Research Program
in Finance
WORKING PAPER SERIES

WORKING PAPER NO. 76

A NEW CLASSIFICATION
OF OPTION POSITIONS

by

Mark Rubinstein

Research Program in Finance Working Papers are preliminary in nature; their purpose is to stimulate discussion and comment. Therefore, they should not be cited or quoted in any publication without the permission of the author. Single copies of a paper may be requested from the Institute of Business and Economic Research.
RESEARCH PROGRAM IN FINANCE AT THE
WALTER A. HAAS SCHOOL OF BUSINESS,
UNIVERSITY OF CALIFORNIA, BERKELEY

The Research Program in Finance in the Walter A. Haas School of Business at the University of California has as its purpose the conduct and encouragement of research in finance, investments, banking, securities markets, and financial institutions. The present reprint and working paper series were established in 1971 in conjunction with a grant from the Dean Witter Foundation.

INSTITUTE OF BUSINESS AND ECONOMIC RESEARCH
Carl Shapiro, Director

The Institute of Business and Economic Research is an organized research unit at the University of California, Berkeley, whose mission is to promote research by faculty and graduate students in the fields of business and economics. The Institute carries out its mission by organizing programs and activities that enrich the research environment, administering extramural research awards, publishing working papers, and making direct grants for research.
A NEW CLASSIFICATION OF OPTION POSITIONS

by

Mark Rubinstein*

September 1978

*Mark Rubinstein is Associate Professor at the Graduate School of Business Administration, University of California, Berkeley. The author acknowledges support from the National Science Foundation (SOC 77-18087). Thanks are also extended to John Cox and Hayne Leland for useful discussions.
ABSTRACT

In response to the recent development of liquid secondary markets for puts and calls, a new classification of option positions is derived from the Black-Scholes option pricing analysis. The new terminology is then applied to the problem of screening out the "best" neutral covered position from all opportunities. Although the measures of position risk arising out of the new classification are derived as local measures, it is shown that they typically provide a useful and close approximation to position risk over finite holding periods.

I. Introduction

The conventional hedge diagram method of classifying option positions for warrants, puts, and calls has been described by Thorpe and Kassouf [1967] and Gastineau [1975]. This method begins by grouping together an underlying stock with its associated options. Then each stock-option position is described by a graph relating the profit or loss from the position, if held to expiration, to the price of the underlying stock on the expiration date of the option.

Although this method proves awkward when options with different maturities comprise the position, prior to the recent development of
organized option markets in the United States, this conventional treat-
ment of option positions was adequate for most purposes. Since, for
practical purposes, OTC options had to be either exercised or let expire,
it was reasonable to base a classification on the profits or losses from
holding a position to expiration.

However, with the introduction of liquid secondary markets for op-
tions, a new position classification was needed to cope with the impli-
cations of possibly early closing purchase or sale transactions. By us-
ing information derived from the Black-Scholes [1973] option pricing
analysis, this paper seeks to develop this classification.

In part, this paper is a report on procedures currently
in use by option market participants. Fischer Black, in his Op-
tion Letters of September and October 1976, as a part of his option pric-
ing service, describes many aspects of this new classification. So pop-
ular is the Black-Scholes option pricing formula that many floor traders
routinely use it, not only to value options, but to determine the expo-
sure of option positions to movements in the underlying stock price.

In section II, this new terminology for classifying option posi-
tions is introduced. Section III discusses the important special case
of neutral positions, and section IV shows how the new concepts for de-
scribing option positions can be used to screen neutral positions—both
related to the same underlying stock and across different stocks—for
the one which takes greatest advantage of the relative mispricing of op-
tion contracts. Although the measures of position risk are local meas-
ures, holding exactly only over an epsilon neighborhood around the current
stock price and time, section V shows that these local measures typically provide a useful and close approximation to position risk over finite holding periods.

II. Position Delta, Gamma, and Theta

Let $S$ represent the current underlying stock price, $K$ the striking price of an associated put or call, $t$ the option's time to expiration, $\sigma$ the standard deviation of the stock's continuously compounded rate of return, and $r$ the discrete one plus rate of interest. The current value of an option will be denoted generally by $V$, or by $C$ if it is a call and $P$ if it is a put.

Under certain conditions, which are now too well known to reiterate, Black and Scholes derived the following stochastic second-order partial differential equation to describe the movement of the option value through time:

$$\theta = (\ln r) V - (\ln r) S \Delta - \frac{1}{2} \sigma^2 S^2 \Gamma,$$

where $\Delta \equiv \partial V / \partial S$, $\Gamma \equiv \partial \Delta / \partial S$, and $\theta \equiv -\partial V / \partial t$. The solution of this equation for $V$, subject to boundary conditions, produces the Black-Scholes option pricing formula. The differential equation implies that, given the interest rate, the three quantities, $S \Delta$, $\sigma^2 S^2 \Gamma$, and $\theta$, will completely characterize the local movements in the option value $V$. That is, these quantities will completely determine the change in the value of the position over the next small interval of time.
The delta (Δ) of an option tells us how much the option value will change for a small change in the underlying stock, other things equal. For a call,

$$\Delta = \frac{\partial C}{\partial S} = N(h) \quad \text{where} \quad h = \frac{\ln(K/Sr^{-t})}{\sigma \sqrt{t}} + \frac{1}{2} \sigma \sqrt{t},$$

where $N(z)$ is the standard normal distribution function evaluated at $z$.

The call $\Delta$ is therefore always a positive number between zero and positive one. For a put, $\Delta = \frac{\partial P}{\partial S} = N(h) - 1$, always a negative number between zero and negative one. The stock itself can be regarded as a perpetual payout-protected call with zero striking price. For it, $\Delta = \frac{\partial S}{\partial S} = 1$.

The gamma ($\Gamma$) of an option tells us how much the option delta will change for a small movement in the stock price, other things equal. For a call or a put,

$$\Gamma = \frac{\partial \Delta}{\partial S} = \frac{1}{2 \sigma \sqrt{t}} N'(h),$$

where $N'(z)$ is the standard normal density function evaluated at $z$.

For both puts and calls, $\Gamma > 0$, and for stock, $\Gamma = 0$.

The theta ($\Theta$) of an option tells us how much the option value will change as its expiration date moves slightly closer, other things equal. For a call,

$$\Theta = \frac{\partial C}{\partial t} = -\frac{S \sigma}{2 \sqrt{t}} N'(h) - Kr^{-t}(\ln r)N(h - \sigma \sqrt{t}),$$

and for a put $\Theta = \frac{\partial P}{\partial t} = -\frac{\partial C}{\partial t} + (\ln r)Kr^{-t}$. The call $\Theta$ is
always positive, while the sign of the (European) put \( \theta \) is ambiguous.\(^1\)

For the stock, \( \theta = -\frac{\partial S}{\partial t} = 0 \).

If we now consider a possibly complex position in an underlying stock and its associated options, let \( i \) index the different securities in the position and \( N_i \) the number of units purchased of each security. By convention, \( N_i > 0 \) if securities are bought, and \( N_i < 0 \) if securities are sold short or written.

To locally characterize the position, we need only multiply the corresponding differential equation by \( N_i \) and add the equations together, yielding

\[
(\sum_i N_i \theta_i) = \ln r(\sum_i N_i V_i) - S \ln r(\sum_i N_i \Delta_i) - \frac{1}{2} \sigma^2 S^2 (\sum_i N_i \Gamma_i).
\]

The four parenthetical positions, from left to right, are called, respectively, the position theta, position value, position delta, and position gamma.

To illustrate these concepts, with only two securities comprising the position:

\[
\text{Position delta} = \frac{\partial \text{position value}}{\partial S} = N_1 \left( \frac{\partial V_1}{\partial S} \right) + N_2 \left( \frac{\partial V_2}{\partial S} \right) = N_1 \Delta_1 + N_2 \Delta_2.
\]

Thus, the position delta measures how exposed the position value is to movements in the stock price. If we think the stock price will rise,

\(^1\)For American puts, if there are no arbitrage opportunities, then quite generally, apart from the Black-Scholes analysis, \( P \) must be a nonincreasing function of \( t \).
we want it to be positive; if we think the stock price will fall, we want it to be negative. If we are uncertain about the direction of the stock price movement and want to insulate ourselves from this uncertainty, we want our position delta to be zero. In view of this, we have the following correspondence:

<table>
<thead>
<tr>
<th>Negative Delta</th>
<th>Bearish</th>
</tr>
</thead>
<tbody>
<tr>
<td>Zero Delta</td>
<td>Neutral</td>
</tr>
<tr>
<td>Positive Delta</td>
<td>Bullish</td>
</tr>
</tbody>
</table>

The position gamma measures the change in the position delta for small changes in the stock price. With just two securities,

\[
\text{Position gamma} = \frac{\partial \text{position delta}}{\partial S} = N_1 \left( \frac{\partial \Delta_1}{\partial S} \right) + N_2 \left( \frac{\partial \Delta_2}{\partial S} \right) = N_1 \Gamma_1 + N_2 \Gamma_2.
\]

The sign of the position gamma, particularly for delta-neutral positions, provides additional information. At zero delta, although the delta indicates the magnitude of profits and losses should be relatively small due to stock price movements, the delta tells us nothing about the conditions that yield profits and not losses. The position gamma, on the other hand, summarizes just the information we need: If it is negative, we only profit as long as the stock price remains relatively stable; if it is positive, we only profit from large movements in the stock in either direction.

Even if the stock price remains unchanged, as the expiration date approaches, the mere passage of time creates profits or losses in option
positions. The position theta measures how much the position value will change as time to expiration decreases, other things equal. To illustrate, with just two securities,

\[
\text{Position theta} = -\frac{\partial \text{position value}}{\partial S} = N_1 \left( -\frac{3V_1}{\partial t} \right) + N_2 \left( -\frac{3V_2}{\partial t} \right) = N_1 \theta_1 + N_2 \theta_2.
\]

The graphs in table 1 describe the twenty-seven logically possible delta-gamma-theta positions. The table indicates the effect on the value of a position (V) of a simultaneous change in the stock price (S) and a reduction in time to expiration (t). For example, for a bullish negative gamma, positive theta position, downward movements in the stock price cause the position value to decrease more quickly than upward price movements cause it to increase, and the position value increases for small downward stock price movements.

All elementary naked positions have direct correspondences, as shown in table 2.

<table>
<thead>
<tr>
<th>Position</th>
<th>Δ</th>
<th>Γ</th>
<th>Θ</th>
</tr>
</thead>
<tbody>
<tr>
<td>Long stock</td>
<td>+</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Short stock</td>
<td>-</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Buy call</td>
<td>+</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>Write call</td>
<td>-</td>
<td>-</td>
<td>+</td>
</tr>
<tr>
<td>Buy put</td>
<td>-</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>Write put</td>
<td>+</td>
<td>-</td>
<td>+</td>
</tr>
</tbody>
</table>
Table 1. Delta-Gamma-Theta Positions

<table>
<thead>
<tr>
<th>Time Bias</th>
<th>Bearish $\Delta &lt; 0$</th>
<th>Neutral $\Delta = 0$</th>
<th>Bullish $\Delta &gt; 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta &lt; 0$</td>
<td>$\Gamma &lt; 0$</td>
<td>$\Gamma = 0$</td>
<td>$\Gamma &gt; 0$</td>
</tr>
<tr>
<td>$\theta = 0$</td>
<td>$\Gamma &lt; 0$</td>
<td>$\Gamma = 0$</td>
<td>$\Gamma &gt; 0$</td>
</tr>
<tr>
<td>$\theta &gt; 0$</td>
<td>$\Gamma &lt; 0$</td>
<td>$\Gamma = 0$</td>
<td>$\Gamma &gt; 0$</td>
</tr>
</tbody>
</table>

This table shows the effect of simultaneous stock price movements and reduced time to expiration on position values. There is an interval around the current stock price $S$ and a reduction in time to expiration for which these descriptions are accurate. Caution: They may not apply to sufficiently large movements of the stock price away from its current value, or to sufficiently large reductions in the time to expiration.
III. Neutral Positions

By definition, a delta-neutral position has a zero delta. The neutral position ratio can be determined by setting the position delta to zero. With just two securities, for a neutral position, \( \left( N_1 / N_2 \right) = -\Delta_2 / \Delta_1 \). The neutral position ratio is the negative of the inverse of the ratio of the deltas.

Neutral position strategies are understandably popular among option floor traders. They permit the market-making function and the accumulation of inventories while minimizing risk. Neutral position strategies are also popular among investors who wish to take advantage of the relative mispricing among different options on the same underlying stock. These investors can concentrate their efforts on predicting stock volatility rather than the expected stock rate of return. However, in practice, transactions costs and occasional jumps in the stock price prevent these investors from maintaining a neutral position as the stock price changes. One strategy is to open a position with a neutral position ratio and revise it only when its exposure to movement in the stock price exceeds some critical level. A natural way to measure this exposure is by the absolute value of the position delta.

A reinforcing strategy is to select positions to begin with, which, in addition to neutrality, have deltas that are relatively insensitive to movements in the stock price. The absolute magnitude of the position gamma, measured at the neutral-position ratio, indicates how fast changes in the stock price will push the position delta past the
critical level and force revision of the position ratio. For a delta-neutral position of two securities, we can express the position gamma in terms of the individual security deltas and the "size" of the position.

Since \( N_1 \Delta_1 + N_2 \Delta_2 = 0 \), then the

\[
\text{Position gamma} = N_1 \Delta_1 \left[ \frac{\Gamma_1}{\Delta_1} - \frac{\Gamma_2}{\Delta_2} \right],
\]

where \( N_1 \), the number of shares or options that benefit from a rise in the stock price, represents the size of the position.

Although delta-neutral, zero-gamma positions cannot generally be constructed from only two securities, three or more different options on the same underlying securities provide enough flexibility to do so. For example, consider a complex "butterfly" spread in which a position is taken in three otherwise similar calls with different striking prices, \( K_1 < K_2 < K_3 \). By buying \( N_1 \) and \( N_3 \) of the extreme calls, and writing \( N_2 \) of the intermediate call, it is clearly possible to create a position such that \( N_1 \Delta_1 + N_2 \Delta_2 + N_3 \Delta_3 = 0 \) and \( N_1 \Gamma_1 + N_2 \Gamma_2 + N_3 \Gamma_3 = 0 \). Such a position has very little risk in terms of change in value, at least over the near future.

---

\(^1\)We can see, from the previous equation, that the position gamma (of a neutral position) will be zero if and only if, by happenstance, \( \frac{\Gamma_1}{\Delta_1} = \frac{\Gamma_2}{\Delta_2} \).
IV. Screening Neutral Positions

If we think a stock is significantly over- or underpriced, it may be better to take a naked option position rather than simply buy or short the stock. Since stock valuation is not our subject, we have nothing to say about techniques for locating mispriced stocks and therefore little to say about the advisability of naked option positions. However, we will attempt to identify which options are most over- and underpriced, relative to the market price of the stock, and we will discuss the implied covered option trading strategies.

To take advantage of these relative mispricings and, at the same time minimize risk, delta-neutral positions are a useful strategy. These positions are particularly appropriate if we feel fairly confident about our ability to forecast volatility but wish to avoid the more difficult task of estimating expected or mean rates of return on the underlying stock. As the derivation of the Black-Scholes formula illustrates, only if our position ratio is delta-neutral will we be able to insulate ourselves from inaccurate predictions of likely trends in stock prices.

Presuming we can identify mispriced options using the Black-Scholes formula for European options or a numerical technique for American options, how do we determine which options are most over- or underpriced? Of all the possible neutral positions in the same underlying security, how do we select the most promising? How do we compare neutral positions across different stocks? In this section, we apply our new terminology to answer these questions.
An obvious way to identify the most over- or underpriced option on
the same underlying security is to measure the absolute difference between
its value and price. Table 3 provides an illustration. By this criter-
ion, the JUL/40 puts are most underpriced (indeed, the only underpriced)
options. Since we will be buying underpriced options, we compare their
ask price to their value. Comparing the bid price to value, the JUL/40
calls are the most overpriced. This simple measure will tend to favor
long maturity, at-the-money options, and discriminate against short matur-
ity, out-of-the-money options. Moreover, the absolute-difference rule
implicitly assumes the number of options we buy or write will be the same
for each option.

Instead of the absolute difference, we might measure the relative
difference between value and price. This is just the absolute differ-
ence divided by the price. This will tend to favor out-of-the-money
options, such as the JAN/35 put. As before, this rule does not consider
the number of options we will buy or write, an important aspect of our
strategy.

We really want to compare options, holding fixed the effect of a
dollar movement in the stock on the option position. That is, we want
to measure the profit from option positions of equivalent risk. Ignor-
ing taxes, margin, and commissions, the adjusted position with the
greatest or lowest difference between value and price is the one we will
select. To standardize for risk, we use the option delta. Recall that
the delta measures the expected dollar change in the option price if the
stock price increases by a small amount. For example, to equalize the
** COVERED OPTION POSITION SCREEN **

DATE: 77/12/21
FORMULA: PC1

<table>
<thead>
<tr>
<th>EXPIRATION</th>
<th>ANN INT</th>
<th>ANN VOL</th>
<th>EX-DIV DATE</th>
<th>DIVIDEND</th>
</tr>
</thead>
<tbody>
<tr>
<td>78/01/20</td>
<td>.050</td>
<td>.300</td>
<td>78/01/05</td>
<td>.50</td>
</tr>
<tr>
<td>78/04/21</td>
<td>.050</td>
<td>.300</td>
<td>78/04/05</td>
<td>.50</td>
</tr>
<tr>
<td>78/07/21</td>
<td>.050</td>
<td>.300</td>
<td>78/07/05</td>
<td>.50</td>
</tr>
</tbody>
</table>

** TYPE SERIES BID ASK ** VALUE DELTA GAMMA THETA ** ER

<table>
<thead>
<tr>
<th>STOCK</th>
<th>/00</th>
<th>40.00</th>
<th>40.00</th>
<th>40.00</th>
<th>1.00</th>
<th>.00</th>
<th>.00</th>
</tr>
</thead>
<tbody>
<tr>
<td>CALL</td>
<td>JAN/35</td>
<td>4.63</td>
<td>4.75</td>
<td>.74</td>
<td>.92</td>
<td>.04</td>
<td>.00</td>
</tr>
<tr>
<td>CALL</td>
<td>JAN/40</td>
<td>1.25</td>
<td>1.38</td>
<td>1.20</td>
<td>.47</td>
<td>.11</td>
<td>.11</td>
</tr>
<tr>
<td>CALL</td>
<td>JAN/45</td>
<td>.06</td>
<td>.13</td>
<td>.11</td>
<td>.00</td>
<td>.04</td>
<td>.00</td>
</tr>
<tr>
<td>CALL</td>
<td>APR/35</td>
<td>5.50</td>
<td>5.63</td>
<td>5.44</td>
<td>.77</td>
<td>.04</td>
<td>.00</td>
</tr>
<tr>
<td>CALL</td>
<td>APR/40</td>
<td>2.50</td>
<td>2.75</td>
<td>2.52</td>
<td>.50</td>
<td>.04</td>
<td>.00</td>
</tr>
<tr>
<td>CALL</td>
<td>JUL/40</td>
<td>3.50</td>
<td>3.75</td>
<td>3.34</td>
<td>.51</td>
<td>.04</td>
<td>.00</td>
</tr>
<tr>
<td>PUT</td>
<td>JAN/35</td>
<td>.13</td>
<td>.19</td>
<td>.10</td>
<td>.07</td>
<td>.04</td>
<td>.31</td>
</tr>
<tr>
<td>PUT</td>
<td>JAN/40</td>
<td>1.50</td>
<td>1.63</td>
<td>1.54</td>
<td>.52</td>
<td>.11</td>
<td>.11</td>
</tr>
<tr>
<td>PUT</td>
<td>JAN/45</td>
<td>5.38</td>
<td>5.50</td>
<td>5.43</td>
<td>.91</td>
<td>.04</td>
<td>.00</td>
</tr>
<tr>
<td>PUT</td>
<td>APR/35</td>
<td>.75</td>
<td>.88</td>
<td>.87</td>
<td>.20</td>
<td>.04</td>
<td>.00</td>
</tr>
<tr>
<td>PUT</td>
<td>APR/40</td>
<td>3.00</td>
<td>3.13</td>
<td>2.88</td>
<td>.47</td>
<td>.06</td>
<td>.26</td>
</tr>
<tr>
<td>PUT</td>
<td>JUL/40</td>
<td>3.50</td>
<td>3.63</td>
<td>3.71</td>
<td>.45</td>
<td>.04</td>
<td>.18</td>
</tr>
</tbody>
</table>

BEST POSITION IN XYZ: PROFIT: 2009

DOLLAR DELTA: 12 DOLLAR GAMMA: 47 THETA: 26

SELL 96 PUT APR/40 @ 3.00
BUY 100 PUT JUL/40 @ 3.63
action in the JAN/45 and APR/40 calls, for every 12.5 JAN/45 contracts, we must hold 2 APR/40s. These positions have the same dollar action as one round lot of stock. Therefore, to compare positions of equivalent risk, for each option we calculate its **equalizing ratio**:

\[
ER = \frac{V - M}{|\Delta|},
\]

where \( M \) represents the market price of the option.\(^1\)

Since the stock has been assumed fairly priced for the purpose of screening, its equalizing ratio is always zero. For an option, let \( M_b \) be its bid price and \( M_a \) its ask price. Either (1) \( V < M_b \), (2) \( M_b < V < M_a \), or (3) \( M_a < V \). In the first case, if anything, we plan to sell the option at the bid price, therefore its \( ER = \frac{(V - M_b)}{|\Delta|} \). In the second case, we can neither buy nor sell at an expected profit, so \( ER = 0 \). In the third case, since we consider buying the option, \( ER = \frac{(V - M_a)}{|\Delta|} \). As a result, options with negative equalizing ratios are possible sells; those with positive equalizing ratios are possible buys; and those with zero equalizing ratios are considered fairly priced. For XYZ, the best sell is therefore the JAN/35 put and the best buy the JUL/40 put. The \( ER \) of \(-.32\) for the JAN/35 put means that selling \( \frac{1}{|\Delta|} = 14 \) puts is equivalent in expected dollar profit and dollar risk to selling one share of stock at $40.32.

\(^1\)|\(\Delta|\) is the absolute value of the delta. It insures that put deltas enter the definition of the equalizing ratio as positive numbers.
For a neutral basic position (i.e., one with a zero position delta that combines only two separately profitable related securities), the total equalizing ratio (ER) is simply the sum of the absolute values of the two security equalizing ratios. That is,

$$\overline{ER} = |ER_1| + |ER_2|.$$ 

Because the value-price differential for each security has been weighted by the inverse of its position ratio, the total equalizing ratio applies only to covered positions that are neutral.

At first glance, the best delta-neutral basic position would appear to be the one with the highest total equalizing ratio. On this basis, we would write a combination in JUL/40 calls and JAN/35 puts. To keep it neutral, we must write about $1 \div .51 = 2$ JUL/40 call contracts for every $1 \div .07 = 14$ JAN/35 put contracts written. Indeed, if our values are correct and we can costlessly revise our position with sufficient frequency, a position of exactly these amounts is equivalent in dollar expected profit and dollar risk to buying one round lot of stock at $39.69 and selling one round lot at $40.32. This riskless arbitrage opportunity has a certain profit of $.63.

However, transactions costs and occasional jumps in the stock price make very frequent revision impractical. Not only do we want to initiate positions with a zero delta, but we also prefer positions with deltas as insensitive as possible to the stock price. This will mean that changes in the stock price will not produce position deltas far from zero. The position gamma provides a natural measure of the delta
sensitivity. To minimize risk, we should compare zero delta positions with the same position gamma in absolute value. While the zero position delta constraint determines the ratio in which the two covering securities are to be held, the equal position gamma constraint determines the comparable "size" of the position.

As developed previously, the formula for the position gamma of a delta-neutral position in two securities is

$$\text{Position gamma} = N_1 \frac{\Delta_1}{\Delta_1} \left[ \frac{\Gamma_1}{\Delta_1} - \frac{\Gamma_2}{\Delta_2} \right].$$

To compare positions with equal gamma in absolute value, we can require

$$\left| N_1 \frac{\Delta_1}{\Delta_1} \left[ \frac{\Gamma_1}{\Delta_1} - \frac{\Gamma_2}{\Delta_2} \right] \right| = 1.$$  \hspace{1cm} (1)

Since the position is neutral, for a zero delta

$$N_2 = -N_1 \left( \frac{\Delta_1}{\Delta_2} \right).$$  \hspace{1cm} (2)

The expected profit from a position is simply the difference between position value and cost. That is,

$$\text{Profit} = N_1 (V_1 - M_1) + N_2 (V_2 - M_2).$$  \hspace{1cm} (3)

To measure the profit on a zero delta position with gamma ±1, $N_1$ and $N_2$ must simultaneously satisfy equations (1), (2), and (3). Substituting for $N_1$ and $N_2$ in equation (3),
\[
\text{Profit} = \pm \left| \left( \frac{\Gamma_1}{\Delta_1} - \frac{\Gamma_2}{\Delta_2} \right)^{-1} \right| \left( \frac{V_1 - M_1}{\Delta_1} - \frac{V_2 - M_2}{\Delta_2} \right).
\]

If we only consider positions with separately profitable pairs of securities (i.e., we write overpriced options and buy underpriced options), the sign of the profit must be positive. Moreover, we can then replace the second parenthetical term with the total equalizing ratio. As a result, for positions with positive profit,

\[
\text{Profit} = k(\overline{ER})
\]

where

\[
k \equiv \left| \left( \frac{\Gamma_1}{\Delta_1} - \frac{\Gamma_2}{\Delta_2} \right)^{-1} \right|
\]

The positive constant \( k \) tells us how much to expand the size of a zero delta position to compare positions with a gamma of \( \pm 1 \).

Table 3 permits comparison of all possible delta-neutral basic positions in XYZ stock and options. We select, as the "best position," the pair of securities with the highest profit at zero delta and equal gamma. By this criterion, we should buy a horizontal spread in APR/40 and JUL/40 puts. The number bought and sold are standardized so that we buy or sell 100 contracts on the side with the lowest option delta. Depending on available capital and requirements, this position may be scaled up or down accordingly.

If we are ranking options on different fairly priced stocks, we need to adjust our comparative positions for differences in stock volatility. Since we want to measure risk in dollars, not rates, we multiply...
the volatility by the current stock price. $\sigma_S$ is termed the stock dollar volatility. This converts the volatility of the stock's rate of return to the volatility of the stock price. The option dollar volatility is then the dollar volatility of the stock times the absolute value of the option delta, or $|\Delta|\sigma_S$.

The greater the dollar volatility of an option on one stock compared to an option on another, the more sensitive its value to movements of equal likelihood in its underlying stock. Therefore, when comparing options on different stocks, we normalize their expected profit for positions of equal dollar volatility. That is, we compare

$$\frac{V-M}{|\Delta|\sigma_S} = \frac{(ER)}{\sigma}.$$ 

The option for which this ratio is the highest is considered the best buy, and the option for which this ratio is the lowest the best sell. Just as the inverse of the option delta measures the number of contracts needed to equalize risk across options on the same stock, the inverse of the option dollar volatility measures the number of contracts needed to equalize risk across options to different underlying stocks. This normalized equalizing ratio then measures an option's dollar return per dollar of risk.

For an option position, we define its dollar delta as the product of the position delta times the stock's dollar volatility, and its dollar gamma or "dollar curvature" as the product of the position gamma times the square of the stock's dollar volatility. Unlike ordinary deltas and gammas, these can be compared across stocks.
The best overall delta-neutral basic position is the zero delta and equal dollar gamma position with the highest profit. In this case, it is convenient to equalize the dollar gamma by setting it equal to one. For purposes of screening options across different underlying stocks, we calculate:

\[
\text{Profit} = k(ER)
\]

where \( k = \left| \frac{\Gamma_1}{\Delta_1} - \frac{\Gamma_2}{\Delta_2} \right| \div (oS)^2 \)

Of course, since \((oS)^2\) is the same for all options to the same stock, this revised measure of profit preserves the ordering within positions related to the same stock. To prevent confusion, table 4 summarizes the concepts we have developed for evaluating option strategies.

For the best delta-neutral basic position for each stock, table 3 shows its dollar delta, dollar gamma, and theta per day. In particular:

Dollar delta = \( .3 \times 40 \times [(-9600) \times (-.47) + 10000 \times (-.45)] \div \sqrt{365} = 12 \).

Dollar gamma = \( (.3 \times 40)^2 \times [(-9600) \times .06 + 10000 \times .04] \div 365 = -47 \).

Theta = \[ (-9600) \times (-3) + 10000 \times (-2) \] \div 365 = 26.

The dollar delta is not zero due to rounding the number of contracts bought and sold to integers.
Table 4. Concepts for Analyzing Option Positions

1. Stock value/price: \( S \)
   Option price: \( M \)

2. Option value:
   \( V \)
   Option delta: \( \Delta \equiv \frac{\partial V}{\partial S} \)
   Option gamma: \( \Gamma \equiv \frac{\partial \Delta}{\partial S} \)
   Option theta: \( \Theta \equiv -\frac{\partial V}{\partial t} \)

3. Position value:
   \( N_1 V_1 + N_2 V_2 \)
   Position delta:
   \( N_1 \Delta_1 + N_2 \Delta_2 \)
   Position gamma:
   \( N_1 \Gamma_1 + N_2 \Gamma_2 \)
   Position theta:
   \( N_1 \Theta_1 + N_2 \Theta_2 \)

4. Stock volatility:
   \( \sigma \)
   Stock dollar volatility: \( \sigma S \)
   Option dollar volatility: \( |\Delta| \sigma S \)

5. Dollar delta:
   \( (N_1 \Delta_1 + N_2 \Delta_2) \times \sigma S \)
   Dollar gamma:
   \( (N_1 \Gamma_1 + N_2 \Gamma_2) \times (\sigma S)^2 \)

6. Equalizing ratio:
   \( ER \equiv \frac{(V - M)}{|\Delta|} \)
   Equalizing ratio: \( ER / \sigma S \)
   (across stocks)

7. Total equalizing ratio:
   \( \overline{ER} \equiv |ER_1| + |ER_2| \)
   (for separately profitable pairs of securities)

\(^a\)We might also define the position dollar volatility as the absolute value of its dollar delta.
V. Risk Measured over Finite Holding Periods

Despite our best efforts to choose positions of low dollar gamma, initially delta-neutral positions will inevitably become unbalanced. We now face the difficult trade-off between transactions costs and our exposure to risk if we do not restore neutrality. To access this trade-off, we need a way to measure the risk of a nonneutral position. The dollar delta would seem a natural way to measure this risk. However, by itself, the dollar delta is incomplete, since it does not consider the sensitivity of the dollar delta itself to movements in the stock price. Therefore, we might suspect that a more accurate measure of risk would also incorporate the dollar gamma. Moreover, the expected holding period of the position, before we intend to restore it to neutrality, should also influence a position's risk. The longer this holding period, the greater the risk.

Specifically, suppose we anticipate holding a position for \( T \) years. The position's average dollar volatility over this period will be approximately

\[
\sqrt{(\text{dollar delta})^2 + 0.584(\text{dollar gamma}^2)T}.
\]

\[1\]To derive the approximation for average dollar volatility, let \( V \) be the uncertain value of a position at the end of holding period \( T \). This value will depend on the uncertain price \( S \) of the underlying stock at the end of the holding period and the time then remaining to expiration of the securities constituting the position. Let \( \Delta \) and \( \Gamma \) be the current delta and gamma of the position. Expanding \( V \) in a Taylor series around the current stock price \( S \)

\[
\tilde{V} = V + \Delta(S - S) + \frac{1}{2} \Gamma(S - S)^2 + \ldots,
\]
This measure of risk is fully consistent with our previous analysis. If the position is delta-neutral, then dollar delta = 0 and the risk of positions with the same intended holding period will be ranked by their dollar gamma. The magnitude of this risk, estimated by

\[ 0.764 \text{(dollar gamma)} \sqrt{\tau} \]

will increase with the holding period. As the position becomes non-delta-neutral, the additional risk depends on the dollar delta.

If risk measured on this basis gets sufficiently large, we should take steps to restore neutrality. Just how large this must be depends on our capital and personal tolerance for accepting risk. Each investor must set his own revision point. His decision must balance the risk reduction achieved by restoring the position to neutrality against the transactions costs incurred.\(^1\)

where all terms involving time to expiration and higher-order terms involving the stock price are omitted. Taking the variance of both sides,

\[ \text{Var} \tilde{V} = \sigma^2 \text{Var}(\tilde{S} - S) + \frac{1}{4} \Gamma^2 \text{Var}[(\tilde{S} - S)^2] \]

Since \( \text{Var}(\tilde{S} - S) = \sigma^2 S^2 \tau \) and \( \text{Var}[(\tilde{S} - S)^2] = 2\sigma^4 S^4 \tau^2 \), then

\[ \text{Var} \tilde{V} = (\text{dollar delta})^2 \tau + .5(\text{dollar gamma})^2 \tau^2 \]

Converting this into an average dollar volatility over the holding period,

\[ \sqrt{\frac{\text{Var} \tilde{V}}{\tau}} = \sqrt{(\text{dollar delta})^2 + .5(\text{dollar gamma})^2 \tau} \]

Computer experimentation comparing this approximation to the exact average dollar volatility for holding period \( \tau \) shows this to be a significantly downward biased approximation for positions constructed from typical listed options. Changing the coefficient weighting \( (\text{dollar gamma})^2 \) from .5 to .584 eliminates this bias and effectively considers the impact of the omitted terms in the expansion.

\(^1\)A full solution of the optimal portfolio revision problem in the face of transactions costs, as in other portfolio problems in the theory of finance, appears to be an intractable dynamic programming problem.
If we could revise our positions continuously, by maintaining
delta-neutral position ratios, we could eliminate all the risk of the
position. However, since such continuous revision is impractical, even
initially neutral positions involve risk. This risk will be larger,
other things equal, the longer the intended holding period. These con-
siderations raise an important question: Knowing that we cannot re-
verse continuously, does a delta-neutral position ratio produce posi-
tions of minimum risk over their holding period? More precisely,
suppose that $\alpha^*$ is the position ratio that minimizes the average
dollar volatility of a position over holding period $T$. How much ad-
ditional risk is created by taking the delta-neutral position ratio
rather than $\alpha^*$?

Table 5 compares the position ratios and average dollar volatil-
ities for a hedge consisting of 1 call contract written against $\alpha$
shares. In this case, the delta-neutral hedge ratio (i.e., where the
position delta is zero) is simply the delta ($\Delta$) of the call, and the
dollar gamma of the position is simply the dollar gamma $\Gamma(\sigma S)^2$
of the call. For example, for the APR/40 call, the delta-neutral hedge
ratio is .572, and the dollar gamma is 8.16. In contrast, $\alpha^*$ depends
on the intended holding period. For a short holding period of two days,
$\alpha^* = .573$, almost identical to the delta-neutral hedge ratio. As the
holding period lengthens, the difference between $\alpha^*$ and $\Delta$ increases.
Knowing we will hold the hedge unrevised for 50 days, we can minimize
its average dollar volatility over the holding period by choosing an
initial hedge ratio of .6. Although this hedge ratio is significantly
Table 5. Option Delta and Gamma as Approximation of Risk over Finite Holding Periods

<table>
<thead>
<tr>
<th>K (Days)</th>
<th>JAN</th>
<th>APR</th>
<th>JUL</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>α*</td>
<td>Δ</td>
<td>α*</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>35</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>.949</td>
<td>24</td>
<td>--</td>
</tr>
<tr>
<td>5</td>
<td>.950</td>
<td>38</td>
<td>--</td>
</tr>
<tr>
<td>10</td>
<td>.952</td>
<td>56</td>
<td>--</td>
</tr>
<tr>
<td>20</td>
<td>.954</td>
<td>87</td>
<td>--</td>
</tr>
<tr>
<td>50</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>40</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>.538</td>
<td>87</td>
<td>--</td>
</tr>
<tr>
<td>5</td>
<td>.542</td>
<td>137</td>
<td>--</td>
</tr>
<tr>
<td>10</td>
<td>.547</td>
<td>195</td>
<td>--</td>
</tr>
<tr>
<td>20</td>
<td>.559</td>
<td>281</td>
<td>--</td>
</tr>
<tr>
<td>50</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>45</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>.103</td>
<td>41</td>
<td>--</td>
</tr>
<tr>
<td>5</td>
<td>.105</td>
<td>65</td>
<td>--</td>
</tr>
<tr>
<td>10</td>
<td>.107</td>
<td>98</td>
<td>--</td>
</tr>
<tr>
<td>20</td>
<td>.113</td>
<td>157</td>
<td>--</td>
</tr>
<tr>
<td>50</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

a The hedge consists of 1 call contract (i.e., rights to 100 shares) written against α shares. The expected annual rate of return of the stock is assumed to be 10 percent. This is needed to compute the exact average dollar volatility over the period.

b The January options have one month to expiration, the April's four months, and the July's seven months.
Details regarding the calculation of the exact average dollar volatility are provided. The methodology used to measure this error is mathematically complex and involves the use of the Black-Scholes model. The approximations are used to estimate the values of the stock and calls at dates t, t', and t, where t > t' > t. For a detailed calculation, see Rudd [1978].

The dollar volatility of the neutral hedge is negligible. At 50 days, the average dollar volatility of the delta-neutral hedge is 219, and the average annual dollar volatility of the risk-minimizing hedge is 1.6.

Table 5 illustrates that even for relatively long intended holding periods, initiating a position at a delta-neutral position ratio close to the value predicted for a position of one year, about two-thirds of the time the total change in the value position would be less than 20% in absolute value.
For the same hedges, table 6 shows the sensitivity of the average dollar volatility to the neutrality of the hedge. For example, for the APR/40 call, the delta-neutral hedge ratio is .572, which from table 5 results in an average annual dollar volatility of 43 over a 2-day holding period. If the hedge ratio is instead .562 (i.e., .01 less), then the risk will be $45. Similarly, a hedge ratio of .522 produces risk of $75, and a hedge ratio of .472 produces risk of $129. That is, a 10 percent error in the hedge ratio triples the average dollar volatility of the hedge for a 2-day holding period. However, this sensitivity is diminished the longer the holding period.

Table 6 also illustrates the error in estimating average dollar volatility from using our approximation formula,

\[ \sqrt{(\text{dollar delta})^2 + .584(\text{dollar gamma})^2} \]

For example, for the APR/40 call, a 10-day holding period and a hedge ratio of \( \Delta = .05 = .522 \), the formula estimates the annual dollar volatility to be 120, just 3 more than the exact annual dollar volatility.
Table 6. Sensitivity of Average Dollar Volatility to Hedge Ratio and Holding Period$^a$

<table>
<thead>
<tr>
<th>K (Days)</th>
<th>JAN$^b$</th>
<th></th>
<th>JAN$^b$</th>
<th></th>
<th>JAN$^b$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Hedge $</td>
<td>$ Volatility</td>
<td>Error in $\hat{\Delta}$</td>
<td>Hedge $</td>
<td>$ Volatility</td>
</tr>
<tr>
<td>2</td>
<td>27 65 123</td>
<td>1 -- -1</td>
<td>31 67 124</td>
<td>1 -- -1</td>
<td>27 65 123</td>
</tr>
<tr>
<td>5</td>
<td>40 73 128</td>
<td>1 -- -2</td>
<td>46 76 130</td>
<td>2 -- -1</td>
<td>40 72 128</td>
</tr>
<tr>
<td>10</td>
<td>58 85 136</td>
<td>-2 -- -4</td>
<td>64 89 139</td>
<td>4 -- -2</td>
<td>55 83 135</td>
</tr>
<tr>
<td>20</td>
<td>89 110 155</td>
<td>-10 -- -11</td>
<td>90 111 156</td>
<td>5 -- -4</td>
<td>76 100 148</td>
</tr>
<tr>
<td>50</td>
<td>-- -- --</td>
<td>-- -- --</td>
<td>143 161 200</td>
<td>7 -- -8</td>
<td>120 141 182</td>
</tr>
<tr>
<td></td>
<td>$\Delta = 0.949$</td>
<td>$\Gamma(\sigma)^2 = 4.38$</td>
<td>$\Delta = 0.829$</td>
<td>$\Gamma(\sigma)^2 = 5.27$</td>
<td>$\Delta = 0.794$</td>
</tr>
<tr>
<td>40</td>
<td>2</td>
<td>88 107 150</td>
<td>6 4 2</td>
<td>45 75 129</td>
<td>3 1 --</td>
</tr>
<tr>
<td>5</td>
<td>138 153 187</td>
<td>10 7 3</td>
<td>69 93 141</td>
<td>4 2 --</td>
<td>53 81 133</td>
</tr>
<tr>
<td>10</td>
<td>196 208 236</td>
<td>13 9 5</td>
<td>98 117 159</td>
<td>6 3 --</td>
<td>73 97 144</td>
</tr>
<tr>
<td>20</td>
<td>282 293 316</td>
<td>14 9 3</td>
<td>138 154 191</td>
<td>9 3 --</td>
<td>104 123 165</td>
</tr>
<tr>
<td>50</td>
<td>-- -- --</td>
<td>-- -- --</td>
<td>221 236 267</td>
<td>10 2 --</td>
<td>165 182 218</td>
</tr>
<tr>
<td></td>
<td>$\Delta = 0.536$</td>
<td>$\Gamma(\sigma)^2 = 16.52$</td>
<td>$\Delta = 0.572$</td>
<td>$\Gamma(\sigma)^2 = 8.16$</td>
<td>$\Delta = 0.594$</td>
</tr>
<tr>
<td>45</td>
<td>2</td>
<td>43 74 128</td>
<td>1 -- -1</td>
<td>41 73 127</td>
<td>2 -- --</td>
</tr>
<tr>
<td>5</td>
<td>67 91 140</td>
<td>-- -- -2</td>
<td>63 88 138</td>
<td>3 1 --</td>
<td>52 80 133</td>
</tr>
<tr>
<td>10</td>
<td>100 119 160</td>
<td>-6 -7 -8</td>
<td>89 110 154</td>
<td>4 1 --</td>
<td>73 97 144</td>
</tr>
<tr>
<td>20</td>
<td>158 172 206</td>
<td>-24 -- -27</td>
<td>127 145 183</td>
<td>4 -- -5</td>
<td>104 124 166</td>
</tr>
<tr>
<td>50</td>
<td>-- -- --</td>
<td>-- -- --</td>
<td>211 226 257</td>
<td>-3 -- -11</td>
<td>168 186 221</td>
</tr>
</tbody>
</table>

\[ \Delta = \frac{\text{Hedge \$ Volatility}}{\text{Error in } \hat{\Delta}} \]

\[ \Gamma(\sigma)^2 = \text{Volatility}^2 \]

$^a$The hedge consists of one call contract (i.e., rights to 100 shares) written against $\alpha$ shares. The expected annual rate of return of the stock is assumed to be 10 percent. This is needed to compute the exact average dollar volatility over the period.

$^b$The January options have one month to expiration, the April's four months, and the July's seven months.
REFERENCES


