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ON THE USE OF RISK-ADJUSTED DISCOUNT RATES

by

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A number of early financial theory texts suggest that the value of a random stream of returns through time can be expressed as a risk-adjusted discounted value of expected returns. That is, if \( \langle \tilde{X}_t \rangle \) (\( t = 1, \ldots \)) denotes a sequence of random returns, then the value \( P_X \) of such a stream can be expressed as:

\[
P_X = \sum_{t=1}^{\infty} E(\tilde{X}_t)/(1+d)^t,
\]

where \( d \) is the "risk-adjusted discount rate."

While practitioners continue to use the risk-adjusted valuation technique widely, it has become passé as a theoretical model of valuation. More generally, models of market equilibrium suggest that price should reflect the certainty equivalent returns at each time period, discounted by the riskless rate. A full analysis of such an approach is available in Rubinstein [1976].

But the question remains open as to when the more general valuation approach is consistent with the existence of nontrivial risk-adjusted discount rates. Scott [1979] has recently posed this question. His results are not particularly encouraging, in that rather strong and nonintuitive assumptions on asset returns are required for congruence of the two methods. Furthermore, his model involves elements that are not present in most models of financial equilibrium.

In this note, we show that risk-adjusted discount rates are consistent with correct valuation under conditions that are assumed in many financial models:
(a) investors' utility functions exhibit constant proportional risk aversion (CPRA) and are additively separable over time, and

(b) the rate of return on the market is intertemporally independent.

We prove that the value of any asset whose rates of return \( X_{t+1}/X_t \) are intertemporally independent can be expressed as the sum of risk-discounted expected payoffs, given assumptions (a) and (b).

It should be noted that our results are implicit in the work of Rubinstein [1976], Ohlson and Garman [1978], and, indeed, in any general equilibrium model of asset valuation. Our objective here is to make explicit an environment consistent with risk-adjusted discounting and to examine the nature of the risk adjustment.

**Theorem:**

**Assume:** a) **Stationary CPRA tastes:**

Investors \( j = 1, \ldots, J \) choose portfolio and consumption strategies to maximize

\[
E \left[ \sum_{t=1}^{\infty} \beta^t U(C_{jt}) \right],
\]

where \( U'(C_{jt}) = C_{jt}^{-\sigma}, \ \sigma > 0 \)

(constant proportional risk aversion CPRA);

b) **Intertemporally independent market returns:**

Define the rate of return on the market \( r_{mt} \) by
\[ r_{mt} = \frac{W_{t+1}}{W_t} - 1, \]

where \( W_t \) is aggregate wealth at time \( t \).

The rates of return \( r_{mt} \) are assumed to be intertemporally independent. That is, aggregate wealth follows a (perhaps nonstationary) multiplicative random walk.

Then: (A) The value of any asset \( i \) whose payouts \( X_{it} \) follow a multiplicative random walk (i.e. whose rates of return

\[ e_{it} = \frac{X_{it,t+1}}{X_{it}} - 1 \]

are intertemporally independent) can be expressed as:

\[ p_i = \sum_{t=1}^{\infty} \frac{E(\tilde{X}_{it})}{\prod_{\tau=1}^{t} (1 + d_{i\tau})}, \]

with

\[ (1 + d_{i\tau}) = (1 + r_{i\tau}) \left[ \frac{E(1 + \tilde{r}_{mt})^{-\sigma}E(1 + \tilde{e}_{it})}{E[(1 + \tilde{r}_{mt})^{-\sigma}(1 + \tilde{e}_{it})]} \right] \]

\[ = (1 + r_{i\tau}) \left[ \frac{1}{\text{Cov}[(1 + \tilde{r}_{mt})^{-\sigma}, (1 + \tilde{e}_{it})]} \right] \]

\[ + \frac{1}{E[(1 + \tilde{r}_{mt})^{-\sigma}E(1 + \tilde{e}_{it})]} \]

where \( r_{i\tau} \) is the riskless return at time \( t \).
(B) If in addition the market's and asset's returns are identically distributed through time, and the riskless rate \( r_t \) is constant, then:

\[
P_i = \sum_{t=1}^{\infty} \frac{E(\tilde{X}_{it})}{(1+d_i)^t}
\]

where \( (1+d_i) \) is related to \( (1+r) \) by (2) and (3), noting that time subscripts can be omitted.

(C) If investors are risk-neutral or asset returns \( \tilde{r}_{it} \) are independent of \( \tilde{r}_{mt} \), then

\[
d_{it} = r_t.
\]

Proof:

From Rubinstein [1974], we know CPRA implies the existence of an aggregate investor maximizing

\[
E \sum_{t=1}^{\infty} \tilde{z}^t U(\tilde{c}_t),
\]

where \( U'(\tilde{c}_t) = \tilde{c}_t^{-\gamma} \), and \( \tilde{c}_t \) is aggregate consumption at time \( t \). Let \( \tilde{w}_t \) denote aggregate wealth at time \( t \). CPRA also implies:

\[
\tilde{c}_t = k_t \tilde{w}_t,
\]

where \( k_t \) is independent of \( t \) if returns are identically distributed through time. (See Håkansson [1970] or Leland [1968].)
In equilibrium, the value of an asset with random returns \( \tilde{X}_{it} \) will be equal to its (normalized) expected marginal utility:

\[
P_i = E \sum_{t=1}^{\infty} \beta^t U'(\tilde{C}_t) \tilde{X}_{it}
\]

\[
= E \sum_{t=1}^{\infty} \beta^t \tilde{C}_t^{-\sigma} \tilde{X}_{it}
\]

\[
= E \sum_{t=1}^{\infty} \beta^t \frac{\tilde{k}_t}{\tilde{w}_t} \tilde{X}_{it}
\]

\[
= E \sum_{t=1}^{\infty} \beta^t \frac{\tilde{w}_t}{\tilde{w}_0} \left( \prod_{\tau=1}^{t} (1+\tilde{r}_{mt}) \right)^{-\sigma} \left[ \frac{\tilde{X}_{i0}}{\tilde{w}_0} \prod_{\tau=1}^{t} (1+\tilde{r}_{mt}) \right].
\]

Using the intertemporal independence assumption, the last line can be rewritten:

\[
P_i = \sum_{t=1}^{\infty} \beta^t \frac{\tilde{w}_t}{\tilde{w}_0} \tilde{X}_{i0} \left( \prod_{\tau=1}^{t} E[(1+\tilde{r}_{mt})^{-\sigma}(1+\tilde{r}_{mt})] \right).
\]

Since, at time \( t \), the value of \$1 invested in the riskless asset at time \( t = 1 \) is \( \prod_{\tau=1}^{t} (1+r_{\tau}) \), we have from (5):

\[
1 = \beta^t \frac{\tilde{w}_t}{\tilde{w}_0} \left( \prod_{\tau=1}^{t} E(1+\tilde{r}_{mt})^{-\sigma} \right) \left( \prod_{\tau=1}^{t} (1+r_{\tau}) \right).
\]

Using this to substitute for \( \beta^t \frac{\tilde{w}_t}{\tilde{w}_0} \) in (5) yields:
(6) \[
P_i = \sum_{t=1}^{\infty} X_{i0} \left( \prod_{\tau=1}^{\infty} \frac{E[(1+\bar{\tau}_{\tau I})^{-\sigma}(1+\bar{\epsilon}_{\tau I})]}{E(1+\bar{\tau}_{\tau I})E(1+\bar{\epsilon}_{\tau I})^{-\sigma}} \right).
\]

Noting \( E(\bar{X}_t) = X_{i0} \prod_{\tau=1}^{\infty} E(1+\bar{\epsilon}_{\tau I}) \) gives

(7) \[
P_i = \sum_{t=1}^{\infty} E(\bar{X}_t) \prod_{\tau=1}^{t} \left( \prod_{\tau=1}^{t} \frac{E[(1+\bar{\tau}_{\tau I})^{-\sigma}(1+\bar{\epsilon}_{\tau I})]}{E(1+\bar{\tau}_{\tau I})E(1+\bar{\epsilon}_{\tau I})^{-\sigma}} \right),
\]

\[
= \sum_{t=1}^{\infty} \frac{E(\bar{X}_t)}{\prod_{\tau=1}^{t} (1+\delta_{\tau I})}.
\]

where \( 1 + \delta_{\tau I} \) is given by equation (2) of Part A of the theorem. Equation (3) is an immediate consequence of (2), recalling that:

\[
E(\tilde{a} \cdot \tilde{b}) = E(\tilde{a})E(\tilde{b}) + \text{Cov}(\tilde{a}, \tilde{b}),
\]

for any random variables \( \tilde{a} \) and \( \tilde{b} \).

Part B of the theorem is an immediate consequence of the assumption of identical distributions through time, which permits the product of expectations to be expressed as a power.

Part C follows from noting that \( \text{Cov}[(1+\bar{\tau}_{\tau I})^{-\sigma}(1+\bar{\epsilon}_{\tau I})] = 0 \) if \( \sigma = 0 \), or if \( \bar{\tau}_{\tau I} \) and \( \bar{\epsilon}_{\tau I} \) are independently distributed.

Q.E.D.
Concluding Observations

We have shown that risk-adjusted discounting is quite appropriate, given the assumptions of CPRA investors and intertemporally independent rates of return. If, in addition, rates of return are identically distributed through time, the risk-adjusted rate will be a constant.

Examination of formula (3), describing the discount rate, shows that it will be higher as the covariance between the asset's return and the market's return raised to the power $-\sigma$ becomes more negative. If market and asset returns are positively correlated, this generally will imply a negative covariance between the asset's return and the market return raised to a negative power. Thus, greater correlations between asset and market returns imply higher discount rates, as we would expect. 4

Our examination also reveals the cases where risk discounting is inappropriate. Most notably, this is when the rates of return of asset payoff are not intertemporally independent. While market efficiency may suggest that the rate of return to marketed assets are intertemporally independent, where rate of return includes price appreciation (or depreciation), it need not be the case that unmarketed assets (such as investment projects chosen by firms) exhibit intertemporally independent returns. Values could not be described by risk-discounted returns in such cases.

Finally, we note that the continuous time valuation model of Breeden and Litzenberger [1978] also suggests that risk-discounting is appropriate for assets whose returns follow a logarithmic diffusion process (implying rates of return that are intertemporally independent). They
assume that the market follows a logarithmic diffusion process, and they show that their assumptions imply CPRA tastes (Theorem 3, p. 647).