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WELFARE ASPECTS OF OPTIONS AND SUPershARES

by

Nils H. Hakansson

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WELFARE ASPECTS OF OPTIONS
AND SUPERSHARES*

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I. INTRODUCTION

The last few years have witnessed a substantial proliferation of new securities in extant financial markets. In particular, the possibilities laid open by the creation of option contracts have captured the imagination of both investors and academics. The result has been an explosive growth in option trading in organized markets and in the literature dealing with the equilibrium relationship between the value of an option and the value of the underlying security. At the same time, the welfare implications of various option contracts have been comparatively neglected, receiving the attention of only a few writers, e.g. Schrems (1973), Ross (1976), and Hakansson (1977).

In the absence of market friction, the "complete market" concept stands as the symbol of ultimate efficiency. A financial market is complete when investors can, in effect, trade in Arrow-Debreu (1964, 1959) certificates; this would be the case when the number of linearly independent securities is the same as the number of visualized time-states of the world and short sales are unrestricted. The much-heralded allure of a complete market rests on the unique ability of such a market to achieve a Pareto-efficient allocation of resources among investors under fully heterogeneous preferences and beliefs. Despite this decided virtue there are at least four important reasons why complete markets have not come into existence, giving rise to the view that the Arrow-Debreu model is "... in the nature of a magnificent tour de force, enriching our insight, but with a somewhat strained relation to reality" [Koopmans (14, p. 327)]. The first obstacle is the difficulty of reaching agreement (a priori) on a set of acceptable time-states. The second problem has to do with the subsequent determination of which state actually occurred, by no means a trivial task. The third obstacle haunting any implementation of a complete market has aptly been dubbed "the moral hazard": the temptation, not unknown in the insurance industry, to bring about by illicit means a
particularly fortuitous state. The final major problem has to do with the
enormity of the number of states, and consequently the number of securities,
that, by most calculations, must be available for such a market to function
properly, with its concomitant demands on information processing, calculation,
and transactions.

The preceding alone suggests that a full-fledged analysis of efficiency
must take transaction costs, including the limited information processing
abilities of individuals, into account. As Hahn (1971) for example has observed,
recognition of transaction costs may imply that certain potentially valuable
markets are better kept closed.

The purpose of this paper is to address some of the welfare issues asso-
ciated with option contracts, particularly options written on the market
portfolio. [Market portfolio options have recently captured the interest not only
of organized exchanges contemplating the opening of regular markets in such options,
but also of academics searching for operational rules in the area of capital
budgeting—see e.g. the recent papers of Banz and Miller (1977) and of Breeden
and Litzenberger (1977)]. While transaction and information costs are not
modelled in a full-fledged manner, much of the analysis is motivated by the
existence of such costs. Section II reviews some of the basic implications of
options written on individual securities and on portfolios while Section III
summarizes the central welfare implications of options keyed to the market
portfolio. The latter may take several forms, and the key differences between
puts and calls and supershares, which are more elementary in nature, are
identified in Section IV. Section V then moves the analysis to a multi-period
context and considers the economies of trading available, in terms of the
required number of securities, by moving to a sequence of markets from an
efficient market opening only once. An attempt to identify the conditions
under which sequential trading in current consumption and supershares to
end-of-period wealth alone is sufficient to achieve full efficiency is also
made. Section VI examines some of the welfare effects that arise when investors employ different partitions of the state-space and focus on subsets of securities, and Section VII summarizes the paper.

II. ON THE POWER OF PUTS AND CALLS

Consider first a single period setting. There is a set \( J \) of primary securities in positive supply (issued by firms), a set \( I \) of investors, and a set \( S \) of \( n \) different states of the world \( s \). Production decisions have been fixed and the payoff per share from security \( j \) in state \( s \) is \( a_{js} \geq 0 \); the \( m \) by \( n \) matrix \( [a_{js}] \) will be denoted by \( A \) and its rank by \( r(A) \). When the financial market is augmented by additional securities issued by investors and/or financial intermediaries, the augmented set of securities will be denoted \( J' \) and the resulting payoff matrix will be called \( A' \). If \( Z_j \) is the number of shares of security \( j \) outstanding, aggregate wealth in state \( s \) is

\[
W_s = \sum_{j \in J'} Z_j a_{js} = \sum_{j \in J} Z_j a_{js} > 0, \quad \text{all } s.
\]

In this section, we assume that investors distinguish only among events associated with differential payoffs on securities. An analysis of the case when beliefs and preferences may be tied to other descriptors, such as war and peace and different degrees of inflation, will be taken up in Section III.

As an example, suppose there are two firms, each with 10 shares outstanding of a single security paying either 1, 2, or 3 per share, with each payoff combination being possible. We must then distinguish between 9 states and the situation can be pictured as in the upper part of Table I. Since there are 9 states and only 2 securities, we have a clear case of an incomplete and thus (with arbitrary beliefs or preferences) inefficient market.

As is well known, options provide a means whereby the market can be augmented. Consider first options on single securities. As shown by Ross (1976), whenever \( a_{js} > 0 \) for all \( s \), any set of such options is equivalent to a (full) set of calls or a (full) set of puts; thus, it suffices to consider calls \(^2\) (if \( a_{js} = 0 \)
TABLE I

Matrix of Payoffs

<table>
<thead>
<tr>
<th>j ^a</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>2_j</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>2</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>3</td>
<td>2</td>
<td>3</td>
<td>10</td>
</tr>
</tbody>
</table>

\[ W_8 = \begin{array}{cccccccc}
30 & 30 & 40 & 40 & 40 & 50 & 50 & 60 \\
\end{array} \]

Superstates s* 1 2 3 4 5

---

Individual security options:

On security 1:

\[ e=1: \begin{array}{cccccccc}
0 & 0 & 0 & 1 & 0 & 1 & 2 & 1 & 2 & 2 \\
\end{array} \]

\[ e=2: \begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
\end{array} \]

On security 2:

\[ e=1: \begin{array}{cccccccc}
0 & 0 & 0 & 1 & 0 & 2 & 1 & 0 & 2 & 1 & 2 \\
\end{array} \]

\[ e=2: \begin{array}{cccccccc}
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
\end{array} \]

Market portfolio options: (1/10):

\[ e=2: \begin{array}{cccccccc}
0 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 4 \\
\end{array} \]

\[ e=3: \begin{array}{cccccccc}
0 & 0 & 0 & 1 & 1 & 2 & 2 & 3 \\
\end{array} \]

\[ e=4: \begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 \\
\end{array} \]

\[ e=5: \begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{array} \]

Supershares: (1 1 0 0 0 0 0 0 0 0)

\[ 2: \begin{array}{cccccccc}
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array} \]

\[ 3: \begin{array}{cccccccc}
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
\end{array} \]

\[ 4: \begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
\end{array} \]

\[ 5: \begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{array} \]
for some $s$, puts alone are sufficient). In the example, there are only 2 non-trivial calls on each security, those with exercise prices $e = 1$ and $e = 2$. Their respective payoffs are shown in the second part of Table I. The augmented market $A'$ now consists of 6 securities, with rank $\text{rank}(A') < 6$. But since there are 9 states, a full set of options on individual securities is clearly incapable of bringing about a fully efficient allocation under arbitrary probability beliefs and/or preferences.

Note that the number of aggregate wealth levels $W_s$ in our illustration is only 5 even though there are 9 states. We will refer to states which differ only in aggregate wealth as superstates, denoted $s^k$; $S^k$, with $n^k$ elements, is thus a partition of $S$. A fraction of the $W_s$ vector now represents the payoff on the "market portfolio"; the 4 non-trivial (call) options on the "market" have exercise prices of 2, 3, 4, and 5 (see third part of Table I). Thus, augmenting the regular securities with options on the market portfolio leaves us with 6 securities and also fails to achieve full allocational efficiency under arbitrary beliefs and/or preferences.

For further perspective, let me generalize slightly on the above numerical example. Suppose each security has $v$ different payoffs, that these payoffs are equidistant and adjacent (as in Table I), and that each payoff combination is possible. Under the state-space assumption of this section, the number of states $n$ is then $v^m$ ($m$ is the number of regular securities). The number of non-trivial options on individual securities is $(v - 1)^m$ so that the augmented market would have $v^m$ securities. The number of superstates is $v^m - (m - 1)$, giving rise to $v^m - m$ non-trivial options on the market portfolio. Thus, with only 10 primary securities, there will be $10^{10}$ or 10 billion states, yet by creating a full set of individual security options or a full set of market portfolio options the augmented market will have only $10 \times 10 = 100$ securities. The power of puts and calls on individual shares or on the market, or both, thus appears limited indeed in terms of completeness.
even if the bulk of the 10 billion combinations should have zero probability of occurrence.

In this perhaps gloomy picture, one bright spot should be noted. By permitting options on portfolios, one can indeed bring $A'$ to full rank ($r(A') = n$). In particular, one can, under the state-space assumption of the present section, construct a portfolio $z$ of regular securities such that $zA = b$ where $b_s \neq b_s'$, all $s \neq s'$, i.e. such that each state has a unique payoff. If $b_s > 0$ for all $s$, this in turn makes it possible to write $n - 1$ different calls (or puts) on this portfolio, bringing the rank of $A'$ to $n$.

More formally,

Theorem 1 [Ross (1976)]. A fully efficient market can be achieved by supplementing the primary securities with options written on a single portfolio of primary securities if (and only if) no two columns in $A$ are the same, i.e. investors' preferences and beliefs distinguish only among states with differential payoffs among primary securities.

Note that in the preceding illustration involving 10 primary securities, each assuming one of 10 values, the primary market can be supplemented with $10^{10} - 1$ non-trivial options on the critical portfolio even though, as noted, there is only room for 90 non-trivial options on individual securities. As a practical matter, the critical portfolio may of course be difficult to identify.

III. OPTIONS ON "THE MARKET"

We now drop the assumption that investors distinguish only among events associated with differential payoffs on securities. For example, assume that there are 2 securities each paying only 1 or 2 and that investors also distinguish between "war" and "peace", with each combination viewed as possible. There are then 8 states, as shown in Table II. Unfortunately, it is no longer possible to create a complete market via calls (or puts); such instruments can at most bring the rank of $A'$ to 4, the number of different columns in $A$ based on regular securities only. Standard options, whether based on individual securities or on portfolios, can only distinguish events associated with


TABLE II

Matrix of Payoffs

\[
\begin{array}{cccccccccc}
& 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & z_j \\
J_2 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 10 \\
J_3 & 2 & 1 & 1 & 2 & 2 & 1 & 1 & 2 & 2 & 10 \\
\text{Additional dimension} & P & W & P & W & P & W & P & W \\
WS & 20 & 20 & 30 & 30 & 30 & 40 & 40 & 40 \\
Superstates s^* & 1 & 2 & 3 \\
\end{array}
\]

Market portfolio options (1/10):

\[
\begin{array}{cccccccc}
e=2 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 2 \\
e=3 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
\end{array}
\]

differential payoffs on securities.

We now turn to the conditions under which augmentation by options on the market portfolio alone yields full efficiency.

**Theorem 2** [Hakansson (1977)]: A fully efficient market can be achieved by supplementing the primary securities with (a full set of) calls (or puts) written on the market portfolio if (and only if)

(i) investor preferences depend only on wealth (but are otherwise arbitrary) and

(ii) probability beliefs \( \pi_{is} \), conditional on aggregate wealth \( W \), are homogeneous, i.e. \( \pi_{is|W} = \pi_{s|W} \) for all investors \( i \in I \).

What this theorem says is that if preferences depend only on wealth but are otherwise fully heterogeneous and there is full heterogeneity of beliefs with respect to the superstates but homogeneity of beliefs conditional on each superstate, then the optimal holdings \( w_{is}^* \) in a complete market will be such that

\[
(2) \quad w_{is}^* = w_{is}^*, \quad \text{for all } i \text{ whenever } W_s = W_s. 
\]
But a market augmented with options on the market portfolio is sufficient to obtain this very allocation; the market portfolio itself plus the \((n^*-1)\) non-trivial options that can be written on it in effect constitutes a complete market with respect to the superstates. Thus, the scale of the market can be drastically reduced when the "superstate conditions" (i) and (ii) hold without loss of efficiency. More generally, state distinctions not based on different payoffs on primary securities can be ignored to the extent that their probability assessments conditional on aggregate wealth are homogeneous whenever (i) holds.

It should also be noted that moving to homogeneity of beliefs with respect to superstates as well does not further reduce the need for securities:

**Corollary 1.** Given (i), augmenting the primary securities with a full set of calls (or puts) on the market portfolio is necessary to achieve full efficiency even when beliefs are completely homogeneous.\(^5\)

In effect, then, Theorem 2 represents an important intermediate situation between "strong homogeneity" and "full heterogeneity". "Strong homogeneity" here refers to the case when two securities, one riskless and one risky, are sufficient to achieve the benefits of a complete market; in the single-period context, this situation occurs under either of the following conditions:

(a) homogeneous probability beliefs and utility belonging to the HARA-class,\(^6\) with the exponent the same for all investors (unless utility is negative exponential);

(b) homogeneous beliefs belonging to the separating class and utility functions that are compatible [Ross (1975)].

The reason for this result, as is well known, is that the above conditions cause all investors to hold the "market" combined with borrowing or lending. Thus, referring to our earlier 10-security illustration of Section II, while "strong homogeneity" necessitates trading in only two markets and full heterogeneity with respect to differential security payoffs requires trading in \(10^{10}\)
securities, the intermediate case of the rather limited homogeneity of Theorem 2 requires trading in only 100 (or 91) securities. What is ruled out by the very nature of conditions (i) and (ii), of course, is differential exposure to particular developments at the individual security level.

IV. CALLS ON THE MARKET VS. SUPERSHARES

Referring back to Tables I and II, the market augmentation accomplished by calls on the market portfolio is not unique; we have already mentioned that an equivalent augmentation can be achieved via (a full set of) puts. The simplest equivalent augmentation is by use of \( n^* - 1 \) "supershares", shares which pay 1 if a given superstare \( s^* \) occurs and nothing otherwise (see bottom of Table I for an example). More precisely, the subspace spanned by \( A_{CM} \) (the primary securities plus a full set, or \( n^* - 1 \), calls on the market portfolio) is identical to the subspace spanned by \( A_S \) (the primary securities plus \( n^* - 1 \) or \( n^* \) supershares).

Supershares, like options, may of course be written by investors themselves or they may be issued by an intermediary holding the market portfolio (see e.g. Hakansson (1977)). Thus, even though market portfolio options and supershares are completely equivalent in a world without friction, there may well be significant welfare differences when transaction costs and collateral tied to resources are taken into account. Unfortunately, our theory is rather incomplete in the presence of imperfect markets and a serious step forward in this respect, however desirable, is not visualized in this paper. Nevertheless, certain differences which clearly relate to the collateral question and to transaction costs are readily identified. In what follows, it will be convenient, though not necessary, to assume that the superstare conditions (i) and (ii) of Theorem 2 hold so that both markets are fully efficient in the ideal (frictionless) sense we have considered.

In the real world, the moral hazard is alive and well, causing, among other things, trust in those who owe to be less than perfect. As a result, the
proceeds from a short sale plus a margin must generally be held in a non-earning or restricted escrow account. Unfortunately, short positions are typically necessary in \( A'_{CM} \) (primary securities plus calls on the market portfolio) as well as in \( A'_{SI} \) (primary securities plus supershares issued by investors themselves). To see this, let

\[(3) \quad w_{is^*} > 0\]

be investor i's optimal payoff in superstate \( s^* \) and \( z_{CM} \) and \( z_{SI} \) his optimal portfolio of securities in the respective markets. Then

\[
 z_{CM} A'_{CM} = z_{SI} A'_{SI} = w_i^* > 0 .
\]

Note, however, that both \( z_{CM} \) and \( z_{SI} \) will generally have negative components, i.e. require the investor to go short in some assets; the main exception occurs in the extreme homogeneity case in which the demand for options is zero and all investors desire the market portfolio.

When (3) holds, short positions on the part of individual investors can be entirely avoided, however, by the simple device of having a financial intermediary, or "superfund", acquire a fraction of the market portfolio against which it issues a matching set of supershares. The investor now simply makes a long investment in some portfolio \( z_i \) of regular securities such that

\[
z_i A' = b_i < w_i^* ,
\]

supplementing this portfolio with

\[
w_{is^*} - b_{is^*} > 0
\]

supershares of type \( s^* \). More formally,

Theorem 3 [Hakansson (1977)]. Whenever (3) holds, there exists a fraction \( \delta < 1 \) of all primary securities (the market portfolio) which, if held by a superfund that issues supershares against them, eliminates the need for individual investors to go short in any security.

If the real costs of the contemporary constraints on short sales and of the monitoring of short positions exceeds the real costs of operating a superfund,
then the augmented market structure \( A'_{SS} \) (primary securities plus supershares issued by a superfund) would be preferable to either of the augmented market structures \( A'_{CM} \) and \( A'_{SI} \). In this evaluation, one of the offsetting difficulties (costs) associated with \( A'_{SS} \) is the initial pricing of supershares issued by the fund under the market processes in contemporary use. However, recent applications of option pricing theory to options on the market portfolio appear to offer considerable promise in this regard [see e.g. Garman (1976), Banz and Miller (1977), and Breeden and Litzenberger (1977)].

Let us now reorient our thinking in the direction of transaction costs. Our knowledge of the welfare impact of such costs is scant, partly because the appropriate functional form is unclear. Loosely speaking, real-world transaction fees tend to increase with quantity but also to differ greatly among types of securities (fees for a given dollar transaction in bonds are substantially less than for the corresponding dollar value in options, for example). We shall therefore be content to make some fairly modest observations.

Let us return to the augmented market structures \( A'_{CM} \), \( A'_{SI} \), and \( A'_{SS} \), where transaction costs were assumed to be zero. Under endowment neutrality (i.e. endowments across states are the same in each market for all investors), an equilibrium in \( A'_{CM} \) is an equilibrium in \( A'_{SI} \) and in \( A'_{SS} \), and conversely (this condition may be called strong market equivalence).\(^{10}\) For any equilibrium, then, the investor must trade from \( w_{iS^*} \) to \( w_{IS^*} \) in superstate \( s^* \). If supershares are available, as in \( A'_{SI} \) and \( A'_{SS} \), he can do this by simply buying \( A_{IS^*} \) supershares (selling if \( A_{IS^*} \) is negative), where

\[
A_{IS^*} = w_{IS^*} - w_{iS^*}.
\]

But in \( A'_{CM} \), he can only do this by trading \( z_i \) for \( z_i \), where \( z_i \) is such that

\[
\sum_j (z_{ij} - \overline{z_{ij}}) a_{js} = A_{IS^*} \quad \text{all } s^*
\]

where, except in unusual cases, we would also have
\[ \sum_{j} |z_{ij} - \bar{z}_{ij}| a_{js} > |A_{is*}| \quad \text{some } s^*, \]

which implies a need for offsetting additions and reductions in superstate holdings. Since (implicit and explicit) superstate prices are the same in the three markets, we obtain

**Theorem 4.** Consider the (strongly equivalent) market structures \( A_{CM}^i, A_{SI}^i, \) and \( A_{SS}^i. \) Then the minimal dollar amount of trading of an investor \( i, T_i = \sum_{j \in J} |z_{ij} - \bar{z}_{ij}| p_j, \) in reaching a given equilibrium satisfies

\[ T_{CM} \geq T_{SI} = T_{SS} \quad \text{all } i. \]

Moreover, \( T_{SI} \) and \( T_{SS} \) represent the minimum of trading necessary in any market structure under the superstate conditions (i) and (ii).

Thus, to the extent that the real costs of market transactions tend to increase with the volume of trading, supershares appear to offer a clear advantage over the use of puts or calls on the market portfolio.

V. MARKET PORTFOLIO OPTIONS IN SEQUENTIAL TRADING

So far, our analysis has been limited to a single period setting. I shall now attempt to identify conditions under which the preceding results concerning options on the market and supershares generalize to a multi-period setting. In particular, I will try to focus on the case in which similar economies of trading are realizable in each period via a sequential opening of selected markets.

Each consumer-investor \( i \) now chooses consumption levels \( c_{its} \) for each time-state pair according to his conditional preference functions

\[ U_{i \alpha} (c_{ils}, c_{i2s}, \ldots, c_{its}) = \sum_{t} u_{it} (c_{its}), \]

where utility in period \( t \) depends only on the level of consumption in that period and \( c_{its} \) denotes his endowment. There are \( n_t \) different states \( s \) at time \( t \) and \( n'_t \leq n_t \) different levels of aggregate consumption \( C_{ts} \) in period \( t \).

For simplicity, assume a complete market and let \( T = 3. \) The situation may then be described as follows:
Market Structure M1:

At $t = 1$, each investor $i$ solves

$$\text{(5)} \quad \max \{ u_{i1}(c_{i1}) + \sum_{s \geq 2} u_{12}(c_{i2s}) + \sum_{s \geq 3} u_{13}(c_{i3s}) \},$$

subject to the budget constraint

$$\text{(6)} \quad c_{i1} + \sum_{t > 2} \sum_{s \geq t} c_{its} P_{ts} = \overline{c}_{i1} + \sum_{t > 2} \sum_{s \geq t} \overline{c}_{its} P_{ts} = w_i,$$

as a price-taker, where the prices $P_{ts}$ (with $P_1$ as numeraire) are such that the optimal holdings satisfy the market clearing conditions

$$\text{(7)} \quad \sum_{i} c_{its} = C_{ts} \quad \text{all } t, s.$$

In the case of an interior allocation, the first-order conditions are

$$\text{(8)} \quad u_{i1}'(c_{i1}) = \lambda_i \quad \text{all } i,$$

$$\text{(9)} \quad \pi_{its} u_{it}'(c_{its}) = \lambda_{it} P_{ts} \quad \text{all } i, s, t \geq 2,$$

where the $\lambda_i$ are Lagrange multipliers.

Theorem 2 now generalizes directly to the present case, that is, the optimal allocation satisfies

$$\text{(10)} \quad c_{its'} = c_{its} \quad \text{for all } i \text{ and for all } s' \text{ such that } C_{ts'} = C_{ts} \quad \text{if and only if}$$

$$\text{(11)} \quad \pi_{its}|C = \pi_{ts}|C \quad \text{all } i, t, \text{ and } s.$$

Applying (10) and (11) to (9), it then follows immediately that the equilibrium prices $P_{ts}$ are such that (for any $i$)

$$\text{(12)} \quad P_{ts} = \pi_{its}|C P_{tC},$$

where $P_{tC} = \sum_{s \geq t} P_{sC}$. Using (12) to value endowments for which $\overline{c}_{its} \neq \overline{c}_{its'}$ for all $s$ and $s'$ such that $C_{ts} = C_{ts'}$, we obtain

Theorem 5 [Breeden and Litzenberger (1977)]. Market structure M1 is equivalent to a market structure which only distinguishes among states that differ in aggregate consumption at each date, i.e. the number of securities traded can be
reduced from $1 + n_2 + n_3$ to $1 + n'_2 + n'_3$ without loss of welfare if and only if (11) holds.

As in Theorem 2, the reduction in the number of markets encompassed by Theorem 5 may be very substantial.

In general, $n_{t+1} > n_t$ and $n'_{t+1} > n'_t$ since the state description at $t + 1$ would typically incorporate the actual state occupied at $t$ -- presumably $\pi_{i,t+1,s}$ depends on where the economy was at $t$, and perhaps earlier. In particular $\pi_{i,t+1,s}$ may be zero for some $s$ given that the economy was in state $\bar{s}$ at $t$. But this also means that the number of securities in which trading is needed can, more generally, be reduced by sequential opening of selected markets instead of opening the complete market $M_1$ once and for all at the beginning.

For example, market structure $M_1$ can be exactly duplicated in terms of realized allocations by first opening, at $t = 1$, a market in which first period consumption plus claims to $t = 2$ wealth are traded for each $(t = 2)$ state, followed, at $t = 2$, by a second trading round in second period consumption and the now relevant claims to third-period consumption (or, in general, $t = 3$ wealth). The crucial requirement for equivalence is that consumer-investors have "rational expectations", at $t = 1$, with respect to the $t = 2$ prices of third-period consumption claims, etc. More formally, the sequential market case corresponding to $M_1$ can be described as:

Market Structure $M_2$:

Round 1: At $t = 1$, each investor $i$ solves

$\text{(13)} \quad \max \{ u_{i1}(c_{i1}) + \sum s \pi_{i2s} \bar{f}_{i2s}(w_{i2s}) \} \equiv f_{i1}(w_{i1})$

subject to the budget constraint

$\text{(14)} \quad c_{i1} + \sum s w_{i2s} p_{2s} = c_{i1} + \sum s \bar{w}_{i2s} p_{2s} \equiv w_{i1}$

as a price-taker, where the prices $p_1 \equiv 1$ and $p_{2s}$ are such that all markets clear, i.e.
\(E \sum_{i} c_{i1} = C_{1}, E \sum_{i} w_{i2s} = E \sum_{i} \bar{w}_{i2s} \quad \text{all } s; \)

In (15), initial wealth claims \(\bar{w}_{i2s}\) are given by

\(\bar{w}_{i2s} = \sum_{s} \phi_{s\mid s} c_{i3s} \quad \text{all } i,\)

where \(\phi_{s\mid s}\) is the anticipated price, given state \(s\) at \(t=2\), of state \(s\), \(t=3\) wealth with state \(s\), \(t=2\) consumption as numeraire.

**Round 2:** At \(t=2\), given that \(s\) has occurred, each investor \(i\) solves

\[\begin{align*}
\max \{u_{i2}(c_{i2}) + \sum_{s} \pi_{13s|s} u_{i3}(c_{i3s})\} = f_{i2s}(w_{i2s})
\end{align*}\]

subject to the budget constraint

\(c_{i2} + \sum_{s} P_{3s} c_{i3s} = w_{i2s}\)

as a price-taker, where the prices \(P_{2s} = 1\) and \(P_{3s}\) are such that all markets clear, i.e.

\(\sum_{i} c_{i2} = C_{2s}, \sum_{i} c_{i3s} = C_{3s} \quad \text{all feasible } s|s,\)

and the expectations \(\phi_{s\mid s}\) in round 1 are fulfilled, i.e.

\(P_{3s} = \phi_{s\mid s}\)

With respect to round 2, note that if endowment \(w_{i2s}\) in (18) is equal to \(c_{i2s} + \sum_{s} P_{3s}/P_{2s} c_{i3s|s}\) in (6), then the allocations to periods 2 and 3 in M2 will be the same as in M1 since the allocation \(\{c_{i2s}, c_{i3s|s}\}\) fact chosen in M1 via the last two utility of consumption components. Function \(f_{i2s}(w_{i2s})\) then represents consumer-investor \(i\)'s induced utility of wealth at \(t=2\) in state \(s\) since it measures the value, in equilibrium, of an optimal strategy from \(t=2\) on given the resource allocation \(\{w_{i2s}\}\) among investors.

The round 1 problem in M2 is to allocate current resources to current (\(t=1\)) consumption \(c_{i1}\) and end-of-period (\(t=2\)) wealth, \(w_{i2s}\), using the indirect (equilibrium) utility function \(f_{i2s}\) to value the latter. But
to do so, that part of the endowment which is realizable after \( t = 2 \) must be valued, somehow, via anticipated future prices; if the anticipated conditional prices \( \phi_s | \bar{s} \) are in fact equal to the realized price ratios \( P_{3s} / P_{2s} \) in (6), then wealth \( w_{11} \) will be the same in (6) and in (14), along with prices \( P_{2s} \) and the allocation \( c_{11} \). Summarizing, we can state

Lemma. To each equilibrium in a complete market which opens only once with trading in current consumption and \( n_2 + n_3 + \ldots + n_T \) different claims to future consumption levels, there corresponds a sequence of markets with trading, at each \( t < T \), only in current consumption and in claims to end-of-period wealth based on \( n''_{t+1} \) securities, where \( n''_{t+1} \leq n'_{t+1} \), all \( t \), which yield the same allocation given rational expectations of future prices. Moreover, if

\[
\pi_{its} | \bar{s} = 0 \quad \text{some } s, \bar{s}, \text{ for all } i
\]

we also have

\[
n''_t < n'_t \quad t > 2 .
\]

Observe that (22) holds for any stochastic process which satisfies (21), which in turn does not rule out processes in which future movements of the economy are strongly dependent on past realizations. Note also that no consumer-investor in \( M_\ell \) wishes to alter his holdings via a new round of trading once a state has been realized, as the Lemma indeed verifies (again, this requires a sufficiently rich state description in \( M_\ell \), of course).

Finally, it is clear that the anticipation of future prices must be "correct" for equivalence between \( M_\ell \) and \( M_2 \) to hold. This is indeed a strong assumption but one which also appears difficult to dispense with in any analysis in which markets open more than once.

The significance of the Lemma, of course, rests on the fact that \( n''_t \) will tend to become much smaller than \( n'_t \) as \( t \) increases for the kinds of stochastic processes that the economy might be expected to obey. For example, suppose that preferences depend on the level of consumption in that period only (as in (4)), that (11) holds, and that aggregate consumption obeys a Markov process;
assume further that there are 50 (adjacent) states \( s \) such that \( C_{t+1,s} > C_{t,s} \) and another 50 such that \( C_{t+1,s} \leq C_{t,s} \) with positive probability for all \( t \) and \( s \). Then \( n'_t = 100(t-1), \) \( t > 2 \), by Theorem 5, while \( n''_t = 100 \), all \( t > 2 \). Thus when \( T = 20 \), for example, a one-time market would require trading in 19000 securities (plus consumption at \( t = 1 \)) while a sequential market, opening 19 times, can achieve the same allocation via trading in current consumption and 100 securities, or 101 items, in each round. Formally,

**Theorem 6.** Assume that

(i) investor preferences for consumption in any period depend only on the consumption level in that period

(ii) probability beliefs, conditioned on aggregate consumption \( C_t \), are homogeneous, and

(iii) aggregate consumption \( C_1, C_2, C_3, \ldots \) is regarded by each investor as obeying a Markov process.

An optimal allocation can then be obtained via sequential trading, in each period \( t \), in current consumption and \( n''_{t+1} \mid C_t \leq n'_{t+1} \) securities, where \( n''_{t+1} \mid C_t \) is the number of values aggregate consumption can assume at \( t + 1 \) given that period \( t \) aggregate consumption is \( C_t \).

(When (i) and (ii) hold but not (iii), it is of course possible for \( n''_t \) to be greater than \( n'_t \).)

Of perhaps greater interest than mere state reduction is the question of whether the \( (n''_{t+1}) \) relevant future states (in the time \( t \) market) are in fact supersets as we defined them in Section II. Aggregate wealth in state \( s \) at \( t + 1 \), \( W_{t+1,s} \) equals, using (15) say,

\[
W_{t+1,s} = \sum_i W_{1,t+1,s}.
\]

Thus, if there is a one-to-one mapping from the set of relevant states at \( t + 1 \), given \( s \) at \( t \), to the set of aggregate wealth levels \( W_{t+1,s} \) given \( s \) at \( t \), the claims traded at \( t \) are in effect supershares, i.e. claims that distinguish only among aggregate wealth levels at \( t + 1 \). But if
\begin{equation}
W_{t+1,s|s} = W_{t+1,s'|s}
\end{equation}
for some \( s \) and \( s' \),
then supershares would not distinguish \( s \) and \( s' \) at \( t+1 \) so that a market limited to supershares may give rise to inefficiencies. More formally, we can state

**Theorem 7.** Consider a sequence of markets that is equivalent to a complete market opening once and denote the relevant state space for claims to \( t+1 \) wealth, given state \( \bar{s} \) at \( t \), by \( S''_{t+1}(\bar{s}) \). Then the claims to \( t+1 \) wealth are equivalent to supershares if and only if there is a one-to-one mapping from \( S''_{t+1}(\bar{s}) \) to the set of aggregate wealth levels \( W_{t+1,s|s} \).

Comparing, for simplicity, \( W_{2s} \) for two states \( \bar{s} \) and \( \bar{s}' \) when \( T = 3 \), we get, using (23), (15) and (16),

\begin{equation}
W_{2s} = C_{2s} - \sum_{s} \phi_{s|s} C_{3s}
\end{equation}

and

\begin{equation}
W_{2s'} = C_{2s'} - \sum_{s} \phi_{s|s'} C_{3s}.
\end{equation}

It is apparent that (24) (equality of (25) and (26)) is not likely to be avoided unless conditions (i), (ii) and (iii) (of Theorem 6) hold. But (i), (ii) and (iii) are not sufficient to escape (24); one can readily construct a Markov process for aggregate consumption with the right kind of (negative) serial correlation for (24) to be valid. Consider, however, the case in which all investors view aggregate consumption as obeying a random walk (measured in growth rates, say). Then \( C_{2s} \neq C_{2s'} \), by Theorem 6; let \( C_{2s} > C_{2s'} \). For any growth rate \( g \), \( C_{3s} \) in (25) is then greater than the corresponding \( C_{3s} \) in (26) and any difference between the anticipated prices \( \phi_{s|s} \) and \( \phi_{s|s'} \), for fixed \( g \), now derives solely from the preference functions.

Whenever these latter differences are small, (25) would be greater than (26). Thus, sequential trading in current consumption and in supershares based on end-of-period wealth is most likely to assure full efficiency when conditions (i) and (ii) hold and aggregate consumption obeys a Markov process with non-
negative or weak negative serial correlation.

The above results should not be viewed as restricted to an exchange economy. There is nothing in the preceding which is inconsistent with the presumption that consumer-investors, in defining the applicable state-space, take the production decisions of firms into account.

VI. HETEROGENEOUS STATE PARTITIONS

It is customary in examining the efficiency of markets to view the state space and the set of securities as fixed (i.e. to assume that everyone is a priori interested in all of the securities, and agrees with a particular description of the state space and the payoffs that the various securities yield in each state) and to address the efficiency question in terms of investor beliefs (over states) and preferences (over outcomes). But this is clearly not necessary for markets to function or to be efficient, nor is it likely to be valid empirically. In particular, everyone is certainly not keeping track of the thousands of securities traded in extant markets or the enormous number of states encountered even in the simple examples of previous sections—it would be surprising if anyone is.

Returning to the single-period setting of Section III, let \( J_i' \) be the set of (augmented) securities considered by investor \( i \), let \( S_i \) be the state-space he takes as relevant, where \( S_i \) is a partition of \( S \) (with \( n_i \) "states"), and denote his estimated (scalar) payoff on security \( j \) in state \( s \) by \( a_{ij,s} \). We assume that each security is of interest to some investors, i.e.

\[
\bigcup_{i \in I} J_i' = J'
\]

and that there is no non-empty \( I' \subset I \) such that

\[
\bigcup_{i \in I'} J_i' \cap \bigcup_{i \notin I'} J_i' = \emptyset,
\]

i.e. totally segmented markets are ruled out. We also assume that the matrices \( A_i' \) are sufficiently "regular" in the sense of Hart (1974) for an equilibrium to exist, i.e. for arbitrage profits to be prevented. The resulting market
structure can then be described as

Market Structure M3:

$$\max \sum_{i \in S} w_i \left( \sum_{j \in J_i} a_{ij} \right)$$

subject to

$$\sum_{j \in J_i} z_{ij} p_j = \sum_{j \in J_i} z_{ij} p_j$$

for all $i$, where the prices $p_j$ are such that all markets clear, i.e.

$$\sum_{i} z_{ij} = z_j \quad \text{all } j \in J' .$$

Note that preferences have been made perfectly general in that they may depend on state $s$ as well as the level of wealth accruing to the investor. Note also that an investor's problem in the above formulation is non-trivial as long as there are at least two securities in $J_i$ and at least two states in $S_i$. An example at the extreme would be an investor who limits himself to the market portfolio (in the form of an index fund) and a risk-free asset and partitions $S$ into only two events ("market up" and "market down").

The necessary and sufficient condition for full efficiency is now clearly that each investor perceives the set of securities $J_i$ to be complete with respect to state partition $S_i$. Formally,

Theorem 8. M3 achieves full efficiency for arbitrary beliefs and preferences if and only if each matrix $A_i$ has full rank ($r(A_i) = n_i$).

The following corollary also has a bearing on Sections II and III.

**Corollary 2.** Let investor $i$'s state partition $S_i$ be such that it distinguishes only among a) the $v_j$ different levels of payoff on a single primary security $j$ (where $a_{ij} = a_{js}$ for all $i$ such that $j \in J_i$) or b) the $n*$ different levels of aggregate wealth $W_s$. Then M3 is fully efficient for arbitrary preferences, beliefs, and payoff matrices if and only if there is a full set of (put) options on each primary security and a full set of (call or put) options on the market portfolio.
Thus, the corollary says that only when investor attention is focused solely on the different payoffs on a single security or on the market portfolio can one be sure, under arbitrary beliefs, preferences, and payoff structures, that a full set of options on individual securities and on the market portfolio will achieve full efficiency.

The following result is also straightforward.

**Corollary 3.** Suppose \( J_i = J \) for all \( i \) and let \( S \) be the coarsest partition such that \( S \) is finer than \( S_i \) for all \( i \). Then augmenting the market with elementary options which distinguish among the \( \bar{n} \) elements in \( S \) is sufficient to achieve full efficiency under arbitrary preferences and beliefs.

A determination of partition \( \bar{S} \) and the number of its elements \( \bar{n} \) is clearly an empirical question. But it would not be surprising if the number of elements \( \bar{n} \) of \( \bar{S} \) is several orders of magnitude smaller than our \( n \) (of order \( v^m \)) of Section II, for example. In any case, Theorem 8 and Corollary 3 do seem relevant to the design of instruments for augmenting the set of primary securities. Partitions \( \bar{S}_f \) containing few elements and having the property that \( \bar{S}_f \) is finer than \( S_i \) for a large subset of investors \( i \) appear particularly relevant in view of Theorem 4. Whether supershares, for example, have this property is unclear but it would perhaps not be surprising if they do since they provide the only direct means of allocating aggregate wealth.

One factor which may cause investors to group a number of basic states into larger events is the degree of similarity among the payoffs. For example, some investors may only care to distinguish returns to the nearest 1/2%. As an example, if each security is seen as going up or down at most 50% (or remain unchanged) in a given period, only 200 calls would be necessary to establish a complete market with respect to the payoffs on any one security considered by itself. With 1000 securities, there would, assuming that all combinations can occur, technically be 200,001 superstates \((1000 \times 201 - 1000 + 1)\)--but a differentiation to the nearest 1/2% implies that 200 (or 201) supershares are
adequate, independently of the number of individual securities.

Two additional observations are in order. First, to the extent that the state-space partitions of some investors are finer than those of others, a hierarchical structure of securities would permit investors to limit their market participation in such a way that the number of securities in which they trade is commensurate with the coarseness of their state-space partitions. Second, there is no particular reason to believe that the state descriptors which are viewed as relevant would not go beyond the differential payoffs on the primary securities (recall the example in Section III). To the extent that such descriptors are associated with "aggregate" economic indicators, which might be difficult to influence, the concomitant moral hazard would appear to be significantly alleviated. Thus supershares which pay off in real rather than nominal terms, for example, do not appear to present a serious problem in this regard.\textsuperscript{13}

VII. SUMMARY

The preceding discussion, set in discrete time, may be summarized as follows:

1. Given a rich set of payoff combinations among regular securities, market augmentation via calls (or puts) on individual securities will fall far short of achieving a complete market even when states are solely associated with differential payoffs on primary securities. However, a full set of puts provides a simple means of completing the market with respect to state-space partitions differentiating only among individual security payoffs.

2. Calls (or puts) on portfolios have considerably greater power than options on individual securities in enriching the allocational possibilities. In particular, as shown by Ross (1976), there exists a single portfolio which, when a full set of options is written on it, completes the market provided all states are solely associated with differential payoffs on the primary securities.

3. Options on the market portfolio also have certain noteworthy properties.
The following holds in a state-space of arbitrary richness.

a. Calls (or puts) on the market portfolio provide a simple means (not available via options on individual securities) to complete the market with respect to the state-space partition based on superstates (states which differ in aggregate wealth) or what may loosely be termed "social risk".

b. A full set of calls (or puts) on the market leads to full efficiency under surprisingly unrestricted conditions, namely if (some) investors (base their decisions on the superstate partition of the state-space and the rest) have preferences that depend only on end-of-period wealth (but are otherwise arbitrary) and beliefs which, conditional on aggregate wealth only, are homogeneous [Hakansson (1977)].

4. Given rational expectations of future prices, a sequence of markets in current consumption and in claims to end-of-period wealth can achieve the same allocation as a complete market opening only once with, under plausible assumptions, a substantial reduction in the number of securities traded. When the mapping from the applicable (end-of-period) state-space to the corresponding set of aggregate (end-of-period) wealth levels is one-to-one, a full set of options on the end-of-period market portfolio is sufficient to guarantee full efficiency. This situation is most likely to occur when aggregate consumption obeys a Markov process with nonnegative or weak negative serial correlation, provided that in addition (i) preferences for consumption in any period depend only on the level of consumption in that period, and (ii) probability beliefs conditional on aggregate consumption are homogeneous.

5. In terms of feasible allocations, a given set of supershares is equivalent to an equal number of call (or put) options on the market portfolio. However, the following differences appear relevant in a world with transaction costs and default risk:

a. In trading to a new position, the amount of trading (measured by, for
example, the dollar value of shares bought and sold) is never greater and generally substantially smaller for supershares than for (call or put) options on the market portfolio.

b. Assuming positive desired holdings in each superstate, short positions can be entirely avoided with supershares by having them issued by a financial intermediary holding the market portfolio rather than by investors themselves. With call (or put) options on the market portfolio, short positions are generally necessary.

6. When investors make decisions on the basis of (personal) partitions of the state space, the need for additional securities may be drastically reduced, especially if there is a tendency for "similar" states to be lumped together, although there is every reason to believe that there will be substantial demand for securities based on several partitions of the overall state space. At the same time, to the extent that "relevant states" are associated with aggregate economic phenomena, the moral hazard tends to be alleviated so that states not based on differential payoffs on securities (for example, supershares based on real rather than nominal returns [Hakansson (1976)]) may in many cases be operational.
1. See e.g. the seminal paper by Black and Scholes (1973) and the surveys by Cox and Ross (1976) and by Smith (1976).

2. A call on security \( j \) with exercise price \( e \) provides a vector of proceeds with elements given by \( \max(0, a_{js} - e) \) all \( s \), while a put with exercise price \( e \) yields \( \max(0, e - a_{js}) \) in state \( s \).

3. \( \bar{S} = \{\bar{s}_1, \ldots, \bar{s}_K\} \) is a partition of \( S = \{s_1, \ldots, s_n\} \) if \( \bar{s}_k \subseteq S \), \( k = 1, \ldots, K \), \( \bar{s}_j \cap \bar{s}_k = \emptyset \) if \( j \neq k \), and \( \bigcup_{k=1}^{K} \bar{s}_k = S \).

4. For an interior allocation, the equilibrium conditions equate the marginal rates of substitution between \( w_{is} \) and \( w_{is'} \) for all investors, i.e., for any investors \( i \) and \( k \):

\[
\frac{\pi_{is}'u'_i(w_{is})}{\pi_{is}'u'_i(w_{is'})} = \frac{\pi_{ks}'u'_k(w_{ks})}{\pi_{ks}'u'_k(w_{ks'})},
\]

which, given (ii), immediately yields (2).

5. This is best seen with reference to footnote 4.

6. The HARA-class (hyperbolic absolute risk aversion) consists of the following utility functions of wealth with the properties \( u'(w) > 0 \), \( u''(w) < 0 \) (the first over at least a finite range of positive wealth):

\[
u(w) = \begin{cases} 
(1/y)(w + \alpha)^y, & y < 1 \\
-exp(yw), & y < 0 \\
-(a - w)^y, & y > 1, \ a \ large 
\end{cases}
\]

(decreasing absolute risk aversion)

(constant absolute risk aversion)

(increasing absolute risk aversion)

7. Spanning of the superstates can be achieved either with \( n^k \) supershares or with \( n^k - 1 \) supershares plus the market portfolio.

8. Sufficient conditions for (3) to hold are infinite marginal utility at some non-negative wealth level plus positive probabilities attached to each superstate.

9. In the sense that there exists a redistribution under \( A_{SS}' \) which leaves everyone better off than under \( A_{CM}' \) or \( A_{SI}' \).

10. For further details, see Hakansson (1978).

11. In a complete market (such as \( M \)), any dependencies on past histories must of course be incorporated into the state descriptions—such dependencies tend to greatly expand the number of states that must be distinguished.

12. This corresponds to the "no-easy-money condition" in Hakansson (1970).

13. For further details, see Hakansson (1976).
References


