Research Program in Finance
Graduate School of Business Administration

WORKING PAPER NO. 66

THE YIELD/BETA/
RESIDUAL RISK TRADE-OFF

by

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and
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THE YIELD/BETA/RESIDUAL RISK TRADE-OFF

Barr Rosenberg* and Andrew Rudd**

The study examines the residual risk that must be undertaken to obtain a stock portfolio with dividend yield ($y$) and/or beta ($\beta$) which differ from the equity market norms. There are two cases, depending on whether or not the investor is constrained by minimum position sizes (short-selling exclusions) and maximum position sizes. The unconstrained case is easily analyzed but unrealistic. The solution expresses the minimal residual risk portfolio for any yield/beta ($y/\beta$) policy as a mixture of three fixed "portfolios."

The more difficult but realistic case is where short-selling is excluded and maximum position sizes are imposed by law or by the impossibility of obtaining a disproportionately large share of any one company. A portfolio-optimization program, using quadratic programming, finds the minimum variance portfolio at each specified $y/\beta$ policy. Solutions for a range of $y/\beta$ policies and for varying universes and position constraints are obtained.

The solutions show the extent to which nonstandard $y/\beta$ policies necessarily incur residual risk due to imperfect portfolio diversification.

The study also considers capital market equilibrium when there are different taxes on capital gains and dividends. When the Capital Asset Pricing Model with differential taxation is modified to admit constrained investment, the equilibrium takes the form of an allocation of the market into subportfolios: portfolios held by tax-exempt investors are shaded toward high-yielding stocks; portfolios held by taxable investors are shaded toward low-yielding, high-capital-gain stocks. For a posited relationship between total return and yield, the families of optimal portfolios for taxable and tax-exempt investors are computed.

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1. INTRODUCTION AND SUMMARY

This paper examines the residual risk which must be undertaken to construct equity (common stock) portfolios having dividend yield (y) and/or beta (β) which differ from the equity market norms. Minimal attainable residual risk is found for the feasible range of combinations of yield and beta, or yield/beta (y/β) policies.

Yield, defined as the ratio of cash dividends received to portfolio market value, is an important desideratum of equity portfolios in several instances. For endowment funds and some personal trust accounts, spending policies may be tied to cash dividends, so that there is an important distinction between yield and capital gains. For taxable investors, the different tax treatment of yield and capital gains is a major issue in maximizing after-tax revenue. The existence of tax law which favors lower-yielding stocks leads to a perturbed equilibrium in the Capital Asset Pricing Model. Brennan (1970) has shown the modifications to the Capital Asset Pricing Model that occur when holdings are unrestricted as to short-selling and borrowing. In this case, the equilibrium results in higher total return for higher-yielding stocks, so that the tax-exempt investor finds dividend yield an attractive feature.

Portfolio beta is the critical determinant of investment risk and expected return. An equity portfolio with beta less than unity is not commonly required, since lower beta can be naturally attained by admixture of bonds or cash with equity. However, an equity portfolio beta greater than 1 is a practical necessity for an investor whose below-average risk aversion encourages him to bear above-average systematic risk, since leverage is generally ruled out as a means to increase beta. Hence, an important question in investment policy is the attainability of equity portfolio betas greater than 1.

Any portfolio that differs from the market’s proportions is exposed to residual (unsystematic) risk. Residual variance is a measure of perfect diversification. For the risk-averse investor with portfolio
yield and beta targets, the optimum portfolio is the one that achieves these targets with lower residual risk. Consequently, the range of investment opportunities can be determined by the function which gives minimum residual risk obtainable for each $y/\beta$ policy, $R(y, \beta)$.

The form of the residual-risk function $R$ depends on whether or not the investor is constrained by minimum position sizes (short-selling exclusions) and/or maximum position sizes. The easier case to analyze is the case where no constraints are present. Then $R$ is a quadratic function which attains its minimum—equal to zero—at the yield and beta of the market portfolio. It can be visualized as an ellipsoidal bowl erected above the yield/beta plane, with the lowest point touching the market portfolio. The optimal portfolio for any $y/\beta$ policy is a mixture of three portfolios. These three portfolios are related, respectively, to the market portfolio, the minimum residual risk portfolio constrained by beta, and the minimum residual risk portfolio constrained by yield. These are weighted in each optimal portfolio so as to attain the associated $y/\beta$ policy. The general problem is solved analytically by identifying the two constrained minimum residual risk portfolios.

The more difficult but far more realistic case is that where short-selling is excluded and where maximum position sizes may also be present. Maximum position sizes may be imposed by law or by the impossibility of obtaining a disproportionately large share of any one company. In this case, the portfolio-optimization problem becomes a quadratic program. Residual variance, which is a quadratic function of holdings, is minimized subject to these constraints on the holdings and subject to obtaining the desired yield and beta. Thus, the optimal portfolio at each $y/\beta$ policy must be obtained as the solution of a separate quadratic programming problem. Solutions are found for a grid of $y/\beta$ policies, and $R$ is found by interpolation between the points defined by these solutions.
$R$ is computed in this fashion in section 3, according to procedures discussed in section 2. The entire function is obtained for two cases corresponding to different maximum position bounds on the individual assets. In the first case, the bounds are relatively loose and are realistic for small actively managed funds; in the second case, the bounds are more stringent and therefore more realistic for large pools of capital. Certain important cases are recomputed with still more stringent bounds applied to more restrictive universes. Solutions are found for the case where a 100% equity portfolio is desired, as well as for the case where equities can be mixed with the risk-free asset (cash).

Total portfolio variance, $\sigma_p^2$, is given by $\sigma_p^2 = \beta^2 \sigma_m^2 + R$, where $\sigma_m^2$ is the variance of market return. The Capital Asset Pricing Model suggests that portfolio expected excess return is a linear function of beta. Brennan's modification suggests that expected excess return is a function of beta and also of yield. The after-tax return for a taxable investor is also a function of beta and yield. With $R$ available, it is therefore possible to consider the problem of the optimal portfolio trade-off between expected return (as a function of $\beta$ and $y$), and total portfolio variance (as a function of $\beta^2$ and $R(\beta,y)$). For an investor with a given mean-variance utility function, there will be a unique $y/\beta$ policy that optimizes this trade-off. If the risk aversion of the investor is unknown, the exact solution is indeterminate. However, for given expected return and total-variance functions, the locus of all optimal solutions can be computed. The optimal solution for every investor will lie somewhere along this locus, with the exact location determined by his risk aversion.

Loci of optimal $y/\beta$ policies are computed for four cases. The cases correspond to the possible combinations of two factors: first, whether the investor is taxable or tax-exempt; second, whether expected return is not a function of yield, or is a linear function of yield, with coefficient equal to .395, as estimated by Rosenberg and Marathe (1978).
The study has immediate implications for portfolio management with nonstandard $y/\beta$ policies. The minimum residual risk at various policies is easily found from the plotted functions, and optimal policies with various tax situations and assumptions on capital asset pricing can be inferred. The results state the cost, in terms of undesirable residual risk, that must be incurred to achieve $y/\beta$ targets.

The study also has broader ramifications. An important issue in institutional investment concerns the effects of differing taxation of capital gains and dividends on capital market equilibrium. Brennan's treatment assumed that investors could short-sell unrestrictedly without margin requirements, and that unlimited borrowing at the risk-free rate was available. It is more realistic to assume that both short-selling and borrowing in large amounts are impossible. When this modification is made, the equilibrium takes the form of an allocation of the market into subportfolios: those held by tax-exempt investors are shaded toward high-yielding stocks; those held by taxable investors are shaded toward low-yielding, high-capital-gain stocks. The sum of these portfolios is the market portfolio. The two groups of investors are joining in a cooperative game against the Internal Revenue Service, the purpose of which is to reduce tax payments on earned dividends and capital gains. The limitation on this game is the growing residual risk induced in the subportfolios as they are increasingly shaded toward different kinds of firms and away from the market portfolio.

The present analysis, by identifying residual risk as a function of yield, suggests how far these portfolios might reasonably go. Thus, it is a step toward determining the distribution of assets that might prevail under existing tax law. Ultimately, the equilibrium would be computed by identifying those prices that would clear the market, given differing tax positions and risk aversion of groups of investors. The equilibrium prices, in turn, would imply a relationship between total return and yield. Thus, this study is a first step in the direction of an analytical derivation of the equilibrium compensation for yield.
2. PROCEDURE

2.1. The Asset Universe

The universe comprises approximately 3,600 common stocks, of which about 3,250 are drawn from the COMPUSTAT universe as of February 1977, and the balance are larger non-COMPUSTAT firms of interest to institutional investors. The common stocks were valued on the last trading day of April 1977. Two "market" portfolios were used: the S&P 500 as of that date, and the aggregate of all common stocks in the universe, weighted by the market value (capitalization) of outstanding shares. The results for both market portfolios were qualitatively similar, and so only those corresponding to the S&P 500 as the market portfolio are reported.

The 399 firms with largest capitalization are included as purchase opportunities for the portfolios. The remaining companies are grouped into composites. The 152 composite assets are constructed as follows. For each of 39 industry groups, the common stocks in the group are divided into four categories: a high-yield, high-beta group; a high-yield, low-beta group; a low-yield, low-beta group; and a low-yield, high-beta group. Of the 156 groups which result, one containing only a single large firm is treated as an individual asset, and three containing fewer than four companies are merged with another group in the same industry. Within each group, the member companies are weighted in proportion to this capitalization to form a composite. The outcome is 400 individual companies and 152 composite "pooled funds."

2.2. Predictions of Asset Properties

The betas of individual assets and the variances and covariances of residual returns of all assets are drawn from the predictive model of investment risk in Rosenberg and Marathe (1975, 1976), updated to April 1977 by the Fundamental Risk Measurement Service. Predictions of common stock yield were obtained by dividing the indicated annual dividend reported by the Interactive Data Corporation on April 30, 1977, by the
price on that date. Optimized portfolios are constructed with respect to these predictions.

Portfolio risk is checked by a retrospective simulation over sixty months of historical data. The investment weights of the portfolio in April 1977 are applied at the beginning of each of the preceding sixty months, to compute the portfolio return that would have been obtained with these weights. The sixty monthly returns thus obtained are then regressed upon similarly reconstructed market portfolio returns. (The market portfolio return is also defined using current investment proportions, rather than those that prevailed in the past.) The residual variance of this regression provides a means of verifying the predictions of the investment-risk model.

2.3. The Unconstrained Case

For a portfolio to be 100% invested in equities, in the ideal case where there are no bounds on asset holdings, it can be shown (see, e.g., Long (1977)) that the optimal holdings $h_P$ are given by:

$$h_P = k_0 y^{-1} \mu + k_1 y^{-1} \gamma + k_2 y^{-1} 1,$$

where $V$ is the covariance matrix of excess returns, $\mu$ is the vector of asset mean excess returns, $\gamma$ is the vector of asset yields, and $1$ is a vector of ones. The $k$'s are the solutions to the following system:

$$A \begin{pmatrix} k_0 \\ k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} \mu_P \\ \gamma_P \\ 1 \end{pmatrix},$$

where

$$A = \begin{bmatrix} \mu' y^{-1} \mu & \mu' y^{-1} \gamma & \mu' y^{-1} 1 \\ \gamma' y^{-1} \mu & \gamma' y^{-1} \gamma & \gamma' y^{-1} 1 \\ 1' y^{-1} \mu & 1' y^{-1} \gamma & 1' y^{-1} 1 \end{bmatrix}.$$  

and $\mu_P$ and $\gamma_P$ are the portfolio target mean excess return and yield, respectively. The portfolio is optimal in the sense of minimizing total variance subject to satisfying these targets. Total portfolio variance is given by:
\[ \sigma_p^2 = (\mu_p : y_p : 1) \Sigma^{-1} \begin{pmatrix} \mu_p \\ y_p \\ 1 \end{pmatrix}. \]

Under the assumption that the asset mean excess returns satisfy the basic Sharpe-Lintner-Mossin Capital Asset Pricing Model, \( \mu = \tilde{\beta} \mu_M \), where \( \tilde{\beta} \) is the vector of asset betas and \( \mu_M \) is the mean excess return on the market. Since \( \mu_p = \tilde{\beta}_p \mu_M \), from (3) total variance is:

\[ \sigma_p^2 = T(y_p, \beta_p) = (\beta_p : y_p : 1) \Sigma^{-1} \begin{pmatrix} \beta_p \\ y_p \\ 1 \end{pmatrix}, \]

where \( \Sigma = \begin{pmatrix} 1 & 0 & 0 \\ \mu_M & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \). \( \Sigma^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ \mu_M & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \).

Thus, minimum total variance at portfolio targets \( y_p \) and \( \beta_p \) is given by the quadratic function \( T \): contours of constant total variance are concentric ellipses centered at the global minimum variance portfolio.

Portfolio residual risk is, by definition, total risk less systematic risk and is given by:

\[ \omega_p^2 = R(y_p, \beta_p) = (\beta_p : y_p : 1) \Sigma^{-1} \begin{pmatrix} \beta_p \\ y_p \\ 1 \end{pmatrix} - \beta_p^2 \mu_M. \]

The contours of residual risk may be shown to be concentric ellipses centered on the market portfolio.

The formulas for the variance functions \( R \) and \( T \) are straightforward. However, inversion of the variance matrix \( \Sigma \) is required, and
\( Y \) is a square matrix of dimension 552. This otherwise difficult inversion operation is greatly simplified because the investment risk model is a multiple-factor model. \( Y \) may be decomposed as the sum of an easily inverted matrix of full rank and another matrix of rank 46. The latter reflects the variance due to the market component plus 45 residual factors. A matrix inversion procedure that exploits this structure reduces computational cost by a factor of 100 and allows \( R \) and \( T \) to be computed easily.

2.4. Investment Constraints

Let \( h \) denote a typical holding, expressed as a proportion of the portfolio. For each stock in the S&P 500, let \( w_{SP} \) denote the capitalization proportion of the stock, defined as the ratio of market value of common stock of that company to the total value of all companies. For companies not in the S&P 500, \( w_{SP} = 0 \). Also, let \( w_U \) denote the capitalization proportion of every company or composite asset in the universe of all assets considered in this study.

For much of the analysis, the investment constraints on every asset (whether an individual common stock or a composite portfolio) are as follows: minimum holding, 0; maximum holding equal to \( w_{SP} + .05 \). For example, the allowable range of holdings for AT&T, which is about 6% of the S&P 500, is from 0 to 11% of the portfolio. The allowable range for a composite asset that contains no common stocks in the S&P 500 is from 0 to 5% of the portfolio.

These restrictions are reasonable ones to apply for a small pool of funds that is being invested without arbitrary legal restrictions and may even be conservative in this case. However, when the value of the portfolio is large, a 5% portion of the portfolio may be a dollar value that is an unreasonably large fraction of the outstanding capitalization of a small company. To reflect this limitation, we considered three other possible upper-bound constraints. First, a restrictive upper bound equal to 2\( w_U \). The universe has aggregate capitalization of approximately $890 billion, so a company with outstanding equity
having market value of $1.5 billion would be approximately 0.17% of
this universe and would experience an upper bound of 0.34%. The next
upper bound was the slightly looser restriction of $u_{y}$. The final up-
per bound, also extensively analyzed, is equal to Minimum($w_{SP} + 0.05, 4u_{y}$).
This upper bound, for example, results in a maximum holding of 11% for
AT&T and 0.68% for the small company with $1.5 billion capitalization.

An optimization program developed by the authors (Rudd and Rosen-
berg, 1978), based on an algorithm of Von Hohenbalken, is used in this
study. The algorithm approaches the optimal solution through the in-
terior of the feasible region, with each improvement from one trial
portfolio to the next termed a cycle. At the terminal cycle, the trial
portfolio is optimal. The algorithm demonstrates excellent convergence
properties, permitting control of computational costs by means of the
optimality or convergence criterion.

For this project, the convergence criterion is established as
follows. A linear mean/variance utility function is hypothesized, and
the risk-aversion parameter was set so as to cause the optimum sys-
tematic risk exposure to occur at $\beta_{p} = 1$. The expected utility of the
market portfolio is therefore one-half of the expected excess market
return (assumed to be $\mu_{M} = 6\%$) or 3% per annum. The convergence cri-
terion is that the increase in the utility function from one cycle to
the next be less than 1/100 of a basis point (1/10,000 of a percent).
Thus, the algorithm terminates when the increase in utility function
is less than 1/30,000 of the utility of the market portfolio. Con-
vergence is always obtained. The number of cycles varies from 13 to
25.

2.5. Developing the Residual-Risk Function

One method of identifying the residual-risk function is to choose
arbitrary $y/\beta$ combinations and compute minimum residual risk at those
points. Another approach, which has several comparative advantages, is
to establish a target with regard to only one parameter (beta or yield)
and to leave the other free. Alternatively, a target can be established with regard to some linear combination of beta and yield. In this case, the optimization program sets the free parameter, as well as residual risk, so as to obtain the minimum residual-risk portfolio, subject to the established target. Whatever the y/β policy is that is obtained as a solution, the residual risk at that point establishes one value of R. Moreover, that y/β policy is the lowest residual-variance policy, subject to the established target. It follows that the isovariance contour at that point (the line of equal residual variance passing through that y/β policy) is tangent to the line established by the target. Thus, we obtain not only a point on the residual-variance function, but also establish the slope of its contour lines at that location. We use this method to find the important regions of the residual-variance function and then fill out the remainder of the function with a grid of specified y/β policies.

3. THE RESIDUAL-RISK FUNCTION

Figure 1 graphs the residual-risk function, as predicted by the model, for the case of 100% investment in equities with asset holdings unconstrained. The vertical axis is the predicted annual percentage yield on the equity portfolio. The horizontal axis is the predicted portfolio beta. Residual variance corresponds to a third dimension, which is described by contours of equal residual variance drawn on the graph. The amount of variance at each isovariance contour is shown by a label.

The central point on the figure is the market portfolio, here defined as an S&P 500 index fund. That portfolio has a beta of 1, by definition, and predicted yield of 4.61%. The portfolio has no residual risk. The straight line sloping downward to the left extends to the risk-free asset, which is taken as a 90-day Treasury bill, with beta equal to 0 and yield of 4.37%. If mixtures of the risk-free asset and the market portfolio were permitted, then every y/β policy on this straight line could be attained at zero residual risk. However, in this
FIGURE 1
RESIDUAL VARIANCE
Holdings Unconstrained
figure, 100% investment in equities is required, and thus residual risk inevitably increases as the \( y/\beta \) policy moves away from market yield and beta values. Hence, the residual-risk function takes the shape of a bowl rising from the point of the market portfolio.

The contours are perfect ellipses. The principal axes of the congruent ellipses are shown. The orientation of the ellipses is not quite vertical, running from upper left to lower right. The majority of assets lie in the lower-right (high-beta, low-yield) or upper-left (low-beta, high-yield) quadrants, so that \( y/\beta \) policies in these quadrants are achieved at lower residual risk.

Figure 2 graphs the residual risk function for the case where short-sales are excluded and upper bounds on holdings are the S&P 500 percentage plus 5%. The figure is analogous to figure 1, but with added complexities reflecting the less regular form of the function.

The dashed line rising upward and slightly to the left in the graph is the locus of portfolios that attain various yields with minimal residual variance, disregarding beta. These portfolios would be appropriate for an investor who was able to control overall portfolio beta by the bond/stock mix, and who was concerned with shading the yield of the equity portfolio upward, so as to earn an extraordinary return he believed to be attached to high-yielding stocks.

For example, suppose that the investor desires an equity portfolio with a yield of 7%. This is substantially higher than the market yield of 4.6%. To do this at minimum residual risk, the optimization program builds a portfolio with characteristics that can be read from the graph. The portfolio is located by reading across from the yield of 7% on the vertical axis to the locus of the minimum-variance portfolios for given yield goals. Then, extending a line downward to the horizontal axis, beta is found to be .93. The residual variance of the portfolio is found to be 14.5, by interpolation between the contours labeled 8.2 and 18.3. The square root of this is the residual standard deviation at an annual rate, or 3.81% per annum. Thus, the lowest
FIGURE 2

RESIDUAL VARIANCE

Holdings Constrained: $0 \leq h \leq v_{SP} + .05$

Locus of min. residual var. at various yields

Boundary of Feasibility

To coordinates $(0, 4.37)$

Locus of min. residual var. at various betas
residual risk equity portfolio with a yield of 7%, subject to the constraints on holdings, has a residual standard deviation of 3.8% per annum and is attained at an equity portfolio beta of .93.

Following up the locus, the portfolio beta is seen to fall almost linearly as desired yield rises, while the residual variance begins to increase rapidly. At a target yield of 8%, residual variance has risen to 48.1, corresponding to a residual standard deviation of 6.93%. At a yield goal of 9% per annum, residual variance has risen to 106.1, with corresponding standard deviation of 10.3% per annum.

The other prominent dashed line, extending downward and to the right, is the locus of portfolios that achieve various beta goals at minimum residual risk. For these portfolios, yield is incidental and is set at whatever value allows optimal diversification. This locus can be used in the same way as that for yield goals. For example, a portfolio beta of 1.3 can be achieved with residual variance of 22.5 at a portfolio yield of 3.8%.

Figure 3 shows the residual risk function, as predicted by the model, for the case of 100% investment in equities and a more stringent maximum position. Here, holdings are bounded by the minimum of the S&P 500 percentage plus 5%, and four times the percentage of company capitalization in total universe capitalization. The results are similar to figure 2; the major difference is that residual risk increases more dramatically the farther the portfolio is positioned from the market portfolio. However, the increase is relatively small in comparison to the increase between figures 1 and 2. The exclusion of short-sales, underlying both figures 2 and 3, greatly increases minimal residual variance when compared to the unconstrained case reported in figure 1. Also, the more stringent bounds make it more difficult to attain a portfolio that is highly divergent from the market portfolio. For instance, it is impossible to achieve a portfolio beta of 1.5 in this case, whereas portfolio betas up to 1.7 were achievable for the less restrictive bounds.
FIGURE 3
RESIDUAL VARIANCE

Holdings Constrained: \( 0 \leq h \leq \min\{w_{SP} + 0.05, 4w_U\} \)

Locus of minimum residual variance at various yields

Locus of minimum residual variance at various betas
3.1. Comparison of Model Predictions with Simulated Results

Figure 4 shows contours of residual variance computed for the same portfolios analyzed in figure 2, but computed by means of a historical simulation. The shapes of the contours are relatively similar, and the simulations produce similar levels of residual variance to the model predictions.

The model predictions of residual variance due to common factors (or extra-market covariance) are based on the period from 1966 to August 1974, and the simulations are based on the period from May 1972 to April 1977. Thus, there is an overlap between the model period and the simulation period, although it is relatively small. The model is fitted to all common stocks and then extrapolated to the collection of stocks in a particular portfolio. Although fitting the model to all stocks incorporates far more information than the simulation, the extrapolation of typical risk characteristics to the characteristics of the portfolio may not be perfect.

In particular, for a portfolio that has been optimized with respect to the model, one would expect a downward bias of unknown magnitude in the model variance predictions. This follows because the optimization will tend to exploit model errors, where present, to obtain spurious reductions in variance. (Of course, these spurious reductions are attained along with true reductions where the optimization algorithm identifies opportunities for variance reduction that are correctly offered by the model.) The fact that simulated values do not always exceed model predictions suggests that the bias is not dominant.

3.2. Portfolios Mixing Equities with the Risk-Free Asset

The analysis thus far requires that the portfolio be 100% invested in equities. When admixture with the risk-free asset is allowed, portfolios with beta below 1 can be achieved with lower residual risk. Figure 5 shows the modification to figure 2 resulting from inclusion of the risk-free asset. The heavy, curved line is the frontier for inclusion of the risk-free asset: to the left of the frontier, the risk-free asset
FIGURE 4
SIMULATION RESIDUAL VARIANCE
Holdings Constrained: $0 \leq h \leq w_{SP} + .05$

Minimum model residual risk

Yield %

0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0 1.1 1.2 1.3 1.4 1.5 1.6 1.7

2.7 10.9 41.0 95.6
MODEL RESIDUAL RISK--RISK-FREE ASSET ADMISSIBLE

Holdings Constrained: $0 \leq h \leq w_{SP} + 0.05$

Locus of minimum residual variance portfolios at various yields

Frontier for inclusion of risk-free asset

Locus of zero residual variance

Yield

0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0

β
appears in minimum-variance portfolios, while to the right it does not. To the right of the frontier, the residual variance function coincides with figure 2. The straight line extending to the left toward the risk-free asset is the locus of \( y/\beta \) policies that can achieve zero residual risk through a mixture of the risk-free asset with the market portfolio. The contour lines of constant residual risk are roughly parallel to this. They are extensions of ellipsoidal contours to the right of the frontier that are shown in figure 2. Thus, when the risk-free asset is admissible, the tight curves in figure 2 that close the ellipses leftward of the frontier are replaced by nearly horizontal lines.

3.3. Alternative Upper Bounds and Restricted Universes

The upper bounds on holdings in the analysis thus far were either \( w_{sp} + .05 \) or \( \min\{w_{sp} + .05, 4w_U\} \). All assets, including the large companies and the composites of smaller companies, were available for investment. The result was a fairly wide range of achievable yields and betas, with tolerable residual risk. Furthermore, the \( y/\beta \) policy could be pushed quite far from the market value before the limit of infeasibility was attained. Approximate boundaries of feasibility are shown in figures 2 and 3: \( y/\beta \) policies beyond these limits can only be obtained with larger holdings than allowed by the upper bound.

Table 1 shows the results of limitations on the universe of attainable assets, or more restrictive upper bounds on holdings. First, many institutional investors find it difficult to maintain positions in small companies, so it is interesting to ask what residual risk levels can be maintained if the composite assets are unavailable and the holdings are limited to the 400 largest companies in the sample. Second, it is interesting to consider whether the more restrictive upper bounds introduced in section 2.5 seriously hinder attainment of the nonstandard \( y/\beta \) policies.
TABLE 1
EFFECT OF CHANGED UPPER BOUNDS AND RESTRICTED UNIVERSE ON RESIDUAL VARIANCE

<table>
<thead>
<tr>
<th></th>
<th>Target $\beta = 1.3$</th>
<th>Target $\gamma = 7%$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>A. All Assets Admissible</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Upper bound:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$2w_U$</td>
<td>INFS</td>
<td>INFS</td>
</tr>
<tr>
<td>$3w_U$</td>
<td>39.3</td>
<td>INFS</td>
</tr>
<tr>
<td>$\min{w_{SP} + .05, 4w_U}$</td>
<td>31.1</td>
<td>30.2</td>
</tr>
<tr>
<td>$w_{SP} + .05$</td>
<td>22.5</td>
<td>14.5</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>B. 400 Largest Companies Only</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Upper bound:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$2w_U$</td>
<td>INFS</td>
<td>INFS</td>
</tr>
<tr>
<td>$3w_U$</td>
<td>INFS</td>
<td>INFS</td>
</tr>
<tr>
<td>$\min{w_{SP} + .05, 4w_U}$</td>
<td>38.9</td>
<td>32.2</td>
</tr>
<tr>
<td>$w_{SP} + .05$</td>
<td>24.4</td>
<td>15.4</td>
</tr>
</tbody>
</table>

Note: The table gives residual variance of annual logarithmic return ($\times 10,000$). Infeasible portfolios are indicated by "INFS." $w_U \equiv (\text{company capitalization}) \div (\text{total capitalization of sample}).$ $w_{SP}$ is the proportion in the S&P 500 index.

Two interesting cases were selected for detailed analysis. These are a target yield of 7%, with no preference as to beta, and a target beta of 1.3, with no preference as to yield. The minimum residual variance solutions to these two problems, with the previous upper bounds and all assets admissible for purchase, appear in the third and fourth rows of the table. These solutions correspond to the residual risk function given in figures 2 and 3. Reading upward in the table, the effect of restrictive upper bounds is seen. Residual variance rises, and the goals become infeasible as the holding restrictions tighten. The increase in residual risk can be regarded as the cost of the restriction on holdings.
Panel B of table 1 shows the attainable solutions when investments are made only in the 400 largest companies. Achievable residual risk rises still more, and infeasibility occurs at less restrictive bounds on holdings.

4. TOTAL PORTFOLIO VARIANCE AND THE RISK/REWARD TRADE-OFF

Figures 6, 7, and 8 show the total portfolio variance function, as predicted by the model for 100% equity portfolios. Figure 6 reports the case where holdings are not constrained. Figures 7 and 8 introduce increasingly restrictive upper bounds. Total portfolio variance is computed as:

\[ \sigma_p^2 = \beta_p^2 (20.8)^2 + \omega_p^2, \]

where \( \omega_p^2 \) is the portfolio residual variance, and 20.8% is the forecast of S&P 500 annual standard deviation. Figure 9 shows simulated total variance for the portfolios reported in figure 7.

Figures 7 and 8 also examine optimal portfolios for investors in four cases. These portfolios are optimal in the sense of achieving an efficient trade-off between total return and total variance. Before considering these aspects of the figures, preliminary issues connected with the determinants of expected portfolio return must be discussed.

4.1. Four Cases of Expected Reward

Consider two alternative forms of the Capital Asset Pricing Model:

\[ E(r_p) = r_F + \beta_p (E(r_M) - r_F), \]

\[ E(r_p) = r_F + \beta_p (E(r_M) - r_F - p_y (y_M - r_F) + p_y (y_F - r_F)), \]

where \( r_p \) is the rate of return on an asset or portfolio, \( r_M \) is the rate of return on the market portfolio, \( r_F \) is the yield on the risk-free asset (equal to the risk-free rate), \( p_y \) is the increase in expected total return per unit increase in yield, or the "abnormal"
FIGURE 6
TOTAL VARIANCE WITH UNCONSTRAINED HOLDINGS

Yield %

75

63.8

62.1

Global minimum variance portfolio

Beta
TOTAL VARIANCE WITH VARIOUS FAMILIES OF ISOREWARD LINES AND ASSOCIATED EXPANSION PATHS

Holdings Constrained: 0 ≤ h ≤ w_{SP} + .05
FIGURE 8
TOTAL VARIANCE

Holdings Constrained: $0 \leq h \leq \min\{w_{sp} + .05, 4w_u\}$
SIMULATED TOTAL VARIANCE

Holdings Constrained: \( 0 \leq h \leq w_{SP} + 0.05 \)
compensation to yield. Equation (7) is the basic Capital Asset Pricing Model, in the absence of taxation, and the latter model is Brennan's (1970) generalization to allow for differential taxation of yield and capital gains. Since the CAP:M is the special case of CAP:T, with \( p_y = 0 \), it will be identified by the notation \( p_y = 0 \). For the CAP:T, we will take the estimated coefficient \( p_y = .395 \), obtained by Rosenberg and Marathe (1978), under the assumption of zero abnormal compensation for specific variance, for the period 1931–1966.

A tax-exempt investor experiences after-tax return equal to pretax return. A taxable investor, on the other hand, experiences after-tax return that may be derived as follows. Let \( t_c \) be the tax rate on capital gains. Let \( t_y \) be the tax rate on yield. Let \( c_p = i_p - y_p \) be portfolio capital gains. Let \( \mu_M = E[i_M] - i_F \) be expected excess market return. Then, after-tax return, \( i_p^A \) is:

\[
i_p^A = (1-t_y)y_p + (1-t_c) c_p .
\]

Taking expectations and substituting the difference between expected return and yield for capital gains, one obtains:

\[
E[i_p^A] = (1-t_y)y_p + (1-t_c)(E[i_p] - y_p)
\]

Substitution for \( E[i_p] \) from (8) results in:

\[
= (1-t_y)y_p + (1-t_c)(i_F - y_p + (\mu_M - p_y y_M - i_F))\beta_p + p_y (y_p - i_F)
\]

\[
= (1-t_y)y_p + (1-t_c)(p_y - 1)(y_p - i_F) + (1-t_c)(\mu_M - p_y y_M - i_F)\beta_p
\]

\[
= (1-t_c)(1-p_y)i_F + (t_c t_y + p_y (1-t_c))y_p + (1-t_c)(\mu_M - p_y y_M - i_F)\beta_p .
\]

At this time of flux in tax law, appropriate values for the tax rates are not clear. For illustration, we will assume \( t_c = .35 \) and \( t_y = .7 \), the highest federal tax rates. These rates do not include state tax and do not reflect the effective reduction in capital gains tax due
to potential deferral of payment until realization of the capital gain. As mentioned above, \( i_F = 0.0437 \) and \( y_M = 0.0461 \).

The implied effective portfolio returns for tax-exempt and taxable investors under the two capital asset pricing hypotheses are collected in table 2, together with their respective effective portfolio variances. This shows the adjustment for taxation of the uncertain capital gains,

\[
V = (1 - t_c)^2 \sigma_p^2,
\]

and assumes that yield is certain.

Table 2, panel 2, reflects the substitution of expected market excess return, \( \mu_M = 0.06 \), or 6% per annum. With this substitution, expected reward in all four cases is a linear function of a beta and yield.

### Table 2

<table>
<thead>
<tr>
<th>Four cases of Risk and Reward</th>
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| 1. Tax-exempt, \( p_y = 0 \) | \( 0.0437 + \beta_p \mu_M \) | \( \sigma_p^2 \) |
| 2. Taxable, \( p_y = 0 \)   | \( 0.0284 + 0.65\beta_p \mu_M - 0.35y_p \) | \( 0.4225 \sigma_p^2 \) |
| 3. Tax-exempt, \( p_y = 0.395 \) | \( 0.0264 + \beta_p (\mu_M - 0.00095) + 0.395y_p \) | \( \sigma_p^2 \) |
| 4. Taxable, \( p_y = 0.395 \) | \( 0.0172 + \beta_p (0.65\mu_M - 0.00062) - 0.0933y_p \) | \( 0.4225 \sigma_p^2 \) |

Assuming \( \mu_M = 0.06 \)

| 1. Tax-exempt, \( p_y = 0 \) | \( 0.0437 + 0.06\beta_p \) | \( \sigma_p^2 \) |
| 2. Taxable, \( p_y = 0 \)   | \( 0.0284 + 0.039\beta_p - 0.35y_p \) | \( 0.4225 \sigma_p^2 \) |
| 3. Tax-exempt, \( p_y = 0.395 \) | \( 0.0264 + 0.0591\beta_p + 0.395y_p \) | \( \sigma_p^2 \) |
| 4. Taxable, \( p_y = 0.395 \) | \( 0.0172 + 0.0384\beta_p - 0.0933y_p \) | \( 0.4225 \sigma_p^2 \) |
Each case can be represented by a family of "isoreward" lines: each line in a family is a locus of \( y/\beta \) policies which return the same expected reward. For example, under the tax-exempt case, \( v_y = 0 \), the expected return does not depend on yield and, hence, isoreward lines are vertical.

4.2. The Risk/Reward Trade-off

Figure 10 is useful in interpreting figures 7 and 8. In the figure, point A is a \( y/\beta \) policy that can be obtained with a mixture of the risk-free asset and the equity market portfolio. Consequently, it has zero residual risk. The curved line passing through point A, labeled \( V_0 \), is a contour of constant total variance. The straight vertical line passing through point A is a line of constant expected return for the tax-exempt \( v_y = 0 \) case. Point A, lying at a tangency of the two curves, is the \( y/\beta \) policy which offers maximum expected return at risk \( V_0 \).

Point B corresponds to a higher yield target, \( y_1 \), with the same level of excess return \( (\mu_0) \) as portfolio A. Portfolio B is the optimal portfolio, with yield \( y_1 \) and excess return \( \mu_0 \): It has the lowest possible risk among such portfolios. Its risk, \( V_1 \), is higher than the risk, \( V_0 \), of portfolio A. The cost of the increase in yield, expressed in terms of greater risk, is \( V_1 - V_0 \).

Portfolio C is the optimal portfolio that achieves yield \( y_1 \) at the same level of total risk \( V_0 \) as the original portfolio. Since risk is held constant, to achieve the increase in income we must sacrifice total return in the amount of \( \mu_0 - \mu_1 \). The nonstandard yield, \( y_1 \), is either obtained by an increase in risk, \( V_1 - V_0 \), or a decrease in total return, \( \mu_0 - \mu_1 \), or some combination of these.

Now the contents of figures 8 and 9 should be readily comprehensible. Both figures refer to the situation where the investor is 100% invested in equity, so that admixture with the risk-free asset is not available. In case 1, where the isoreward lines are vertical,
FIGURE 10

THE TRADEOFF AMONG TOTAL RETURN, RISK, AND YIELD
FOR THE TAX-EXEMPT INVESTOR, ASSUMING $p_y = 0$

- $M$ is the market portfolio, with yield $y_M$ and expected excess return $\mu_M$.
- $A$ is the optimal portfolio with excess return $\mu_0$ (and total risk $V_0$) when yield is not relevant.
- $B$ is the optimal portfolio with the same excess return $\mu_0$ at higher yield $y_1$: $V_1 - V_0$ is the necessary increase in total risk.
- $C$ is the optimal portfolio with the same total risk as the original at higher yield $y_1$: $\mu_1 - \mu_0$ is the necessary reduction of expected return.
the expected return is proportional to beta. Consequently, among those portfolios with given total variance, the one with the highest expected reward is the one with the greatest beta. As the risk aversion of the investor changes, he will be ready to accept greater and greater variance in pursuit of the expected reward offered by the market. Without knowing the risk aversion of the investor, we cannot determine the exact optimum portfolio. However, this portfolio is known to lie along the "expansion path" that passes through all points of tangency between the isovariance contours and the vertical isoreward lines. Conversely, every such point of tangency will be the optimal portfolio for an investor in case 1 with some degree of risk aversion. This expansion path is none other than the locus of minimum residual variance portfolios at different beta targets, which was already shown in figures 2 and 3 and which is repeated in figures 8 and 9. (If cash were admissible, the expansion path would be the line of zero diversifiable risk.)

Next, isoreward lines are shown for each of the remaining three cases, as are the expansion paths applicable to those cases. For example, a tax-exempt investor, believing that \( p_y = .395 \), would wish to select some portfolio on the expansion path at the top of the figures. This path is the locus of portfolios that maximize the reward in case 3 at varying levels of total risk. The expansion paths for the other cases can be interpreted similarly.

4.3. Discussion

Consider, first, the situation of a taxable investor with a small pool of funds, who is precluded from short-selling. With a small dollar amount to invest, large positions in individual assets can be attained easily, and a maximal attainable holding 5% greater than the S&P 500 proportion is, if anything, unreasonably restrictive. With short-sales excluded, the minimum holding is zero. Hence, the constraints underlying figure 7 are appropriate. If the investor accepts the CAP:M, with \( p_y = 0 \), then case 2 (of table 2) applies. If \( p_y = .395 \), case 4 applies. The expansion paths for these two cases (the loci of solutions
for differing risk aversion) are the two horizontal curves at the bottom of figure 7. Each path is drawn through the points of tangency between the isovariance contours and the isoreward contours for that case. (The portfolio lying along each locus are found by the optimization algorithm, with targets for expected return established as the isoreward lines.)

The locus for case 4 lies at higher yields, since the substantial positive coefficient, \( p_y = 0.395 \), almost offsets the tax disadvantage, reducing the after-tax inferiority of yield from 0.35 in case 2 to 0.0933 in case 4. A different value of \( p_y \), intermediate between these two, would result in an expansion path lying between the two paths that are shown.

Continuing the discussion of the taxable investor, suppose that the pool of funds to be invested is large. This situation would arise, for instance, if we were concerned with equilibrium behavior for the aggregate of all investors in the highest tax bracket. The aggregate pool would then amount to many billions of dollars and would be a significant proportion of the aggregate capitalization of all equities. Restrictive upper bounds on holdings of assets are now appropriate. For example, if the total pool is one-twentieth of aggregate capitalization (roughly $45 billion), then the largest conceivable holding is twenty times the proportion of the asset in total capitalization; this limit is reached when all outstanding shares are held by the investors in question. It is then quite reasonable to impose a still more restrictive upper bound, since it is improbable that the group will obtain all shares. A maximum of four times proportional capitalization seems to be a representative choice. This constraint is implemented in figure 8.

When figure 8 is compared with figure 7, the isovariance contours are seen to curve more sharply to the left. The upper bounds are binding for some assets and force the use of less perfectly diversifying assets to obtain the \( y/\beta \) policy. Residual variance increases at any nonstandard \( y/\beta \) policy where the constraints are binding, and increases at an increasing rate as more atypical policies are sought. The increased curvature of the isovariance contours, with no change in
the isoreward lines, causes tangency at higher yields. The optimal solutions are closer to the market portfolio in figure 8, and the expansion paths lie higher. The upward shift is roughly 30 basis points of yield.

Interestingly, the situation is far different in figure 6, where short-sales are allowed. For the model of table 2 to apply in this situation, the dividends advanced by the short-seller to the buyer must be treated as an ordinary business expense, deductible from ordinary income and resulting in tax savings of .7 times yield. When short-sales are allowed under this assumption, the expansion paths for case 2, and particularly for case 4, exist at negative yield, off the bottom of the figure. The interpretation of this result is as follows. When unlimited short-sales are available, portfolios with negative total yields are readily constructed (without substantial residual variance), and it is beneficial for a taxable investor to maintain large short-sales of high-yielding stocks. The taxable investor turns himself into a dividend-issuing entity, thus sheltering income from other sources while earning extraordinarily high capital gains, on average, from his large positive holdings in low-yielding, earnings-retaining companies. The tax savings from the resulting ordinary expenses offset the increased residual variance. This is not a practicable outcome for a participant in the real economy. However, it is an interesting consequence of the assumption of unlimited short-selling.

Turning to the tax-exempt investor, similar but reversed reasoning applies. Yield is now indifferent or desirable rather than undesirable. For \( p_y = 0 \) (case 1), the tax-exempt investor is indifferent between capital gains and dividends, isoreward lines are vertical, and the expansion path is the locus of minimum residual risk portfolios at various betas that passes through the market portfolio. For \( p_y = .395 \) (case 3), yield is highly preferred, and the expansion path lies very high: above the top of the figure for the unrealistic unconstrained case in figure 6, quite high for the looser constraints in figure 7, and somewhat lower for the tighter constraints in figure 8.
4.4. Equilibrium

The pair of expansion paths for taxable and tax-exempt investors illustrates the nature of market equilibrium in the presence of unequal taxation of dividends and capital gains when short positions are excluded. One can visualize the supply of all equities apportioned between taxable and tax-exempt investors. The most highly taxed investors lie on expansion paths below the market portfolio; the tax-exempt investors lie well above. Investors in other tax brackets lie on expansion paths intermediate between the two extremes.

The complete derivation of equilibrium may be sketched as follows. The upper bounds on holdings, instead of being assigned arbitrarily, are derived from market-clearing conditions: for each asset, aggregate demand must equal the outstanding supply. Equilibrium prices are such that the market is cleared in each asset. Equivalently, expected return for each asset is set so as to equilibrate demand and supply. For the equilibrium set of expected returns, only the loosest bounds are relevant: no portfolio may hold more than 100% of supply, and no portfolio may hold a negative position. When a holding of an investor group lies at the upper bound, the group must have bid the price of the asset so high that all other investors find it optimal to hold zero positions. This will rarely occur. Generally, the market-clearing prices will be such that the available supply is apportioned among many investor groups. For low-yielding assets, larger proportions are normally held as the tax bracket increases; for high-yielding assets, the reverse is expected.

The subtlety of the problem lies in the fact that the exact distribution of an asset among investor groups depends upon its contribution to the residual variance of their portfolios. One example may be useful to clarify this point. Suppose that one industry, uncorrelated with other industries and exhibiting substantial residual variance, makes up a significant portion of the market portfolio. Hence, any portfolio not holding the market proportion in that industry will be exposed to residual variance. Suppose that all but one of the companies in that industry are high yielding, while the exceptional company is low yielding. Then, the exceptional company is unusually valuable to taxable
investors, as the sole asset offering this form of diversification at a low yield. One would expect its price to be bid up more than the typical low-yielding company, resulting in lower equilibrium expected return; consequently, this asset would be concentrated in the portfolios of the most highly taxed investors. Other low-yielding assets, without the same unique diversifying role, might be distributed throughout the portfolios of taxable investors. A company that is, itself, a large proportion of the market, so that any portfolio not holding it is exposed to substantial specific risk, might well appear in all portfolios--taxable and tax-exempt--particularly if its yield is average. This can occur because the tax advantage associated with its yield is not great enough to offset the specific variance that would arise from a zero position.

In principle, the computation of equilibrium would be possible as soon as data were available to partition the investment community by tax bracket and risk aversion, so that the wealth of relevant investor groups is established. However, the computation is two orders of magnitude more difficult than the ones undertaken in this paper: the optimization problems of the several investor groups must be solved simultaneously for a posited vector of expected rewards, and the reward vector must then be modified iteratively until the solution converges to the point where all markets are cleared. It is to be hoped that this project will be undertaken at some later date.

4.5. A Caveat

The results in this paper are indicative of those which would be obtained with perfect predictive models for expected return, variance of return, and yield, but it must be remembered that the models are not perfect. The predictive model for expected return uses an approximate consensus figure for expected excess market return ($\mu = .06$) and assumes a premium for yield that is arbitrarily asserted to be a linear function of yield, with estimated coefficient $p_y = .395$ having a substantial standard error. The predictive model for investment risk
has been carefully estimated and employs Bayesian adjustments that should minimize bias; nevertheless, it incorporates certain simplifying assumptions, is estimated subject to error, and is limited—as all models must be—as by the validity of the fundamental and market-related descriptors that are used to characterize assets.

The prediction rule for yield is the most arbitrary, being equal to the previous twelve months' dividends divided by current price. This rule is not a minimum-mean-square-error predictor and lacks the Bayesian adjustment that would remove prediction bias for atypical assets. A Bayesian adjustment to the prediction rule would take the form \( \hat{y} = y_M + c(y - y_M) \), where \( y \) is yield as reported in this paper, and \( \hat{y} \) is the best prediction. The adjustment coefficient "c" is ordinarily less than unity. Its effect is to draw the predictions in toward the market norm. We have not estimated this coefficient but conjecture, from other experience, that its value is roughly \( .8 \). Thus, the best prediction for individual asset and portfolio yield may diverge from the market norm by only four-fifths the amount stated in the tables. Consequently, in interpreting the figures, the yield scale should be contracted inward toward the market norm to obtain minimum-mean-square-error predictions of subsequent yield. For example, if the correct contraction is a factor of \( .8 \), then a reported yield of \( 9.6\% \), which is \( 5\% \) above the market norm, corresponds to a predicted yield of \( 8.6\% \), \( 4\% \) above the norm.

The best test of the validity of the results is prospective study of experienced future returns for the constructed portfolios. This study must await the accumulation of a long enough data sample to provide reasonable estimates of experienced beta, yield, and residual variance. The historical simulation provides partial verification of beta and residual variance, since the simulation period (May 1972-April 1977) does not greatly overlap the model-fitting period (April 1966-August 1974). However, although the coefficients of the variance
model were estimated in this prior period, the predictions are based on the product of these coefficients with descriptors of the companies that were computed as of April 1977. These descriptors do reflect information through April 1977 and therefore overlap the simulation period. Hence, there is some tendency for the simulation to confirm the model prediction, and it cannot be regarded as an independent confirmation.

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