PAYOUTS AND INVESTMENT POLICIES FOR UNIVERSITY ENDOWMENTS

by

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1. Introduction

The financial health of private colleges and universities is an item of major concern. Do the shortness of income, the expanded budgets, and the atrophied endowments indicate a temporary seizure, or are they the first symptoms of a terminal illness? The answer to this important question depends upon the external economic factors and internal policies that determine a university's vital signs.

The long-range financial health of a university depends upon government support, the growth in real disposable income, the performance of the stock market, and several other uncertain external economic factors. Financial well-being also depends on a university's internal policies regarding tuition, student aid, salaries, and endowment management. This paper will focus on endowment investment and payout policy under the assumption that tuition, government support and other decisions affecting income are fixed; in this case the level of endowment payout will fix the university's rate of budget expenditures.
The objective guiding policy choice is the expected discounted return on budget expenditures. We postulate that there are diminishing marginal returns on budget expenditures and that a minimum level of expenditures is necessary to keep the university operating.

The main result of this paper is that the endowment can be partitioned into two funds. The first, a trust fund, is invested in risk-free assets and has a payout that is sufficient to close the gap between the minimum level of budget expenditures and the income from tuition and other sources. The second fund, called the discretionary fund, has a fixed fraction ($\beta$) of its assets in the market portfolio and the remainder in risk-free assets. The payout from the discretionary fund is a fixed percentage of the assets in the fund. Both funds have fixed investment and payout policies; however, when the funds are combined the observed investment and payout policies will change as the level of assets in each fund changes. Thus we shall see that higher levels of the total endowment will increase the overall endowment's portfolio beta and, in general, that higher levels of the endowment will result in higher payout rates.

The underlying model is based on three factors. The first is the university's valuation of the time stream of budget expenditures as represented by an impatience factor that puts future and present consumption on an equal footing and a coefficient of risk aversion which indicates the general shape of returns function. The second factor is the external economic environment described by the return on risk-free investments, and the mean and variance of return on the market portfolio. The third
factor is the university's operating environment as described by the current level of endowment and forecasted gap between minimum budget expenditures and the nonendowment sources of university income.

The paper examines the sensitivity of investment and payout policies to changes in the parameters that characterize the three underlying factors; preferences, external economic environment, and operating conditions. In addition, we also describe the evolution of the university when the optimal policy is used.

The vehicle for this analysis is stochastic dynamic programming in continuous time. After much pulling and tugging the problem is transformed into an analogous individual's consumption investment decision. Analytical solutions to this type of problem have been presented by Bellman and Dreyfus [2], Levhari and Srinivasan [11], Merton [14], and Hakansson [8]. The continuous time case was first formulated by Mirrlees [15], and solved in its current form by Merton [14]. The continuous time model has the advantage of yielding an optimal policy that is quite easy to interpret in terms of the parameters that describe the valuation of expenditures, the investment environment, and the operating conditions of the university. A discrete time model yields a similar closed form solution; but the equations that define the optimal payout and investment policies are much more difficult to analyze.

In [3] Black noted that the guidelines for management of university endowments should be tied to the overall objectives of the institution. This has not been the case in much of the literature on endowment
management which generally analyzes either payout or investment policy in isolation from each other and from the underlying objectives. Payout policy is continually dogged by the notion of spending only the dividends and interest earned by the endowment. Among others, Meckling and Jensen [13], Williamson [19], Blume [4], and Litvak et al. [12] have debunked this classic notion of income, yet it persists. The studies mentioned above generally simulate the effects of using a particular payout policy. A typical policy either pays out a fixed fraction of the endowment or a fixed fraction of some weighted average of past endowment values.

Investment policies are examined in Williamson [18], Blume [4], and Meckling and Jensen [12]. These studies, in general, choose different risk levels of the endowment and simulate the effect of using that investment policy over a period of several years.

This model is not intended to be an operational guide to setting endowment policy, rather it is to give some insight into the structure of that policy and to provide guidance in constructing realistic operating models. For a more practical approach to university financial management and academic planning, see the forthcoming book by Hopkins and Massy [9], and references cited there.

2. The Objective

The university's objective is assumed to be the maximization of expected discounted return on budget expenditures. This section provides a motivation for that choice and describes the particular functional form of the return that is used in the paper.
The university has a fixed size (enrollment, staff, etc.) and a minimum rate \( b_0^- \) of annual expenditures that will support that fixed size. Each dollar of budget expenditure per year above that minimum level has a positive but decreasing return. Let \( b \) represent the rate of expenditure and \( r(b) \) the marginal return. The total return is denoted as \( R(b) \). The particular form selected for \( R(b) \) is

\[
R(b) = \begin{cases} 
\frac{(b - b_0)^{1-\gamma}}{1-\gamma} & \text{if } \gamma > 0, \gamma \neq 1 \\
\ln(b - b_0) & \text{if } \gamma = 1 
\end{cases}
\]

The marginal return is

\[
r(b) = (b - b_0)^{-\gamma}.
\]

Future returns are discounted at rate \( \rho \), thus if \( b(t) \) is the budget at time \( t \), the total discounted return will be

\[
\int_0^\infty e^{-\rho t} R(b(t)) dt.
\]

The objective of this paper is to find endowment investment and payout policies that will maximize the expected value of (2.3). Before proceeding to this principal task, we shall give some support to the form of objective we have selected.

The first, simplest, and perhaps most convincing argument in favor of decreasing marginal return is that the university rationally orders
its budget priorities; thus higher return expenditures will be made before the lower return expenditures. A second argument supporting decreasing marginal return is built on the notion of consumer surplus. Suppose the excess \( b - b_0 \), above the minimum budget level is used to lower tuition; this would create a consumer's surplus for the students since they would be getting the same good at a lower price. Another alternative is for the university to fix tuition and use the excess \( b - b_0 \) to enrich the program. In this case consumer's surplus would increase since the students would be receiving a higher quality good with no increase in price. Any combination of tuition cutting and enrichment would also lead to an increase in consumer's surplus. A third support of diminishing marginal returns is similar to the consumer surplus argument. If one defines returns in terms of quality of the student body, then it is apparent that lowering tuition and enriching the program will increase the pool of qualified applicants and thus improve the quality of the student body.

The sections that follow construct a simple framework in which to study the maximization of the expected value of our objective (2:3).

3. The Budget and Control Variables

This section defines the budget and control variables that are used in the endowment model. Let \( b \) be the rate of expenditure. There are two sources of budget funds: income, \( i \), from tuition and other sources, and payout \( z \) from endowment. The budget-balancing equation is

\[
(3:1) \quad b = i + z.
\]
There is an external flow of gifts \( g \) into the endowment at any time; thus the net flow out of the endowment is \( x = z - g \).

We shall assume that the flow of income \( i \) and gifts \( g \) are known constants.\(^2\) This will allow us to concentrate on endowment policy and also allows us to express budget expenditures in terms of \( x \), the net endowment outflow.

The minimum rate of expenditures is \( b_0 \); this expenditure rate corresponds to a net rate of withdrawal of \( k \) dollars per year from the endowment. Therefore

\[
(3:2) \quad b_0 = i + g + k.
\]

Since \( b_0 \), \( i \), and \( g \) are constants, \( k \) must also be constant. From (3:3) and (3:2) we find that the rate of spending \( b \) above the minimum level \( b_0 \) is simply equal to the net rate of withdrawal from the endowment \( x \) above its minimum level \( k \): i.e.

\[
(3:3) \quad b - b_0 = x - k.
\]

When (3:3) is combined with our specification of return in (2:3) it allows us to express the return in terms of net payout \( x \).

\[
(3:4) \quad R(x) = \begin{cases} 
\frac{(x-k)^{1-\gamma}}{1-\gamma} & \text{if } \gamma \neq 1, \gamma > 0 \\
\ln(x-k) & \text{if } \gamma = 1
\end{cases}
\]

and the marginal return is

\[
(3:5) \quad r(x) = (x-k)^{-\gamma}.
\]
4. Endowment Investment and System Evaluation

We assume the endowment can be invested in two assets; a risk-free asset that earns an instantaneous rate of return \( r_{dt} \) and the market portfolio of risky assets that earns an uncertain rate of return

\[
dm = \mu dt + \sigma dM.
\]

The definition of \( dm \) becomes less painful when the term \( dM \) is interpreted as a random variable that is equally likely to take on the values of plus or minus the square root of \( dt \). Thus the expected value of \( dm \) is \( \mu dt \) and (to first order), the variance of \( dm \) is \( \sigma^2 dt \).

Let \( e \) be the endowment level and \( y \) the amount invested in the market portfolio and thus \( e - y \) is the amount invested in the risk-free asset. The instantaneous inflow from this investment policy is \( (e - y)rdt + ydm \). From section 3 we learned that the outflow from the payout policy is \( xdM \), thus the total change in endowment level is

\[
de = -xdt + (e - y)rdt + ydm.
\]

If we let \( \psi = \mu - r \) designate the excess expected return on the market portfolio, then the random change in the endowment is given by

\[
de = (re + \psi y - x)dt + CydM.
\]

5. Dynamic Programming Formulation

The state of the university in our model is summarized by the size of the endowment \( e \). All other exogenous variables such as \( g \) and \( i \) are known and constant; the only variable that changes due to payout \( x \) and
investment \( y \) is the future level of the endowment. We define the value \( V(e) \) of any level of endowment \( e \) to be the expected discounted value of all future budget returns that can be generated from that endowment and the use of optimal payout and investment policies. Thus \( V(e) \) is the expected value of (2:3) when the initial endowment is \( e \) and the expenditures \( b(t) \) are determined by an optimal policy.

The function \( V(e) \) is, of course, unknown. We can, however, use standard dynamic programming arguments to characterize \( V(e) \). We do this by breaking the problem into two phases: an initial phase of length \( \Delta t \) and a subsequent phase of indefinite length. Given \( e \), the decisions \( x \) and \( y \), and the uncertain return \( d\lambda \) the return in these two stages will be

\[
(5:1) \quad R(x)dt + \exp(-\rho dt)V(e+de)
\]

where \( de \) is given by (4:3). The optimal expected value \( V(e) \) must be greater than or equal to the expected value of (5:1) for any choice of \( x \) and \( y \), since (5:1) is the value obtained by using a particular \( x \) and \( y \) over the interval \( \Delta t \) while \( V(e) \) corresponds to using an optimal policy over the initial interval.

The key to relating (5:1) and \( V(e) \) is to expand (5:1) and then take its expectation. The first-order expansion of \( \exp(-\rho dt) \) is straightforward.

\[
(5:2) \quad \exp(-\rho dt) = 1 - \rho dt .
\]

The first-order expansion of \( V(e+de) \) requires the use of Ito's lemma.
\[(5.3) \quad V(e + de) = V(e) + \dot{V}(e)de + \frac{\ddot{V}(e)(de)^2}{2}\]

where

\[(5.4) \quad (de)^2 = \sigma^2 y^2 dt.\]

Equations (5.3) and (5.4) can be combined to yield

\[(5.5) \quad V(e + de) = V(e) + \left[ (re + \psi y - x)\dot{V}(e) + \frac{\ddot{V}(e)\sigma^2 y^2}{2} \right] dt + \sigma y \dot{V}(e) dN.\]

When we take the expected value of \(V(e + de)\) the last term vanishes. We can now combine (5.1), (5.2), and (5.5). To ease notation the argument \(e\) of \(V, \dot{V}\) and \(\ddot{V}\) is suppressed; to first order the expected value of (5.1) equals

\[(5.6) \quad V = \rho V dt + \left[ R(x) dt + (re + \psi y - x)\dot{V} dt + \frac{\ddot{V} \sigma^2 y^2}{2} dt \right].\]

Since \(V\) must be greater than (5.6) for all \(x\) and \(y\) it follows that \(V\) equals the maximum of (5.6) over all \(x\) and \(y\). Notice that the terms \(V\) and \(-\rho V dt\) in (5.6) are independent of this maximization; thus we restate the optimality condition as \(\rho V dt\) equals the maximum of the term in brackets. The infinitesimal \(dt\) is a common factor of \(\rho V dt\) and the bracketed expression in (5.6); therefore we can cancel the \(dt\). This leaves us with the relationship:

\[(5.7) \quad \rho V = \max_{x,y} \left[ R(x) + (re + \psi y - x)\dot{V} + \frac{\ddot{V} \sigma^2 y^2}{2} \right].\]
Equation (5:7) is the key to discovering the form of $V$ and the nature of the optimal policy for payout and investment.

6. The Optimal Policy

The form of the optimal policy and value functions can be derived from (5:7). We do this in three steps: first we lay out the optimality conditions for maximization in (5:7), second we make an informed guess about the optimal value function $V$ and optimal policy functions $x$ and $y$, and third we show our guesses are consistent with (5:7) and the optimality conditions.

The first-order conditions are obtained by differentiating the argument of (5:7) with respect to $x$ and $y$. The result is

(6:1) \[ r(x) = \dot{V} \]

and

(6:2) \[ \psi V = - \sigma^2 \ddot{V}_y. \]

Equation (6:1) states that the marginal return of expenditures, $r(x)$, is equal to the marginal value of an increase in endowment. This is easy to explain. An additional unit of endowment would be split between additional payout and additional funds available for investment. However, the optimal policy will have equated the marginal return on expenditure with the marginal return on investment; thus a small addition to endowment could be divided in any way between investment and expenditure and still yield the same marginal return as if it was devoted
entirely to expenditures. From (5:2) we see that the investment policy \( y \) depends entirely upon the risk tolerance of the induced value function \( V \) and the parameters \( \psi \) and \( \sigma^2 \) that describe the market.

\[
y = (-\dot{V}/\ddot{V})(\psi/\sigma^2)
\]

The term \( (\psi/\sigma^2) \) is the market's reward for bearing risk whose risk is measured by the covariance of portfolio return with return on the market portfolio of all risky assets and where the reward is the excess return required by investors for assuming that level of risk. The expression \((-\dot{V}/\ddot{V})\) is the absolute risk tolerance of the university if \( V(e) \) is viewed as the university's indirect utility function.\(^7\) The reciprocal of the absolute risk tolerance is known as the absolute risk aversion.

The beta or normalized risk level of the portfolio is simply \( y/e \) since that is the fraction invested in the market portfolio. The beta is given by

\[
\beta = y/e = (-\dot{V}/e\ddot{V})(\psi/\sigma^2) .
\]

The first term in brackets is the relative risk tolerance, i.e. the inverse of the relative risk aversion; the second term is market's reward for bearing risk.

Now that we have derived the optimality conditions from (5:7) our next task is to guess the form of the optimal value function. This guess can be guided by analogy with the consumer's optimal investment and consumption problem. The analog to consumption here is the endowment expenditures above minimum level: i.e. \( x-k \). The analog of wealth is the
discretionary portion of the endowment. The total endowment size is \( e \), but a minimum outflow \( k \) is necessary to support the university. The present value of this required outflow is \( k/r \). If we call \( k/r \) the committed endowment funds then the discretionary endowment \( e - k/r \) is the source of all future consumption: i.e. all endowment expenditures above minimum level. Then by analogy with other models we guess that

\[
(6.5) \quad (i) \quad \nu = \frac{\phi w^{1-\gamma}}{1-\gamma}
\]

where

\[
(ii) \quad w = e - k/r
\]

and \( \phi \) is a constant.\(^8\) The first two derivatives of \( \nu \) and the risk tolerance are given below

\[
(6.6) \quad (i) \quad \dot{\nu} = \phi w^{-\gamma}
\]

\[
(ii) \quad \ddot{\nu} = -\gamma \phi w^{-\gamma}/w
\]

\[
(iii) \quad (-\dot{\nu}/\nu) = w/\gamma
\]

When (6.6) is combined with (6.3) we have

\[
(6.7) \quad \gamma = \psi/\gamma \sigma^2
\]

or

\[
(6.8) \quad \beta = \frac{1}{\gamma} \left(1 - \frac{k}{re}\right)(\psi/\sigma^2).
\]

We shall discuss (6.8) in some detail in the next section.
From (6.1), (3.5), and (6.6) we can solve for \( x \) in terms of \( w \).

\[
(6.9) \quad x = k + \theta w - \frac{1}{\gamma} \theta
\]

where \( \theta = \phi \frac{1}{\gamma} \).

Our final task is to use the functional form (6.5) and the optimal policies (6.7) and (6.9) in order to verify that the functional equation (5.7) is satisfied.

When the optimal policies (6.7) and (6.9) are substituted in (5.7) the maximization is unnecessary; we must thus verify that

\[
(6.10) \quad \rho V = R(x) + (re + \psi y - x)\dot{V} + \frac{\sigma^2 y^2 \dot{V}}{2}.
\]

Note, first, that (6.7) and (6.6) imply

\[
(6.11) \quad \rho y \dot{V} + \frac{\sigma^2 y^2 \dot{V}}{2} = \frac{\sigma}{\gamma} (\psi^2 / \gamma \sigma^2) w^{1-\gamma}.
\]

Our second simplification of (6.10) is based on \( re = rw + k \), thus from (6.9) \( re - x = (r - \theta)w \). This and (3.4) imply

\[
(6.12) \quad R(x) + (re - x)\dot{V} = \frac{\phi \theta w^{1-\gamma}}{1-\gamma} + (r - \theta)\phi w^{1-\gamma}.
\]

The sum of (6.12) and (6.11) must equal

\[
(6.13) \quad \rho V = \frac{\rho \phi}{1-\gamma} w^{1-\gamma}.
\]

The common factor \( \phi w^{1-\gamma} \) in (6.11-13) cancels, which yields.
\[ \frac{c}{1-\gamma} = \frac{\theta}{1-\gamma} + (r - \theta) + \frac{1}{2} \left( \frac{\psi^2}{\gamma\sigma^2} \right). \]

This identity will be satisfied when \( \theta \) is given by
\[ \theta = \frac{1}{\gamma} \left[ \rho + (\gamma - 1) \left( r + \frac{\psi^2}{2\gamma\sigma^2} \right) \right]. \]

Notice that \( \theta \) and therefore \( \phi = \theta^{-\gamma} \) will be positive if and only if the bracketed expression in (6:15) is positive. Since \( r, \psi, \) and \( \sigma^2 \) are exogenous market parameters, the boundaries of the region that lead to positive values of \( \theta \) is
\[ \rho \geq (1-\gamma) \left( r + \frac{\psi^2}{2\gamma\sigma^2} \right). \]

This region is shown in figure 1 for \( r = 0.03, \psi = 0.056, \) and \( \sigma = 0.17. \) Since \( \rho \) is always nonnegative the restriction will always be satisfied for \( \gamma > 1. \) When \( \gamma < 1, \) the total return is unbounded above, thus the integral (2:3) that describes the total discounted reward may not be finite. Condition (6:15) insures that it will be finite by requiring a sufficiently high discount rate, sufficiently small expected returns \( r \) and \( \psi, \) and a sufficiently large uncertainty in return \( \sigma. \) In fact we shall see that the term \( r + \frac{\psi^2}{2\gamma\sigma^2} \) is a certainty equivalent return on the discretionary portion of the endowment.

We close this section with a brief analysis of the case where \( \gamma = 1: \) i.e. where \( R(x) = \ln x. \) In that case
Acceptable region when \( r = 0.03, \psi = 0.056, \) and \( \sigma = 0.17. \)
(6.17) \[ V(e) = \frac{1}{\rho} \ln \left( e - \frac{k}{r} \right) + h \]

where

(6.18) \[ h = \frac{1}{\rho} \left[ \ln \rho + \frac{(r-\rho)}{\rho} + \frac{1}{\rho} \frac{\psi^2}{2\sigma^2} \right] \]

The optimal policies are

\[ x = k + \rho \left( e - \frac{k}{r} \right) \]

and

\[ y = \frac{\psi}{\sigma^2} \left( e - \frac{k}{r} \right) . \]

7. The Investment Policy

The optimal investment and payout policies were derived in the previous section. This section will interpret the investment policy and examine the sensitivity of investment policy to changes in the exogenous variables \((r, \psi, \sigma)\), changes in preferences \((\gamma, \rho)\), and changes in system operating variables \((k, e)\).

For convenience, the optimal investment policy is repeated here. The optimal dollar amount \(y\) invested in the market portfolio and the \(\beta\) (beta) of the investment portfolio are

(7.1) \[ y = w \cdot \frac{\psi}{\gamma \sigma^2} , \]

and

\[ \beta = \left( \frac{w}{e} \right) \frac{\psi}{\gamma \sigma^2} , \]

where \( w = e - k / r \).
For a detailed analysis of the investment policy it is best to partition the endowment, e, into two funds. We shall call the first a trust fund; it contains \( \frac{k}{r} \) dollars and is invested at the risk-free rate \( r \). The trust fund provides the income flow \( k \) that is required to meet the minimum level of expenditures. We call the second the discretionary fund; it contains \( w = e - \frac{k}{r} \) dollars. The discretionary fund provides the surplus of income over and above the minimum level \( k \).

The risk level of each fund is constant. The trust fund is invested entirely in fixed return securities while the discretionary fund maintains constant beta;

\[
\beta_* = \frac{\psi}{\gamma \sigma^2}.
\]

However, the size of the trust fund is fixed while the size of the discretionary fund will vary with the uncertain market returns, thus the overall risk level of the endowment will vary with the level of the discretionary fund. The aggregate beta is

\[
\beta = 0 \frac{k/r}{e} + \beta_* \frac{w}{e} = \left(1 - \frac{k/r}{e}\right) \frac{\psi}{\gamma \sigma^2}.
\]

We can see immediately from (7:3) that the aggregate beta will increase as the level of endowment increases; for very large discretionary endowments \( w \approx e \) the aggregate beta will approach its upper limit \( \beta^* \). This behavior is shown in figure 2.
AGGREGATE INVESTMENT RISK

Data: $\psi = 0.035$
$r = 0.02$
$\sigma' = 0.17$
$\gamma = 0.85$
k = 5
$\beta^* = 1.424$
k/r = 250
We shall first consider the sensitivity of investment policy to the preferences as expressed by the risk aversion $\gamma$ and time preference $\rho$. This is straightforward. Investment policy is independent of the time preference $\rho$; time preference will affect payout rate and thus the size of endowment available for investment. However, once a decision has been made to invest the funds, the risk level (beta) will be independent of the time preference. The risk level will, of course, depend on the coefficient of relative risk aversion $\gamma$. We see that increased risk aversion unambiguously reduces the risk level of the portfolio.

The effect of changes in the exogenous parameters $\psi$, $\sigma^2$, and $r$ on investment policy are only slightly more complicated. Increases in the excess return $\psi$ or decreases in market risk increase $\beta^*$ and thus $\beta$. An increase in the risk-free rate $r$ has a negative substitution effect due to decreases in $\psi$, the excess market return, and a positive wealth effect since an increase in $r$ shifts funds from the trust to the discretionary fund. The total effect is

$$\frac{\partial \beta}{\partial r} = -\left(\frac{w}{e}\right) \frac{\beta}{\psi} + \left(\frac{e-w}{e}\right) \frac{\beta}{r}. \quad (7:4)$$

Notice that $\left(\frac{w}{e}\right)$ and $\left(\frac{e-w}{e}\right)$ are the fractions of endowment in the discretionary and trust funds, respectively. For large value of $w$ the substitution effect will tend to dominate the wealth effect. The sensitivity of $\beta$ to the risk-free rate $r$ is shown in figure 3. The risk level, $\beta^*$, of the discretionary fund falls as $r$ increases since the excess return $\psi$ on market investments is decreasing. At the same time increasing $r$ will
FIGURE 3

EFFECT OF \( r \) ON INVESTMENT RISK

Data:
\[ \mu = 0.06 \]
\[ \sigma = 0.17 \]
\[ e = 400 \]
\[ k = 5 \]
\[ \gamma = 1.0 \]
decrease the present value of the obligation to supply funds at rate \( k \); the fraction \( \frac{w}{e} \) increase. The combined effect makes the aggregate beta initially increase, then decrease.

We have already seen that \( \beta \) increases as \( e \) increases since the size of the discretionary fund will increase. The effect of changes in \( k \) link investment policy and operating policy. For example, suppose a later negotiation increases the minimum budget level more than an offsetting increase in tuition can cover. This will cause \( k \) to increase. The increase in \( k \) has a negative wealth effect; funds are shifted from the discretionary fund to the trust fund. This will cause a decrease in aggregate beta. This effect can be significant. If viewed in terms of an elasticity we see that

\[
\frac{k}{\beta} \frac{\partial \beta}{\partial k} = -\frac{(e-w)}{w}. \tag{7.5}
\]

For example, when \( k = 5 \), \( e = 300 \), and \( r = 0.03 \), then \( w = 133.3 \) so the elasticity is \(-1.25\); a 10 percent increase in \( k \) will cause a 12.5 percent decrease in \( \beta \).

The next section will discuss the sensitivity of the endowment payout to these same exogenous and system parameters.

8. Analysis of Payout Policy

The previous section analyzed the investment policy. This section continues in the same style and examines the payout policy in detail.

Recall that we partitioned the endowment into two funds: a trust fund of fixed size \( k/r \) and a discretionary fund of random size
\[ w = e - k/r. \] Each fund has a fixed investment policy, and, as we see below, a fixed payout policy. The total annual flow from the endowment is \( z \) (see section 3); while our optimal payout policy is expressed in terms of \( x = z - g \), the net outflow. The payout policy is more neatly described in terms of \( x \). From (6:9), we have

\[(8:1)\]
\[ x = r \left( \frac{k}{r} \right) + \theta w. \]

The payout rate is \( r \) for the trust fund and \( \theta \) for the discretionary fund. Let \( p \) be the payout rate from the endowment; \( p \) is given by

\[(8:2)\]
\[ p = r \left( \frac{e - w}{e} \right) + \theta \left( \frac{w}{e} \right). \]

The payout rate is simply a weighted average of the two payout rates, and it is easily seen that \( p \) will increase as total endowment \( e \) increases if and only if \( \theta > r \); in fact, \( \theta > r \) will be valid for all sufficiently large values of the risk aversion coefficient \( \gamma \) (if \( \rho > r \)).

In order to analyze the sensitivity of payout to changes in the problem's parameters it is first necessary to examine \( \theta \) in some detail. Let \( \alpha \) be defined by

\[(8:3)\]
\[ \alpha = r + \frac{\rho^2}{2 \gamma \sigma^2}. \]

This constant plays the role of certainty equivalent return. If we consider a consumption and savings model with the deterministic savings return \( \alpha \), then one will consume a fraction \( \theta \) of wealth where
(8:4) \[ \theta = \frac{1}{\gamma} \left[ \rho + (\gamma - 1)\alpha \right]. \]

Any change in the parameters determining \( \theta \) has an associated value effect since the optimal value function is

(8:5) \[ V(w) = \frac{e^{-\gamma}}{\theta \gamma} \left( \frac{w^{1-\gamma}}{(1-\gamma)} \right). \]

If one maintains a constant value, then changes in \( \theta \) must be compensated for by corresponding changes in \( w \). If a parameter \( d\lambda \) is changed by

(8:6) \[ dw = \frac{-\gamma w}{(\gamma - 1)\theta} \left( \frac{\partial \theta}{\partial \lambda} \right) d\lambda \]

in order to keep value \( V \) constant. The change in payout \( 8w \) from the discretionary fund will be the combination of a substitution effect (consumption vs. savings) and a wealth effect (due to changes in the savings possibilities). For changes in \( \alpha \) we get

(8:7) \[ \frac{\partial (8w)}{\partial \alpha} = -\frac{w}{\gamma} + w = \frac{(\gamma - 1)}{\gamma} w \]

where the first term reflects the substitution of investment for payout and the second term is wealth effect. The substitution effect decreases as risk aversion (\( \gamma \)) increases; substitution dominates the income effect for aggressive investors (\( \gamma < 1 \)) and is dominated by the wealth effect for cautious investors (\( \gamma > 1 \)). Equations (8:7) and (8:3) show how payout from the discretionary fund will vary with changes in \( \psi \) and \( \sigma^2 \). Note
that

(8:8) \[ \frac{\partial \alpha}{\partial \psi} = \beta_x \]

\[ \frac{\partial \alpha}{\partial \tau} = (1 - \beta_x) \]

and

\[ \frac{\partial \alpha}{\partial \sigma} = - \sigma \beta_x^2 \]

where \( \beta_x \) is the risk level of the discretionary fund (see (7:2)). A change in \( \tau \) has the added effect of changing the value in the fund. Thus

(8:9) \[ \frac{\partial (6w)}{\partial \tau} = \frac{\gamma - 1}{\gamma} \left( \omega(1 - \beta_x) + \frac{\alpha}{\tau} (e - \omega) \right) , \]

since increases in \( \tau \) shift funds from the trust to the discretionary fund as well as increasing the measure of investment productivity (\( \alpha \)).

A change in the impatience measure \( \rho \) has the anticipated direct impact on the payout.

A change in the fixed payout \( k \) from the trust decreases payout if \( \theta > \tau \). This follows because \( \theta > \tau \) implies a higher payout rate in the discretionary fund. A unit increase in payout requirement causes a unit increase in payout from the trust fund, yet it also decreases the value of the discretionary fund by \( \frac{1}{\tau} \), thus causing a decrease of \( \theta/\tau \) in payout from the discretionary fund; the net is

(8:10) \[ \frac{\partial x}{\partial k} = \left( \frac{\tau - \theta}{\tau} \right) . \]
9. Optimal Evolution of Endowment and Budget

This section examines the evolution of the endowment and budget under the optimal investment and payout policies.

The development is simpler if we concentrate on the discretionary portion of the endowment \( w = e^{-k/r} \). Since \( k \) and \( r \) are considered as constants, the differentials \( dw \) and \( dw \) will be identical.

When (4:3), (6:7), and (6:9) are combined we obtain

\[
\frac{dw}{w} = \left( r + \frac{\psi^2}{2 \gamma \sigma^2} - \delta \right) dt + \frac{\psi}{\gamma \sigma} dM.
\]

The parameter \( \delta \) is given by (6:15). Substitution of \( \delta \) into (9:1) yields

\[
\frac{dw}{w} = \left( \delta + \frac{\eta^2}{2} \right) dt + \eta dM,
\]

where

(i) \( \delta = \frac{1}{\gamma} [\alpha - \rho] \),

and

(ii) \( \eta = \frac{\psi}{\gamma \sigma} \), and (iii) \( \alpha = r + \frac{\psi^2}{2 \gamma \sigma^2} \).

This stochastic differential implies that the \( \ln(w(t)/w(0)) \) will be normally distributed with mean \( \delta t \) and variance \( \eta^2 t \). The ratio \( w(t)/w(0) \) will be lognormally distributed with mean
(9:3) \[ \exp \left( \delta t + \frac{\eta^2 t}{2} \right), \]

and variance

(9:4) \[ \left( \exp (2\delta t + \sigma^2 t) \right) \cdot \left( \exp (\eta t) - 1 \right). \]

This insures the \( w(t) \) will remain positive and also implies that the standard deviation of \( w(t) \) will grow exponentially.

If \( \alpha < 0 \), \( w(t) \) will most likely be smaller than \( w(0) \). In fact,

(9:5) \[ \Pr[w(t) \leq w(0)] = N \left( \frac{\ln w(0) - \alpha t}{\eta t} \right), \]

where \( N(*) \) is the cumulative normal distribution. In extreme cases it is possible that \( \delta + \frac{\eta^2}{2} \) will be negative. In that instance the expected value of \( w(t) \) will decrease exponentially and the variance of rise or decline exponentially depending on the sign of \( \delta + \frac{3\eta^2}{2} \). The endowment \( e \) is given by \( w + \frac{k}{r} \). Thus \( e \) is also lognormal and will stay above its minimum value \( k/r \) with probability one.

The rate of budget expenditures \( b(t) \) depends directly on \( w(t) \). From (3:3) and (6:9)

(9:6) \[ b(t) = b_0 + \theta w. \]

Thus \( b(t) \) will be lognormally distributed with expected value

(9:7) \[ b_0 + \theta w(0) \exp \delta t + \frac{\eta^2 t}{2}. \]
The variance of $b(t)$ will be $\theta^2$ times the variance of $w(t)$ given in (9:4). Thus $\theta$ acts as a buffer between the uncertainty in endowment and the uncertainty in the crucial operating variable $b(t)$. Since $\theta$ is typically between 0.01 and 0.04, see figure, we see it is effective, although the standard deviation and variance of $b(t)$ do grow exponentially.

10. A Useful Extention

It is not necessary to treat the minimum flow of funds $k$ from the endowment as a constant. Suppose $b_o(t), g(t), and i(t)$ are the known minimum level of spending, gifts, and income flows, then $k(t)$ is defined by

$$k(t) = b_o(t) - g(t) - i(t).$$

Let $K(t)$ be the size of an endowment needed to supply the flow $k(z)$ for $z > t$;

$$K(t) = \int_t^\infty e^{-r(z-t)}k(z)dz.$$

The analysis holds as before with $w = e - K(t)$ defined as the size of the discretionary fund.

11. Summary and Conclusions

We have constructed and solved a model that captures the essential trade-offs between endowment spending policy, endowment investment policy, and the objectives of the university. An analysis of the optimal policy
shows that universities should take a two-fund approach when it comes to setting endowment payout and investment policies. The first fund should be invested in risk-free assets and be large enough to cover future endowment payouts that are needed to keep the university above water. The second fund should have a fixed proportion of its assets in the market portfolio (i.e. a fixed β) and should pay out a constant fraction of its value each year. The aggregate policy of the endowment will differ from the form that is usually assumed: i.e. a constant β and a fixed payout rate (or a smoothed payout rate).

The model has many shortcomings. The objective does not fully grasp the university's aversion to sudden shifts in payout policy. The assumption that future gifts and income are known is unrealistic; however, the degree of uncertainty in these variables is of much less consequence, see [6], than the uncertainty in endowment return. Inflation is another matter considered in [6]; our conclusions from working with that investment planning model are that inflation does not have a major impact on university financial policies. Perhaps the most significant assumption is ignoring the special tax status of university endowments and the university's ability (in some cases) to issue bonds whose interest is tax free. Despite these limitations we believe the framework presented here is and will continue to be a useful vehicle for analyzing policy choices and will be a useful aid in constructing operating models for endowment management.
FOOTNOTES

1. We do not suppose that gifts are spent as they arrive. The gifts enter the endowment front door and a like amount of funds is extracted from the back door.

2. This assumption is eased in section 10.

3. See Merton [14] for background on this type of stochastic differential.

4. This procedure can be found in Dreyfus [5], pp. 215-25.

5. See Kushner [10], pp. 309-10, for a straightforward statement of Itō’s lemma.


7. See Pratt [16] and Arrow [1] for definitions of absolute and relative risk aversion. Risk tolerance is simply the inverse measure.

8. We examine the special case $\gamma = 1$ at the close of this section.

9. In what follows we ignore the portion of the payout due to the inflow of gifts. The ratio $g/e$ should be added to (8:2) to get the actual payout fraction.

10. This analysis follows Merton [14], pp. 252-54. An alternate analysis can be found in Hahn [7].
REFERENCES


