WELFARE COMPARISONS OF FINANCIAL MARKETS AND THE BASIC THEOREMS OF VALUE CONSERVATION

by

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ABSTRACT

This paper analyzes the impact, on both welfare and equilibrium prices, of movements from one financial market structure to another in a general equilibrium, two-period context. Previous papers have focussed on the "securities effect", tending to essentially ignore the equally important "endowment effect" that arises when market structure changes are implemented. Strong and weak endowment neutrality and market structure changes of Types I, II and III differentiate the main theorems, which are based on arbitrary preferences and beliefs and substantially extend and modify extant results; in particular, earlier statements identified with value conservation are sharply moderated. Very roughly, the paper yields the following implications for some of the more common changes in the market structure: nonsynergistic corporate spinoffs and the opening of option markets have, on balance, strongly positive welfare effects; nonsynergistic mergers tend to have strong negative welfare effects, while the welfare effects of alternative risky debt structures tend to be ambiguous. All of the preceding, however, may under certain conditions be redistributive.
I. INTRODUCTION AND SUMMARY

Changes in the structure of the financial market have long been of interest to financial economists. While such changes may take many forms, it is perhaps surprising that only a few of the more common ones have been systematically studied. Foremost among these are changes which involve the firm's capital structure (the relative amounts of debt and equity), mergers, and other special recapitalizations.

This paper analyzes the impact, on both welfare and equilibrium prices, of movements from one market structure to another in general in a two-period context. It differs from previous studies primarily in that it is based on a fully integrated (general equilibrium) approach. Thus, while the resulting mosaic contains previous studies as clearly recognizable fragments, such as the classic paper of Modigliani and Miller (1958), it also provides the necessary framework and tools for an evaluation of market structure changes of any type, such as changes involving subordinated debt, convertibles, and warrants, and the opening of option markets.

Three key features characterize the approach used. First, market structure changes are grouped into three types: those which preserve the space of feasible allocations (Type I), those which expand (or contract) it (Type II), and those which cause nontrivial shifts in the set of feasible allocations (Type III). Second, the focus of attention is on "global" statements, i.e. statements which can be made independently of preferences and beliefs. Previous studies, in contrast, have concentrated almost exclusively on specialized preference and belief structures [e.g. Mossin
(1973), Rubinstein (1974)] and on Type I cases. Finally, endowment effects play a crucial role in the analysis; the broader ramifications of such effects have, with few exceptions, notably Litzenberger and Sosin (1977) and Sosin (1978), previously been ignored or overlooked. When taken into account, they tend to sharply moderate earlier statements identified with value conservation, for example. Thus, when the set of securities in the market is altered, there are two separable and equally important effects: one resulting from the change in feasible portfolio choices and one from the (generally unavoidable) substitution of endowment patterns.

In a larger sense, the present paper may be viewed as an attempt to shed further light on the demand for financial assets, a subject which is yet to be integrated with the supply side of financial markets. While the supply conditions have not received nearly as much attention as the demand conditions, a promising beginning has been made by Jensen and Meckling (1976), who examine how "agency costs" affect the issuance of financial securities. In any event, it is in the context of eventual integration with costly supply conditions that (demand) comparisons between (perfect) markets that achieve less than full allocational efficiency assume particular relevance.

The results of the paper may be sketched as follows: In the presence of at most "small" endowment effects, the following "weak" results obtain (Theorems 1, 5, 7): Only market structure changes of Type I preclude Pareto-dominance (of one market structure over another) in both directions, only those of Type II rule out Pareto-inferiority (superiority), while those of Type III, not surprisingly, have unrestricted welfare effects. In general, price effects and welfare effects appear to be essentially independent of each other, that is, wealth may for example decrease as everyone becomes better off.
Contrary to common beliefs, Type I changes in the market do not imply value conservation and investor indifference but may be redistributive. The attainment of investor indifference and value conservation requires the imposition of much stronger conditions: there must also be weak endowment neutrality accompanied by unique equilibria (in both markets) or there must be strong endowment neutrality plus endowments in the first market that constitute an equilibrium in that market (Theorems 3, 4, 3', and 4'). In the preceding, weak endowment neutrality means that the equilibrium value (in the first market) of the disappearing (endowment) securities equals the equilibrium value (in the second market) of the substitute (endowment) securities for all investors. Strong endowment neutrality is even more demanding: it requires that everyone's state by state payoff from the endowment securities given up be preserved by the (endowment) securities received in their place. Returning to the redistributive possibilities, only when strong endowment neutrality (which is often unattainable) holds is each equilibrium in market 1 also an equilibrium in market 2 (Theorem 2). In consequence, value and welfare conservation in market structure changes of Type I should probably be viewed as the exception rather than the rule.

As far as I am aware, market structure changes of Type II were first examined by Borch (1968, pp. 95-103) but have not previously been subjected to systematic analysis. While, as noted, Pareto-inferior allocations are always precluded, redistributive effects are certain to be avoided only if endowments in market 1 constitute an equilibrium in that market and strong endowment neutrality holds (Theorem 6). In moving to a more constrained market structure (Type II reversed), Corollary 4 shows that (values and) welfare can be conserved only under very specialized conditions. Finally, "value
continuity with new instrument" is shown to be linked to Type I changes in the market structure (Theorem 3).

Very roughly, the preceding yields the following implications for some of the more common changes in the market structure: nonsynergistic corporate spinoffs and the opening of option markets have, on balance, strongly positive welfare effects; nonsynergistic mergers tend to have strongly negative welfare effects, while the welfare effects of alternative risky debt structures tend to be ambiguous. All of them, however, may under certain conditions be redistributive.

The paper proceeds as follows. The basic model and its equilibrium properties are specified in Section II. Section III formalizes the relationship between market structures and feasible allocations. Section IV defines endowment neutrality, which, as noted, plays a crucial role in the analysis. Several of the basic issues that arise in comparing equilibria are analyzed in Section V, which also contains some of the underpinning results. The main theorems on weak and strong market equivalence are given in Section VI. Market structure changes of Type II are addressed in Section VII, while Type III changes are examined in Section VIII. Section IX illustrates the tenuous relationship between prices and welfare. The conditions which imply value conservation, and "value continuity" in the presence of new investment, are summarized in Section X. Section XI applies the findings of the paper to corporate spinoffs, mergers, option markets, and a sprinkling of corporate capital structure decisions while Section XII contains some concluding comments.

II. PRELIMINARIES

We consider a pure exchange economy with a single commodity which lasts for two periods under the standard assumptions. That is, at the end of
period 1 the economy will be in some state $s$, where $s = 1, \ldots, n$. There
are $I$ consumer-investors indexed by $i$, whose probability beliefs over the
states are given by the vectors $\pi_i = (\pi_{i1}, \ldots, \pi_{in})$, where

$$\pi_{is} > 0, \quad \text{all } i, s.$$  \hspace{1cm} (1)

Thus, while probability assessments are permitted to differ among investors,
none assigns probability 0 to any of the identified states $1, \ldots, n$.
The preferences of consumer-investor $i$ are represented by the (conditional)
functions $U_{is}(c_i, w_{is})$, where $c_i$ is the consumption level in period 1
and $w_{is}$ is the consumption level in period 2 if the economy is in state
$s$ at the beginning of that period. These functions are defined for
c_i > \alpha_i > 0 \text{ and } w_{is} > \beta_{is} > 0 \text{ and are assumed to be such that}

$$\frac{\partial U_{is}}{\partial c_i} > 0 \text{ for } c_i > \alpha_i, \quad \frac{\partial U_{is}}{\partial w_{is}} > 0 \text{ for } w_{is} > \beta_{is}, \text{ all } i, s$$

and

$$U_{is} \text{ is strictly concave, } \text{ all } i,$$

that is, consumer-investors prefer more to less and are risk-averse. The
numbers $\alpha_i$ and $\beta_{is}$ may be thought of as minimal subsistence levels that
must be satisfied. That is, consumer-investors in effect self-impose the
constraints

$$c_i > \alpha_i, \quad w_{is} > \beta_{is}, \quad \text{all } i, s.$$  \hspace{1cm} (2)

At the beginning of period 1 (time 0), consumer-investors may allocate
their resources among current consumption $c_i$ and a portfolio chosen from
a set $J$ of securities indexed by $j$. Security $j$ pays $a_{js} > 0$ per share
at the end of period 1 and the total number of outstanding shares is \( Z_j \).

Let \( z_{ij} \) denote the number of shares of security \( j \) purchased by investor \( i \) at time 0; his portfolio \( z_i = (z_{i1}, \ldots, z_{ij}) \) then yields the payoff

\[
w_{is} = \sum_{j \in J} z_{ij} a_{js},
\]

available for consumption period 2, if state \( s \) occurs at the end of period 1. Investor endowments are denoted \((\overline{c}_i, \overline{z}_i)\) and financial markets, as is usual, are assumed to be competitive and perfect.\(^3\) Securities, however, are not assumed to be "competitive", that is the number of securities need not be large (although this is not ruled out). If the rank of matrix \( A = [a_{js}] \) is full (equals \( n \)), the financial market will be called complete; if not, it will be called incomplete.

Aggregate wealth in state \( s \) is given by

\[
W_s = \sum_{j \in J} z_{ij} a_{js}, \quad \text{all } s.
\]

Total resources are assumed to be bounded and to be more than adequate to meet the minimal-subsistence levels, that is

\[
\sum_{i} \overline{c}_i > \sum_{i} \alpha_i, \quad W_s > \sum_{i} \beta_{is}, \quad \text{all } s.
\]

Finally, we also assume that everyone's endowment \((\overline{c}_i, \overline{z}_i)\) is such that it satisfies

\[
\overline{c}_i > \alpha_i + \varepsilon, \quad \overline{w}_{is} > \beta_{is} + \varepsilon, \quad \text{all } i, s
\]

for some \( \varepsilon > 0 \). Condition (5) insures that each consumer-investor faces a feasible and nontrivial decision problem.
Under our assumptions, each consumer-investor \( i \) maximizes

\[
\begin{align*}
  u_i &= \sum_{s} \sum_{i_s} U_i \left( c_{i_s}, z_{ij}, a_{js} \right) \\
\end{align*}
\]  

(6)

with respect to the decision vector \((c_{i_s}, z_{ij})\), subject to (3) and to his budget constraint

\[
\begin{align*}
  c_{i_s}^{P_0} + \sum_{j \in J} z_{ij} P_j &= c_{i_s}^{P_0} + \sum_{j \in J} z_{ij} P_j \\
\end{align*}
\]

as a price-taker, where \( P_0 \) is the price of a unit of period 1 consumption and \( P_j \) is the price (at \( t = 0 \)) of security \( j \).

The object of this paper, as already noted, is to compare the allocation possibilities provided by, and the welfare implications of, various market structures, also referred to simply as markets, market situations, and market arrangements. For our purposes the focal point of a given market structure in these comparisons will be the set of instruments available for investment. Thus, even though in a larger sense the totality of the participants is an integral part of a complete description of a market arrangement \( M \), it is generally sufficient to think of the distinctive feature of \( M \) as the matrix \( A \) of security payoffs; for this reason we will in effect view \( M \) and \( A \) as interchangeable.

In view of assumption (1)-(5), an equilibrium will exist but need not be unique [see e.g. Hart (1974)]. The equilibrium conditions for any market structure \( A \) may be written
\[ \frac{\partial U_i}{\partial s_{is}} (c_i, \Sigma z_{ij} a_{js}) \frac{\Sigma \pi_{is}}{s_{is}} \frac{\partial s_{is}}{\partial c_i} + \delta_{i0} = \lambda_1 \quad \text{all } i \quad (7) \]

\[ \frac{\partial U_i}{\partial s_{is}} (c_i, \Sigma z_{ij} a_{js}) a_{js} \frac{\Sigma \pi_{is}}{s_{is}} \frac{\partial s_{is}}{\partial w_{is}} + \Sigma \delta_{is} a_{js} = \lambda_{ij} \quad \text{all } i, j \quad (8) \]

\[ c_i + z_{ij} P = \bar{c}_i + z_{ij} P \quad \text{all } i \quad (9) \]

\[ \Sigma c_i = \bar{c}_i, \Sigma z_{ij} = Z_j \quad \text{all } j \quad (10) \]

\[ c_i \geq a_i, z_{ij} A \geq \beta_i \quad \text{all } i \quad (11) \]

\[ \delta_{i0} > 0, \delta_{is} > 0, \delta_{i0} (c_i - \alpha_i) = 0, \delta_{is} (w_{is} - \beta_i s_{is}) = 0 \quad \text{all } i, s \quad (12) \]

where the \( \lambda_i \) and \( \delta_{is} \) are Lagrange multipliers, (10) represents the market clearing equations, and \( P_0 \) has been chosen as numeraire, i.e., \( P_0 \equiv 1 \).

In the system (7)-(12), \( \delta_{is} = 0 \) corresponds to an allocation which is interior with respect to the relevant subsistence level.

We note that an allocation \((c^*, z^*)\) which constitutes a solution to system (7)-(12) (along with a price vector \( P \), a vector \( \lambda \) and a matrix \( \delta \)) is Pareto-efficient with respect to market structure \( A \), that is no other allocation \((c, z)\) obtainable within market structure \( A \) can make some consumer-investors better off without making others worse off. However, there may exist trades outside the market (e.g. by the invention of new
securities) which yield allocations that Pareto-dominate the market allocation \((c^*, z^*)\). When this is not the case, i.e. when \((c^*, z^*)\) is Pareto-efficient with respect to all conceivable allocations inside and outside the existing market, \((c^*, z^*)\) will be said to be fully Pareto-efficient or fully allocationally efficient.

To be more precise, let

\[
P_{is} = \frac{1}{\lambda_i} \left\{ \frac{\partial U_{is}(c_i, \sum_{j \in J} z_{ij}^a_{js})}{\partial w_{is}} + \delta_{is} \right\}.
\]

It is well known that (7) and (8) plus

\[
P_{is} = P_{is} \quad \text{all } i \geq 2, \text{ all } s
\]

is a necessary and sufficient condition for the market allocation \((c^*, z^*)\) to be fully Pareto-efficient, because (13) insures that the marginal rates of substitution of wealth between any two states are the same for all investors \(i\).

Our assumptions also imply that \(P_{j}^* > 0\) for all \(j\). However, even though \(z_{i}^A > \beta_{i} > 0\), some of the \(z_{ij}^*\) will generally be negative, i.e. the equilibrium solution generally calls for short selling of some securities by some investors. (The equilibrium solution will of course also be dependent on the endowments \((\bar{c}, \bar{z})\) as well as the utility functions and the probability assessments.) The equilibrium value of a feasible
second-period payoff vector \( w \) will be denoted \( V(w) \); thus if \( w \) is obtainable
via portfolio \( z \), we obtain \( w = zA \) and hence

\[
V(w) = V(zA) = zP .
\]  

III. FEASIBLE ALLOCATIONS

In this section, we characterize the sets of possible second-period
allocations \( w_i \) obtainable via different market arrangements. Recall that
a market structure \( A \) is any "full" set of instruments, i.e. any set of
instruments capable of allocating aggregate wealth \( W = (W_1, \ldots, W_n) \).
Thus, a market may have as few as one instrument.\(^5\) The set of feasible
second-period consumption allocations \( w = (w_1, \ldots, w_I) \) obtainable via
market structure \( A \) will be denoted \( F(A) \), i.e.

\[
F(A) \equiv \{w \mid w_i = z_i A, \sum_{i=1}^{I} z_{ij} = z_j, \text{ all } j\} .
\]

Note that this definition makes no reference to, and is therefore completely
independent of, endowments. It merely summarizes all conceivable allocations
of second-period consumption via the securities that comprise \( A \).

In comparing two market structures \( A' \) and \( A'' \) with respect to feasible
allocations, we first observe that \( F(A') \cap F(A'') \) is always non-empty since
holding the "market portfolio", i.e. an equal proportion of every instrument,
for example, is always feasible.\(^6\) Thus, there are three possibilities;
either

\[
F(A') = F(A'')
\]

or

\[
\begin{array}{c}
F(A'), F(A'') \quad \text{(Type I)} \end{array}
\]
\[ F(A') \subseteq F(A'') \text{ (or the converse)} \]

\[ \{F(A') \cap F(A'')\} \subseteq F(A') \]

\[ \{F(A') \cap F(A'')\} \subseteq F(A'') \]

A necessary, but not sufficient, condition for a market structure change to be of Type I is that \( A' \) and \( A'' \) have the same rank, i.e. \( \text{rank} (A') = \text{rank} (A'') \).

Similarly, a necessary, but not sufficient, condition for a comparison to be of Type II is that \( \text{rank} (A') < \text{rank} (A'') \). However, if \( \text{rank} (A) < n \), then clearly \( F(A) \subseteq F(A_n) \), where \( A_n \) is a complete (financial) market.

A Type III comparison may be illustrated by a simple example:

\[
\begin{array}{ccc}
A' & & A'' \\
1 & 2 & 2 & 1 & 1 & 3 \\
2 & 2 & 3 & 2 & 3 & 2 \\
W & 300 & 400 & 500 & W & 300 & 400 & 500 \\
\end{array}
\]

For instance, a payoff \( w = (15, 18, 24) \) for the three states is attainable in \( A' \) (buy 3 and 6 shares, respectively, of the two instruments) but not in \( A'' \). Conversely, a payoff \( w = (15, 18, 33) \) is feasible in \( A'' \) (buy 9 and 3 shares, respectively, of the two instruments) but not in \( A' \).

A sure way to increase the allocation possibilities (to obtain a Type II change) is to make a finer and finer breakdown of existing instruments into an ever larger set of linearly independent securities. Type III changes, however,
remind us that sheer numbers are not the end-all: two instruments alone may be able to accomplish some of the things that a million (linearly independent) instruments cannot. The relation $\subseteq$ among feasible allocations induces only a partial ordering of financial markets.

In the sequel, an instrument $j$ will be termed reproducible, redundant, or spanned in market $A$ if its payoff pattern $a_j$ is obtainable via a linear combination (or portfolio) of other instruments in $A$. If an instrument is not reproducible, it will be termed non-reproducible or unique in relation to market $A$.

IV. ENDOWMENT NEUTRALITY

The welfare implications of moving from market structure $A'$ to market structure $A''$, or the comparative implications of moving from $A^0$ to $A'$ versus from $A^0$ to $A''$, depend not only on the feasible allocations $F(A')$ and $F(A'')$ but also on whether there are "endowment effects". Our concern here is with endowment effects which, avoidably or unavoidably, arise from the replacement of existing instruments with new ones. For example, in a simple merger of two companies $X$ and $Y$, the separate payoff patterns of the common of $X$ and of $Y$ become combined into a new single pattern. But the accompanying exchange of shares generally also affects the endowment pattern $(c_i, w_i)$.

More precisely, a change in the market structure under pure exchange is brought about by the substitution, removal, and/or addition of securities, including the numeraire commodity (cash), in such a way as to leave the supply of aggregate consumption, i.e. $CE_i$ and $W$, unchanged. All such changes are
consider first a change from \( A' \) to \( A'' \). In general, some instruments in \( J' \) will then be continued in the new set \( j' \) and some will disappear. Denote the disappearing subset by \( J' \subseteq J \) and the subset of new instruments by \( J' \subseteq J \), since endowment holdings of securities \( j' \not\in J \) continue unchanged in \( A'' \), we observe that (15) can now be guaranteed, for an arbitrary endowment pattern, only if the payoff from each security \( j \) in \( J \) can be replaced by a linear combination of instruments in \( J' \).

In each state be the same in the two markets. Any departure from (15) constitutes an endowment shift and will in general (as we shall see) have significant welfare effects.

### Appendix

As defined, strong endowment neutrality requires not only that each consumer-investor's initial claims to current consumption remain unchanged between the two markets but that his initial claims to end-of-period wealth will be the opening or closing of option markets. Decentralized market structure changes, besides being the more natural (and possibly convenient), will have the property that they affect only a relatively small subset of the securities available in the market. It will be useful to distinguish between two kinds of endowment effects in connection with market structure changes. To identify the first type of effect, we define

\[
(c', w') = (c', w')
\]
Formally,

**Lemma 1.** A necessary and sufficient condition for \( E'' \) sn \( E' \) to be attainable (for arbitrary endowments) is that there exist exchange ratios \( e_{jk} \) such that

\[
a'_{js} = \sum_{ks \in J' \cap J} e_{jk} a''_{ks} \quad \text{all } j \in J', \text{ all } s,
\]

that is, \( F(A'_C) \subseteq F(A''_C) \) must hold.

Note that \( F(A'') = F(A') \) does not guarantee \( E'' \) sn \( E' \). For example, if two instruments are exchanged for one (e.g. through a 100% non-synergistic merger) and one (but not the other) of the old instruments is unique, then the new instrument is also unique and \( F(A'') = F(A') \).

But only if all investors happen to hold each of the old (replaced) instruments in the same proportion can the "new" firm preserve endowment neutrality in this particular case. Similarly, strong endowment neutrality is certain to be symmetric only when \( F(A'_C) = F(A''_C) \). Summarizing,

**Remark 1.** \( E'' \) sn \( E' \) is certain to be attainable if the ratios \( \frac{z''_i}{z'_j} \) for all \( j \in J' \cap J \) depend only on \( i \).

**Remark 2.** Both \( E'' \) sn \( E' \) and \( E' \) sn \( E'' \) are certain to be attainable only if \( F(A'_C) = F(A''_C) \).

One implication of the preceding is that movements from one complete market to another, via decentralized changes in the market structure, are generally not accompanied by strong endowment neutrality. In the same vein, we observe that \( F(A') \subseteq F(A'') \) may or may not be accompanied by \( E'' \) sn \( E' \); moreover, \( E' \) sn \( E'' \) would now occur only in rare cases.
The following are examples of market structure changes that generally do not satisfy strong endowment neutrality:

1. 100% nonsynergistic mergers. 9
2. The refinancing of bonds (prior to maturity) with new bonds of a different maturity.
3. The calling of convertible bonds and replacing them with regular bonds of the same (or a different) maturity.
4. The conversion of convertible securities into common stock. 10
5. The closing of option markets, with settlements in cash and/or the underlying securities.

The following types of market structure changes, on the other hand, do preserve strong endowment neutrality:

1. Nonsynergistic corporate spinoffs.
2. The issuance, on a pro rata basis, of bonds (risk-free or subordinated) or preferred stock in exchange for common stock placed in treasury.
3. The opening of option markets.

Strong endowment neutrality is clearly a very demanding requirement.

A less restrictive notion of neutrality which more closely captures our intuitive sense of this concept is that the exchange of endowment securities and "cash" transfers be "even" in terms of value. More precisely,

**Weak endowment neutrality.** Market structure A'' is weakly endowment neutral with respect to market structure A' (written E'' wn E') if and only if
\[
\bar{c}''_{ic} + V'(\bar{w}''_{is}) = \bar{c}'_{ic} + \sum_{j \in J'} \bar{z}'_{ij} I''_j,
\]
all \(i\), \hspace{1cm} (16)

where

\[
\bar{w}''_{is} = \sum_{j \in J''} \bar{z}''_{ij} a''_{js}
\]

and

\[
\sum_{i} \bar{c}'_{ic} = \bar{c}'_{ic} = 0.
\]

(17)

In the preceding, \(\bar{c}''_{ic}\) is the net commodity (cash) addition to consumer-investor \(i\)'s holdings involved in the exchange. Under our assumption of pure exchange, all numeraire transfers must cancel out in the aggregate, i.e. (17) must hold. Since \(V'(\bar{w}''_{is})\) is only defined for payoffs that are feasible in \(\Lambda'\), the notion of weak endowment neutrality is of somewhat limited usefulness. It is, however, by no means restricted to Type I changes.

Finally, when endowment neutrality is absent, the notion of "small" endowment shifts will be useful. An endowment shift will be said to be of size \(\eta > 0\) when

\[
(\bar{c}'' - \bar{c}'_{i}) \neq (\bar{c}', \bar{w}'), |\bar{c}''_{i} - \bar{c}'_{i}| \leq \eta, |\bar{w}''_{is} - \bar{w}'_{is}| \leq \eta, \text{ all } i, s, \hspace{1cm} (18)
\]

(5) is preserved, and equality holds somewhere in (18).

V. COMPARISONS OF EQUILIBRIA

In comparing different market structures, the comparison which is ultimately relevant is that which compares allocations actually obtained, i.e. equilibrium allocations. Furthermore, our primary concern is with the welfare implications of such (equilibrium) allocations. Using (6),

\[

\]
we denote investor $i$'s equilibrium expected utility in market structure $A''$ by $u''_i$ and his equilibrium expected utility in market structure $A'$ by $u'_i$. A comparison of any given equilibrium in market $A''$ with some equilibrium in some other market $A'$ must then yield one of four cases:

\[ u''_i > u'_i, \text{ all } i, \quad u''_i > u'_i, \text{ some } i, \quad (i) \]

or

\[ u''_i = u'_i, \text{ all } i, \quad (ii) \]

or

\[ u''_i > u'_i, \text{ some } i, \quad u''_i < u'_i, \text{ some } i, \quad (iii) \]

or

\[ u''_i < u'_i, \text{ all } i, \quad u''_i < u'_i, \text{ some } i. \quad (iv) \]

The task at hand, then, is to identify conditions under which each of these cases, as well as combinations of these cases, will occur. The reader will recognize that (i), and its converse (iv), correspond to the usual definition of Pareto-dominance.

The subset of consumption allocations $(c, w)$ which, for a given collection of beliefs and preferences $(\pi, U)$, are obtainable as competitive solutions to system (7)-(12) for all feasible endowments (i.e., all endowments satisfying (5)) will be denoted $F^* (A)$. (Note that in view of the difference between (3) and (5), $F^* (A)$ is a proper subset of all allocations that are Pareto-efficient with respect to collection $(\pi, U)$.)

Observing that $F(A'') = F(A')$ implies that the set of feasible endowments are the same in the two markets, this in turn leads directly to
Lemma 2. If $F(A') = F(A'')$, then

$$F^*(A') = F^*(A'')$$

for every collection $\{\pi, U\}$ satisfying (1) and (2).

We now state a useful intermediate result.

Lemma 3. Let $A''$ and $A'$ be financial markets such that

$$F(A') \subset F(A'') \quad \text{(Type II)}$$

Then there exist collections $\{\pi, U\}$ satisfying (1) and (2) and endowment shifts satisfying (18) for some $\eta > 0$ (and (15) when feasible) such that

$$u''_i > u'_i, \text{ all } i, \quad u''_i > u'_i, \text{ some } i,$$

(i)

i.e. such that all investors are better off under $A''$ than under $A'$, and other collections and endowments for which either

$$u''_i = u'_i, \text{ all } i \quad \text{(ii)}$$

or

$$u''_i > u'_i, \text{ some } i, \quad u''_i < u'_i, \text{ some } i \quad \text{(iii)}$$

An illustration which yields (i) may be found in Section IX; while based on strong endowment neutrality, it is readily altered to accommodate endowment shifts. An example for which (ii) or (iii) obtains is that of homogeneous probability assessments and $U_{is}(c_i, w_{is}) = u_i(c_i) + \alpha_i w_{is}^\gamma$ for all $i$ (with $\alpha_i > 0$ and $\gamma < 1$); in this case, all investors, as is well-known, always hold "the market portfolio" of all assets and every market $A$ is fully allocationally efficient. Under weak endowment neutrality, "the market portfolio" proportions under $A'$ and $A''$ are unchanged for any given investor, yielding (ii), while shifts in endowments give rise to (iii).
The following corollary is immediate:

Corollary 1. Let $A'$ and $A''$ be financial markets such that

$$\{F(A') \cap F(A'')\} \subset F(A') \quad \text{(Type III)}$$

$$\{F(A') \cap F(A'')\} \subset F(A'') .$$

Then there exist collections $\{\pi, U\}$ satisfying (1) and (2) and endowment shifts satisfying (18) for some $\eta > 0$ (and (15). when feasible) such that

$$u''_i > u'_i, \text{ all } i, \ u''_i > u'_i, \text{ some } i , \quad \text{(i)}$$

and other collections for which

$$u''_i \leq u'_i, \text{ all } i, \ u''_i < u'_i, \text{ some } i . \quad \text{(iv)}$$

VI. EQUIVALENT MARKETS

When two market structures $A'$ and $A''$ provide the same allocational possibilities, i.e. $F(A') = F(A'')$, they may be thought of as equivalent. But allocational equivalence does not necessarily imply that everyone’s expected utility will be the same in the two markets. For example, we have already observed that an endowment $(c_i, \bar{w}_i)$ may not be transferable from one market to another that is allocationally equivalent. In addition, multiple equilibria may occur and must be dealt with. Finally, welfare equivalence may occur for some collections of preferences and beliefs but not others. The end result of these realities is that as we move into the sphere of welfare, we will need to distinguish between several types of equivalence: weak, semi-strong, and strong, global and selective.
Our first result concerns weak, global market equivalence, i.e. what we can say independently of endowments, preferences, and beliefs.

**Theorem 1.** (Weak global market structure equivalence.) In comparing an equilibrium under market structure $A''$ with an equilibrium under market structure $A'$, it is sufficient to obtain, for every collection $\{\pi, U\}$ satisfying (1) and (2), either

$$u''_i = u'_i, \text{ all } i \quad \text{(ii)}$$

or

$$u''_i > u'_i, \text{ some } i, \quad u''_i < u'_i, \text{ some } i \quad \text{(iii)}$$

that

$$F(A'') = F(A') \quad \text{(Type I)}$$

a Type I change is also necessary for (ii) or (iii) alone to occur provided endowment shifts, when present, are sufficiently small.

**Proof:** The equilibrium allocations $(c', w')$ and $(c'', w'')$ are such that $(c', w') \in F^*(A')$ and $(c'', w'') \in F^*(A'')$. But in Type I comparisons, $F^*(A') = F^*(A'')$ for each collection $\{\pi, U\}$ satisfying (1) and (2) by Lemma 2, which in turn implies (ii) or (iii) independently of endowments. This establishes sufficiency.

To show necessity, suppose to the contrary that $F(A'') \neq F(A')$, which in turn means that the comparison is either of Type II or of Type III. But by Lemma 3 and Corollary 1, there now exist collections $\{\pi, U\}$ for which (i) and (iv) obtain not only under endowment neutrality but for sufficiently small endowment shifts. Thus, $F(A') = F(A'')$ is indeed required for (ii) and (iii) alone to be possible when preferences and beliefs are unrestricted.
Theorem 1 reminds us that while Pareto-dominance cannot occur in market comparisons of Type I, redistributive effects are not ruled out. The next goal is to identify conditions which insure that each equilibrium allocation \((c', w')\) in \(A'\) is also an equilibrium allocation in \(A''\) and conversely. This is a much stronger implication than Theorem 1 offers and requires the presence of strong endowment neutrality as well.

**Theorem 2.** (Semi-strong global equilibrium correspondence.) In moving from market structure \(A'\) to market structure \(A''\), we obtain, for every collection \(\{w, U\}\) satisfying (1) and (2), that each equilibrium in \(A''\) is also an equilibrium in \(A'\), and conversely, if and only if

\[
F(A'') = F(A') \quad \text{(Type I)}
\]

and

\[
E'' \equiv E' \quad \text{(19)}
\]

**Proof:** Let \((c', w')\) be an equilibrium allocation under \(A'\). \(F(A'') = F(A')\) then implies that there exists a security portfolio \(z''_i\) in \(A''\) such that

\[
(c''_i, z''_i) = (c'_i, w'_i), \quad \text{all } i \quad \text{(20)}
\]

and that there exist weights \(e_{jk}\) such that

\[
a'_{js} = \sum_{k \in J'} e_{jk} a'_{ks} \quad \text{all } s, \text{ all } j \in J''.
\]

But this in turn implies that the allocation \((c'', z'')\) satisfies equilibrium conditions (7), (8), and (10)-(12), with

\[
\lambda'' = \lambda', \delta'' = \delta' \quad \text{(21)}
\]

\[
p''_j = \sum_{k \in J'} e_{jk} p'_k \quad \text{all } j \in J'', \quad \text{(22)}
\]
and

\[ P''_i = P'_i, \quad \text{all } i \text{ and } s. \] (23)

In view of (20), (23) also means that total expenditures are the same in the two markets, i.e.

\[ c''_i + z''_i P''_i = c'_i + z'_i P'_i, \quad \text{all } i, \] (24)

while (19) guarantees that endowments as well are unchanged in value.

That is,

\[ \bar{c}''_i + \bar{z}''_i P''_i = \bar{c}'_i + \bar{w}'_i P'_i = \bar{c}'_i + \bar{z}'_i P, \quad \text{all } i \] (25)

for each equilibrium that may occur in \( A' \). By symmetry, each equilibrium in \( A'' \) is also an equilibrium in \( A' \).

The necessity of \( F(A'') = F(A') \) derives from the fact that, without it, \((c', w')\) need not be feasible in \( A'' \) and \((c'', w'')\) need not be feasible in \( A' \). Suppose now that (19), or equivalently (15), does not hold and consider the feasible allocation \((c'', z'')\) satisfying (20); as noted, this allocation also satisfies (7), (8), (10)-(12), (23), and (24). But in view of (23), the inner equality in (25) need not hold in the absence of (19); even if it should hold for some equilibrium pair of (implicit) price vectors \( P''_i \) and \( P'_i \), it would not hold for other pairs in the case of multiple equilibria (which are not ruled out since \((\pi, U)\) is unrestricted).

This concludes the proof.

Recall that strong endowment neutrality in one direction does not imply strong endowment neutrality in the other. Thus, the conditions of
Theorem 2 only guarantee the existence of duplicate equilibria in the two markets for uni-directional movements (such as from A' to A' but not from A'' to A'). In order to be able to move interchangeably from one market to another, while at the same time guaranteeing that each equilibrium in one market is also an equilibrium in the other, strong endowment neutrality must be present in both directions. Formally, giving recognition to the equilibrium valuations $V'$ and $V''$ as well, this gives

**Corollary 2.** (Strong global equilibrium and value correspondence.) In moving from A' to A'', or from A'' to A', we obtain, for every collection $\{\pi, U\}$ satisfying (1) and (2), that each equilibrium in A'' is also an equilibrium in A' with the property

$$V''(w) = V'(w), \quad \text{all feasible } w,$$  \quad (26)

and conversely, if and only if

$$F(A'') = F(A')$$  \quad (Type I)

and

$$E'' \cap E' \text{ and } E' \cap E''$$  \quad (27)

i.e. strong endowment neutrality holds in both directions.

In view of Theorem 2, the only way to guarantee a unique equilibrium in A' is to assume that

$$(c', w') = (\overline{c'}, \overline{w'})$$  \quad (28)

i.e. that endowments in market A' constitute an equilibrium allocation in that market. Furthermore, the only way that same allocation can now be
guaranteed in market \( A'' \) in Type I comparisons, independently of preferences and beliefs, is for it to be brought about via the new endowment \((c'', w'')\), i.e. by \( E'' \) on \( E' \), since trading cannot be counted on to re-generate \((c', w')\) when multiple equilibria are possible. In Type II reverse and in Type III comparisons, \((c', w')\) need not be feasible in \( A'' \) for some collections \((\pi, U)\); in Type II changes, \((c' \bar{w}')\) need not be optimal in \( A'' \) by Lemma 3. Consequently, \( F(A') = F(A'') \) is also necessary to insure \((c'', w'') = (c', \bar{w}')\). This gives

**Theorem 3.** (Semi-strong global market structure equivalence.) In moving from market structure \( A' \) to market structure \( A'' \), we obtain

\[
(\text{ii}) \quad u_i'' = u_i' \quad \text{all } i
\]

for every collection \((\pi, U)\) satisfying (1) and (2) if and only if

\[
F(A'') = F(A') \quad \text{(Type I)}
\]

\[
(c', w') = (c_i, \bar{w}_i) \quad \text{(28)}
\]

and

\[
E'' \text{ on } E' \quad \text{(19)}
\]

Perhaps the most noteworthy thing about Theorem 3 is that, given both \( F(A') = F(A'') \) and (28), the presence of weak endowment neutrality may still bring about a redistribution of welfare when there are multiple equilibria.  

In the proof of Theorem 2, we found that \( F(A'') = F(A') \) implies (10)-(24) but not necessarily (25). Suppose, however, that we have weak endowment neutrality, i.e. that (16) holds, and that equilibrium in \( A' \) is unique for the collection \((\pi, U)\) at hand. Then (25) also holds so that \((c', w')\) is an equilibrium in \( A'' \).
If equilibrium in \( A' \) were not unique, (16) would not hold for \( \bar{w}^{sw} \) for a different price vector \( P' \) unless (19) were also valid. Similarly, unless the constructed equilibrium in \( A'' \) based on the chosen (weakly neutral) endowment is unique, there would not necessarily be an equilibrium in \( A' \) for each equilibrium in \( A'' \). This gives,

**Theorem 4.** (Selective market structure equivalence.) In moving from market structure \( A' \) to market structure \( A'' \), we obtain

\[
\bar{u}_i'' = \bar{u}_i', \text{ all } i
\]  

(ii)

for every collection \( \{\bar{v}, U\} \) consistent with the requirement that equilibrium in each market be unique, if and only if

\[
F(A'') = F(A') \tag{Type I}
\]

and

\[
E'' \equiv E' .
\]  

(29)

As noted earlier, the exchange ratio of securities which guarantees weak endowment neutrality is based on the equilibrium prices that prevail in the first market (\( A' \)). These would not necessarily be known prior to trading when the substitution of securities must take place. Weak endowment neutrality is thus somewhat difficult to visualize. One possible scenario would be to assume that endowments in \( A' \) already represent an equilibrium allocation achieved in "earlier" trading that revealed prices \( P' \). (16) and (22) then provide the mechanism to implement weak endowment neutrality so as to achieve an equivalent equilibrium in \( A'' \) for Type I changes.

This leads to the following (weaker) statement of existence
Corollary 3. Assume that \((\overline{c}', \overline{w}')\) represents an equilibrium allocation in \(A'\) supported by the price vector \(P'\). In moving to \(A''\), there then exists an equivalent equilibrium allocation in that market for every collection \((\pi, U)\) satisfying (1) and (2) if and only if

\[
F(A'') = F(A')
\]

and

\[
E'' \subseteq E',
\]

(29)

where (29) is based on \(P'\).

As already noted, (29), as opposed to (19), does not guarantee that the equivalent equilibrium will be attained when multiple equilibria exist. However, Corollary 3, by in effect side-tracking the endowment effect, seems to come closest to capturing previous results on capital structure indifference [e.g. Modigliani and Miller (1958), Stiglitz (1969), Fama (1978)] which appear to have paid minimal attention to endowment effects. But to assume that no trading is required in \(A'\), which is what is needed for \(P'\) to be known, is a strong assumption indeed. Relaxing these requirements, however, means, as we have seen, that equivalent allocations are far from guaranteed (how do we get \(P'\)?) and this, in turn, implies that redistributive effects generally do accompany even those market structure changes that leave the allocational possibilities unaltered. Decentralized substitutions of (subsets of) securities will thus generally have redistributive effects, in complete as well as incomplete markets.
VII. MARKET STRUCTURE DOMINANCE

The central feature of Type I changes in the market structure is that welfare effects are at most redistributive—and that redistribution is unavoidable more commonly perhaps than previously thought. When we turn to Type II changes, i.e. the case when \( F(A') \subset F(A'') \), Pareto-dominance is of course possible, as we saw in Lemma 3. Our first result confirms that everyone cannot be worse off in \( A'' \).

**Theorem 5.** (Weak global market structure dominance.) In comparing equilibrium under market structure \( A'' \) with an equilibrium under market structure \( A' \), we obtain, for every collection \( \{ \pi, U \} \) satisfying (1) and (2) provided endowment shifts, when present, are sufficiently small, either

\[
 u''_i > u'_i, \text{ all } i, u''_i > u'_i, \text{ some } i \quad (i)
\]

or

\[
 u''_i = u'_i, \text{ all } i \quad (ii)
\]

or

\[
 u''_i > u'_i, \text{ some } i, u''_i < u'_i, \text{ some } i \quad (iii)
\]

if and only if

\[
 F(A') \subset F(A'') . \quad \text{(Type II)}
\]

To show sufficiency, we first observe that \( (c'', w'') \in F^*(A'') \) and that \( (c', w') \in F(A'') \) since \( F(A') \subset F(A'') \). Thus, (iv) is ruled out since \( (c'', w'') \) cannot be Pareto-dominated by an allocation in \( F(A'') \). In addition, by Lemma 3, there are indeed collections \( \{ \pi, U \} \) under which each of situations (i)-(iii) obtains given sufficiently small endowment shifts.
The necessity of $F(A') \subseteq F(A'')$ follows from the observation that if this condition does not hold, we must either have a Type I comparison, which cannot generate (i) by Theorem 1, or a Type III comparison, which by Corollary 1 does not rule out (iv) when endowment shifts are sufficiently small.

Similarly, Theorem 3 translates directly to

**Theorem 6.** (Strong global market structure dominance.) In moving from market structure $A'$ to market structure $A''$, we obtain, for every collection $\{w, U\}$ satisfying (1) and (2), either

\[ u''_i > u'_i, \text{ all } i, \quad u''_i > u'_i, \text{ some } i \] (i)

or

\[ u''_i = u'_i, \quad \text{all } i \] (ii)

if and only if

\[ F(A') \subseteq F(A''), \] (Type II)

\[ (c', w') = (\overline{c'}, \overline{w'}) \] (28)

and

\[ E'' \cap E'. \] (19)

A comparison of Theorems 5 and 6 reveals that the only way that redistributive effects can be avoided with certainty in moving to an allocationally richer market is to guarantee that the equilibrium allocation in $A'$ is the initial position in $A''$. This requires both (28) and (19); if either is absent, $(\overline{c''}, \overline{w''}) \neq (c', w')$ and there will then be some preferences and beliefs for which not only $(c'', w'') \neq (c', w')$ but which leave some consumer-investors worse off in $A''$ than they would be in $A'$. 
As in Theorem 4, strong endowment neutrality can be relaxed to weak endowment neutrality in Theorem 6 but only if 1) the price structure of the replaced securities turns out to be the same as for the forfeited ones and 2) those prices are unique, i.e. equilibrium in \( A'' \), given (28), is unique (since weak endowment neutrality generally does not hold, for given endowments, for more than one price system). Formally

**Remark 3.** Suppose that

\[
F(A') \subset F(A'') \quad \text{(Type II)}
\]

Then, in moving from market structure \( A' \) to market structure \( A'' \), we obtain, for any collection \( \{ \pi, U \} \) satisfying (1) and (2) that is also consistent with the requirement that equilibrium in \( A'' \) be unique and that \( v''(w) = v'(w) \) for all jointly feasible \( w \) obtainable from the securities in \( J'_c \) and \( J''_c \), either

\[
\begin{align*}
&\ u''_i > u'_i, \text{ all } i, \ u''_i > u'_i, \text{ some } i \quad \text{(i)} \\
\quad \text{or} \\
&u''_i = u'_i \quad \text{(ii)}
\end{align*}
\]

whenever

\[
(c', w') = (c', \overline{w'}) \quad \text{(28)}
\]

and

\[
E'' \equiv E'. \quad \text{(29)}
\]

Finally, the following result considers movements in the opposite direction, from less constrained to more constrained markets.

**Corollary 4.** (Welfare preservation.) In moving from market structure \( A'' \) to market structure \( A' \), where \( F(A') \subset F(A'') \), we obtain

\[
u''_i = u'_i, \text{ all } i \quad \text{(ii)}
\]
whenever

\[(c''', w''') = (\tilde{c}'''', \tilde{w}''')\]  

(30)

and

\[E' \cap E''\].  

(31)

Proof: Assumptions (30) and (31) guarantee that \((\tilde{c}', \tilde{w}') \in F^*(A'')\). Since \(F(A') \subseteq F(A'')\), every feasible \((\tilde{c}', \tilde{w}') \in F^*(A'')\) is such that \((\tilde{c}', \tilde{w}') = F^*(A')\) and hence no trading will occur.

It should be noted that when \(F(A') \subseteq F(A'')\), the condition \(E' \cap E''\) will rarely be satisfied because the greater richness of market structure \(A''\) will be reflected in the equilibrium allocations \((c''', w''')\). In fact, only for those collections \(\{\pi, U\}\) which don't take advantage of the additional opportunities provided by \(A''\), by holding the market portfolio, say, and certain singular cases will we find that \(E' \cap E''\) is not totally ruled out. The implication of this is that market structure changes which limit feasible allocations will almost always either have redistributive effects or leave everyone worse off.

VIII. TYPE III CHANGES

Market structure changes of Type I can, as we have seen, never lead to Pareto-dominance. Similarly market structure changes of Type II preclude Pareto-dominance in one (but not both) directions. In view of Corollary 1, it is then both necessary and sufficient, for sufficiently small endowment effects, to be in the domain of Type III changes in order for any one of the outcomes (i) through (iv) to be feasible. That is,
Theorem 7. (Unconstrained outcomes.) In comparing an equilibrium under market structure $A''$ with an equilibrium under market structure $A'$, we obtain, for every collection $\{\pi, U\}$ satisfying (1) and (2) provided endowment shifts are sufficiently small, either (i), (ii), (iii), or (iv) if and only if

$$(\text{Type III})$$

$$\begin{align*}
\{F(A') \cap F(A'')\} & \subset F(A') \\
\{F(A') \cap F(A'')\} & \subset F(A'')
\end{align*}$$

When preferences and beliefs are unrestricted and endowment shifts are partially restricted, Theorems 1, 5, and 7 reveal that the welfare implications for each type of market structure change are not only sharply delineated but substantively different. The basic mapping from Types I–III to welfare space is both simple and informative. The absence of strong results for Type III comparisons, of course, does not imply that Type III changes should be viewed as rare—quite the contrary.

IX. WELFARE AND PRICES

Apparently beginning with Hirshleifer (1970, p. 275), a number of authors have observed that value maximization by firms need not be an optimal rule in incomplete markets. While the impact on prices resulting from market structure changes have not been addressed directly, there appear to be at least some suggestions to the effect that the equilibrium prices and welfare improvements attained by market enrichment are positively related. It may therefore be worthwhile to examine briefly the relation
between welfare and prices in the current framework. Suppose that, for simplicity, there are two consumer-investors and three states. A' has two securities and A'' three, as follows:

<table>
<thead>
<tr>
<th></th>
<th>A'</th>
<th></th>
<th>A''</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>100</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>100</td>
</tr>
<tr>
<td>W</td>
<td>200</td>
<td>300</td>
<td>400</td>
<td></td>
</tr>
</tbody>
</table>

W 200 300 400

The third security in A'' may be recognized as a call option on security 2 with an exercise price of 2. Since A'' has full rank, F(A') ⊂ F(A'').

Suppose beliefs are given by \( \pi_1 = (.3, .4, .3) \) and \( \pi_2 = (1/3, 1/3, 1/3) \) and that preferences are additive, with second-period utilities \( U_i \) and marginal utilities \( U'_i \) for selected consumption levels given by:

### Individual 1:

<table>
<thead>
<tr>
<th>Cons. level</th>
<th>126</th>
<th>130</th>
<th>150</th>
<th>159</th>
<th>165</th>
<th>170</th>
</tr>
</thead>
<tbody>
<tr>
<td>( U_1 )</td>
<td>0</td>
<td>5.7</td>
<td>28.5</td>
<td>36</td>
<td>40.05</td>
<td>42.75</td>
</tr>
<tr>
<td>( U'_1 )</td>
<td>1.5</td>
<td>1.4</td>
<td>.9</td>
<td>.75</td>
<td>.6</td>
<td>.5</td>
</tr>
<tr>
<td>( U'_{1a} )</td>
<td>1.5</td>
<td>1.4</td>
<td>.9</td>
<td>.7425</td>
<td>.582</td>
<td>.5</td>
</tr>
<tr>
<td>( U'_{1b} )</td>
<td>1.5</td>
<td>1.4</td>
<td>.9</td>
<td>.7575</td>
<td>.618</td>
<td>.5</td>
</tr>
<tr>
<td>( U'_{1c} )</td>
<td>1.5</td>
<td>1.4</td>
<td>.9</td>
<td>.7575</td>
<td>.618</td>
<td>.5</td>
</tr>
</tbody>
</table>
**Individual 2:**

<table>
<thead>
<tr>
<th>Cons. level</th>
<th>70</th>
<th>74</th>
<th>141</th>
<th>150</th>
<th>230</th>
<th>235</th>
</tr>
</thead>
<tbody>
<tr>
<td>$U_2$</td>
<td>0</td>
<td>5.7</td>
<td>81</td>
<td>88.53</td>
<td>142.5</td>
<td>145.41</td>
</tr>
<tr>
<td>$U_{2a}$</td>
<td>1.44</td>
<td>1.35</td>
<td>.9</td>
<td>.72</td>
<td>.63</td>
<td>.54</td>
</tr>
<tr>
<td>$U_{2b}$</td>
<td>1.44</td>
<td>1.35</td>
<td>.891</td>
<td>.72</td>
<td>.63</td>
<td>.5238</td>
</tr>
<tr>
<td>$U_{2c}$</td>
<td>1.44</td>
<td>1.35</td>
<td>.909</td>
<td>.72</td>
<td>.63</td>
<td>.5562</td>
</tr>
</tbody>
</table>

where $U_{1a}$, $U_{1b}$, and $U_{1c}$ represent three different examples (cases) of marginal utilities consistent with the given $U_1$ and property (2).

If endowments are $z_1' = (110, 20)$ and $z_2' = (-10, 80)$, there would be no trading in $A'$ since the endowment allocation is in fact an equilibrium supported by the price vector $P' = (.93, 1.59)$. This yields the second-period consumption allocations $w_1' = (130, 150, 170)$ and $w_2' = (70, 150, 230)$; the resulting expected utilities are $u_1' = k_1 + 25.935$ and $u_2' = k_2 + 77.01$ (where $k_1$ represents the utility of first-period consumption). Note that the preceding is true for each of the cases a, b, and c.

If we now move to market structure $A''$ (leaving endowments unchanged, which is feasible and of course insures strong endowment neutrality), individual 1 will alter his endowed portfolio $z_1'' = (110, 20, 0)$ by trading to $z_1'' = (93, 33, -27)$, which in turn yields consumption levels $w_1'' = (126, 159, 165)$ in the three states. Analogously, individual 2 will switch his portfolio from $z_2'' = (-10, 80, 0)$ to $z_2'' = (7, 67, 27)$, which gives $w_2'' = (74, 141, 235)$. First-period consumption levels remain unchanged. Expected utilities are $u_1'' = k_1 + 26.415 > u_1'$ and $u_2'' = k_2 + 77.37 > u_2'$ so that both individuals are better off in $A''$. 
Note that the preceding is true in each of the three cases a, b, and c. The differences in marginal utilities which characterize the three cases do become reflected in the equilibrium prices, however. In Case a, the equilibrium price vector is \( \mathbf{p}''_a = (0.93, 1.59, 0.18) \) which means that securities 1 and 2 have the same equilibrium prices in the two markets—even though everyone is better off in \( A'' \). Aggregate wealth is also valued the same, i.e. \( V''(W) = V'(W) \).

In Case b, however, the equilibrium price vector \( \mathbf{p}''_b = (0.9216, 1.5678, 0.1746) \) so that securities 1 and 2 as well as aggregate wealth have lower equilibrium values in \( A'' \) than in \( A' \) (recall that a unit of first-period consumption serves as numeraire in both markets). Thus, in moving from \( A' \) to \( A'' \), welfare and equilibrium prices move in opposite directions in this case.

In Case c, the price vector which supports the indicated equilibrium is \( \mathbf{p}''_c = (0.9384, 1.6122, 0.1854) \) which implies \( V''(a_1) > V'(a_1), V''(a_2) > V'(a_2) \) and \( V''(W) > V'(W) \), i.e. that equilibrium prices and welfare move in the same direction when \( A' \) is replaced with \( A'' \).

While \( A'' \) is a complete market in the preceding example, the result does not depend on that circumstance and is readily generalizable.

To summarize, we state

Remark 4. (Independence of prices and welfare.) In comparing equilibrium under \( A'' \) with equilibrium under \( A' \), equilibrium prices are independent of welfare in the sense that we may have

\[
  u''_i > u'_i, \quad \text{all } i
\]
accompanied by either
\[ V''(w) > V'(w) \]
or
\[ V''(w) = V'(w) \]
or
\[ V''(w) < V'(w) . \]

The moral of this tale of course is that the relationship between prices and welfare is a tenuous one at best. Wealth and prices are not reliable measures of welfare.

X. VALUE CONSERVATION

Conservation of values across different types of financial markets has been observed to hold for two types of situations: comparisons involving strong homogeneity and highly specialized preferences and/or beliefs, and for market structure changes which leave the feasible set of allocations unaltered, i.e. changes of Type I. The former category is exemplified by equilibrium models in which all investors end up combining a position in the "market portfolio" with borrowing or lending, as in the mean-variance capital asset pricing model [e.g. Hamada (1969), Martin (1973)] and in models based on subsets of time-additive, state-independent, HARA-class preferences and homogeneous beliefs [Rubinstein (1974)]. In these models, investors are indifferent between all financial markets that contain a risk-free asset provided there is weak endowment neutrality since equilibrium (in terms of \((c, w)\)) is unique. In addition, \(V'(w) = V''(w)\), i.e. values are conserved for (jointly) feasible \(w\), in the HARA-class case even in the absence of weak
endowment neutrality as long as all investors have the same time-discounting of utilities.

The second group of writings providing value conservation results have focused on Type I cases, i.e. on changes which imply that \( F(\Lambda') = F(\Lambda'') \). Included in this group are the works of Williams (1938), Modigliani and Miller (1958), Hirshleifer (1966), Robichek and Myers (1966), Stiglitz (1969, 1972, 1974), Schall (1972), Mossin (1973), Kim, McConnell and Greenwood (1977), Fama (1978), and Nielsen (1978). Perhaps the most striking aspect about these analyses is their inadequate discussion of endowment effects, especially since, as noted in Section IV, such effects are often unavoidable. (This is not to say that endowment effects are always a problem, however.) Previous results, therefore, provide a somewhat incomplete picture.

By reference to Sections VI, VII, and VIII, we observe that value conservation cannot occur independently of preferences and beliefs unless there is welfare conservation, i.e. (ii) holds. By Theorem 3, this requires not only \((c', w') = (\bar{c}', \bar{w}')\) but strong endowment neutrality, which we know may be impossible to attain in any given market structure change. In effect, then, Theorem 3 may be restated as

Theorem 3'. (Global value conservation.) In moving from market structure \( \Lambda' \) to market structure \( \Lambda'' \), we obtain, for every collection \((\pi, U)\) satisfying (1) and (2), that

\[
V''(w) = V'(w), \quad \text{all (jointly) feasible } w
\]

(26) if and only if

\[
F(\Lambda'') = F(\Lambda'), \quad \text{(Type I)}
\]

\[
(c', w') = (\bar{c}', \bar{w}')
\]

(28)
We are forced to conclude, then, that (global) value conservation is a rare phenomenon, and apparently much less common than previously suggested. If we limit ourselves to unique equilibrium situations, however, we can dispense with the need to start in equilibrium and relax strong endowment neutrality to weak neutrality provided the latter insures that the same valuation function is operative in both markets—which is only true for arbitrary beliefs and preferences in Type I changes. In other words, Theorem 4 can also be restated as

*Theorem 4'.* (Selective value conservation.) In moving from market structure $A'$ to market structure $A''$, we obtain, for every collection $(\pi, U)$ satisfying (1) and (2) and consistent with the requirement that equilibrium in each market be unique, that

\[ V''(w) = V'(w), \quad \text{all feasible } w \]  

if and only if

\[ F(A'') = F(A') \]  

(Type I)

and

\[ E'' \equiv E'. \]  

(29)

It may be worthwhile to relate Theorems 3' and 4' to the usual notion of "competitive securities". To say that securities are "competitive" presumably means that each one has very little influence on the total picture so that changes in a small subset of $J$ would not (significantly)
affect the prices of the other securities. As Theorem 4' shows, this is certainly true in connection with Type I changes accompanied by weak endowment neutrality, even when the number of securities is small. But consider a situation of Type II. In this case, for example, the introduction of a unique zero-supply option could easily have dramatic impact on the prices of all existing securities, no matter how numerous, if it opened up desired but previously unavailable allocation opportunities among certain states. To capture the notion of "small" price changes, we therefore introduce the following

**Definition.** Value continuity with new investment will be said to occur if, given that a new project \( k \) is financed by issuance of a new security, there exists a price \( P''_k \) of that security such that

\[
\begin{align*}
&z''_{ik} = 0, \quad \text{all } i \\
&P''_j = P'_j, \quad \text{all } j \in J'
\end{align*}
\]  

(32) (33)

for all collections \( \{\pi, U\} \) satisfying (1) and (2) and for each equilibrium, where \( A' \) is the market structure without \( k \) and \( A'' \) is the market structure with \( k \).

Not surprisingly, value continuity as defined above, can be directly linked to market structure changes of Type I.

**Theorem 8.** Value continuity with new investment occurs, for every collection \( \{\pi, U\} \) satisfying (1) and (2), if and only if the (per share) payoff \( a_k \) of the proposed project is reproducible in \( A' \), i.e. there exists a portfolio \( z^* \) such that

\[
a_k = z^*A'.
\]  

(34)
Proof: Let the cost per share of project $k$ be $c_k = P''_k$. Since project $k$ is initially unfinanced, we have $E''$ on $E'$ and (34) implies that $F(A'') = F(A')$. Thus $z''_i = 0$ for all $i$ if

$$P''_k = z^* P'$$

(35)
since there is no unsatisfied demand for portfolio $z^*$ in $A'$ when (35) holds. Thus (32) and (33) both obtain for every collection $\{\pi, U\}$ when $c_k = z^* P'$.

Suppose now that (34) does not hold. This implies that $F(A') \subset F(A'')$ and that $A'$ is incomplete. Consequently, there exist collections $\{\pi, U\}$ such that $z''_i \neq 0$ for some $i$ for every price $P''_k > 0$ by Lemma 3.

The import of the above is that even when aggregate demand is zero, i.e. $\Sigma z''_i = 0$, there will generally be an impact on existing prices in the sense that $P''_j \neq P'_j$ for $j \in J'$, an impact which may be "large" independently of the number of securities in $A'$, unless (34) holds. If somewhat differently, if project $k$ is (validly) undertaken when (34) is satisfied, there may well be a perturbation of prices by that fact alone. But if project $k$ is (validly) undertaken when (34) is not satisfied, there will in effect be two perturbations: one from the introduction of a genuinely new security, a perturbation which may be large, and a second one from the undertaking of the project itself. "Value continuity with new investment", then, is the absence of the former perturbation and may be thought of as a basic theorem of value conservation in that it is independent of preferences and beliefs. That is, value conservation occurs only if the new investment does not simultaneously introduce a fundamentally new type of security into the market.
XI. APPLICATIONS

At this point, it may be useful to compare the conventional wisdom with what the preceding results have to offer concerning some of the more commonly encountered situations involving changes in the structure of the financial market. For this purpose, attention will be limited to a handful of illustrations: corporate spinoffs, mergers, option trading, and the possible use of risky debt in corporate capital structure decisions.

Spinoffs

In a corporate spinoff, a segment of an existing corporation is set up as an independent company, and existing shareholders are simply given new shares, representing their ownership in the new entity, on a pro-rata basis (such as 1 new share and .05 bonds for 2 previous shares). Thus, strong endowment neutrality is automatically satisfied in the absence of synergy and, since no portfolio choices disappear, the market structure change is either Type I or II. More precisely,

A. If both the old and all the new securities\textsuperscript{15} are reproducible, or the old common was unique and exactly one of the new securities is unique, the market structure change is of Type I. In this case, there is "no impact", that is, either

1. both values and welfare are unaffected if endowments were in fact equilibrium allocations before the new securities arrived (Theorem 3), or

2. if the endowments were not equilibrium allocations when the new securities arrived, the set of possible equilibria after trading is the same as it was before (Theorem 2).
B. If the old common was reproducible and two or more of the new securities are unique, or if the old common was unique and two or more of the new securities are unique, the market structure change is of Type II. In this case, we obtain 1. that the possibility of redistributive effects occurs only if endowments were not equilibrium allocations before the new securities arrived (Theorema 5, 6); otherwise we find 2. that investor welfare is either unchanged or improved.

With the exception of possibility B.1., we can thus conclude that nonsynergistic spinoffs are at worst neutral and generally beneficial in terms of investor welfare in the context of the present model. One could certainly argue that, ceterus paribus, they should be encouraged rather than discouraged.

Mergers

A merger may be thought of as the opposite of a spinoff. Since we are dealing with pure exchange, it will be necessary to restrict our attention to nonsynergistic mergers that involve an exchange of securities; any cash payments must be among investors only (the exchange need not involve 100% of the outstanding quantity of any given security, however). In terms of feasible allocations, a merger involving at most common stock and risk-free debt implies either a market structure change of Type I or of Type II in reverse. If other types of securities are involved [as in Stiglitz (1972), Azzi (1977) and Scott (1977)], Type III changes are clearly possible while a Type II change will result only if the new entity issues a richer set of securities than that which had been floated by the companies in question as separate entities. Most previous studies, however
[e.g. Mossin (1966), Myers (1968), Mueller (1969), Levy and Sarnat (1970),
and Nielsen (1978)], have focussed on 100% mergers between companies having
only common stock and risk-free debt outstanding, concluding, in the main,
that the consequences of mergers are "neutral". For this reason, we shall
limit the present discussion to that case as well. However, even in this
case, the number of factors affecting the welfare consequences is perhaps
surprising. Strong endowment neutrality, for example, would be very rare,
generally occurring only when investor endowments contain the merged
securities in their market portfolio proportions.

C. The market structure change is of Type I only if 1) the old
securities are reproducible (in which case the new security is
also reproducible) or 2) if exactly one of the previous securities
is unique (in which case the new security is also unique). To
conserve values and welfare we must have either
1. endowments that represent a pre-merger equilibrium and
   strong endowment neutrality (Theorem 3), or
2. weak endowment neutrality and a unique equilibrium in both
   the pre- and post-merger market (Theorem 4).

In all other cases, the merger will have redistributive effects
and value conservation will typically be absent.

D. The market structure change will be of Type II reversed if two
(or more) of the old securities are unique (in which case the
new security will either be reproducible or unique). Value and
welfare conservation can now be assured only if
1. pre-merger endowments represent a pre-merger equilibrium
   and strong endowment neutrality is achieved (Corollary 4).
In all other cases, the merger will, except in special cases, have redistributive effects or leave everyone worse off, all accompanied by an absence of value conservation.

The preceding presents a considerably gloomier picture of the welfare effects of nonsynergistic mergers than previous studies, in which "neutrality" and value conservation have been the central features. Most of these studies have made assumptions with respect to preferences or beliefs that are sufficiently strong to yield two-fund separation, a property which leaves investors indifferent between all market structures as long as they contain a risk-free asset and which also generally conserves values [e.g. Mossin (1966), Levy and Sarnat (1970). Other studies have been limited to Type I mergers [e.g. Nielsen (1978)]. It is noteworthy that the necessity of either condition C.1. or condition C.2. for this case has been either overlooked or implicitly assumed.

Since simple mergers have zero welfare effect under investor heterogeneity only in the presence of C.1., C.2., or D.1., the neutrality of non-synergistic mergers would appear to be the exception rather than the rule. Pareto-improvements being impossible, the prevalent result will be one of welfare redistribution or the achievement of an allocation that is inferior to the one that would have been attained without the merger. In view of the tendency to overcompensate the "target" company's shares in terms of pre-merger prices, redistribution may be the typical outcome, with those who have proportionately large pre-merger holdings in the target company's common tending to benefit, primarily at the expense of those who have large proportionate holdings in the "acquiring" company's shares.
Option Markets

The possible welfare implications from opening markets in (European) options are identical to those resulting from corporate spinoffs. That is, strong endowment neutrality is trivially satisfied, and the resulting market structure change will be either of Type I or of Type II. Thus the introduction of option trading leads either to zero change in welfare, to redistribution, or to a Pareto-improvement, with redistribution possible only if endowments are not equilibrium allocations when options are introduced.

Capital Structure Changes

One of the most enduring propositions of modern finance, in the kind of perfect market environment we have assumed, is that the value of the firm is independent of its capital structure [Modigliani-Miller (1958), Stiglitz (1969)]. In application, this statement has been limited to mixtures of debt and common stock (with the debt assumed to be risk-free whenever the financial market is incomplete or investors exhibit reasonable heterogeneity). Even so, as Theorems 3 and 4 remind us, the validity of this classical statement depends critically on (as far as I know) previously unidentified preconditions: either the market must be in equilibrium before the exchange of (equivalent) securities (in such a way that state-by-state payoffs are preserved) or, if not, the equilibrium must be unique and weak endowment neutrality must be attained.

The findings of the present paper, of course, enable one to make comparisons between capital structures of unlimited variety, i.e. involving not only common and regular debt, but also preferred stock, risky debt, subordinated debt, convertibles, and warrants, including various and sundry protective covenants. A full analysis of these possibilities is
well beyond the scope of this paper. However, consider for a moment the choice of the level of risky debt to be combined with equity in an incomplete market setting, which has been the subject of a large number of studies (e.g. Smith (1972), Baron (1974, 1976), Milne (1975, 1976), and Hagen (1976)). In this case, moving to the firm's alternative financing method may lead to a market structure change of Type I (if none of the (four) securities is unique), or Type II (if one of the securities in the alternative is unique), or Type II reversed (if one of the securities in the existing capital structure is unique instead) or Type III (if at least one security in each alternative is unique). On top of this, there are endowment effects that must be considered. Clearly, predicting the welfare effects of various capital structure alternatives is not always a simple task.

XII. CONCLUDING COMMENTS

Since the central findings were summarized at the beginning of the paper, they will not be repeated here. However, it may be worth summarizing the three principal qualitative conclusions generated by this exercise.

1. The requirements which give rise to value conservation in comparisons between different market structures are considerably more demanding than previously recognized. This is partly due to the fact that

2. Decentralized changes in the structure of the capital market often lead to unavoidable redistributive effects, even in movements between complete markets.
3. The opening of option markets, while not immune to redistributive effects, represents perhaps the simplest and easiest means to achieve Pareto-improvements in the welfare of consumer-investors. Nonsynergistic corporate spinoffs also tend to have positive effects while nonsynergistic mergers on balance have negative welfare effects.

The next step is to bring taxes into the picture, a process which is currently under way.
Footnotes

1. Most previous studies have assumed a single commodity in a one- or two-period setting. It should be noted, however, that the introduction of multiple commodities or more than two periods is a nontrivial step which may bring about additional complications, such as Pareto-dominated equilibria, as shown by Hart (1975).

2. Assumption (1) can be slightly relaxed without jeopardizing the existence of equilibrium (see below). However, by employing (1), we avoid confronting the possibility that state $s$ actually occurred even though some individuals assessed $\pi_{is}$ to be $0$—which, in turn, raises issues beyond the scope of this paper.

3. That is, consumer-investors perceive prices as beyond their influence, there are no transaction costs or taxes, securities and commodities are perfectly divisible, and the full proceeds from short sales can be invested.

4. It is sufficient (but not necessary) for (13) to obtain that the security market be complete, i.e. that the rank of $A$ be n or that the market be equivalent to an Arrow-Debreu (1964, 1959) market. Clearly, (13) also occurs when all consumer-investors are completely identical. A third set of well-known sufficient conditions which imply (13) is the following: at least two securities in the market, one of which is risk-free; homogeneous beliefs; additive utility, with the second period's utility function being a member of the HARA-class, the exponent being the same across all consumer-investors (except in the negative exponential case). (See footnote 12 for a specification of the HARA-class.)

5. Note that since we assume that the subsistence levels $\alpha_i$ and $\beta_i$ are consistent with the single-security case, they are in fact constrained beyond the requirement (4).


7. Recall that tender offers, for example, are typically announced after markets close, on weekends, or in conjunction with a suspension of trading.

8. A number of institutional arrangements designed to counteract endowment effects can be observed. For example, bond covenants and various "me-first" rules are designed to protect debt-holders against certain actions by shareholders and/or management, such as the issuance of new debt with the same priority as existing debt [see e.g. Fama and Miller (1972), Jensen and Meckling (1976), and Fama (1978)]. While such protection is clearly possible in special cases, endowment shifts in general are rather difficult to insure against, as exemplified by some of the illustrations which follow.
9. Suppose the per share payoffs for Companies X and Y in market A' are (2, 3, 4) and (1, 3, 6), respectively and that each firm has 10 shares outstanding. If Company X now offers one of its shares, say, in exchange for one share of Company Y and a nonsynergistic merger is consummated, the new market A" will have 20 shares of Company X, each with payoff (1.5, 3, 5), and no shares of Company Y. But the endowment of an investor with two shares of X and one share of Y in A', a total of (5, 9, 14), has now, through the exchange of shares, been shifted to an endowment of (4.5, 9, 15) in market A".

10. Note that examples 3 and 4 also typically affect the payoff pattern on the common stock and other derivative securities.

11. Whether the economy can be "nudged" toward a particular equilibrium (in the presence of alternate equilibria) is an intriguing question about which the current theory of adjustment processes has little to offer.


13. The HARA-class (hyperbolic absolute risk aversion) consists of the following utility functions of wealth with the properties $u'(w) > 0$, $u''(w) < 0$ (the first over at least a finite range of positive wealth):

$$
u(w) = \begin{cases} 
\frac{1}{\gamma (\gamma + a)^{\gamma}} & \gamma < 1 \text{ (decreasing absolute risk aversion)} \\
-\exp(\gamma w) & \gamma < 0 \text{ (constant absolute risk aversion)} \\
-(a-w)^{\gamma} & \gamma > 1, \text{ a large (increasing absolute risk aversion)}
\end{cases}$$

14. The reader will recognize (34) as the "spanning property", a condition which, if present, cause consumer-investors to be unanimous as to whether project k should be undertaken (at the margin) or rejected [Leland (1973), Ekern and Wilson (1974)].

15. Note that if Company X spins off a division Y by issuing common shares, the "new" shares include not only Company Y common but Company X common and possibly other securities issued by Company X--in fact all securities whose gross payoff vectors $Z_{j,k}$ are affected are in fact "new" securities after the spinoff and "old" securities prior to the spinoff.

16. For empirical studies providing varying degrees of support for this statement, see Mandelker (1974), Dodd (1976), and Dodd and Ruback (1977).

17. For previous studies on this subject, see Schrems (1973), Ross (1976) and Hakansson (1977, 1978).

18. It is noteworthy that the assumptions behind the option pricing models based either on arbitrage arguments [e.g. Black and Scholes (1973), Cox and Ross (1976)] or on special preference functions [e.g. Rubinstein (1976)] also imply that the options in question are redundant, i.e. their introduction give rise to a Type I change, that is, no change, in the market structure--which in turn leaves welfare either unchanged or redistributed.
References


