Valuation of Risky Assets in Arbitrage-Free Economies with Transactions Costs

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Abstract. This paper analyzes the equilibrium valuation of risky assets in the case where transactions costs are present. The methodology involves applying "theorems of the alternative" (Farkas' Lemma) as a consequence of arbitrage-free markets. Under relevant assumptions, it is found that the price of an asset having transactions costs is the corresponding price that would obtain in a perfect market, plus a "fudge factor." This latter factor is provided explicit bounds.

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1. INTRODUCTION

The theory of asset valuation in markets with transactions costs (i.e., "imperfect markets") is virtually non-existent. By sharp contrast, the theory in which markets are assumed atomistic and frictionless ("perfect markets") is well developed. The core of the problem is easy enough to identify: in an economy with transactions costs, it is generally exceedingly difficult to derive the demands for (risky) assets as a function of transactions costs. Nevertheless, "intuition" suggests that in some appropriate sense transactions costs should have only a limited effect on equilibrium values. Again somewhat loosely speaking, it should be possible to identify equilibrium prices in a transactions-costs economy as being a function of the prices that prevail in a corresponding transactions-costs-free economy plus some "fudge factor," and apparently the latter factor should depend upon the magnitude of the transactions costs. Meaningful results would then further require the specification of appropriate bounds on the "fudge factor."

The ideas become quite concrete if one considers the valuation of a call option. The introduction of transactions costs with respect to the option could presumably change its equilibrium price. However, the impact should be limited by the magnitude of the transactions costs, provided that the underlying stock price remains unchanged. This perspective is particularly compelling in the two-state world of Cox-Ross-Rubinstein (1979): too much of a deviation of the option value from its price in the no-transactions-costs environment would clearly lead to arbitrage opportunities. Thus, in a world with transactions costs the value of the option may no longer be uniquely identifiable without the explicit introduction of preferences/beliefs, but the difference between this price and that of the no-transactions-costs case should nevertheless be bounded by some function of transaction cost magnitudes.

It is the purpose of the present paper to make the above ideas precise, and, more generally, to develop a theory which assesses the impact of transactions costs upon equilibrium values. The methodological approach is simply one which precludes arbitrage opportunities; this permits an application of duality theory (viz. Farkas' Lemma) and the demonstration of the existence of a linear operator that values the transactions-costs-adjusted income stream across different states. Transactions costs are incurred at both dates (going "in" and "out" of the security) and the costs of long positions may differ from those of short positions. The representation theorem that will be derived is a generalization of Beja's (1967), which was developed in the context of perfect markets. The full power of duality theory can be brought to bear, and, as will be seen, under appropriate circumstances the effects of transactions costs on equilibrium prices are surprisingly easy to deduce. Indeed, the imperfect-markets case demonstrates clearly the indispensable role that duality theory may play in analyzing issues of equilibrium valuation.

2. NOTATION AND BASIC ASSUMPTIONS

All of the analysis will pertain to a standard one-period (i.e., two-date) setting. The following notation and basic assumptions will be used throughout the paper. Let

\[ \theta = \text{the state at the end of the period; the state is random and unknown as of the beginning of the period. It is assumed that there are a finite number of states: } \theta = 1, \ldots, n. \]

\[ P_j = \text{the pre-transactions-costs price of security } j \text{ at the beginning of the period.} \]

\[ a_j(\theta) \equiv P_j(\theta) + D_j(\theta) = \text{the (exogenous) pre-transactions-costs gross end-of-period payoff of security } j \text{ given state } \theta; \text{ where} \]

\[ P_j(\theta), D_j(\theta) \text{ denote, respectively, the end-of-period price and dividend across different states.} \]

\[ c^f_j(\theta) = \text{the transaction cost incurred if one unit of security } j \text{ is sold at the end of the period in state } \theta. \]

\[ c^p_j(\theta) = \text{the transaction cost incurred if one unit of security } j \text{ is bought at the end of the period in state } \theta. \]

\[ c^f, c^p = \text{same as above except that the transactions costs pertain to positions taken at the beginning of the period.} \]
From the above one infers that an individual who acquires a long position in asset $j$ at the beginning of the period will pay a total of $P_j + c_j^f$, and when he liquidates at the end of the period the realized proceeds in state $\theta$ are $a_j(\theta) - c_j^l(\theta)$, i.e. net of transactions costs. In a similar fashion, a short position at the beginning of the period yields $P_j - c_j^f$, and there will be a $\theta$-contingent negative cash flow of $a_j(\theta) + c_j^v(\theta)$ at the end of the period.

Implicitly, it has been assumed that transactions costs are "impersonal" in that they do not reflect circumstances unique to the individual (such as taxes). However, without elaborating on the point, it is also possible to extend the present analysis to personal markets by the introduction of additional methodological devices.¹ We further note that the assumptions ignore any scale issues having to do with the "number of shares traded," in the sense that transactions costs are linear in the number of shares traded; this assumption is crucial and appears difficult to relax.² The same applies to the situation where shares are other than infinitely divisible. Finally, we note that the transactions costs are state-contingent, so nothing precludes transactions costs from being dependent upon the price of the asset.

3. ANALYSIS AND RESULTS

In an economy with transactions costs, the absence of (riskless) arbitrage opportunities requires that:

[NA-IM] There exists no portfolio $y_j, z_j, j = 1, \cdots, J$, such that

\[
\sum_j y_j (P_j + c_j^f) - \sum_j z_j (P_j - c_j^f) < 0
\]

and

\[
\sum_j y_j (a_j(\theta) - c_j^l(\theta)) - \sum_j z_j (a_j(\theta) + c_j^v(\theta)) \geq 0 \text{ for all } \theta,
\]

such that $y_j, z_j \geq 0$ for all $j$.

In the above problem NA-IM, which is to be read as "no arbitrage in imperfect markets," interpret $y_j$ as the number of units purchased and $z_j$ as the number of units sold of the $j$-th asset in some portfolio ranging over the $J$ available assets. The first constraint, (1), now assumes positive proceeds, while the second constraint, (2), provides that no future obligations are incurred in any state; existence of $[y_j, z_j]$ satisfying problem NA-IM would thus constitute an arbitrage opportunity, and this would clearly be inconsistent with any reasonable notion of equilibrium.

The special case of no transactions costs implies that no-arbitrage reduces to:

[NA-PM] There exists no portfolio $y_j, j = 1, \cdots, J$ such that

\[
\sum_j y_j P_j < 0
\]

and

\[
\sum_j y_j a_j(\theta) \geq 0 \text{ for all } \theta,
\]

with $y_j$ unconstrained.

The so-formulated NA-PM ("no arbitrage in perfect markets") problem has a well-known implication which has been extensively exploited in modern valuation theory (e.g., see Ross (1976), Rubinstein (1976), and Garman (1977)) via the following representation theorem:

¹The key idea would be one of simultaneously denying arbitrage opportunities to all classes of investors, and then deriving results similar to those of the present paper, as functions of the differing personal characteristics. These results involve a common implicit price system and a set of "fudge factors" which are indexed over investor classes, so that the minimum fudge factor over this index forms the valuation bounds.

²A treatment of the relaxed assumption would lead into nonlinear duality theory, which is well beyond the scope of the current paper. Of course, the same comment applies to permitting prices and dividends to be affected by the number of shares traded.
Proposition I (Beja, 1967). Let \( \{P_j\} \) be some price system; then, NA-PM obtains if and only if there exists some nonnegative vector \( V \equiv (V(1), \cdots, V(\theta), \cdots, V(n)) \) such that
\[
P_j = \sum_a V(\theta) a_j(\theta)
\]
for all \( j = 1, \ldots, J \).

The proof follows as a direct application of Farkas’ Lemma\(^3\) (see Mangasarian, 1969).

As is well known, the significance of the proposition is that it proves the existence of a linear operator (not necessarily unique) that maps the state contingent payoffs into equilibrium prices. Conversely, the existence of such a nonnegative operator precludes arbitrage. The \( V \)-operator is generally referred to as the implicit price system of the economy since it assigns values to future contingent payoff units. (If markets are complete, then of course \( V(\theta) \) is unique and equal to the price of an Arrow-Debreu security which pays off one unit if and only if state \( \theta \) occurs.)

We shall next concern ourselves with NA-IM as a generalization of the NA-PM economy. Analogously to Proposition I, the existence of appropriate operators must encompass both necessity and sufficiency across all price systems that exclude arbitrage. Additionally, and this aspect is unique to the transactions-costs economy, it becomes necessary to identify the component in the representation which depends on the asset-specific structure of transactions costs. The desired representation should thus be written as:
\[
\hat{P}_j = \sum_a \hat{V}(\theta) a_j(\theta) + \epsilon_j,
\]
where \( \epsilon_j \) is the component which depends directly on the transactions costs. Observe that the notation \( \hat{V} \) is to indicate that it can differ from the operator \( V \) derived in the PM economy; this must be so since the price systems \( \{P_j\}, \{\hat{P}_j\} \), can generally differ between the two economies. The question now becomes, what can be said about \( \epsilon_j \) if arbitrage is absent? The answer is as follows:

Proposition II. Let \( \{\hat{P}_j\} \) be some price system, and let \( \{c^p_j, c^c_j(\theta), c^c_j(\theta)\} \) be some structure of transactions costs; then NA-IM obtains if, and only if, there exists some nonnegative vector \( \hat{V} \equiv (\hat{V}(1), \cdots, \hat{V}(\theta), \cdots, \hat{V}(n)) \) and \( 2J \) nonnegative numbers \( \{w_j, u_j\} \) such that
\[
\hat{P}_j = \sum_a \hat{V}(\theta) a_j(\theta) + \epsilon_j,
\]
where
\[
\epsilon_j = c^c_j + \sum_a \hat{V}(\theta) c^p_j(\theta) - u_j
\]
\[
= -[c^p_j + \sum_a \hat{V}(\theta) c^c_j(\theta) - w_j].
\]

Proof. The second set of constraints in NA-IM, (2), combined with \( y_j, z_j \geq 0 \) can be written in matrix form as
\[
\begin{bmatrix}
A_s & -A_b \\
I & 0 \\
0 & I
\end{bmatrix} x \geq 0
\]
where \( x \equiv (y_1, \cdots, y_J, z_1, \cdots, z_J) \), \( A_s \equiv [a_j(\theta) - c^c_j(\theta)] \), and where \( j \) is the index for columns, \( \theta \) is the one for rows. In a similar fashion, one defines the matrix

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\(^3\)A slightly more general approach to no-arbitrage assumes that there exists no portfolio \( \{y_j\} \) such that \( \sum_j y_j \hat{P}_j \leq 0 \) and \( \sum_j y_j a_j(\theta) \geq 0 \) for all \( \theta \) and strict inequality holds for at least one of the foregoing \( n+1 \) equations. No-arbitrage now implies that \( V \) is strictly positive for all \( \theta \). (The proof is an application of Steinke’s Theorem of the Alternative; see Mangasarian (1969, p.34).) Subsequent results could be modified using this stronger notion of arbitrage. In the imperfect markets case, this would involve another alternative theorem, namely that of Tucker (see Mangasarian (1969, p.34)). The extension is modest for the present context, and is therefore omitted.
$A_r \equiv \{c_r(\theta) + c_r(\theta)\}$. The two identity matrices $I$ have $J$ rows and columns each, so that the matrix in (5) has $n + 2J$ rows (recall that $n$ is the number of possible states at the end of the period) and $2J$ columns. Now, using Farkas' Lemma, (5) and (1) cannot occur for any $x$ if and only if there exists some nonnegative vector

$$Q \equiv (\hat{V}(1), \ldots, \hat{V}(n), w_1, \ldots, w_J, u_1, \ldots, u_J)$$

such that

$$Q \begin{bmatrix} A_r & -A_r \\ I & 0 \\ 0 & I \end{bmatrix} = [(P_1 + c_1^f), \ldots, (P_j + c_j^f), -(P_1 - c_1^f), \ldots, -(P_j - c_j^f)].$$

Simplifying and rearranging terms then yields the proposition. $\blacksquare$

As a special case of Proposition II, it is readily verified that the no-transactions-costs case implies $w_r = u_r = e_r = 0$; the observation is immediate since $w_r$ and $u_r$ are always nonnegative. However, there is the broader issue of establishing bounds on the terms $e_r$.

**Proposition III.** The term $e_r$ in (3) is bounded by

$$e_r^f \geq e_r \geq -e_r^f;$$

moreover,

$$|e_r| \leq \frac{1}{2}|e_r^f - e_r^f| + \frac{1}{2}|e_r^f + e_r^f|,$$

where $e_r^f$ and $e_r^f$ are defined by

$$e_r^f \equiv c_r^f + \sum_{s} \hat{V}(s)c_r^f(s),$$

$$e_r^f \equiv c_r^f + \sum_{s} \hat{V}(s)c_r^f(s).$$

If it is further assumed that $c_r^f - c_r^f \equiv c_r$ and $c_r^f \equiv c_r^f$, then

$$|e_r| \leq |e_r^f| = |e_r^f| \leq c_r + \rho \max_{\theta} \{c_r(\theta)\},$$

where $\rho \equiv \{\sum_{s} \hat{V}(s)\}$ is the reciprocal of the one plus rate of return obtained on a (no-transactions-costs) risk-free asset. Hence, if it is additionally assumed that $c_r(\theta) = c_r$ for all $\theta$, then

$$|e_r| \leq c_r \{1 + \rho\} \quad (\leq 2c_r \text{ if } \rho \leq 1).$$

**Proof.** The terms $w_r$, $u_r$ are nonnegative. The first inequality therefore follows directly from (4a) and (4b). The remainder of the proposition is now immediate, since

$$\max \{\{e_r\}, \{e_r\}\} = \frac{1}{2}|e_r^f - e_r^f| + \frac{1}{2}|e_r^f + e_r^f| \quad \blacksquare$$

The last two propositions provide for the straightforward development of the transactions-costs case of arbitrage-free markets. The notion of an implicit price system is still present, and the transactions-costs component acts as an adjustment to the valuation implied by the existing implicit price system. Although this is a structurally simple result, analysis of its implication and interpretation require some care. It must be emphasized that $\hat{V}$ does generally (indirectly) depend on the structure of transactions costs. The point is readily appreciated if one simply notes that, in general, the structure of equilibrium prices $\hat{P}_i$ depends upon the structure of transactions costs $\{c_r^f, c_r^f, c_r^f(s), c_r^f(\theta)\}$ in a fairly general fashion depending on the specifics of individuals' preferences/beliefs and endowments. Hence, if one changes the transactions-costs structure, then this might require a change of not only the $\{w_r, u_r\}$ vector, but also the implicit price system $\hat{V}$. It follows specifically that if one compares a no-transactions-costs economy to one which includes such costs, one cannot assume a priori that there exists some $\hat{V}$, $\{w_r, u_r\}$ which solves NA-IM and where $\hat{V}$ also solves NA-PM. (i.e.
\( \hat{V} = V \). The latter can be given a simple illustration. Suppose that there are a complete set of Arrow-Debreu securities, and, further, that the transactions-costs economy has a transactions costs structure \( c^i, c^j, c^i(\theta), c^j(\theta) > 0 \) and \( c^i, c^j, c^i(\theta), c^j(\theta) = 0 \) for all \( j \geq 2 \). \( V \) is now unique, and it is readily seen that it cannot also serve as a solution in the transactions-costs economy if \( P_j (\equiv \hat{V}(j)) \neq \hat{P}_j \) for some \( j \geq 2 \); the latter relationship is completely innocuous in a general equilibrium setting, and cannot, of course, be ruled out on prior grounds.

The above merely boils down to saying that in general equilibrium everything depends on everything. With no particular knowledge about the exogenous specifications of the economy (such as preferences/beliefs), even a small change in the transactions costs has a completely unpredictable effect on the structure of security prices and thus, implicitly, the (set of) implicit price system(s). In more concrete terms, to compare the value of an option having transactions costs to one without these costs is an almost meaningless exercise if the value of the underlying stock and the interest rate change when such transactions costs are introduced.

The analysis clearly reveals that the impact of transactions costs upon equilibrium values can be assessed directly only for a fixed implicit price system. In other words, transactions costs are permitted to "perturb" individual asset prices, but the economy must remain the same on its most fundamental level. Whether this is true or a reasonable approximation in any one case is a difficult empirical matter. However, as previously indicated, the point is that the question of assessing transactions costs is not really meaningful unless it is assumed that the implicit price system is (locally) insensitive to the transactions costs. Further, it should be clear that in the context of partial equilibrium analysis, there is no logical alternative to that of taking the implicit price system as exogenously established.

Given a fixed implicit price system, of special interest is the relationship between prices in the perfect and imperfect markets. With \( V = \hat{V} \), expression (3) reduces to

\[
\hat{P}_j = P_j + \epsilon_j. \tag{3''}
\]

The price of the \( j \)-th asset is equal to its price in the frictionless economy plus the "fudge factor" \( \epsilon_j \), which can be bounded via Proposition III by the implicit price system and the transaction cost structure unique to the \( j \)-th asset. The additional assumption of proportional transactions costs, \( c^i = fP_j \) and \( c^j(\theta) = f\alpha_j(\theta) \), where \( 100f > 0 \) is the percentage of the price (payoff) which is incurred as a transactions cost, leads to especially concrete results. Equations (4a) and (4b) combined with (3) and (3') (and of course, \( V = \hat{V} \)) then reduce to

\[
\frac{1-f}{1+f} \leq \frac{\hat{P}_j}{P_j} \leq \frac{1+f}{1-f},
\]

which further implies that

\[
\left| \frac{\hat{P}_j - P_j}{P_j} \right| \leq \frac{2f}{1-f}.
\]

In words, the percentage price deviation between the frictionless value and the transactions-costs value is bounded by the roundtrip percentage transactions costs plus second order terms which will be insignificant when \( f \) is small.

Finally, it might be of some interest to apply Proposition III and (3'') to the pricing of a call option using the two-state framework developed by Cox-Ross-Rubinstein (1979). If it is assumed that the risk-free asset and the underlying stock can always be acquired (sold) with no transactions costs, then it is readily established that \( V = \hat{V} \) regardless of the transactions cost structure associated with the call option, provided that the price of the stock and the risk-free rate remain unchanged. Specifically, the implicit price system can be identified as (Cox-Ross-Rubinstein, 1979, p. 240):

\[
V(1) = \rho q
\]
\[
V(2) = \rho(1-q),
\]
where

\[ q \equiv \frac{\rho^{-1}P - a(1)}{a(2) - a(1)} \]

and \( P \) is the stock price. \( a(1), a(2) \) are its end-of-period payoffs in each of the two states. (Also, for simplicity, assume no dividends.) Given exogenous \( a(1) \) and \( a(2) \), it is readily verified\(^*(\) that not only is \( V \) fully determined, but also \( V = V \) if \( P \) and \( \rho \) remain unchanged; thus (3') applies and closed-form bounds can be established on the price of the call option for any specific scheme of transactions costs. Roughly speaking, the exact equilibrium price of the call option cannot be determined through arbitrage arguments alone when transactions costs are present, but the value cannot deviate "too much" from its value in a frictionless economy, since then there would be arbitrage opportunities. The "too much" notion is given explicit treatment in Proposition III and the fact that the implicit price system is completely specified in the above two-state world.

4. CONCLUDING REMARKS

This paper has developed a generalization of the classical representation theorem of the no-arbitrage condition to include transactions costs. The extension has demonstrated the existence of an implicit price system along with new nonnegative variables which give rise to the "fudge factor." The classical valuation result was shown to be approximated to within the "fudge factor" by fixing the implicit price system and comparing the imperfect-market and perfect-market results. Moreover, this factor was explicitly identified and bounded for any implicit price system and transactions-costs structure. Finally, the relative ease with which the valuation and bounding results are obtained illustrates the vital role of duality theory as a tool in the analysis of asset market equilibrium.

REFERENCES


\(^*\)Three equations are implied by (3), one each for the stock, bond, and option. The first two of these equations have zero fudge factors; hence the implicit price system is immediately inferred from the first two equations alone.