_i = \frac{b_i \rho_i^2}{\rho_i^2} = \frac{E[\Omega_{iA}] \div \left( N-1 \left( \frac{\lambda}{\kappa} \right)^2 \left( \frac{1}{W_i} \right) \right) }{E[w_i^2] \div \left( N-1 \left( \frac{\lambda}{\kappa} \right)^2 \left( \frac{b_i}{W_i} \right)^2 \right)} \]
\[ = \frac{E[\Omega_{iA}] \left( b_i^2 / W_i \right) }{E[w_i^2]} \]
\[ \Rightarrow \frac{E[w_i^2] W_i}{E[\Omega_{iA}]} = b_i \]

Finally, (T13.10) is obtained by substitution into (T11.4) of (T13.9):
\[ E[\alpha_i] = \left( \frac{\kappa}{\lambda} \right) \frac{W_i}{b_i} E[w_i^2] = \left( \frac{\kappa}{\lambda} \right) \frac{E[\Omega_{iA}]}{E[w_i^2]} E[w_i^2] = \frac{\kappa}{\lambda} E[\Omega_{iA}] \]
A.3.8. Appropriate Investment Proportions and Management Fees

Thus far, the investment proportions \( W_j \) have been taken as given. This led to no loss of generality, for the subportfolio investments were able to adjust fully to compensate for any change in investment proportions. If the proportion \( W_j \) were decreased, the optimal aggressiveness of manager \( j \)'s portfolio would increase inversely, and the consequence would be the same contribution to the overall portfolio as before. Thus, the investment proportions were not material to the overall portfolio outcome.

How, then, should the portion allocated to a manager be determined? One principle, reflecting current practice, is that the allocation of funds be a means of rewarding managers for their contribution to overall portfolio utility. The greater the contribution of the manager, the greater the portion allocated to him. In fact, management fees are not usually variable in consequence of performance, so that the only way of changing the total fee paid to a manager is to change the amount of funds under management. This principle might be idealized as follows: Proportions under management should be the same as the proportional contributions to portfolio utility. Assuming that the dependence-adjusted explained variance is used as a measure of contribution, the funds under management are then allocated in proportion to \( b_j \sigma_j^2 \).

A second approach, also reflecting current practice, is to allocate funds so that each active manager maintains approximately the same level of unsystematic risk in his portfolio. This approach is justified by the existence of legal restrictions on subportfolio holdings. These restrictions—principally the prohibition of short sales, and occasionally upper bounds on proportionate holdings—begin to reduce the portfolios' contribution to utility below the optimum as soon as the optimal active holdings run up against the bounds. Portfolio unsystematic standard deviation \( \omega \) increases in proportion to the aggressiveness of the active holdings. Therefore, if the unsystematic standard deviations are the same in different portfolios, the absolute magnitude of the typical active holding will
also be roughly the same. This suggests that constituent portfolios will suffer equally from legal restrictions if equal residual risks are present.

Theorem 14 shows that the reward and equal-risk principles are in conflict.

**Theorem 14. Appropriate Investment Weights**

Two conflicting principles may be used to determine managers' investment portions:

I. Proportional to Contributed Utility (Reward Principle)

If investment weights are established in the same relative proportions as the managers' dependence-adjusted contributions to the aggregate Sharpe Ratio (and to CEER with a linear LMV utility function), then

\[
W_j = b_j \sigma_j^2 / \sum_{i=1}^{J} b_i \sigma_i^2 .
\]

II. Portions That Equate Expected Unsystematic Risk (Equal-Risk Principle)

If investment weights are established so that all managers, on average, maintain the same unsystematic variance and the same aggressiveness in active policies, then

\[
W_j = b_j \sigma_j / \sum_{i=1}^{J} b_i \sigma_i .
\]

Weights are proportional to the dependence-adjusted squared correlation in the first instance and to the correlation in the second. Under principle I, the root mean square unsystematic return (the square root of the variance) and the root mean square active holding decrease in inverse proportion to \( \rho_{j} \):

\[
\sqrt{E(\omega_j^2)} = \frac{a}{\rho_j} \sqrt{E(\omega_A^2)} \quad j = 1, \ldots, J .
\]

Thus, the reward principle results in smaller levels of unsystematic risk for managers who contribute more to utility.
In contrast, if principle II is followed, the expected utility contributed per dollar under management increases in proportion to the manager's correlation coefficient:

\[(T14.4)\]

\[
\frac{\left(\frac{1}{W_j}\right) E[z_{AU}^2(\delta_j)]}{\left(\frac{1}{W_i}\right) E[z_{AU}^2(\delta_i)]} = \frac{\rho_j}{\rho_i}.
\]

Thus, if the equal-risk principle is used to assign investment proportions, and if reward is to be proportional to utility contributions, the unit management fee, per dollar under management, should be set for each manager \( j \) in proportion to his \( \rho_j \). Total reward then increases as the product of dollars managed times unit fee, or \( W_j \rho_j \propto (b_j \rho_j) \rho_j = b_j \rho_j^2 \), as it should.

Proof: From (T13.4)

\[(3.17)\]

\[
E(\omega_j^2) = E(\delta_j^T R \delta_j) = (N-1) \left(\frac{\lambda}{\kappa}\right)^2 \left(\frac{b_j}{W_j}\right)^2 \rho_j^2.
\]

Since \( \delta_j^T R \delta_j \) is a weighted sum of squares of the active holdings, weighted in proportion to their variance contributions, it is a natural measure of the squared magnitude of the active holdings. Consequently, \( \omega_j \) is an indicator of the root-mean-square active holding, as well as being the unsystematic standard deviation.

From (T12.1) the expected contribution to the portfolio unsystematic Sharpe Ratio from manager \( j \) is:

\[(3.18)\]

\[
E(z_{AU}^2(j)) = (N-1)b_j \rho_j^2,
\]

and, under linear MV utility, the expected contribution to portfolio CEER is, from (T12.2):

\[(3.19)\]

\[
E(U_{AU}(j)) = \frac{1}{2} \left(\frac{\lambda}{\kappa}\right) (N-1)b_j \rho_j^2.
\]

Hence, if the investment weights are proportional to contributions to \( z_{AU}^2 \), then \( W_j \propto b_j \rho_j^2 \). The requirement that \( \sum_{j=1}^{J} W_j = 1 \) then gives (T14.1).
In this case, substituting (T14.1) for $w_j$ in (3.17),

\[(3.20) \quad E(\omega^2_j) = \left( \frac{1}{\lambda c} \sum_{i=1}^{J} (b_i \rho_i^2) \right)^2 (N-1) \left( \frac{\lambda}{\kappa} \right)^2 \frac{b_i^2}{(b_j \rho_j^2)^2} \rho_j^2. \]

But,

\[(3.21) \quad E(\omega^2_A) = E(\delta_A^2) = (N-1) \left( \frac{\lambda}{\kappa} \right)^2 \rho_c^2 = (N-1) \left( \frac{\lambda}{\kappa} \right)^2 \left( \frac{1}{\sum_{i=1}^{J} b_i \rho_i^2} \right). \]

By substituting (3.21) into (3.20) and simplifying, one obtains:

\[(3.22) \quad E(\omega^2_j) = E(\omega^2_A) \rho_c^2 / \rho_j^2, \]

which implies (T14.3).

Under the equal-risk principle, expected systematic risk must be equal across managers, which will hold if $E(\omega^2_j) = E(\omega^2_1)$, or from (3.17)

\[(3.23) \quad (N-1) \left( \frac{\lambda}{\kappa} \right)^2 \left( \frac{b_i^2}{W_j^2} \right) \rho_j^2 = (N-1) \left( \frac{\lambda}{\kappa} \right)^2 \left( \frac{b_i^2}{W_i^2} \right) \rho_i^2 \quad i \neq j, \]

which will hold if $W_j = b_j \rho_j$, $j = 1, \ldots, J$. The requirement that $\sum_{j=1}^{J} w_j = 1$ then gives (T14.2). In this case,

\[(3.24) \quad \frac{E(\omega^2_{AU}(j))}{W_j} = \frac{(N-1)b_j \rho_j^2}{b_j^2 \rho_j} \left( \frac{1}{\sum_{i=1}^{J} b_i \rho_i} \right) = \frac{(N-1)\rho_j}{\sum_{i=1}^{J} b_i \rho_i} \]

which gives (T14.4).

Thus, the two principles lead to conflicting formulas: both imply greater investment portions for those managers who contribute more to overall
performance. But the "reward" principle sets \( W_j = b_j \rho_j^2 \), and the "equal risk" principle sets \( W_j = b_j \rho_j \). It is important for the sponsor to recognize this discrepancy and resolve it. If the reward principle is followed, the smaller portions should be operating at higher risk levels. If the equal-risk principle is followed, the funds under management by the more valuable managers will increase less than in proportion to their contributions, and an increasing unit management fee—exactly the opposite of current practice—might be applied to compensate for this.

A.3.9. The Extension to Distinguish Specific Events and Common Factors

Thus far, to simplify the notation and derivations, the CPE assumption has been applied uniformly across all aspects of residual returns. However, the intercorrelation among managers is generally different with regard to the forecasting process for specific events of individual firms, on the one hand, and common factors of residual returns, on the other. This necessitates a distinction between these two components of unsystematic return. For clarity, the analysis of this section is carried out in terms of these two particular components. The same approach, with little modification, can be applied to any number of components that have the properties of (a) zero correlation between components, and (b) collectively exhausting all elements of unsystematic return.¹

¹We have already distinguished between extraordinary market return and unsystematic return. The reader may wonder whether that distinction is formally identical to the present additional distinction between two components of unsystematic return. The answer is in the negative: systematic and unsystematic return, when regarded as two components of total return, do possess properties (a) and (b), above. In addition, they possess a third property: (c) the \( N \times N \) variance matrices, \( \Sigma_s \) and \( \Sigma_u \), giving the contribution of each to asset variance, are orthogonal in the sense that \( \Sigma_s \Sigma_u^{-1} = 0 \). Hence, the "overlap" measure in (T15.20), below, is zero. If the two components of unsystematic returns possessed zero overlap, which they do not, then they could be analyzed entirely separately, just as systematic and unsystematic returns have been.
This section parallels subsections A.3.1 through A.3.7 and may be used as a summary. The proofs of results are omitted where they may be easily worked out as slight generalizations of the proofs presented above. The two components will be represented by the superscripts \( r \) and \( s \), with generic index \( \lambda = r, s \).

**Theorem 15. Decomposition of Unsystematic Returns**

**Decomposition of Unsystematic Return (the Components Are Exhaustive):**

\[(T15.1) \quad u = u_r^r + u_s^s .\]

**Zero Expectation for Unsystematic Components:**

\[(T15.2) \quad E(u_r^r) = E(u_s^s) = 0 .\]

**Variance and Covariance of Unsystematic Components (Zero Covariance):**

\[(T15.3) \quad \text{VAR}(u_r^r) = R_r^r ; \quad \text{VAR}(u_s^s) = R_s^s ; \quad \text{COV}(u_r^r, u_s^s) = 0 .\]

Therefore,

\[(T15.4) \quad R = R_r^r + R_s^s .\]

**Appraisals for Each Component by Each Manager:**

\[(T15.5) \quad \alpha_{j}^r \text{ and } \alpha_{j}^s , \quad j = 1, \ldots, J .\]

**Information Properties of Appraisals, Generalized CPE Assumption:**

\[(T15.6) \quad \text{VAR}(\alpha_{j}^l) = \text{COV}(\alpha_{j}^r, u_{j}^l) = (\sigma_{j}^l)^2 ; \quad \lambda = r, s \quad j = 1, \ldots, J .\]

**No Confusion Between Components Assumptions:**

\[(T15.7) \quad \text{COV}(\alpha_{j}^r, u_{j}^s) = \text{COV}(\alpha_{j}^s, u_{j}^r) = 0 , \quad j = 1, \ldots, J .\]

**Covariance Between Managers' Appraisals (CPE Assumption):**

\[(T15.8) \quad \text{COV}(\alpha_{i,j}^l, \alpha_{i,j}^l) = \sigma_{i,j}^r R_{i,j} = \rho_{i,j}^l \sigma_{i,j}^l \sigma_{i,j}^r ; \quad \lambda = r, s \quad i, j = 1, \ldots, J .\]
No Covariance Between Different Component Appraisals Assumption:

(T15.9) \( \text{COV}(\alpha^r_i, \alpha^s_j) = 0 \) for all \( i, j \).

Best Combined Forecast for Unsystematic Return Is the Sum of Best Forecasts for the Two Components, and Explained Variance Is the Sum of Explained Variances:

(T15.10) \( \alpha = \alpha^r + \alpha^s \); \( \Sigma = \Sigma^r - (\rho^r)^2 \Sigma^r - (\rho^s)^2 \Sigma^s \).

Best Combined Forecasts for Each Component:

(T15.11) \( \alpha^\ell_C = \sum_{j=1}^J b^\ell_j \alpha^\ell_j \), where \( b^\ell_j = \frac{\sum_{i=1}^I \rho^\ell_{ij} \rho^\ell_i}{\rho^\ell_j} \) \( \ell = r, s \), \( j = 1, \ldots, J \).

Explained Variance for Each Component:

(T15.12) \( (\rho^\ell_j)^2 = \sum_{j=1}^J b^\ell_j (\rho^\ell_j)^2 \), \( \ell = r, s \).

(Hereafter, the approximation \( \Sigma = \Sigma^r \) will be made.)

MV Optimal Aggregate Portfolio, Active Holdings:

(T15.13) \( \delta_A = \left( \frac{\lambda}{\kappa} \right)^2 \Sigma^{-1} \left( \sum_{j=1}^J b^r_j \alpha^r_j + \sum_{j=1}^J b^s_j \alpha^s_j \right) \).

MV Optimal Aggregate Active Holdings, Properties:

(T15.14) \( \alpha_A = \left( \frac{\lambda}{\kappa} \right)^2 \Sigma_{AU}^{-1} \); \( \omega_A^2 = \left( \frac{\lambda}{\kappa} \right)^2 \Sigma_{AU}^{-1} \); \( U_{AU} = \frac{1}{2} \alpha_A \).

MV Optimal Aggregate Portfolio, Sharpe Ratio:

(T15.15) \( \tilde{z}_{AU} = \left( \sum_{j=1}^J b^r_j \alpha^r_j + \sum_{j=1}^J b^s_j \alpha^s_j \right)^{-1} \left( \sum_{j=1}^J b^r_j \alpha^r_j + \sum_{j=1}^J b^s_j \alpha^s_j \right) \).

MV Optimal Decentralized Portfolios, Active Holdings:

(T15.16) \( \delta_j = \lambda^*_j R^{-1} \left( b^r_j \alpha^r_j + b^s_j \alpha^s_j \right) \), where \( \lambda^*_j = \left( \frac{\lambda}{\kappa w_j} \right) \).
MV Optimal Aggregate Portfolio, Required Alphas:

\[ \alpha_A = \left( \frac{\kappa}{\lambda} \right) \omega_A^2 ; \alpha_n = \left( \frac{\kappa}{\lambda} \right) k_n \quad n = 1, \ldots, N. \]

MV Optimal Decentralized Portfolios, Required Alphas:

\[ b_{rj}^r \alpha_j^r + b_{sj}^s \alpha_j^s = \frac{1}{\lambda_j^2} \omega_j^2 \quad ; \quad b_{rj}^s \alpha_j^r + b_{sj}^s \alpha_j^s = \frac{1}{\lambda_j^2} k_j^s \quad n = 1, \ldots, N, \]

where \( \alpha_j^r = \delta_j \alpha_j^r \quad \lambda = r, s. \)

Portfolio Covariances of Unsystematic Return:

\[ \omega_j^2 = (\omega_j^r)^2 + (\omega_j^s)^2, \quad \text{where} \quad (\omega_j^r)^2 = \delta_j \omega_j^r \omega_j^r \quad \lambda = r, s. \]

\[ \Omega_{ij}^r = \Omega_{ij}^r + \Omega_{ij}^s, \quad \text{where} \quad \Omega_{ij}^r = \delta_i \omega_j^r \omega_j^r \quad \lambda = r, s. \]

\[ \Omega_{ij}^r + \omega_{ij}^s, \quad \text{where} \quad \omega_{ij}^s = \delta_i \omega_j^s \omega_j^s = \sum_{t=1}^T \omega_{ij} \Omega_{ij}^s. \]

Measure of Importance of Components of Unsystematic Variance:

\[ y^s = \text{TRACE}(R_t^t R_t^s) \quad ; \quad y^r = \text{TRACE}(R_t^t R_t^r) = (N-1) - y^s. \]

Measure of Lack of Orthogonality or "Overlap" Between Specific and Common Factors:

\[ y^{sr} = \text{TRACE}(R_t^t R_t^s R_t^r R_t^r). \]

Expected Contributions to Aggregate Sharpe Ratio:

\[ E[z_{AU}^2] = y^r (\rho^r_c^2 + y^s (\rho^s_c^2 = \sum_{j=1}^J \left( y^r b_{rj}^r (\rho^r_j)^2 + y^s b_{sj}^s (\rho^s_j)^2 \right). \]

Expected Contribution to Aggregate CEER, Given LMV Utility:

\[ E[U_{AU}] = \frac{1}{2} \left( \frac{\lambda}{\kappa} \right) E[z_{AU}^2]. \]

Expected Values of Sample Covariances:

\[ E[\Omega_{ij}^r] = \lambda_i \lambda_j \left( b_{rj}^r b_{ij}^r \omega_j^r + b_{sj}^s b_{ij}^s \omega_j^s \right). \]
\[ E \begin{bmatrix} \Omega^r_{ij} \\ \Omega^s_{ij} \end{bmatrix} = \lambda^*_i \lambda^*_j \begin{pmatrix} (y^r - y^{sr}) : y^{sr} \\ (y^{sr} : y^s - y^{sr}) \end{pmatrix} \begin{pmatrix} b_i^r b_j^r \sigma^r_{ij} \\ b_i^s b_j^s \sigma^s_{ij} \end{pmatrix}. \]

\[ E[\Omega^r_{iA}] = \lambda_i^* \left( \frac{\lambda}{\kappa} \right) \left( \sigma_i^r (\sigma_i^r)^2 Y^r + \sigma_i^s (\sigma_i^s)^2 Y^s \right). \]

\[ E[\Omega^s_{iA}] = \lambda_i^* \left( \frac{\lambda}{\kappa} \right) \begin{pmatrix} (y^r - y^{sr}) : y^{sr} \\ (y^{sr} : y^s - y^{sr}) \end{pmatrix} \begin{pmatrix} (\sigma_i^r)^2 \\ (\sigma_i^s)^2 \end{pmatrix}. \]

Expectations Relating to Dependence-Adjustment Factors:

\[ b_i^L = \frac{\kappa E[K^L_i]}{E[L_i]} \quad L = r, s \quad i = 1, \ldots, j \]

where:

\[ \left( \begin{array}{c} K_i^r \\ K_i^s \end{array} \right) = \begin{pmatrix} (y^r - y^{sr}) : y^{sr} \\ (y^{sr} : y^s - y^{sr}) \end{pmatrix}^{-1} \begin{pmatrix} \Omega^r_{ii} \\ \Omega^s_{ii} \end{pmatrix}. \]

\[ \left( \begin{array}{c} L_i^r \\ L_i^s \end{array} \right) = \begin{pmatrix} (y^r - y^{sr}) : y^{sr} \\ (y^{sr} : y^s - y^{sr}) \end{pmatrix}^{-1} \begin{pmatrix} \Omega^r_{iA} \\ \Omega^s_{iA} \end{pmatrix}. \]

Expectation Relating to Required Alpha:

\[ E[\alpha^L_i] = \left( \frac{\kappa}{\lambda} \right) E[L_i^L] \quad L = r, s \quad i = 1, \ldots, j. \]

Proof: Equations (T15.1-9) define the two components of return, establish reasonable assumptions concerning the forecast properties, and define the covariance matrices of forecasts. Since all covariances between "r" and "s" forecasts are assumed to be zero, a derivation analogous to Theorem 7 demonstrates that the two forecast components are treated distinctly, resulting in the optimal combined forecast (T15.10), which is the sum of the two optimal combined forecasts of components (T15.11-12).
When this expression for the optimal combined forecast is substituted for (T7.1), the properties of the centralized and decentralized portfolios follow from Theorems 9-11. Equations (T15.13, 14, 15) correspond to Theorem 9, (T15.16) corresponds to Theorem 10, and (T15.17, 18) correspond to Theorem 11.

Equations (T15.19-21) break unsystematic return covariance into components. These follow directly from (T15.4) and Lemma 3. For example:

\[(3.25) \quad \Omega_{ij} = \delta_i R_{ij} \delta_j = \delta_i (R^R + R^S) \delta_j = \delta_i R^R \delta_j + \delta_i R^S \delta_j = \Omega^R_{ij} + \Omega^S_{ij}.\]

The next step is to compute expectations of portfolio covariance components, so as to obtain required appraisal premia for the components. Here the problem becomes much more complex than before, because the two components do not remain distinct. The judgmental forecasts are uncorrelated between components, but the forecasts for any one component may necessitate exposure to the other in the optimal portfolio. For example, security analysis may result in favorable forecasts of specific returns for two automobile manufacturers, despite a neutral forecast for the industry. In this case, the optimal portfolio would show a concentration in the automobile industry and, hence, exposure to extra-market covariance due to the automobile-industry factor. Thus, an exposure to XMC arises as the indirect consequence of a specific return forecast.

This happens because there is an "overlap" between the patterns of holdings that cause exposure to the two components: a random realization of forecasts for any one component will tend to cause a certain amount of exposure to the other as well. To measure the importance of this tendency, the first step is to compute the expected total unsystematic covariance between any pair of portfolios i and j. Let the notation TRACE be abbreviated "TR," since the traces of various matrices appear frequently in these equations.
\[ E[\Omega_{ij}] = E[\delta'_R \delta_j] = E[\lambda^* (b^+_1 \alpha^+_i + b^+_i \alpha^+_j - \alpha^+_i \alpha^+_j - \alpha^+_j \alpha^+_i) (b^+_1 \alpha^+_i + b^+_i \alpha^+_j)] \\
= \lambda^* \lambda^* TR[\delta'_R \delta_j] E[(b^+_j \alpha^+_j + b^+_j \alpha^+_j)(b^+_j \alpha^+_j + b^+_j \alpha^+_j)'] \\
= \lambda^* \lambda^* TR[\delta'_R \delta_j] (b^+_j b^+_j c^+_j R^+_j + b^+_j b^+_j c^+_j R^+_j) \\
= \lambda^* \lambda^* (b^+_j b^+_j c^+_j TR[\delta'_R R^+_j] + b^+_j b^+_j c^+_j TR[\delta'_R R^+_j]). \]

(3.26)

Also, the expected covariance for either component, identified by index \( \ell \), becomes:

\[ E[\Omega^\ell_{ij}] = E[\delta'_R \delta_j^\ell] = \lambda^* \lambda^* (b^+_j b^+_j c^+_j \delta'_R \delta_j^{\ell} + b^+_j b^+_j c^+_j \delta'_R \delta_j^{\ell}). \]

(3.27)

Thus, the moments depend on the traces of six matrices: \( y^R = TR(R^+_R) \), \( y^S = TR(R^+_R) \), \( y^{\ell R} = TR(R^+_R \delta'_R \delta_j^\ell) \), \( y^{\ell S} = TR(R^+_R \delta'_R \delta_j^\ell) \), \( y^{RS} = TR(R^+_R \delta'_R \delta_j^\ell) \), and \( y^{SR} = TR(R^+_R \delta'_R \delta_j^\ell) \). One of the elementary properties of the trace is that \( TR(A) = TR(B) \). Hence,

\[ y^{RS} = TR((R^+_R)(R^+_R)) = TR(((R^+_R)(R^+_R)) = y^{SR}. \]

(3.28)

Further,

\[ y^R + y^S = TR(R^+_R (R^+_R + R^+_R)) = TR(R^+_R) = (N-1) \]  

(3.29)

Similar reasoning leads to:

\[ y^{RR} + y^{RS} = y^R; \quad y^{SR} + y^{SS} = y^S. \]

(3.30)

When these restrictions are substituted, it is found that only two terms are free, which have been chosen as \( y^S \) and \( y^{RS} \) in (T15.22, 23). The other traces are given in terms of these by:

\[ y^R = (N-1) - y^S; \quad y^{SS} = y^S - y^{RS}; \quad \text{and} \quad y^{RR} = y^R - y^{RS}. \]

\( y^S \) is a measure of the importance of specific variance, and \( y^{RS} \) is a measure of the extent to which exposure to common factors tends to cause exposure to specific variance, and vice versa. It measures the "overlap" between these two aspects of security returns.
When the expectation of $z_{AU}^2$, as given in (T15.15), is evaluated as in (3.26), the result is (T15.24). Expected utility in (T15.25) then follows by substitution of (T15.24) into the first and third equations of (T15.14). (T15.26) is (3.26), and (T15.27) is (3.27). (T15.28, 29) then follow from these equations by the same aggregation that led to (T13.8). The balance of the theorem corresponds to (T13.9-10), and is derived in the same manner.

It is beyond the scope of this appendix to go into details concerning the computation of required appraisal premia for the specific return and XMC. Values in the text were computed based on the multiple-factor model in [3], [4]. Some rather complicated matrix decompositions are needed to evaluate $y^s$ and $y^{sr}$. 