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THE OPTION VALUE OF RESERVES OF NATURAL RESOURCES

by

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THE OPTION VALUE OF RESERVES OF NATURAL RESOURCES

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1. INTRODUCTION

This paper addresses the problem of finding the value of a reserve of a natural resource when the future price of the resource is uncertain.

There are many reasons for being interested in this valuation problem. From the point-of-view of resource allocation in a market economy, there are two major questions that its solution can help answer. The first one is to decide how much to invest in exploration. The optimal amount clearly depends on the value of the reserve that will eventually be found. The second question is to determine which reserves to extract and when. This can be answered by obtaining the extraction decision rule, which is an integral part of solving the valuation problem. This rule will also provide information about the pattern of resource availability and about the timing of expenditures for extraction. On the other hand, from the private point-of-view, the value of the reserve is also important information. It is crucial, for example, for the owner of a reserve in deciding on for how much to sell his asset. It is also
useful for investors considering holding reserves for speculative motives.

Of interest are not only the explicit formulas for the value of the reserve, the optimal extraction rule and the optimal investment in exploration, but also the comparative statics results. In particular, for policy purposes, there could be strong interest in predicting changes in the owners' willingness to extract in reaction to changes in the variance and in the level of the price of the resource.

To find the value of the reserve under uncertainty, the discussion here will depart from the usual formulations in two major aspects. First, it will assume, a priori, the existence of a market for the extracted resource where short-selling is allowed. This highlights the speculative aspect of reserves, when seen as assets. Second, uncertainty is introduced in the model by assuming that the price of the resource follows a continuous-time stochastic process which reflects the uncertainty in supply and demand.

This approach to uncertainty in exhaustible resource problems is novel with respect to other treatments found in the literature. There the problem has usually been approached at a more disaggregated level, where the uncertainty has been assumed to stem from a specific source. The following studies are examples of that type of approach. Dasgupta and Heal [1974] determine the optimal consumption pattern when there is uncertainty as to the time of arrival of a perfect substitute. Weinstein and Zeckhauser [1976] calculate the market equilibrium when the
demand is uncertain. Gilbert [1976] and Gilbert and Stiglitz [1978 chap. 2] discuss the socially optimal path of consumption when there is the possibility that new reserves or a substitute will be discovered. Devarajan and Fisher [1979] allow for a stochastic exploration production function and discuss the relation between marginal discovery costs and rent. Arrow and Chang [1978] assume that the location of the mines and the level of reserves are uncertain and calculate the socially optimal path. Findyck [1979] studies the interaction of stochastic demand with a stochastic level of reserves in a market where the objective is the maximization of expected discounted profits. In this paper I take instead a more aggregated approach and argue that the price, affected by many random shocks of several different origins, behaves randomly. Under these conditions I study the market equilibrium for the value of reserves.

The two main assumptions just described will permit us to address the questions posed above by using the methodology for pricing options developed in the Finance literature. The main principle to be employed for valuation will be the elimination of riskless arbitrage opportunities.

The plan of the paper is as follows. In the next section the implications of assuming the existence of a market for the extracted resource are developed under certainty. These constitute a "benchmark" for the uncertainty models which are summarized in section 3. Section 4 looks at the implications of these valuation formulas in a simple model of exploration. The main findings are summarized in the concluding
section. Finally, I should note that most of the technical details are
relegated to appendices, which may themselves be of considerable
interest to options theorists.

2. THE EXTRACTION PARADOX IN THE CERTAINTY MODEL

The usual equilibrium condition under certainty, when the extrac-
tion cost is fixed, requires that the value of the reserve increase at
the rate of interest\(^1\). The net present value of the difference between
the price of the resource and the extraction cost is constant and the
owner of the reserve is indifferent between extracting and holding. We
can then value it as if it is extracted at the initial time period and
therefore, \( C(S) = \max[0, S-X] \) where \( C \) is the value of a unit of the
reserve, \( S \) is the price of the resource and \( X \) is the marginal extraction
cost\(^2\). The value is zero when extraction would generate a loss, i.e.
when \( S < X \).

The following assumptions are implicit in this argument, and are
listed here for completeness.

(1) There is a known instantaneously riskless interest rate \((r)\) which
is constant through time and equal for borrowers and lenders.
Further, borrowing and lending are allowed to all investors.

(2) The owners of reserves are price-takers.

In this equilibrium no positive stocks of the extracted resource
are held for any period of time, because the return to doing that would
be inferior to the riskless interest rate. To see this, define \( \mu = \frac{ds}{dt} \frac{1}{S} \) and \( p = \frac{dc}{dt} \frac{1}{C} \) and note that if \( p = r \), then \( \mu = r \frac{(S-X)}{S} < r \). In other words, all the resources that are extracted are immediately consumed and any exchange market that could exist for the extracted resource would serve only to redistribute it among consumers within the same time period. No one would hold extracted resources as an asset, even if storage were costless.

However, the organized markets that exist for many extractive commodities do seem to have a speculative nature, as if they were asset markets. The example of the market for gold and other precious metals comes immediately to mind. Another example is that of minerals like aluminum, copper, lead and zinc which are traded in the London Metal Exchange. The question I now want to pose is "What happens if we take the existence of these markets as given?". To address this question I will assume, as is usually done in the study of financial markets, that

(3) Traders are allowed to short-sell the resource.

Of course, allowing short-sales by some investors does not mean that there could exist negative resource holdings in aggregate. For every short position there should exist an offsetting long position. For some commodities for which futures markets exist, the easiest way to build a short position may involve selling in the futures market.

Finally, for simplicity, I will assume that
(4) There are no storage costs for the resource after it is extracted.

The implications of relaxing this assumption will be discussed later, but the arguments below use it, except where stated to the contrary.

While these assumptions may seem reasonable in view of the existence of organized commodity markets, they impose restrictions on the type of equilibria that can be obtained. In particular, the certainty equilibrium described above, with \( S-X \) growing at the rate of interest, is ruled out. As we just saw, it implies that \( \mu < r \), and that would generate an arbitrage profit opportunity through short-selling the resource\(^3\).

In fact, assumptions (3) and (4) above imply that \( \mu = r \). This, in turn, implies that owners of reserves would never want to extract, because the best way to invest \( S-X \) (what they would get if they extracted) would be to hold it in the form of reserves. To prove this, assume it is now optimal to extract i.e. \( C(S) = S-X \), and show that this leads to a contradiction. For this value of the reserve, the instantaneous return to holding it is \( \rho = \frac{rS}{S-X} > r \). But if this is the case, it would not have been optimal to extract.

If there is a time limit for production to begin, as in the case of mineral leases with diligence requirements, the owners of reserves would delay production until the last possible moment. The value of the reserve would then be the present value of the net cash flow at the extraction date: \( C(S) = \text{Max} \left[ 0, S-Xe^{-RT} \right] \), where \( T \) is the time to expiration of the lease.
Alternatively, if there is no time limit, owners of reserves would never want to extract. Note that this very fact could provide an explanation for the existence of the diligence requirements referred to above. Further, if there is no extraction, because all owners of reserves see them as speculative assets, there could not exist a market for the extracted resource. I will dignify this puzzling result by calling it the extraction paradox. One way out of it has already been suggested: perhaps most reserves have an explicit or implicit time limit. Another possibility is that some owners do not or cannot act as pure speculators, perhaps due to some restriction on borrowing. However, what I want to consider here are ways to avoid the paradox while still assuming there is no time limit for extraction and that all owners are speculators. To do this I propose two reasons that might induce the owner of the reserve to extract.

First, there could exist holding costs associated with keeping the reserve, costs that could be avoided by extraction. They arise naturally as rents to be paid on mineral leases, and as opportunity costs associated with alternative uses of the land.

Second, the extraction cost could be rising through time at a sufficiently high rate to make it unprofitable to postpone extraction. One rationale for this could be that extraction uses as inputs some exhaustible resources whose price is rising in time. Another rationale is related to the growth of the economy associated with the passage of time. Cities expand and environmental restrictions on production become tighter, making extraction more costly.\(^4\)
Rising extraction costs can also be seen as the cumulative liability resulting from borrowing to pay the holding costs, with repayment promised for the extraction date. While not formally exact, this interpretation highlights the close connection that exists between the two solutions to the paradox.

If there are holding costs of \( H \) dollars per unit time, it is easy to see (recalling that equilibrium requires \( u = r \)) that \( p = \frac{rS - H}{S - X} \). For owners to be willing to extract, we need that \( p \leq r \), which implies the condition \( \frac{H}{r} \geq X \). We can even derive a more general condition for the case in which there are storage costs of \( K \) dollars per unit time for the extracted resource, in addition to the holding costs. Clearly, the storage costs will make extraction less desirable, so we can expect the condition for extraction to be even more stringent than the condition above. To see that this is actually the case, first note that for the investor to be just indifferent between holding inventories of extracted resources and investing in bonds, it is necessary that the increase in the price, net of storage costs, be equal to the return on the value of the inventory: \( dS - Kdt = rSdt \). This implies that \( u = \frac{rS + K}{S} \). Now, using this expression for \( dS \), the return on the reserve can be calculated to be \( p = \frac{rS + K - H}{S - X} \). For extraction to occur, we need \( p \leq r \), which implies \( \frac{H - K}{r} \geq X \). If holding costs are not high enough to meet this condition, owners would still prefer to hold their reserves.

Now consider the case of extraction costs rising at an exponential rate \( g = \frac{dx}{dt} \frac{1}{X} \), and note that in this case \( p = \frac{rS - gX}{S - X} \). For owners to be willing to extract we need \( p \leq r \), which implies \( g \geq r \). If extraction
costs are not rising at least as fast as the interest rate, owners will be unwilling to extract.

To summarize the foregoing discussion, for equilibrium with assumptions (1) – (3) we need \( g \geq r \) or \( \frac{H - K}{r} \geq X \). If the strict inequality prevails, the reserves which are currently optimal to extract are extracted immediately. If the equality prevails, owners are indifferent about when to extract. In either case, those that cannot be economically extracted in the first period (because \( X_0 > S_0 \)) are never extracted, and therefore the value of the reserve is \( C(S) = \max \{0, S - X\} \). This function is graphed in Figure I. Also as a consequence of this extraction rule, there will never be exploration for reserves which are not currently optimal to extract. We will see that when uncertainty is introduced the value of the reserve will exceed \( \max \{0, S - X\} \), for some values of \( S \). This last result on exploration will also be altered.

Figure I: about here

3. THE UNCERTAINTY MODEL

To calculate the value of the reserve under uncertainty I view it as an option to extract the resource in the future. Seen in this way the reserve is analogous to an option on a stock. We can then draw on the option pricing models developed in the Finance literature to derive
the value of the reserve.

An European call option on a stock gives its holder the right to buy, at a certain future date (the expiration date), one share of stock at an agreed-upon price (the exercise price). At the expiration date the holder of the option should exercise it only if the price of the stock is larger than the exercise price, in which case he receives the difference between these two prices. Otherwise he should let the option expire and receive a zero payoff. This asymmetric pattern of payoffs is characteristic of options and is also displayed by the reserve. If the reserve is not extracted the payoff is zero and if it is extracted the payoff is $S - X$. The reserve then is an option where the underlying asset is the resource. More precisely, the reserve is an American option, which is like the European option described above, except that it allows exercise prior to the expiration date.

In essence, the owner of the reserve has an option because he is not forced to extract if it is not optimal to do so. He can "buy" a certain quantity of the resource by paying an amount equal to the extraction cost. This cost corresponds to the exercise price of the option.

To be able to use the option pricing methodology, I shall assume, in addition to (1) - (4) above, that the spot price of the resource follows a continuous-time stochastic process. The main benefit of the continuous-time analysis is that the reserve will be shown to be a dependent asset, since the returns to holding it can be duplicated by an
appropriate dynamic portfolio of resources and riskless borrowing. Other advantages and some shortcomings of the continuous-time approach have been summarized by Merton [1975].

As Samuelson [1969] points out, there is no a priori reason for the spot price of a commodity to follow a specific probability model, although correctly anticipated futures prices must have unbiased price changes. To justify my approach, note that at a more fundamental level, the spot price of the resource ought to vary with shifts of supply and demand caused by such things as the discovery of new reserves, the foreclosure of known reserves, technological changes in production of the resource and changes in tastes. In this context the stochastic process assumption can be rationalized by noting that these many sources of uncertainty are somewhat independent, and that they do not display any obvious serial correlation. The sum of these many independent random factors could then generate the stochastic process postulated below. A further justification for postulating an independent increments process relates to the efforts of speculators to take advantage, through inventory accumulation, of any serial correlation that might exist in the price series. Their action will cause spot prices to change in such a way as to eliminate the serial correlation. McRae [1978], in analyzing prices for 5 non-fuel mineral ores for the period 1922-1974, concludes that all of his series behave in a fashion consistent with the hypothesis that the processes are purely random. However, because he uses deflated price series, I cannot claim that the study offers direct support for my assumptions.
In the tradition of the continuous-time models in the Finance literature, I assume that

(5) The price follows a Gauss-Wiener diffusion with constant drift ($\mu$) and constant variance ($\sigma^2$).

This has the attractive features that successive price ratios are independent and identically distributed and that the resulting price distribution is log-normal at any point in time, being therefore consistent with nonnegative prices. In addition note that it is necessary for equilibrium that all traders agree on the variance of the process.

Another assumption related to the stochastic process of prices is that

(6) Trading can take place continuously in the market for the resource.

This does not seem to to be an unreasonable assumption due to the existence of organized exchanges for many commodities. The only serious objection is that when there are transactions costs the participants cannot trade continuously, in the strict sense of the word. However, the continuous-time analysis is likely to give tolerable approximations when transactions costs are small. Therefore, I assume that

(7) There are no transaction costs.

As in the study of the certainty model, the main principle used for valuation is the elimination of riskless arbitrage opportunities.
Under uncertainty and with the assumptions (1) - (7), we encounter a paradox analogous to the one found under certainty. To characterize it, use theorem 2 in Merton [1973a] which states that a dividend-protected option with fixed exercise price will never be exercised prior to expiration\(^7\). If there is a time limit for extraction (an expiration date for the option), this theorem would imply that extraction is delayed until the last possible moment. There is, however, a positive supply of resources because the reserves are eventually extracted, and the standard Black-Scholes [1973] option valuation formula can be used to calculate the value of the reserve:

\[
C(S,T) = S N(d) - X e^{-rT} N(d - \sigma \sqrt{T})
\] (1)

where \(d = \frac{\ln(S/X) + [r + \sigma^2/2] T}{\sigma \sqrt{T}}\), \(N\) is the Normal distribution function, \(T\) is the time until the expiration of the option to extract, and the other variables are as before. The graph of this function is shown in Figure II.

\[\text{FIGURE II about here}\]

It is easy to see that when the variance in the price is eliminated, this reduces to the certainty value derived earlier for the case
when there is a time limit for extraction. To verify this, note in (1) that \( C(S,t) \to S - X e^{-rT} \) when \( \sigma^2 \to 0 \). Recalling that the reserve must have a non-zero value, \( C(S,t) \to \max \{0, S - X e^{-rT} \} \), as desired.

If there is no expiration date on the reserve, the no premature exercise condition would suggest that the resource will never be extracted. This paradox cannot be resolved by assuming a particular stochastic process for prices since the theorem would still hold. To obtain an equilibrium we then have to relax either the no-payoffs assumption or allow the exercise price to depend on time.

Premature exercise of an unprotected call option can be optimal when dividends are paid to the stock because the holder of the call has no claim on the dividends. In the models presented here the existence of holding costs for the owner of the reserve would generate an effect analogous to the payment of dividends to the stock. The holder of the reserve incurs a cost that he could have avoided by exercising the option. The exercise price rising with time generates an incentive for early extraction to take advantage of the lower cost.

Analytical formulas for the valuation of perpetual options when either (i) there are constant holding costs per unit time or (ii) the exercise price is rising at an exponential rate, are not available in the literature. They have been developed in Appendixes A and B respectively and the results are summarized in the next subsections. The reader interested in the derivations should refer to the appendices for details.
Results for the case of holding costs

As in the analysis of the certainty case, the type of holding cost considered is a constant \( H \) per unit time. It is assumed that the owner of the reserve can abandon it, and thereby avoid holding costs, if the price of the resource becomes too low. This price that triggers disposal \( (Z) \) is determined in the course of the solution.

The necessary and sufficient condition for extraction of the reserve to eventually occur is that \( \frac{H}{r} > X \). Note that this is almost the same condition encountered in the analysis of the certainty case. The difference here is that when \( \frac{H}{r} = X \) owners prefer never to extract, instead of being indifferent. If the holding cost is high enough for this inequality to be satisfied, the optimal timing for extraction will be when the price \( (S) \) reaches the critical level \( (W) \). This price that triggers extraction is also determined in the course of the solution.

The value of the reserve in this case is:

\[
C(S) = \begin{cases} 
0 & S < Z \\
A_1 S + A_2 S^u - \frac{H}{r} & Z < S < W \\
S - X & S > W
\end{cases}
\] (2)

where \( u = \frac{-2r}{\sigma^2} \) and

\[
A_1 = \left[ 1 - \left( 1 - \frac{Xr}{H} \right)^{-1} \right]^{-1} \quad , \quad A_2 = \frac{-A_1}{u^2 u^{u-1}} = \frac{1-A_1}{u W^{u-1}}
\]
\[ W = \frac{u}{u-1} \left( X - \frac{H}{r} \right) \frac{1}{1-A_1} \quad \text{and} \quad Z = \frac{u}{u-1} \left( \frac{H}{r} \right) \frac{1}{A_1} \]

Figure III displays the general form of this solution.

Several conclusions can be drawn from the analysis of these equations. First, it can be shown that for \( Z < S < W \), \( C(S) > \text{Max}[0, S-X] \) i.e. the value of the reserve exceeds what the owner would get if he extracted immediately. Therefore, when the current price is in that interval, the reserve is more valuable if left in the ground. Since different reserves have different \( W \)'s, these equations can be used to determine which reserves should be extracted, and when. Those for which \( W < S \) should be extracted immediately, while the others should await further price increases.

Second, the effect of an increase in uncertainty is to increase the value of the reserve. This can be proved by showing that an increase in \( \sigma^2 \) causes \( Z \) to decrease and \( W \) to increase, which is consistent with an upward shift of the curve. The increase in \( W \) also has the effect of delaying extraction of resources still in the ground. The price will have to rise higher for extraction to become optimal.

Third, when the holding costs \( (H) \) increase, the value of the
reserve decreases, as expected.

Fourth, an increase in the extraction cost \( X \) causes \( Z \) and \( W \) to increase, but the net effect is to decrease the value of the reserve.

Fifth, in the absence of uncertainty (when \( \sigma^2 \rightarrow 0 \)), the solution in (2) reduces to its certainty counterpart\(^8\) i.e. \( C(S) = \text{Max} \{0, S-X\} \).

The discussion of the case where there are storage costs above ground \( K \) under uncertainty is not included in this paper because the study of the differential equation that is obtained in the analysis requires the use of numerical solutions. However, experiments with these solutions suggest that the effects of an increase in uncertainty, in the extraction cost, or in the holding cost are of the same nature as those described above for when storage costs are zero. The effect of an increase in the storage costs is to increase the value of the reserve. Finally, in this more general case the necessary condition for extraction is \( \frac{H-K}{r} > X \), which is similar to the one in the certainty case. Note also that this reduces to the condition for premature exercise under uncertainty presented earlier when \( K=0 \).

Results for the case of rising extraction costs

It is assumed that the extraction cost is rising at an exponential rate \( g \). The necessary and sufficient condition for extraction of the reserve to eventually occur is that \( (g-r) > \frac{\sigma^2}{2} \). If we interpret \( r + \frac{\sigma^2}{2} \) as a risk-adjusted expected time rate of change of the price of the resource, this condition then requires that the rate of growth of the
extraction cost exceed this risk-adjusted rate. Note however that this does not impose any conditions on the unadjusted rate of growth of the price of the resource ($\mu$). In particular, it is possible that the condition be satisfied even if $S$ is rising relative to $X$, on average. Note also that when $\sigma^2 \to 0$, this condition reduces to $g > r$, which is almost the same condition obtained in the analysis of the certainty case. The difference here is that when $g=r$ there is no extraction.

If the condition described above is met, then there exists a price ($W$) that triggers extraction, and the value of the reserve can then be shown to be equal to:

$$
C(S,t) = \begin{cases} 
\frac{(v-1)^{v-1}S^v}{v^vX^{v-1}} & S < W(t) \\
S-X & S > W(t)
\end{cases}
$$

(3)

where $v = \frac{(g-r)}{\sigma^2/2}$, $X(t)$ is the extraction cost at time $t$, and $W(t) = \frac{v}{v-1}x(t)$.

Note that, in contrast with the model with holding costs, the value of the reserve here depends indirectly on time, through its dependency on the extraction cost. However, dividing both sides of (2) by $X$, it can be seen that the normalized value of the reserve ($C/X$) is independent of time and depends only on the normalized price of the resource ($S/X$). Defining $s = \frac{S}{X(t)}$, $D(s) = \frac{C(S,X)}{X(t)}$, and $w = \frac{v}{v-1}$, we have then:
The graph of the normalized value of the reserve is shown in Figure IV.

Several conclusions can be drawn from these formulas. First, reserves for which $W < S$ should be extracted immediately. The others should await further price increases because $C(S) > \text{Max}[0, S-X]$ for $S < W$ i.e. their value exceeds what the owner would get if he extracted immediately.

Second, an increase in uncertainty, or in the interest rate, increases the value of the reserve. It delays extraction of as yet unextracted resources, because their $W$'s increase.

Third, an increase in the rate of growth of the extraction cost ($g$) causes the value of the reserve to decrease. It may also cause the extraction of some deposits to become desirable, because their $W$'s decrease.

Fourth, an increase in the initial extraction cost ($X(t_0)$) reduces
the value of the reserve and delays the extraction of the remaining reserves, because \( W \) increases.

Fifth, the value of the reserve in (3) tends to the certainty value \( C(S,t) = \text{Max}[0,S-X(t)] \) when uncertainty is reduced\(^9\) \((\sigma^2 \to 0)\).

Comments

The following comments apply to both cases discussed above. First, a surprising feature of the formulas for the value of the reserve and of the trigger prices for optimal extraction, developed in the last 2 subsections, is that they are not functions of the expected time rate of change of the price of the resource. This is clear from above by noting that \( u \) and \( v \) do not depend on the drift of the stochastic process. The expected return on the reserve will, however, depend on the expected return on the resource.

The intuitive explanation of why the drift term does not appear rests essentially on the fact that the reserve can be shown to be a dependent asset in the continuous-time framework. The arbitrage argument shows that the returns to the resource can be duplicated by holding reserves and lending. Since the price of the resource already reflects this stream of returns, there is no need for the expected return to appear, in addition to the price, in the formula for the value of the reserve. The price performs the function of a summary statistic for the drift term in the relative pricing formula.
Second, as illustrated in Figures I, II, III and IV the introduction of uncertainty adds a premium to the economic rent \((\text{Max}[0,S-X])\) generated by the eventual depletion of the resource. This premium, which I will call the option value of the reserve, is the insurance value which protects the owner against a decrease in the price of the resource. It is easy to see that it is at a maximum when the price of the resource is equal to the extraction cost.

Third, in either of the cases above it can be verified that if the price of the resource is consistent with Merton’s [1973b] Intertemporal Capital Asset Pricing Model (ICAPM), then the value of the reserve is also consistent with the ICAPM.

4. IMPLICATIONS

If there are reserves with different extraction costs, at time \(t\) those that cannot be extracted have an option value equal to \(C(S,t) - \text{Max}[0,S-X]\). One of the implications of this option value is that under price uncertainty more will be spent in the search for deposits that will not be extracted immediately, as compared to the amount spent under certainty. I will show this in the context of the following very simplified model of exploration, but I believe the conclusion carries through to more complete models.

Exploration may produce information about the size, the extraction cost and the location of the reserve. For simplicity I will be concerned with the exploration to locate a deposit of known size and
extraction cost. It contains one unit of the resource. Assume that the probability \( p \) of finding the deposit is a function \( f \) of the amount \( I \) spent on exploration. Therefore, \( p = f(I) \). Assume also that \( f' > 0 \) and that \( f'' \leq 0 \) i.e. there are "decreasing returns" to the money spent on exploration. At time \( t \) a decision has to be made on how much to spend. The outcome of the exploration process is known immediately.

Let \( V(S,t) \) denote the value of the reserve. Under certainty it is equal to \( \max [0,S-X] \). It is extracted immediately if it has a positive value. Under uncertainty, according to (2) or (3), \( V(S,t) = C(S,t) \geq \max [0,S-X] \). The reserves for which \( W > S \) are not extracted immediately after discovery and their value is strictly larger than \( \max [0,S-X] \).

It can be argued that the risk in exploration is diversifiable if the typical firm drills many wells and does not make any systematic mistakes in selecting sites. If there is no systematic risk, the appropriate objective is expected profit maximization. Therefore,

\[
\max_{I} E(w) = f(I) V(S,t) - I
\]

The optimal level of exploration is such that \( f'(I) = \frac{1}{V} \). Letting \( I_c \) and \( I_u \) be these optimal levels under certainty and uncertainty respectively, we have that \( f'(I_c) = \frac{1}{\max [0,S-X]} \) and \( f'(I_u) = \frac{1}{C(S,t)} \). Recalling that \( \frac{1}{C(S,t)} \leq \max [0,S-X] \) and that \( f'' \leq 0 \), we see that \( I_u > I_c \). The investment in exploration under uncertainty is strictly
larger than under certainty whenever the option value is strictly positive, and that occurs for deposits that are currently not optimal to extract.

Using the comparative static results summarized earlier, we can see that the more variance there is in the price of the resource, the larger the investment in exploration for deposits that at the time of exploration are not commercially exploitable.

If we do not allow complete diversification of exploration risk and assume there is risk-aversion to the remaining risk, there will be an incentive to invest less. Whether the risk-aversion effect will offset the option effect depends on the risk-aversion coefficient and on the variance of the price of the resource.

5. CONCLUSION

I will first summarize the main findings and then will offer some final thoughts on the role of uncertainty in the models developed here.

In section 2 it was argued that, due to the possibility of short-selling, there would be no extraction unless there were holding costs on the reserve or the extraction cost was rising. Necessary and sufficient conditions for extraction were derived.

In section 3, expressions were obtained for the current value of the reserve under uncertainty, assuming that (i) the holding costs are larger than the product of the interest rate and the extraction cost or
that (ii) the extraction cost is rising faster than the sum of the interest rate and one-half the variance of the price. They have an exponential form and generate a value which exceeds the value derived under certainty, under the same assumptions. This difference between the value under certainty and under uncertainty is a measure of the option value of the reserve. It increases with an increase in uncertainty.

Although in section 3 storage costs for the extracted resource were not taken into account, it was pointed out that the general nature of the results does not change when we include them in the model.

As argued in section 4, the existence of the option value indicates that there could be exploration for deposits that are currently not optimal to extract. This would not occur if, maintaining the same assumptions on the holding and extraction costs, the uncertainty in prices was eliminated. Another interesting implication is that the amount invested in exploration increases with an increase in uncertainty, due to the increase in the option value.

To conclude, I would like to stress the importance of reducing uncertainty (modeled here by the variance of the price) in order to achieve a reduction in the speculative holding of reserves. The models here suggest that, contrary to what one might have expected, the expected return on the price of the resource plays no role on the decision of owners to extract resources. Rather, the resource price that induces extraction depends crucially on the variance of the price series.
APPENDIX A - THE VALUE OF A PERPETUAL CALL WITH HOLDING COSTS

The type of holding cost that will be considered here is a constant (H) per unit time. The proposition below can be established before any distributional assumptions are made.

Proposition 1. A sufficient condition for no premature exercise of the option with holding costs is that

\[ \frac{H}{r} \leq X \]  

(1)

Proof: consider the portfolios in the following table

<table>
<thead>
<tr>
<th>time=t</th>
<th>time=t* (expiration)</th>
</tr>
</thead>
<tbody>
<tr>
<td>S &lt; X</td>
<td>S*</td>
</tr>
<tr>
<td>S*</td>
<td>S*</td>
</tr>
</tbody>
</table>

where \( h = \frac{H}{r} (1 - e^{-rT}) \) is the present value of the stream of holding cost payments and \( t = t^* - t \) is the time to expiration. To avoid dominance, we need \( C \geq S = (Xe^{-rT} + h) \). It is clear that if the last term of this inequality is smaller than X, there will be no premature exercise because the option value will always exceed S-X. Therefore, a sufficient condition for no premature exercise is that \( Xe^{-rT} + \frac{H}{r} (1 - e^{-rT}) < X \) and this implies that \( \frac{H}{r} < X \). Q.E.D.
Assuming that \( dS = \mu S dt + \sigma S dz \), and applying Ito's lemma while accounting for the holding costs, we have:

\[
dC = (C_t + \frac{\sigma^2 S^2 C_{SS}}{2} dt + C_S dS - H dt
\]

Forming the hedge portfolio \( V = dS + C \), and finding the instantaneous return, we have:

\[
dV = (C_t + \frac{\sigma^2 S^2 C_{SS}}{2} - H) dt + (d + C_S) dS
\]

Therefore, if \( \delta = -C_S \) the hedge is riskless and must give a return equal to the riskless rate:

\[
dV = (-C_S S + C) r dt
\]

Equating these two expressions for \( dV \), we have:

\[
\frac{\sigma^2 S^2 C_{SS}}{2} + rSC_S - rC + C_t = H
\]

Since we are dealing with perpetual options, \( C_t = 0 \) and \( C = C(S) \). Letting 'primes' denote ordinary derivatives,

\[
\frac{\sigma^2 S^2 C'}{2} + rSC' - rC = H \tag{2}
\]
The general solution for this differential equation can be found by the standard methods. The homogeneous solution is of the exponential form and the characteristic polynomial has roots equal to unity and

\[ u = \frac{-2r}{\sigma^2} \]  

(3)

The particular solution is \( C(S) = -\frac{H}{r} \). Therefore, the general solution is:

\[ C(S) = A_1 S + A_2 S^u - \frac{H}{r} \]  

(4)

Now, let's consider the boundary conditions. If there is a constant holding cost per unit of time, and the price is very low, the value of the option would be negative, if it could not be abandoned. The remote possibility of a large price increase would not be enough to outweigh the stream of holding costs. In this situation the owner would like to dispose of the option and thereby avoid the holding costs. To account for this I make a free-disposal assumption that restricts the option value to be non-negative. This implies that there exists a stock price \( Z > 0 \) that induces disposal and such that \( C(S) = 0 \) for any \( S \leq Z \) and

\[ C(Z) = 0 \]  

(5)

High-contact can also be required at \( Z \) since the holders of options can be assumed to choose \( Z \) optimally, in which case Merton's [1973a p.
171] argument applies. Since the slope of the boundary condition at \( Z \) is zero,

\[
C'(Z) = 0 \tag{6}
\]

**Proposition 2.** The necessary and sufficient condition for exercise to occur when the appropriate, finite, \( S \) is reached is:

\[
X < \frac{H}{r} \tag{7}
\]

**Proof:** From (1) we know that \( X < \frac{H}{r} \) is a necessary condition for exercise to eventually occur. Sufficiency will be shown latter by exhibiting a solution. However, in this solution when \( X = \frac{H}{r} \), exercise is only optimal for an infinitely high \( S \). Therefore, to obtain a necessary and sufficient condition for exercise with finite \( S \), the strict inequality has to be required. Q.E.D.

From this Proposition we see that if (7) is satisfied, for sufficiently large \( S \), the owner will want to exercise the option. If \( W \) is the price that triggers exercise, then

\[
C(W) = W - X \tag{8}
\]

We can assume owners decide optimally on when to exercise the option, so that we can require high contact at \( W \).
Recalling from (4) the form of the general solution, we have:

\[
C(S) = \begin{cases} 
0 & S < Z \\
A_1 S + A_2 S^u - \frac{H}{r} & Z < S < W \\
S - X & S > W
\end{cases} \quad (4')
\]

Using (4') in (5), (6), (8) and (9), we have a system of 4 equations in 4 unknowns:

\[
A_1 Z + A_2 Z^u - \frac{H}{r} = 0 \quad (5')
\]

\[
A_1 + A_2 u Z^{u-1} = 0 \quad (6')
\]

\[
A_1 W + A_2 W^u - \frac{H}{r} = W - X \quad (8')
\]

\[
A_1 + A_2 u W^{u-1} = 1 \quad (9')
\]

Solving for \(W\) from (8') and (9'), we get

\[
W = \frac{u}{u-1} (X - \frac{H}{r}) \frac{1}{1-A_1} \quad (10)
\]

Solving for \(Z\) from (5') and (6'),
\[ Z = \frac{u}{u-1} \left( \frac{H}{r} \right) \frac{1}{A_1} \]  

(11)

Using (10) and (11),

\[ \frac{W}{Z} = \left( \frac{rX}{H} - 1 \right) \frac{A_1}{1-A_1} \]  

(12)

Solving for \( A_2 \) in (9') and (6') and equating,

\[ A_2 = \frac{1-A_1}{uw-1} = \frac{-A_1}{u_2u-1} \]  

(13)

Therefore,

\[ \frac{A_1}{1-A_1} = - \left( \frac{Z}{w} \right) u-1 \]  

(14)

Using (14) in (12), \( W^u = Z^u \left( 1 - \frac{rX}{H} \right) \). Recalling (7), the term in parentheses is positive and therefore,

\[ W = Z \left( 1 - \frac{rX}{H} \right)^{1/u} \]  

(15)

Now solve for \( A_1 \) in (14) and use (15) to get

\[ A_1 = \left[ 1 - \left( 1 - \frac{X}{H} \right)^{u-1} \right]^{-1} \]  

(16)
Equations (4'), (10), (11), (13) and (16) display the solution.

**Proof of Proposition 2 (continued).** It remains to be shown that when $X = \frac{H}{r}$, exercise occurs only for infinitely high $S$. To prove this, calculate the limit of $W$ when $X \to \frac{H}{r}$. Start by noting in (16) that $A_1 \to 1$. To find the limit in (10), apply L'Hopital's rule:

$$
\lim_{X \to \frac{H}{r}} W = \left( \frac{u}{u-1} \right) \frac{\lim \frac{\partial}{\partial X} (X - \frac{H}{r})}{\lim \frac{\partial}{\partial X} (1 - A_1)} = \left( \frac{u}{u-1} \right) \frac{1}{\lim(- \frac{\partial A_1}{\partial X})}
$$

Taking the partial derivative of (16), it is easy to see that the limit in the denominator is 0 and therefore that $W \to \infty$. Q.E.D.

**Remark.** The limiting solution for when $X \to \frac{H}{r}$ also provides a solution for the case when $X \geq \frac{H}{r}$ and there is no exercise. We have seen above that in the limit, $W \to \infty$. Using this in (4'), we have

$$
C(S) = \begin{cases} 
0 & \text{if } S < Z \\
A_1 S + A_2 S^u - \frac{H}{r} & \text{if } S \geq Z
\end{cases} 
(4'')
$$

In (16), $A_1 \to 1$, and therefore, in (11), $Z \to \frac{u}{u-1} \left( \frac{H}{r} \right)$. Finally, in (13), $A_2 \to \frac{-1}{u^2 u-1}$. This solution is tangent to the line $S = \frac{H}{r}$ when $S \to \infty$ and is independent of $X$. It could have been obtained directly by solving the same differential equation with boundary conditions (5), (6) and $\lim_{S \to \infty} C(S) = S - \frac{H}{r}$. The rationale for this last boundary is that $S \to \infty$. 
if the option is never exercised, then for large $S$ the probability of $S$ being reduced to $Z$ in the future is very small and therefore the free-disposal provision should have little value. The value of the option must then approach $S - \frac{H}{r}$, which is the value without free-disposal, if the option is never exercised. The solution of the resulting system of equations follows along the same lines of the calculations above.

The constraints on $Z$ and $W$

For the calculations below, it is useful to develop explicit expressions for $Z$ and $W$ by substituting for $A_1$ (given by (16)) in (10) and (11):

\[ W = \frac{u}{u-1} \left( x - \frac{H}{r} \right) \left[ 1 - \left( 1 - \frac{rX}{H} \right)^{u-1} \right] \]

\[ Z = \frac{u}{u-1} \left( \frac{H}{r} \right) \left[ 1 - \left( 1 - \frac{rX}{H} \right)^{u-1} \right] \]

It will also be useful to calculate the Taylor series expansion of the function $y(x) = (1-x)^b$ in the neighbourhood of $x=0$ with $\Delta x = a$.

\[ y(a) = (1-a)^b = 1 - ba + \frac{1}{2}b(b-1)a^2 + o(a^3) \]
Proposition 3. If $Z$ is given by (11), then $0 < Z < X$.

Proof: using (11'), the inequality reduces to:

$$1 > \left(1 - \frac{rx}{H}\right)^{u-1} > 1 - \frac{rx}{H} \left(\frac{u}{u-1}\right)$$

(18)

Recalling (7), the first inequality is trivially satisfied. To verify the other inequality, use (17), letting $b = \frac{u-1}{u} > 1$ and $a = \frac{rx}{H} < 1$. Therefore, $y(a) > 1 - ba$ and the second inequality is satisfied.

Q.E.D.

Proposition 4. If $W$ is given by (10), then $W > X$.

Proof: using (10') and dividing by $\frac{H}{r}$, the inequality reduces to:

$$-(1 - \frac{rx}{H}) \left[1 - \left(1 - \frac{rx}{H}\right)^{u-1} \right] > \frac{rx}{H} \left(\frac{u}{u-1}\right)$$

Performing the multiplication in the left-hand side, and simplifying, this reduces to:

$$\left(1 - \frac{rx}{H}\right)^{\frac{1}{u}} > 1 - \frac{1}{u} \left(\frac{rx}{H}\right)$$

(19)

To verify this, use (17) with $b = \frac{1}{u} < 0$ and $a = \frac{rx}{H} < 1$. Q.E.D.
Conditions for optimal exercise and disposal

Here I will show that it is actually optimal to exercise the option when \( S=W \) and to dispose of it when \( S=Z \). The following lemma will be useful in the proof of the next propositions.

**Lemma 1.** The return to holding the reserve when \( Z \leq S \leq W \) is:

\[
dC = [A_1 \mu S + A_2 \mu S^u] \left[ \mu - r \left( \frac{u-1}{u} \right) \right] dt + \sigma [A_1 S + A_2 \mu S^u] \, dz \quad (20)
\]

**Proof:** from the Ito expansion for \( dC \),

\[
dC = (\mu SC_S + \frac{\sigma^2 S^2 C_{SS}}{2} - H) dt + \sigma SC_S \, dz
\]

Taking derivatives in \((4')\),

\[
C_S = A_1 + A_2 \mu S^{u-1} \quad (21)
\]

\[
C_{SS} = A_2 \mu (u-1) S^{u-2} \quad (22)
\]

Substituting for the derivatives in the expression for \( dC \) above, and recalling that \( \frac{\sigma^2}{2} = \frac{r}{u} \), the result follows. Q.E.D.

**Proposition 5.** When \( S<Z \), it is optimal to dispose of the option.

**Proof:** note in \((4')\) that \( C(S)=0 \) for \( S<Z \). Therefore, \( dC = -Hdt \). This negative return can be avoided by disposing of the option. Q.E.D.
**Proposition 6.** When \( S = Z \) the holder of the option is just indifferent between holding and disposing.

**Proof:** we have to show that the expression for \( dC \) in (20) reduces to zero when \( S = Z \). Substitute for \( A_2 \) (from (13)) in (20) and let \( S = Z \). Note that the coefficient for \( dz \) vanishes and therefore,

\[
dC = A_1\mu (\mu - [\mu - r(\frac{u-1}{u})]) dt - H dt
\]

Now, substitute for \( Z \) from (11) to get \( dC = 0 \). Q.E.D.

**Proposition 7.** When \( Z < S < W \), it is optimal to hold the option.

**Proof:** note that \( C(S) > S - X \) in that range. Q.E.D.

**Proposition 8.** When \( S = W \), the holder of the option is just indifferent between holding and exercising.

**Proof:** the returns to holding the option can be calculated by substituting for \( A_2 \) (from (13)) in (20), letting \( S = W \) and using (9'):

\[
dC = [A_1\mu W + (1-A_1)W(\mu - r(\frac{u-1}{u})) - H] dt + \sigma W dz
\]

Simplifying,

\[
dC = [-(1-A_1)W(\frac{u-1}{u}) + \mu W - H] dt + \sigma W dz
\]

Substituting for \( W \), from (10), in the first term:
\[ dC = (\mu W - rX)dt - \sigma W dz \] (23)

If the investor instead borrows the exercise price and exercises the option, he will hold a portfolio of stocks and \( X(t) \) one-dollar bonds: \( V = S - X \). The return will be:

\[ dV = (\mu S - rX)dt + \sigma S dz \] (24)

Letting \( S = W \), (24) reduces to (23), proving the proposition. Q.E.D.

**Proposition 9.** If \( S > W \), it is not optimal to hold the option.

**Proof:** from (24) we have the return if the investor exercises the option. Recalling from (4') that \( C(S) = S - X \) if \( S > W \), the return to holding the option is

\[ dC = (\mu S - H)dt + \sigma S dz \] (25)

Recalling (7), (24) and (25) show that \( dC < dV \) in any state of the world. Q.E.D.

**Comparative statics**

(a) \( \frac{\partial C}{\partial S} > 0 \), \( \frac{\partial^2 C}{\partial S^2} > 0 \) and \( \frac{\partial C}{\partial t} = 0 \).

**Proof:** Using (21) and (13), \( C_S = A_1 \left[ 1 - \left( \frac{S}{2} \right)^{u-1} \right] > 0 \). This last inequality follows by recalling that \( S > Z \), that \( u < 0 \), and that \( A_1 > 0 \).
The second inequality follows from (22) by recalling that $A_2 > 0$.

Q.E.D.

(b) $\frac{\partial Z}{\partial x} > 0$

Proof: from (11'),

$$\frac{\partial Z}{\partial x} = \left(\frac{u}{u-1}\right) \frac{H}{r} \left[\left(-\frac{u-1}{u}\right) \left[1 - \frac{rX}{H}\right]^{u-1} \left(-\frac{r}{H}\right)\right] = \left[1 - \frac{rX}{H}\right]^{u-1} > 0$$

(c) $\frac{\partial W}{\partial x} > 0$

Proof: from (10'),

$$\frac{\partial W}{\partial x} = \left(\frac{u}{u-1}\right) \left[1 - \left(1 - \frac{rX}{H}\right)^{u-1}\right] + \left(\frac{H}{r}\right) \left[\frac{u-1}{u}\right] \left[1 - \frac{rX}{H}\right]^{u-1} \left(-\frac{r}{H}\right)$$

Therefore, noting that $(x - \frac{H}{r})(-\frac{r}{H}) = 1 - \frac{rX}{H}$, we have

$$\frac{\partial W}{\partial x} = \frac{u}{u-1} \left[1 - \left(\frac{1}{u}\right) \left[1 - \frac{rX}{H}\right]^{u-1}\right] > 0 \quad Q.E.D.$$

(d) $\frac{\partial Z}{\partial H} > 0$

Proof: from (11'),
\[ \frac{\partial Z}{\partial H} = \left( \frac{u}{u-1} \right) \left\{ \frac{1}{r} \left[ \left( 1 - \frac{rX}{H} \right)^{u-1} \right] + \frac{H}{r} \left[ -\frac{u-1}{u} \left( 1 - \frac{rX}{H} \right)^{u-1} \frac{rX}{H^2} \right] \right\} \]

\[ \frac{\partial Z}{\partial H} = \frac{u}{u-1} \left\{ \frac{1}{r} \left[ 1 - \frac{rX}{H} \left( u-1 \right) \left( 1 - \frac{rX}{H} \right)^{u-1} \right] - \left( 1 - \frac{rX}{H} \right)^{u-1} \right\} \]

To show that the term in curly brackets is positive, start with (19) and add \(-\frac{rX}{H}\) to both sides:

\[ \left( 1 - \frac{rX}{H} \right)^{u-1} + \frac{1}{u} \left( \frac{rX}{H} \right) - \frac{rX}{H} > 1 - \frac{rX}{H} \]

now, multiply by \(\left( 1 - \frac{rX}{H} \right)^{u-1}\)

\[ 1 - \frac{rX}{H} \left( 1 - \frac{1}{u} \right) \left( 1 - \frac{rX}{H} \right)^{u-1} > \left( 1 - \frac{rX}{H} \right)^{u-1} \quad \text{Q.E.D.} \]

(e) \( \frac{\partial W}{\partial H} < 0 \)

Proof: from (10'),

\[ \frac{\partial W}{\partial H} = \left( \frac{u}{u-1} \right) \left\{ \left( \frac{-1}{r} \right) \left[ \left( 1 - \frac{rX}{H} \right)^{u-1} \right] + \left( \frac{H}{r} \right) \left[ \frac{u-1}{u} \left( 1 - \frac{rX}{H} \right)^{u-1} \frac{1}{rX} \right] \right\} \]

Noting that \( \left( X - \frac{H}{r} \right) \frac{rX}{H^2} = \left( \frac{-1}{r} \right) \frac{rX}{H} \left( 1 - \frac{rX}{H} \right) \).
\[
\frac{\partial W}{\partial H} = \frac{u}{u-1} (-\frac{1}{r}) \left[ 1 - \left( 1 - \frac{rX}{H} \right)^{\frac{u-1}{u}} \left( 1 - \frac{rX}{H} \frac{u-1}{u} \right) \right]
\]

To show that the term in curly brackets is positive, multiply both sides of (18) by \( \left( 1 - \frac{rX}{H} \right)^{\frac{u-1}{u}} \):

\[
1 > \left( 1 - \frac{rX}{H} \right)^{\frac{u-1}{u}} \left( 1 - \frac{rX}{H} \frac{u-1}{u} \right) \quad \text{Q.E.D.}
\]

For the next two proofs it will be useful to expand the function \( y(x) = (1-a)^x \) in the neighborhood of \( x=0 \), with \( \Delta x = b \).

\[
y(b) = (1-a)^b = 1 + [\ln(1-a)]b + \frac{1}{2}[\ln(1-a)]^2 b^2 + o(b^3) \quad (26)
\]

\( (f) \quad \frac{\partial z}{\partial a^2} < 0 \)

Proof: from (11'),

\[
\frac{\partial z}{\partial u} = \frac{H}{r} \left\{ \frac{-1}{(u-1)^2} \left[ 1 - \left( 1 - \frac{rX}{H} \right)^{\frac{u-1}{u}} \left( 1 - \frac{rX}{H} \frac{u-1}{u} \right) \right] + \frac{u}{u-1} \left[ -\left( 1 - \frac{rX}{H} \right)^{\frac{u-1}{u}} \ln(1 - \frac{rX}{H}) \frac{1}{u^2} \right] \right\}
\]

Showing that the term in curly brackets is negative is equivalent to showing that
\[
\left(1 - \frac{rX}{H}\right)^{u-1} < 1 - \frac{u-1}{u} \ln(1 - \frac{rX}{H})
\]

To show that this last inequality holds, use (26) letting \( a = \frac{rX}{H} < 1 \) and \( b = -\frac{u-1}{u} < -1 \), which then gives \( y(b) > 1 + b \ln(1-a) \). This last expression implies the inequality.

Now note that \( \frac{\partial u}{\partial \sigma^2} = \frac{2r}{\sigma^4} \), and finally, \( \frac{\partial z}{\partial \sigma^2} = \frac{\partial z}{\partial u} \frac{2r}{\sigma^4} > 0 \). Q.E.D.

\( (g) \frac{\partial w}{\partial \sigma^2} > 0 \)

Proof: from (10'),

\[
\frac{\partial w}{\partial u} = (x - \frac{H}{r}) \left\{ \frac{-1}{(u-1)^2} \left[ 1 - \frac{rX}{H} - \frac{u-1}{u} \right] + \frac{u}{u-1} \left[ \left(1 - \frac{rX}{H}\right)^{-\frac{u-1}{u}} - \frac{1}{u-1} \ln(1 - \frac{rX}{H}) \right] \right\}
\]

Recall that \( (x - \frac{H}{r}) < 0 \) and note that to show that the term in curly brackets is positive is equivalent to showing that

\[
-\left[ 1 - \frac{rX}{H} - \frac{u-1}{u} \right] \leq -\left( \frac{u-1}{u} \right) \left[ 1 - \frac{rX}{H} - \frac{u-1}{u} \ln(1 - \frac{rX}{H}) \right]
\]

Multiplying by \( -\left(1 - \frac{rX}{H}\right)^{u-1} \), this reduces to
\[
\left(1 - \frac{rX}{H}\right)^{u-1} u > 1 + \frac{u-1}{u} \ln(1 - \frac{rX}{H})
\]

Using (26) with \( a = \frac{rX}{H} \) and \( b = \frac{u-1}{u} > 1 \), we have that \( y(b) > 1 + b \ln(1-a) \), and therefore, \( \frac{\partial W}{\partial u} > 0 \). Finally, \( \frac{\partial W}{\partial \sigma^2} = \frac{\partial Z}{\partial u} \frac{2r}{\sigma^2} > 0 \).

Q.E.D.

Consistency with the CAPM

Merton's [1973b] continuous-time capital asset pricing model (CAPM) applied to the stock, requires that

\[
\mu = r + \lambda \sigma_{SM}
\]

(27)

where \( \sigma_{SM} = \text{Cov}\left[\frac{dS}{S}, \frac{dM}{M}\right] \), and \( M \) is the market portfolio.

Proposition 10. If the stock satisfies the CAPM, the option value in (4') also satisfies the CAPM.

Proof: since the presence of the holding cost does not affect the stochastic part of the option value, we know from Black and Scholes [1973] that \( \sigma_{CM} = \frac{C_S S}{C} \sigma_{SM} \). Using (21),

\[
\sigma_{CM} = \frac{C_S S}{C} \sigma_{SM} = \frac{A_1 S + A_2 u S^u}{C} \sigma_{SM}
\]

(28)

Now we must show that \( p = r + \lambda \sigma_{CM} \). From the expression for \( dC \):
\[ p = E[\frac{3C}{C}] = \frac{1}{C} \left[ \frac{\sigma^2}{2} S^2 C_{SS} + \mu S C_{S} - H \right] dt \]

Substituting for \( \mu \) from (27), for \( C_{S} \) and \( C_{SS} \) from (21) and (22), and recalling that \( \frac{\sigma^2}{2} u = -r \), we have:

\[ p = \frac{1}{C} \left[ (A_1 S + A_2 S^\nu) r + \lambda (A_1 S + u A_2 S^\nu) \sigma_{SM} - H \right] \]

Using (28) and recalling (4) we get \( p = r + \lambda \sigma_{CM} \). Q.E.D.
APPENDIX B - THE VALUE OF A PERPETUAL CALL WITH CHANGING EXERCISE PRICE

The change in the exercise price does not affect the hedging argument that produces a stochastic differential equation of the Black-Scholes [1973] type. It does however introduce time-dependent boundary conditions.

Assume that the exercise price is an increasing function of calendar time: \(X = X(t)\) and \(\frac{dX(t)}{dt} > 0\). Assuming that \(dS = \mu S dt + \sigma S dz\), the value of the call \(C(S,t;X(t))\) has to satisfy the following stochastic partial differential equation.

\[
\frac{1}{2} \sigma^2 S^2 C_{SS} + rSC_S - rC + C_t = 0 \tag{1}
\]

The boundary conditions are:

\[
C(S,t^*;X(t^*)) = \text{Max}[0, S - X(t^*)] \tag{2}
\]

\[
C(S,t;X(t)) \geq \text{Max}[0, S - X(t)] \quad \text{for any } S \text{ and } t \tag{3}
\]

where \(t^*\) denotes the expiration date of the option.

Following a suggestion by Merton [1973a], this equation can be put in a more manageable form by re-scaling the stock price and the call value in units of the exercise price. Define \(s(S,t) = \frac{S}{X(t)}\) and

\[
D(s(S,t),t) = \frac{C(S,t;X(t))}{X(t)} \quad \text{and note that } C_s = D_s, \quad C_{ss} = \frac{D_{ss}}{X} \quad \text{and that } \quad C_t = D_t = D_{ss} \frac{dx}{dt} + D_t X.
\]
Equation (1) reduces to

\[ \frac{1}{2} \sigma^2 s^2 D_{ss} + k s D_s - k D + D_t = 0 \quad (1') \]

where \( k(t) = r - \frac{dX}{X} \frac{1}{dt} \). The boundary conditions reduce to

\[ D(s, t^b) = \text{Max} \ [0, s-1] \quad (2') \]

\[ D(s, t) \geq \text{Max} \ [0, s-1] \quad (3') \]

Note that (1') is just the standard Black-Scholes equation where \( k(t) \) substitutes for the interest rate \( r \). Note that for a fixed interest rate, \( k(t) \) is variable, nonstochastic and can also be negative. For simplicity, assume that \( \frac{dX}{X} \frac{1}{dt} = g \), where \( g \) is a constant. Therefore,

\[ k = r - g \quad (4) \]

We know that if \( k \) is positive, there will be no premature exercise since the solution derived based on that premise never touches the \( \text{Max}[0, s-1] \) frontier. Therefore, a necessary condition for premature exercise is that \( k < 0 \).

To obtain the equation for the perpetual option, note that since the time to expiration is infinite and the boundary conditions of the normalized option are independent of time, \( D(s) \) has to be independent of
time. Therefore,

\[ D_t(s,t) = 0 \]  \hfill (6)

Letting "primes" denote ordinary derivatives, and using (6), (1') reduces to:

\[ \frac{1}{2} s^2 D_1'' + ksD'_1 - kD = 0 \]  \hfill (7)

Note that since \( D \) depends only on \( s \), the decision to exercise must be based on \( s \) reaching a critical level. Let \( w \) be the normalized price that will cause the option to be exercised. The boundary conditions (2') and (3') become:

\[ D(w) = w-1 \]  \hfill (8)

\[ D(s) \geq \text{Max}\{0, s-1\} \quad s \leq w \]  \hfill (9)

The parameter \( w \) will have to be determined in the course of the solution, creating the need for an additional boundary condition.

A "high contact" condition can be required at \( w \) by using Merton's [1973] p. 171 argument, which only requires that investors choose \( w \) optimally. Since the slope of the boundary at \( w \) is 1,

\[ D'(w) = 1 \]  \hfill (10)
Our problem now is to solve (7) with boundary conditions (8), (9) & (10). Rewrite (7) as:

\[ s^2 D'' + \frac{2k}{\sigma^2} sD' - \frac{2k}{\sigma^2} D = 0 \]  \hspace{1cm} (7')

The general form of the solution is exponential. The characteristic polynomial has roots equal to unity and \( v \) where

\[ v = \frac{-2k}{\sigma^2} = \frac{g-r}{(\sigma^2/2)} \]  \hspace{1cm} (11)

and the general solution is:

\[ D(s) = A_1 s + A_2 s^v \quad 0 \leq s \leq w \]  \hspace{1cm} (12)

and the boundary conditions (8) and (10) become:

\[ A_1 w + A_2 w^v = w-1 \]  \hspace{1cm} (13)

\[ A_1 + A_2 w^{v-1} = 1 \]  \hspace{1cm} (14)

**Proposition 1.** A necessary and sufficient condition for premature exercise of the perpetual option is that

\[ g-r > \frac{\sigma^2}{2} \]  \hspace{1cm} (15)
Proof: we can impose economic restrictions on the general solution (12) for premature exercise to be optimal. First note that $A_2 \neq 0$. If $A_2 = 0$, the solution would either not be consistent with premature exercise ($A_1 = 1$) or would not satisfy the high-contact condition ($A_1 \leq 1$).

Second, require that $\lim_{s \to 0} D(s) = 0$. Since $A_2 \neq 0$, this implies:

$$v \geq 0$$

(16)

Third, note that the slope of $D(s)$ at the origin cannot be negative, because of limited liability; or greater than unity, because the option is at most as valuable as the stock. Therefore,

$$0 \leq A_1 \leq 1$$

(17)

Note from (14) that

$$A_2 = \frac{1-A_1}{v\sqrt{v-1}}$$

(18)

Using (16) and (17) and recalling that $A_2 \neq 0$, it follows that:

$$A_2 > 0$$

(19)

Finally, recall that the call must be a convex function of the stock price when the distribution of returns on the stock is independent
of the level of the stock price (theorem 10 in Merton [1973a]). From (12), \( D''(s) = A_2 v(v-1)s^{v-2} \). Using (19), \( D'' > 0 \) implies that \( v > 1 \). The solution with \( v = 1 \) is either not consistent with premature exercise (when \( A_1 + A_2 = 1 \)) or does not satisfy the high-contact condition (when \( A_1 + A_2 \neq 1 \)). Therefore,

\[ v > 1 \quad (20) \]

Using (11) in (20), equation (15) follows, as desired. Q.E.D.

We still cannot get a final solution because we have three unknown parameters \( (A_1, A_2 \text{ and } w) \), and only two boundary conditions \(^1\) ((13) and (14)). There is a family of functions of the form (12) that satisfy the conditions. They can be distinguished by solving for \( w \) from (13) and (14), giving:

\[ w = \frac{1}{1-A_1} \left( \frac{v}{v-1} \right) \quad (21) \]

Proposition 2 below shows that the solution must be the minimum function of this family and Proposition 3 shows that this solution corresponds to the smallest \( w \) consistent with the boundary conditions\(^2\).

**Proposition 2.** The stable equilibrium solution to (7') must be the smallest function of the family defined by (12) and conditions (13) and (14).
Proof: I will show that if the value function is not the minimum function of the family, there is opportunity for riskless arbitrage.

Suppose \( D^2(s) \) is the equilibrium schedule we are examining for stability and suppose it is not the minimum function. Then there exists \( D^1(s) \) belonging to the family of functions and such that \( D^1(s) < D^2(s) \). Let \( C^1(S,t) \) and \( C^2(S,t) \) be the un-normalized option values corresponding to \( D^1 \) and \( D^2 \). Let \( W_1 \) and \( W_2 \) be the respective "trigger" prices. Label the current time \( t_0 \) and let \( t_1 \) and \( t_2 \) be the times when \( S=W_1 \) and \( S=W_2 \) respectively. If \( S_0 < W_1 \), then \( t_1 < t_2 \) since \( W_1 < W_2 \). I show in 5 steps how the arbitrage strategy could be implemented.

1) At time \( t_0 \) we sell short one option at price \( C^2(S_0,t_0) \), thereby acquiring the liability of providing to the buyer of the option one share of stock against a payment equal to the exercise price.

2) Put \( C^2(S_0,t_0) - C^1(S_0,t_0) \) in the bank.

3) Form a portfolio of stocks and bonds that duplicates the return on \( C^1(S,t) \), by following the continuous hedging strategy suggested by Black and Scholes [1973]. They showed that a portfolio which is short on one call and long on \( C^1_S(S,t) \) shares of stock can exactly duplicate the returns on riskless bonds. Therefore we can borrow to implement the hedge and be assured of a return sufficient to cover the interest cost. We follow this strategy until \( t=t_1 \).

4) At \( t=t_1 \) the hedge ratio is \( C^1_S(W_1,t_1) = 1 \), and therefore our position is short one call and long one share of stock. We will want to
retain the long position on one share of stock until \( t = t_2 \). By the boundary condition, \( C^1(W_1, t_1) = W_1 - X(t_1) \). The net value of our portfolio of stocks and the call is then \( X(t_1) \). Therefore to maintain the long position until \( t = t_2 \) we must borrow an amount \( X(t_1) \) at the interest rate \( r \).

5) At \( t = t_2 \) the owner of the option will want to exercise it. He will pay \( X(t_2) \) and we will deliver the share of stock we have held long since \( t = t_1 \). The amount \( X(t_2) = X(t_1)e^{r\Delta t} \) will be more than enough to repay the bond issued in step 4. Its price now is \( X(t_1)e^{r\Delta t} \) which is smaller than \( X(t_2) \) because \( g > r \).

The amount put in the bank in step 2 was never touched and is therefore riskless profit. The argument can be repeated whenever the function \( C^2 \) is not the minimum of the set. Therefore the only stable solution is the minimum function. Q.E.D.

**Proposition 3.** The minimum function of the family defined by (12), (13) & (14) is the one that corresponds to the minimum \( w \).

**Proof:** first I write the equation for the family \( D(s, w) \) and then I show that \( \frac{\partial D}{\partial w} > 0 \) for \( 0 < s < w \). From (13) and (14),

\[
A_2 = \frac{1}{w^{(v-1)}} \quad \text{and} \quad A_1 = 1 - \left( \frac{v}{v-1} \right) \frac{1}{w}
\]

Using this in (12), the family is:

\[
D(s, w) = s - \left( \frac{v}{v-1} \right) \frac{s}{w} + \frac{1}{v-1} \left( \frac{s}{w} \right)^v
\]
Taking the partial derivative,

$$\frac{\partial D}{\partial w} = \left(\frac{v}{v-1}\right) \frac{1}{s} \left[\frac{s}{w}\right]^2 - \left[\frac{s}{w}\right]^{v+1}$$

For $0 < s < w$, $\frac{s}{w} < 1$. The function $f(x) = x^v$ is decreasing for $x < 1$.

Note that since $v > 1$, $v+1 > 2$ and $v-1 > 0$. Therefore, $\left[\frac{s}{w}\right]^2 > \left[\frac{s}{w}\right]^{v+1}$ which then implies that $\frac{\partial D}{\partial w} > 0$. Q.E.D.

To minimize $w$, note from (21) that $\frac{\partial w}{\partial A_1} = \frac{1}{(1-A_2)^2} \left(\frac{v}{v-1}\right)$. Since $v > 1$, $w$ is increasing. By Kuhn-Tucker conditions, the minimum of $w$ is then at the lower boundary of $A_1$, which is given by (17). Therefore, $A_1 = 0$.

Set $A_1 = 0$ in (21), (18) and (12):

$$w = \frac{v}{v-1} \quad (21')$$

$$A_2 = \frac{(v-1)^{v-1}}{v^v} \quad (18')$$

$$D(s) = A_2 s^v \quad \text{for } 0 < s < w \quad (12')$$

Applying the inverse transformation,

$$C(S,t) = \begin{cases} 
\frac{(v-1)^{v-1}s^v}{v^x v^{v-1}} & \text{for } S \leq W(t) \\
S - X & \text{for } S > W(t)
\end{cases} \quad (22)$$
where \( W(t) \) is the price that induces extraction. It is defined by:

\[
W(t) = wX(t) = \frac{v}{v-1}X(t)
\]  

(23)

The following lemma will be useful in the proof of the next propositions.

**Lemma 1.** The return to holding the reserve when \( SK \) is

\[
dC = C[v(\mu-r)+r]dt + \sigma vdz
\]  

(24)

**Proof:** by an Ito expansion of \( C(S,t) \),

\[
dC = (C_t + \mu SC_S + \frac{1}{2}S^2 \sigma^2 C_{SS} )dt + \sigma SC_Sdz
\]

Taking derivatives in (22),

\[
C_t = -g(v-1)c
\]  

(25)

\[
C_S = \frac{vC}{S}
\]  

(26)

\[
C_{SS} = \frac{C}{S^2(v(v-1))}
\]  

(27)

Substituting for the derivatives in the expression for \( dC \) above, and recalling that \( v \frac{\sigma^2}{2} = g-r \), the result follows. Q.E.D.
Optimal exercise time

The following propositions show that the optimal strategy is to exercise the option when \( S = W(t) \).

Proposition 4. When \( S = W(t) \) the holder of the option is just indifferent between holding and exercising.

Proof: the return to holding the option can be calculated by evaluating (24) at \( S=W \), while recalling that \( C(W,t) = W-X(t) \):

\[
\frac{dC}{dt} = (W-X)[(\mu (\mu - r) + r)dt + \sigma dz] \tag{24'}
\]

If the investor instead borrows the exercise price and exercises the option, he will hold a portfolio of stocks and \( X(t) \) one-dollar bonds: \( V = S-X \). The return will be:

\[
\frac{dV}{dt} = (\mu S-rX)dt + \sigma Sz \tag{28}
\]

Letting \( S=W(t) = W(t) \), it is easy to show that

\[
\frac{dV}{dt} = (W-X)[(\mu \nu - r(\nu - 1))dt + \sigma dz] \tag{29}
\]

Comparison of (27) with (29) proves the proposition. Q.E.D.

Proposition 5. If \( S < W(t) \), it is not optimal to exercise the option.
Proof: Note that $C(S,t) > S - X$ when $S < W(t)$. Q.E.D.

Proposition 6. If $S > W(t)$ it is not optimal to hold the option.

Proof: from (29) we have the returns if the investor exercises the option. Recalling that $C(S,t) = S - X(t)$ if $S > W(t)$, the return to holding the option is:

$$dc = (\mu S - gx)dt + 
\sigma Sdz$$  \hspace{1cm} (30)

Comparing (28) and (30) we see that since $g < r$, $dc < dv$, for any state of the world. Therefore it is optimal to exercise immediately. Q.E.D.

Comparative Statics

In this section I analyze the comparative statics of (22).

(a) $\frac{\partial C}{\partial S} > 0$, $\frac{\partial^2 C}{\partial S^2} > 0$, and $\frac{\partial C}{\partial t} < 0$.

Proof: recall from (20) that $v > 1$ and inspect (25)-(27). Q.E.D.

(b) $\frac{\partial C}{\partial X_0} = -(v-1) \frac{C}{X_0} < 0$.

(c) $\frac{\partial C}{\partial g} < 0$, $\frac{\partial C}{\partial r} > 0$ and $\frac{\partial^2 C}{\partial \sigma^2} > 0$.

Proof: Note that $\frac{\partial C}{\partial v} = X(t) \frac{\partial D}{\partial v}$. First I show that $\frac{\partial D}{\partial v} < 0$. Using (12'),

$$\frac{\partial D}{\partial v} = s^v \left( \frac{\partial A_2}{\partial v} + A_2 \ln s \right)$$
Using (18'), it can be shown that \( \frac{dA_2}{dv} = A_2 \ln\left(\frac{1}{w}\right) \). Therefore, \( \frac{dD}{dv} = s' A_2 \ln\left(\frac{3}{w}\right) \). Recalling that \( A_2 > 0 \) and that \( s < w \), we have that \( \frac{dD}{dv} < 0 \). Using this and noting from (11) that

\[
\frac{dv}{d\theta} > 0, \quad \frac{dv}{dr} < 0 \quad \text{and} \quad \frac{dv}{d\sigma^2} < 0
\]  

(31)

the result follows immediately. Q.E.D.

(d) \( \frac{dW}{dg} < 0, \quad \frac{dW}{dr} > 0 \) and \( \frac{dW}{d\sigma^2} > 0 \).

Proof: from (23), \( \frac{dW}{dv} = X(t) \frac{dW}{dv} \). Note from (21') that \( \frac{dW}{dv} = -\frac{1}{(v-1)^2} \). The inequalities follow by using (31). Q.E.D.

(e) \( \frac{dW}{dx_0} > 0 \).

Consistency with the CAPM

Merton’s [1973b] continuous-time capital asset pricing model (CAPM) applied to the stock, requires that

\[
\mu = r + \lambda \sigma_{SM}
\]  

(32)

where \( \sigma_{SM} = \text{Cov}\left[\frac{dS}{S} , \frac{dM}{M}\right] \), and \( M \) is the market portfolio.

**Proposition 7.** If the returns to \( S \) satisfy the CAPM, then the returns to \( C \) also satisfy the CAPM.
Proof: we want to show that \( \rho = r + \lambda \sigma_{CM} \), where \( \sigma_{CM} = \text{Cov}[\frac{dC}{C}, \frac{dM}{M}] \).

Since the differential equation was not affected by the time-dependency of the exercise price, we know from Black and Scholes [1973] that \( \sigma_{CM} = \frac{C_{S}}{C} \sigma_{SM} \). Using (26), we then have that

\[
\sigma_{CM} = \nu \sigma_{SM} \tag{33}
\]

Now, recalling (24),

\[
\rho = \mathbb{E}[\frac{dC}{C}] = [(\nu(\mu - r) + r)dt]
\]

Substituting for \( \mu \) from (32), we have that \( \rho = r + \nu \Lambda \sigma_{SM} \). The proof follows by using (33). Q.E.D.
FOOTNOTES

1. This condition was first derived by Hotelling [1931] in a model with zero extraction costs.

2. Extraction cost is broadly understood to include the unit cost of mining, processing, and transporting, as well as a per unit rental cost of capital equipment.

3. To "lock-in" the profit, sell the resource short for $S$ and lend the full amount. Next period receive $S_{t+1}$, buy the resource in the spot market for $S_{t+1}$ and close the short position. Keep $S_{t}(r-\mu)t$ as pure profit.

4. Note that the extraction cost of a particular deposit is being assumed to rise exogenously. This is in contrast with the model of Solow and Wan [1976] where the extraction cost is a function of the quantity already extracted which then induces the growth of costs over time.


6. The derivations would go through even if the drift is a stochastic variable that could depend on the level of the price and time. The variance cannot be stochastic and can at most be a function of time. In these more general cases the process may not have independent increments or be stationary.
7. Merton [1973a] shows that at any period prior to expiration,
\( C(S,T) \geq \max \{0, S - X e^{-rT}\} \), where \( T \) is the time to expiration. Since for exercise to be optimal we need \( C(S,T) = S - X \), it is clear from the above that no premature exercise will occur. The proof of the inequality is based on the idea that if it did not hold, a speculator would be able, with zero net investment, to build a portfolio composed of the option and \( \alpha \) one-dollar bonds held long, and the stock sold short, that would generate a non-negative payoff at the expiration date.

8. To see this, note that \( u \to -\infty, \frac{u-1}{u} \to 1, A_1 \to \frac{H}{rX}, Z \to X \) and \( W \to X \). Therefore (2) reduces to \( C(S) = S - X \) for \( S > X \) and \( C(S) = 0 \) otherwise, as required. Also, these limiting values for the trigger prices are consistent with the certainty decision rules that recommended (when \( \frac{H}{r} > X \)) immediate extraction of the economical reserves and abandonment of the others.

9. To show that the value for \( S < W \) tends to zero, recall from (3') that
\[
D(s) = \frac{W}{V} \left( \frac{S}{W} \right)^v.
\]
Since \( v \to \infty \) and \( W \to 1 \) and \( s < W \), \( \left( \frac{S}{W} \right)^v \to 0 \) and \( \frac{W}{V} \to 0 \). Therefore, \( D(s) \to 0 \) for \( 0 < s < 1 \), which implies that \( C(S) = S - X \) for \( S > X \) and \( C(S) = 0 \) for \( S < X \). Also, since \( W \to X \), in the limit we have immediate extraction of all the economical reserves. This limiting extraction decision rule is consistent with the rule derived under certainty when \( g > r \).

10. The normalized price also follows a Gauss-Wiener process. To calculate the parameters of the stochastic process, use Itô's Lemma,
\[ ds = s_d S + \frac{1}{2} s_S^2 ds^2 + s_t dt \]

Recalling that \( dS = \mu S dt + \sigma S dz \), and noting that \( s_S = \frac{1}{X} \) and that \( s_t = -sg \), we have then that

\[ \frac{ds}{s} = (\mu-g)dt + \sigma dz \]

11. If we were willing to make a free-disposal assumption, we could assume that the relative price for disposal (z) would be chosen optimally. Using Merton's [1973a p. 171] argument, we would be able to require "high contact" at \( z \) i.e. \( D'(z)=0 \). Since there is no reason for disposal before \( z=0 \), this would generate a third boundary condition: \( D'(0)=0 \), which would immediately allow us to solve the differential equation without going through all the arguments in the text. The approach used there was preferred because it is more general since it does not require this free-disposal assumption.

12. As an aside, propositions analogous to these can be developed in the process of calculating the value of a perpetual option with proportional dividends which are paid out continuously. Merton [1973a] has addressed that problem by using Samuelson's [1969] solution to a differential equation which is of the same form as the one obtained through the hedging argument. As in this Appendix, Samuelson [1969] obtains a family of functions, but he relies on a very difficult appendix to his paper, written by Mokan, to select the lowest function. The argument there does not involve arbitrage, as opposed to the approach used here which
can, through a straightforward adaptation of Proposition 2, be used to complete Samuelson's proof, with great gains in simplicity.

13. By equilibrium I mean that all trades in the market are performed on the assumption that this is the equilibrium valuation function. If the equilibrium is not disturbed by a trader that thinks otherwise, it is self-confirming.
FIGURE III

FIGURE IV
REFERENCES


Pindyck, R.S. [1979], Uncertainty and the pricing of exhaustible resources, Working Paper No. MIT-EL 79-021WP, Massachusetts Institute of Technology.


