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ABSTRACT

Difficulties with the assumptions upon which mean variance (MV) models of portfolio choice are based are well known. Nevertheless, information and computation problems associated with alternative models, such as the growth optimal (GO) model, have seriously impeded attempts to articulate and test them empirically. In an attempt to overcome these problems, this paper develops a technique for generating discrete state probability distributions of security returns from a market model that employs a discrete approximation to normally distributed market returns. The operational feasibility of the suggestion is investigated by computing GO and MV investment policies for various specifications of the state return distributions. The investment policies are then compared to determine whether the GO and MV models are empirically identical when returns are approximately normal, as is often claimed in the literature. (See, for example, Black [1972], Fama and MacBeth [1973, 1974], or Gonedes [1976].) The results indicate that: (a) the naive forecasts appear reasonable when judged by the "no-easy-money" condition; (b) very short computational times are involved in solving the investment problems, particularly the GO problem; and (c) for a given state probability distribution, there are differences in the GO and MV investment policies measured in terms of either the mix of risky assets or the forecasted expected returns and the standard deviations of the optimal unlevered portfolios.
I. INTRODUCTION

The past two decades have seen a proliferation of mathematically sophisticated portfolio selection models. Of these, the mean variance (MV), expected utility, and growth optimal (GO) models have received the bulk of attention.

The expected utility model offers a "completely" rational, and general decision-making framework. The approach yields a complete ordering of investment opportunities but only for those investors who can provide a detailed specification of tastes (that may be represented by a specific utility function) and beliefs (that describe a complete joint probability distribution). Because of these requirements, the model is widely viewed as nonoperational, even though it is employed as the primary criterion for judging the theoretical soundness of other portfolio models.

The growth optimal model of Williams [1936], Kelly [1956], Latane [1959], and Brieman [1960], which calls for maximizing the geometric mean of principal plus return in each period, is consistent with logarithmic utility (only). The unique intuitive appeal of the model is that the optimal investment policy will almost surely lead to more capital under reinvestment in the long run. However, it is apparently also viewed as somewhat nonoperational because of burdensome forecasting and computational requirements. The forecasting problem is burdensome in and of itself, because the whole joint probability distribution must be specified, but as Mossin states, "The main disadvantage--at least for the present--of the growth optimal policy as compared, e.g., with the selection of an E, S [MV] efficient policy, is the formidable computational problem" [1973, p. 41].

-1-
Because investors have been either unwilling or unable to provide a complete specification of their tastes and/or beliefs and have been deterred by the computational complexities, other less burdensome but correspondingly less general models of portfolio choice have appeared and, in the case of the MV model, have gained wide (and almost overwhelming) popularity. Despite serious and well-known shortcomings, the MV model enjoys a strong appeal that derives from an almost perfect balance between elegance and simplicity, both on the individual, or portfolio selection, and social, or market equilibrium, levels. On the social level, the MV capital asset pricing model (MV CAPM) has become the most widely studied model of the equilibrium pricing of risky assets. On the portfolio selection level, an MV decision maker finds it sufficient to consider only the most widely known statistical parameters of any distributions—the means, variances, and covariances. Yet the data requirements are still burdensome. For example, in an n asset problem, the investor must provide \((n \times (n-1))/2\) covariance estimates.

Sharpe's [1963] diagonal model (alternatively, the single index or market model) provided the means for reducing both the information and computation problems. By assuming that returns were linearly related to a market index, Sharpe was able to reduce the variance-covariance matrix to a diagonal matrix containing only variances of error terms generated from the time-series regressions of individual asset returns on a market index. The first purpose of this paper is to modify the market model slightly so that manageable discrete state probability distributions can be generated and used as inputs to expected utility formulated investment problems.
But the market model's appeal does not end with its ability to alleviate the computational problem. Its empirical validity, with respect to common stocks, has been extensively documented in the literature. (See, for example, Blume [1971] and the references contained in this article.) In addition, many MV theorists and empiricists assume that returns are approximately normally distributed. (See, for example, Black [1972] or Fama and MacBeth [1973, 1974].) If it is assumed that market returns are approximately normally distributed and that the modified market model holds, then we have the means of generating fairly realistic probability distributions of security returns. This allows us to address an important issue raised by Fama and MacBeth [1974]. They note that, based on some fairly skewed examples, Hakansson [1971] has left the impression that the MV model may lead to quite different investment policies than the GO model. Then, they claim to have dispelled the impression that the MV and GO investment models differ for the universe of common stocks. However, their methodology did not involve solving the true GO and MV portfolio problems. Thus, the second purpose of this paper is to address the Fama-MacBeth issue directly—specifically, to compute the GO and MV investment policies for a variety of individual securities and portfolios of securities to see whether the two policies differ when return distributions are approximately normal.

The paper proceeds as follows. Section II contains a review of previous studies that have compared the MV and GO models. Section III presents the modified market model as a means of generating fairly realistic discrete state probability distributions that are simple enough to be
employed as input to portfolio selection models containing twenty or more assets. Section IV describes the data. Section V contains an investigation of the operational feasibility of the models. Section VI compares the GO and MV investment policies for individual securities and portfolios. Section VII contains a summary and conclusions.

II. A REVIEW OF PREVIOUS WORK

A number of studies have compared the GO and MV models, both at the individual portfolio selection level and in an equilibrium CAPM framework. On the individual portfolio selection level, Hakansson [1971]--based on some two risky security examples, where the returns were fairly skewed--showed that the MV model led to quite different investment policies than the GO model. In addition, the MV model was shown to be severely compromised. Fama and MacBeth [1974] took issue with the impressions derived from Hakansson's work and specifically attempted to dispel the impression that the MV and GO models differ for the universe of common stocks. Their argument, embedded in an equilibrium framework, runs as follows:

Given normally distributed one-period percentage portfolio returns, the Efficient Set Theorem applies. The growth-optimal portfolio is just the specific mean-variance efficient portfolio that is optimal for the log utility function. Thus the two-parameter and growth-optimal models are mutually consistent (p.859).

In an MV CAPM setting, with a riskless asset, the efficient portfolios are composed of combinations of the riskless asset and the market portfolio. Thus, they argue that, in such a case, the observed GO investment policy can be determined by selecting $\beta_e$ to maximize
\[
\ln(1 + R_e) = \frac{1}{n} \sum_t \ln(1 + R_{et} (1 - \theta_e) + \beta R_{mt}),
\]

where \( R_{et}, R_{ft}, \text{ and } R_{mt} \) are the rates of return on an MV efficient portfolio, the riskless asset, and the market portfolio, respectively; \( n \) is the number of time periods in the sample; and the bar indicates average. Technically, the quoted statements are incorrect. With normal distributions, an expected utility maximizer with logarithmic utility (a growth optimizer) will invest exclusively in the riskfree asset.\(^1\)\(^2\) For that matter, one of the properties of the log and power utility functions, unbounded from below, that makes them particularly interesting is the very fact that they will avoid any possibility of bankruptcy. If, on the other hand, returns are approximately normal, the GO and MV policies may be similar, but that must be determined, and it is, of course, a stated purpose of this paper to investigate whether they are or not.

In an equilibrium framework, Roll [1973] tested the GO and MV CAPMs using weekly stock market data from the 1960s. First, he tested for the GO model's basic validity, using Hotelling's \( T^2 \) statistic. The results of this test indicated that the GO model was well--perhaps even too well--supported. He then attempted to distinguish between the GO and MV CAPMs by comparing the slopes and intercepts of the (generalized) security market lines predicted by the two models. The results indicated that the two models were empirically identical.

In a broader study of linear reward-to-risk trade-offs, which covered the 1934-1971 period and employed monthly data, Grauer [1978a] did not find any statistical difference between linear risk-tolerance models
that included "composite" investors with logarithmic and quadratic utility functions—i.e., GO and MV decision makers—as special cases. However, both Roll [1977] and Grauer [1978c] pointed out fundamental problems in testing the equilibrium implications of the CAPMs using the usual linear risk-return trade-off methodology. In light of their criticisms, it is particularly interesting to see whether GO and MV investors would have chosen the same investment policies if they were presented with the opportunity to invest in the portfolios used in the linear risk-return tests.

Also in an equilibrium framework, Grauer [1978b] attempted to predict the relative market values of risky portfolios of bonds and stocks, including five portfolios of stocks ranked by beta. The study employed expected utility-maximizing techniques, joint frequency distribution estimates of joint probability distribution, and quadratic and logarithmic utility functions (among others). His results indicated that the mix of risky assets differed for the GO and MV models. Thus, the evidence, with respect to the differences between the GO and MV models, is mixed. Moreover, because of the computational problems, the comparison of computed investment policies for more than, say, five securities remains a relatively neglected area.3

III. A SIMPLIFIED MODEL FOR PORTFOLIO SELECTION

A. The Generation of Discrete State Probability Distributions

In general, computational difficulties in solving expected utility formulated investment problems are onerous, to say the least. However,
there are at least two ways that one might proceed to solve them: (a) assume a specific form of continuous return distribution and use numerical analysis and numerical integration routines to solve the problem, and (b) assume a specific discrete return distribution and use a numerical analysis routine to solve the problem. The second alternative will be explored here. There are, however, two key difficulties with this approach. The first is to determine an appropriate number of states of nature, and the second is to forecast the returns in each state. To overcome these two difficulties, a simple twist on the market model is proposed as an intermediate step in fixing the states and determining the returns in a state. First, consider a market index and assume that market returns can be approximated in discrete form (in this specific case, by a discrete approximation to a normal distribution) in terms of, say, $S$ points or states of nature. Second, assume that returns on each security in each state are generated by a market model.

To be more specific, the first step is to consider a discrete approximation to a market index as a means of generating a state set. Let $r_{js}$ be unity plus the rate of return on security $j$ in state $s$, $r_{ms}$ be unity plus the rate of return on the market in state $s$, $\mu_{r_m}$ be the mean market return, and $\sigma_{r_m}$ be the standard deviation on the market. As the construction of a forecasting model of market returns in beyond the scope of the paper, the parameters used in estimating the mean and standard deviation of market returns are taken to be the historic time-series average and standard deviation. Finally, only for concreteness, let $S$ equal eleven. The states are centered at
(1) \[ r_{ms} = u_{rm} + k_s \sigma_{r_m} \quad k_s = -2.5, -2.0, -1.5, \ldots, 2.0, 2.5. \]

The probability of the nine interior states occurring is calculated from normal probability tables to be:

\[
u_{r_m} + (k_s - .25)\sigma_{r_m} < r_{ms} < u_{r_m} + (k_s + .25)\sigma_{r_m} \quad k_s = -2, -1.5, \ldots, 2.
\]

The remaining probability is then swept into the two outlying states.

The second step is to determine the returns in state \( s \). The historical market model is given by:

(2) \[ r_{jt} = \alpha_j + \beta_j r_{mt} + \epsilon_{jt} \quad j = 2, \ldots, J \]

where \( \alpha_j, \beta_j, \) and \( \epsilon_j \) are parameters specific to asset \( j \). It is further assumed that the random disturbances \( \epsilon_j \) have the properties that:

(3a) \[ \mathbb{E}(\epsilon_j) = 0 \]

(3b) \[ \text{cov}(\epsilon_j, \epsilon_k) = 0 \quad j \neq k \]

(3c) \[ \text{cov}(\epsilon_j, r_m) = 0 \quad j = 2, \ldots, J \]

where \( \mathbb{E}(\cdot) \) denotes the expectation operator and \( \text{cov}(\cdot) \) denotes covariance. In this paper, it is also assumed that \( \epsilon_j \sim N(0, \sigma_{\epsilon_j}) \). The return on security \( j \) in state \( s \) is then calculated as:

(4) \[ r_{js} = \hat{\alpha}_j + \hat{\beta}_j r_{ms} + \epsilon_{js} \quad r_{js} > 0 \]

\[ = 0 \quad \text{otherwise} \]
where $\hat{\alpha}_j$ and $\hat{\beta}_j$ are parameters estimated from the time-series regression of $r_{jt}$ on $r_{mt}$, and $e_{js}$ is calculated as a normal zero one random deviate times the standard error of estimate from the time-series regression.\footnote{Equation (4) explicitly recognizes that, even though we are modeling approximate normality, limited liability constrains unity plus the rate of return on common stocks to be greater than or equal to zero. This process of generating an "approximately" normal joint return distribution may be illustrated as follows. The (assumed) normally distributed market returns are massed over eleven states (see figure 1), and a market model is then used to generate returns for the individual securities in the eleven states. The marginal return distributions of the individual securities will also be approximately normal, but the approximation may not be quite as close, in that the spikes of probability will not be centered exactly at each security's mean, plus or minus the relevant number of standard deviations.}

B. The Formulation of the GO and MV Investment Problems

The GO model. Let $v_j$ be the fraction of wealth invested in asset $j$, $j=1,...,J$, $\pi_s$ be the probability of occurrence of state $s$, $r$ be unity plus the riskless borrowing or lending rate associated with asset $1$, and $m_{jt}$ be the margin requirement on asset $j$ at time $t$. The unconstrained GO investment problem can be formulated as: \footnote{Equation (5) can be interpreted as maximizing expected utility, where $\tilde{W}_{t+1} = \sum_j \pi_s \left( \sum_{j=2}^J v_j (r_{js} - r) + w_j \right)$.}

\[
\max_{v_j} \sum_s \pi_s \ln \left( \sum_{j=2}^J v_j (r_{js} - r) + r \right),
\]

subject to:
FIGURE 1

AN ELEVEN-STATE APPROXIMATION TO NORMALLY DISTRIBUTED MARKET RETURNS
(6) \[ \sum_{j=2}^{J} v_j (r_{js} - r) + r \geq 0 \text{ for all } s. \]

With no short sales or margin purchases, the GO problem is given by (5), subject to (6), and

(7) \[ v_j \geq 0, \quad \sum_{j=2}^{J} v_j \leq 1. \]

With margin purchases and no short sales, the GO problem is given by (5), subject to (6), and

(8) \[ v_j \geq 0, \quad \sum_{j=2}^{J} m_j v_j \leq 1. \]

Assume, as is usually done, that the full proceeds of a short sale can be reinvested. For concreteness, assume that short sales are restricted to \(-0.25\) times wealth, or capital, and that a constant margin rate \(m\) applies to all assets. Then the GO problem is given by (5), subject to (6), and

(9) \[-0.25 \leq v_j \leq 1/m_j \quad \text{and} \quad \sum_{j=2}^{J} m_j v_j \leq 1. \]

In all cases, the fraction of wealth borrowed or lent is given by

\[ v_1 = 1 - \sum_{j=2}^{J} v_j. \]

Thus, for example, with \(m\) equal to one-half, borrowing is limited to an amount equal to the investor's initial capital.

**The MV model.** Lintner's [1965] algorithm is employed to solve the MV portfolio selection problem. Let \(x_j\) be the proportion of the amount
invested in asset \( j \) to the total amount invested in risky assets. Maximizing the ratio of excess return to standard deviation on the portfolio leads to the optimal MV portfolio. Formally, the MV portfolio selection problem with a riskfree asset is:

\[
\text{max } \theta = \frac{\sum_{j=2}^{J} x_j E(r_j) - r}{\left( \sum_{j=2}^{J} \sum_{k=2}^{J} x_j x_k \sigma_{jk} \right)^{1/2}}
\]

subject to \( \sum_{j=2}^{J} x_j = 1 \), and, with no short sales, \( x_j \geq 0 \).

IV. THE DATA

Twenty portfolios of securities and twenty individual securities were employed in the comparisons of the GO and MV models. The twenty portfolios consisted of New York Stock Exchange common stocks formed into portfolios on the basis of historical beta estimates. The portfolio return data were taken from a study by Grauer [1978a] and cover the period 1934-1971. The twenty portfolios were constructed from stocks contained in a merged Compustat and Center for Research in Security Pricing data base, using the standard beta-ranking procedure used in equilibrium tests of CAPMs. In detail, the ranking procedure closely follows the technique used by Fama and MacBeth [1973, 1974]. As the portfolio returns were originally stated on a monthly basis, they were transformed to quarterly and annual bases by linking together four and twelve months of wealth relatives, respectively. The twenty individual stocks were chosen from
the universe of thirty Dow Jones industrial stocks. As noted above, the purpose of studying the portfolio data is to provide a rough type of consistency test, or comparative cross-check, on regression studies of CAPMs. On the other hand, the individual stocks were chosen to see what types of investment policies would be recommended for actual securities.

Fisher's Arithmetic Performance Index was employed as the proxy in constructing the market returns. The riskless interest rate was estimated from Treasury bill data, with quarterly and annual rates constructed by linking together the relevant monthly rates. The margin requirements corresponded to initial margin requirements set by the Federal Reserve that were in effect at the beginning of each year. The limits on short sales were arbitrarily set to be -.25 times wealth.

The historical regressions used in constructing the annual state probability distributions covered a twelve-year period that reflected a trade-off between the number of observations and the time period over which the betas might reasonably be assumed to be stable. As the annual data covered the 1934-1971 period, this left twenty-six periods over which investment policies could be chosen. The regressions used in constructing the quarterly state probability distributions covered an eight-year period. The 1944-1971 period was chosen to allow eighty portfolio selection dates from 1952 through 1971. This period was singled out because it yielded a reasonably large number of portfolio selection dates during a recent time period in which the market index showed a fairly wide range of variability.

In summary, lagged historical data were used to create a state probability distribution of security returns—assumed to hold over a specified
decision horizon. Together with the riskfree rate observed at the date of purchase, this allowed investment portfolios to be chosen at prespecified fixed decision points (the first of the year and the first of each quarter, respectively).

V. THE OPERATIONAL FEASIBILITY OF THE FORECASTING AND SOLUTION TECHNIQUES

A. The Criteria

The operational feasibility of the forecasting and solution techniques hinges on the realism of the forecasts and the ease or speed of computing the optimal investment policies, given the forecasts. What constitutes a reasonable probabilistic forecast of security returns is open to debate, but there is one fairly weak condition that many would agree any reasonable forecast should meet. If the forecast is to be consistent with an equilibrium in the financial markets, there should be no opportunity for "easy money." Hakansson [1970] has suggested a formal "no-easy-money" condition, which states that no combination of risky investment opportunities exists which provides, with probability 1, a return at least as high as the riskfree rate of interest; no combination of short sales is available for which the probability is 0 that the loss will exceed the riskfree rate of interest; and no combination of risky investments made from the proceeds of any combination of short sales can guarantee against loss.

B. The Evidence

The no-easy-money criterion. The values of the GO feasibility constraints, equation (6), at the optimum correspond to the ex ante GO
portfolio returns in each state. If the portfolio returns do not stochastically dominate the riskfree return, there are no opportunities for easy money. In the process of conducting the research for this paper, at least 282 GO problems were solved. From this sample of 282, there were only two instances where an opportunity for easy money was forecast. 8

Computational time. The investment problems were solved with the aid of a numerical analysis algorithm described in Best [1975]. The computations were performed on an IBM 370/155 computer. The respective mean, standard deviation, and coefficient of variation of cpu times, in seconds, for the 282 sets of MV and GO investment policies were: MV (2.49, 3.28, and 1.32); GO (1.17, .89, and .76). The times were correlated on the order of .12. But the figures are somewhat misleading, in that the MV times ballooned when the less highly correlated Dow Jones data were employed. The mean cpu times for the Dow Jones data were: MV (8.13); GO (1.24) seconds. Disregarding the Dow Jones data, the respective mean cpu times were: MV (1.39); GO (1.15) seconds.

By historical standards, the MV solutions are fast and, with the possible exception of the Maier, Peterson, and VanderWeide [1977] results, the GO times are extremely fast. By both the criteria of mean-solution time and the stability of that mean-solution time over different investment opportunities, the GO formulation must be judged to have performed very well. Hopefully, then, this package of the forecasting and solution techniques may be a small first step in formulating GO, or more generally expected utility-based investment problems that can be employed by real-world investors. But regardless of the primitive form of the current
forecasts, they do capture the essence of an approximately normal real-world return distribution. Therefore, the comparisons of the GO and MV models hold whether or not one might wish to use this primitive form of the model for actual real-world portfolio selection.9

VI. COMPARISONS OF GO AND MV INVESTMENT POLICIES

A. The Comparisons

There are two ways that the GO and MV investment policies may be compared. First, and most important, consider the mix of risky assets. With a riskless asset and (approximately) normal beliefs, the popular argument (see Fama and MacBeth [1973, 1974], Black [1972], or Gongetes [1976]) is that the mix of risky assets is identical. To determine whether the mix of risky assets is in fact (approximately) the same, the GO and MV investment proportions were computed and compared for a given set of beliefs. There is no sampling problem involved. Given a set of approximately normal beliefs, the policies are either the same or they differ. A second way to gain insight into any difference in policies, which has the advantage of having to reconcile less data, is to compare the ex ante, or forecasted, means and standard deviations of the unlevered GO and MV investment policies calculated from the state return distributions. Again, there is no sampling problem. For a given set of beliefs, the models are either the same or they differ.10

B. The Results

The mix of risky assets. The data in table 1 permits comparisons of GO and MV investment policies, with respect to both the number of risky
<table>
<thead>
<tr>
<th>Type of Data</th>
<th>Number of Portfolio Selections</th>
<th>Model</th>
<th>Frequency with Which</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>No Assets Were Chosen</th>
<th>One Asset Was Chosen</th>
<th>Two Assets Were Chosen</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Annual, portfolio, 11 states, margin, 1946-1971</td>
<td>26</td>
<td>MV</td>
<td>GO</td>
<td>7</td>
<td>12</td>
<td>7</td>
<td>3</td>
<td>6</td>
<td>18</td>
<td>26</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2. Annual, portfolio, 11 states, no margin, 1946-1971</td>
<td>26</td>
<td>MV</td>
<td>GO</td>
<td>7</td>
<td>12</td>
<td>7</td>
<td>3</td>
<td>8</td>
<td>15</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3. Annual, portfolio, 11 states, margin, different random numbers, 1946-1971</td>
<td>26</td>
<td>MV</td>
<td>GO</td>
<td>7</td>
<td>15</td>
<td>4</td>
<td>2</td>
<td>9</td>
<td>11</td>
<td>4</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4. Annual, portfolio, 7 states, margin, 1946-1971</td>
<td>26</td>
<td>MV</td>
<td>GO</td>
<td>7</td>
<td>10</td>
<td>2</td>
<td>5</td>
<td>8</td>
<td>11</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5. Annual, portfolio, 13 states, margin, 1946-1971</td>
<td>26</td>
<td>MV</td>
<td>GO</td>
<td>7</td>
<td>15</td>
<td>4</td>
<td>2</td>
<td>9</td>
<td>11</td>
<td>5</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7. Annual, Dow Jones, 11 states, margin, 1946-1971</td>
<td>26</td>
<td>MV</td>
<td>GO</td>
<td>7</td>
<td>15</td>
<td>4</td>
<td>2</td>
<td>9</td>
<td>11</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>256</td>
<td>MV</td>
<td>GO</td>
<td>142</td>
<td>99</td>
<td>56</td>
<td>13</td>
<td>10</td>
<td>6</td>
<td>1</td>
<td>20</td>
<td>73</td>
<td>139</td>
</tr>
</tbody>
</table>

aThe column records: (i) the length of the decision horizon, annual or quarterly; (ii) the type of assets to be chosen, best-ranked portfolios or Dow Jones securities; (iii) the number of states of nature employed by the discrete market model, 7, 11, or 13; (iv) the constraints imposed on the growth-optimal investment problem, margin or no margin; and (v) the period over which the portfolio selections were made.

bMV = mean variance; GO = growth optimal.

cAt each portfolio selection date, the number of assets chosen ranged from 1 to 7 out of a 20-asset opportunity set.
assets chosen and the mix of risky assets. The first set of columns in the table indicates the type of data being compared. For example, the first comparison (line 1) shows that annual portfolio returns and 11 states of nature were used in constructing the return distributions and that margin purchases were permitted for the GO model during the 26-year 1946-1971 period. The second comparison (line 2) differs from the first in that GO investors were not allowed to buy on margin. The third comparison (line 3) differs from the first in that a different set of random numbers was used in generating the return distributions. The other comparisons vary different sets of parameters: the number of states used in constructing the forecasts, 7, 11, and 13; the length of the holding periods, quarterly versus annual; and the universe of securities, Dow Jones stocks versus grouped beta portfolios.

The second column shows the number of portfolio selections made. For example, line 1 indicates that there were 26 portfolio selections made (at the beginning of each year) during the 1946-1971 period. Line 6 indicates that there were 80 portfolio selections made (one at the beginning of each quarter) during the 1952-1971 period. The final line of the table indicates that an overall total of 256 portfolio selections were made and compared.

The third set of columns shows the frequency with which $n$ assets were chosen, where, for any one portfolio selection, the number of assets actually chosen ranged from 1 to 7 out of a 20-asset opportunity set. The fourth main column shows the frequency with which the mix of risky assets was identical—i.e., the frequency with which the
\[ \left\{ v_k / \sum_{j=2}^{J} v_j \right\} \] of the GO model were equal to the \( \{x_k\} \) of the MV model.

The fifth column shows the frequency with which no assets were chosen in common. This column highlights the fact that not only did the mix of risky assets differ, but that completely different assets were chosen. For instance, the differences were not of the kind that the GO model selects assets 2 and 18 in the proportions 50-50, while the MV model selects assets 2 and 18 in the proportions 60-40. Rather, the differences were of the kind that the GO model selects assets 2 and 18, while the MV model selects assets 4, 7, and 13. The sixth and seventh columns are somewhat similar to the fifth column, in that they report the number of times one or two assets were chosen in common (but not necessarily with the same mix).

Table 1 shows that neither policy calls for a high degree of diversification. This is particularly true of the GO model, which selected only 1 asset, 142, out of the 256 times reported in the table. 11

Now consider the mix of risky assets. Recall that the popular view is that with normal distributions, the GO policy will be a member of the MV efficient set. With riskfree borrowing and lending, this is equivalent to asserting that the mix of risky assets of the two models will be equal. As noted, this cannot be literally true in general, but may be approximately true in some sense if returns are approximately normally distributed. Yet table 1 shows that the mix of risky assets was the same only 20 times (about 8 percent of the time). Alternatively, 92 percent of the time the mix of risky assets differed. Even more strikingly, 29
percent of the time the two models did not choose even one asset in com-
mon. 12

The expected-return, standard-deviation characteristics. The two
models exhibited the same mix of risky assets 20 times. Thus, 20 times
the GO policy was MV efficient. But in the remaining 236 cases, the ex-
pected return/standard deviation ratio of the unlevered GO policy (cal-
culated from the state probability distributions but not explicitly re-
ported) was smaller than for the unlevered MV policy. 13 Moreover, both
the expected return and standard deviation of the unlevered GO portfolio
were strictly greater than their MV counterparts. This means that, in
the cases studied here, the unlevered GO portfolio is inefficient and
systematically lies to the northeast of the efficient unlevered MV port-
folio—i.e., the MV efficient tangency portfolio. Somewhat along the
same line, it was surprising to see the large ex ante variability in re-
turns that the GO model is willing to risk. When the leverage factor is
accounted for, it was not uncommon to see GO policies risking ex ante
losses of 85 to 97 percent (albeit with very small probability). 14

VII. SUMMARY AND CONCLUSIONS

This paper has presented a method for generating fairly realistic
and tractable discrete state probability distributions of security re-
turns from a market model that employed a discrete approximation to nor-
mally distributed market returns. This technique was employed to gener-
ate state return distributions for a variety of actual securities and
portfolios. The operational feasibility of the suggestion was
investigated by computing GO and MV investment policies for various specifications of the state return distribution. The investment policies were then compared to determine whether the GO and MV models are empirically identical when returns are approximately normal, as is often claimed in the literature. The results indicated the following:

(a) The naive forecasts appear reasonable when judged by the no-easy-money condition.

(b) Very short computational times are involved in solving the investment problem, particularly the GO problem.

(c) For a given state probability distribution, there are pronounced differences in the GO and MV investment policies measured either in terms of the assets chosen—i.e., the number of assets chosen, the individual assets chosen, and, hence, the mix of risky assets— or the forecasted expected returns and the standard deviations of the optimal unlevered portfolios.

If the results prove to be robust, there appear to be two rather interesting implications. First, the forecasting and solution techniques suggested for the GO model may prove to be a small first step in formulating expected-utility-based investment problems that can be used by real-world investors. In other words, in terms of operating feasibility, with further refinement the simple twist on Sharpe's diagonal model idea and a good numerical analysis routine may do for the expected utility model what the diagonal model did for the operational feasibility of the MV
model. Second, it is often stated that, with exact multivariate normal- 
ity and a riskfree rate, there is no difference in the unlevered GO and 
MV portfolios. But, given that the statement is technically incorrect,
the relevant empirical issue is whether the portfolios are similar when
returns are approximately normal. The results of this paper indicate
that with approximate normality, the investment policies exhibit pro-
nounced differences. Thus, while linear risk-return tests of the equi-
librium implications of the GO and MV CAPMs by Roll [1973] and Grauer
[1978a] have been unable to detect significant differences between the
models, and no studies to my knowledge have been able to detect statis-
tically significant differences in the ex post geometric mean investment
performance of different portfolio selection models, it does not neces-
sarily mean the models are empirically identical. Rather, it strongly
suggests that our empirical tools are too "blunt" to pick up the differ-
ences. Roll's [1977] and Grauer's [1978c] criticisms of the linear risk-
return tests seem to support this conclusion.
FOOTNOTES

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1 When the opportunity set consists of a number of risky assets, exhibiting an exact multivariate normal return distribution and a riskfree asset, the one and only feasible solution for a growth optimizer (or, more generally, for anyone with tastes described by a power or log utility function unbounded from below) is to place all his capital in a riskfree asset. The reason is simple. An investor facing multivariate normal returns who places any wealth in the risky assets will find that his end-of-period wealth is also normally distributed, which, in turn, implies there is some (perhaps) small probability of realizing a negative wealth level. But the log and power functions are not defined over negative wealth levels and, hence, neither is the expected utility problem. The mathematical formulation of the unconstrained GO problem, equations (5) and (6), below, makes exactly this point.

2 Despite the statements in the text and in footnote 1, Fama and MacBeth have assumed that, with normality, the unlevered GO and MV portfolios are identically equal to the market portfolio. Therefore, they did not really determine the GO policy. Their paper does not address the issue of whether the policies differ; it simply assumes they do and then proceeds to show that there is no difference in the ex post geometric mean performance of a (specially) levered and an unlevered market portfolio proxy.
Two exceptions are Dexter, Yu, and Ziemba [1975] and Maier, Peterson, and VanderWeide [1977].

It is best to consider these as "naive" forecasts. I envision that any investor considering using the model for real-world portfolio selection may wish to modify these naive forecasts to reflect his beliefs more fully.

The investment policies reported in section VI indicate that neither the GO nor the MV models diversify very widely. It was suggested that this may have been caused by the way in which the state return distributions were generated from historical data. Specifically, it was suggested that the historical regressions should have been estimated from rate-of-return data, i.e.,

$$R_{jt} = \alpha_j^* + \beta_j^* R_{mt} + \epsilon_{jt}^*.$$  

Given the rate-of-return formulation, the $\alpha_j^*$ should be zero, on average, if the capital markets are in equilibrium. A nonzero $\alpha_j^*$ simply reflects superior ex post performance. Therefore, the best estimate of $\alpha_j^*$ in the ex ante forecasts should be zero. Otherwise, one might expect that portfolio selection from a return distribution with nonzero $\alpha_j^*$'s built into the forecasts would lead to the selection of the few securities with large, positive $\alpha_j^*$'s. This possibility was checked for a subset of the data. The results, based on the appropriately adjusted rate-of-return forecasts with the $\alpha_j^*$ set equal to zero, were essentially the same as the results reported in the text. Somewhat along the same lines, one may suspect that the small number of states may have contributed to the lack of diversification. While that may (or may not) contribute to the seeming lack of diversification for the GO model, it most certainly will not explain the MV results, as only the forecasted means' variances and covariances were supplied as input to the MV objective function (10).

The unconstrained policy refers to a lack of constraints on the fractions of wealth invested in the risky assets. Makansson [1971] provides the formal derivation that the GO policy is myopic. He shows that
the optimal policy, given by the time-series set of $v_j$'s, is found by solving (5), period by period, subject to any other constraints operating on the $v_j$'s. But technically, with no short sales or margin purchases, and $x_{js} \geq 0$ for all $j, s$, (6) and (7) together are redundant. However, there are good reasons for formulating all three constrained GO investment problems with (6) included. First, (6) was explicitly included in the computer work, as it was the most efficient way to set up the problem for the specific numerical analysis routine employed here. Second, as will be shown below, (6) provides a convenient way to check the forecasts for easy-money opportunities. Third, explicit recognition of (6) may serve to remind the reader that neither the constrained or unconstrained GO policies will risk bankruptcy.

Three points are worth noting. First, the GO model is formulated in terms of fractions of wealth invested in the various securities, and the solution includes the amount of the riskless asset to be borrowed or lent. On the other hand, the MV model is formulated in terms of proportions of the amount invested in asset $k$ to the total amount invested in risky assets, i.e., $x_k = \left\{ \frac{v_k}{\sum_{j=2}^{J} v_j} \right\}$. Thus, the MV investor determines his leverage factor at a second stage, independent of the first-stage problem solved here. Because of the difference in the basic decision variables, it is relatively easy to solve investment problems involving margin and short-sales constraints in a GO but not in an MV framework. Second, in this paper, the market model has been used solely as a means of formulating a tractable form of discrete state probability distribution. Once the probability distribution for the 20 risky assets is formed, it—and it alone—is used in solving the GO and MV investment problems. The market portfolio used in generating the probability distributions is discarded, and the variance-covariance matrix calculated from the probability distribution therefore does not reduce to a diagonal matrix (as in Sharpe's model) that is based solely on ex post estimates. Third, while Lintner's formulation, equation (10), has been employed to
solve the MV problem because it is so well known, no claim is made that this formulation, together with the specific numerical analysis routine used here, provides the most efficient way to solve the MV problem.

A second intuitive way that one might wish to judge the forecasts is by asking: Did the forecasted means roughly approximate the ex post means and, more important, did covariances calculated from the probability distribution roughly approximate the covariances that would have been generated using the simple Sharpe market model, i.e., \( \text{cov}(r_j, r_k) = \beta_j \beta_k \sigma^2 \). Inspection of the forecasted means showed them to be randomly spread around their ex post counterparts. Some means were higher, some lower, than the ex post means, as one might hope. A mathematical appendix is available from the author. It shows that covariances calculated from the forecasted probability distributions will almost surely be smaller than the simple Sharpe market model forecasted covariances. A check of both sets of covariances confirmed this result.

The reasons why one may be somewhat hesitant to employ the model without additional fine tuning are: (a) as noted, the decision maker may wish to alter the forecasts to reflect more specific beliefs that he might hold, and (b) he may wish to allow for more variability in the return distributions. For example, allowing the extreme market returns to range to \( \pm 3 \) standard deviations from the mean, as was done with the 7- and 13-state examples, led to different portfolio choices than with the 11-state case that considered only the range \( \pm 2.5 \) standard deviations from the mean. In general, then, as part of the fine-tuning process, one would wish to consider more closely the effect of changing the number of states, the range of outcomes, the lag period over which forecasts were generated, and the length of the decision horizon.

These statements are perhaps overly strong. It might be argued that if the proportions were "reasonably close," or the mean standard deviation pairs were within some small neighborhood of each other, the policies could be judged sufficiently close to be considered the same.
However, the differences in both measures appear to be large enough to make any arguments regarding what is to be judged reasonably close somewhat extraneous.

11 At least 282 pairs of investment problems were solved, but of these, 26 pairs allowed short sales by the GO model, while short sales were always prohibited in the MV model. As these 26 pairs would have biased the results, only 256 pairs were compared. Incidentally, though, a position was taken in all 20 assets by the GO model when short selling was permitted.

12 It is implicit in lines 1 and 2 of table 1 that the GO model does not mix its risky assets in the same way when margin purchases are permitted as it does when margin purchases are not permitted. The implications are both obvious and important. As noted in footnote 7, MV theorists have argued that an investor can solve the investment problem as a two-stage procedure: first, find the mix of risky assets and, second, lever the risky portfolio to the desired level. Yet the GO results indicate that the mix of risky assets changes, depending on the degree of margin permitted. This result is also implicit in the calculations presented in Hakansson [1971].

13 As the MV policy was calculated by maximizing the ratio of expected portfolio return less the riskfree return to portfolio standard deviation, it is obvious, from the definition of MV efficiency, that the unlevered GO portfolio's expected return/standard deviation ratio must be smaller than the MV portfolio's, unless the GO portfolio is exactly MV efficient.

14 The perhaps somewhat surprising lack of risk aversion displayed by the GO policies can be further illustrated by noting that, for the return distributions investigated in this paper, the GO model always called for borrowing to the margin limits. A possible explanation is that GO investors are less risk averse than the average investor. For
example, Friend and Blume [1975], based on evidence generated from cross-sectional household asset holdings, suggest that a power utility function with a power on the order of \(-1\), or even less, may characterize the "market" utility function, which is consistent with this notion.
REFERENCES


Williams, J. B. 1936. "Speculation and Carryover." Quarterly Journal 
of Economics (May).