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TIME DOMINANCE EFFICIENCY ANALYSIS
BY STEINAR EKERN

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TIME DOMINANCE EFFICIENCY ANALYSIS

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TIME DOMINANCE EFFICIENCY ANALYSIS

ABSTRACT

Building on the stochastic dominance framework, time dominance efficiency analysis provides similar rules for a partial ordering of temporal prospects. Time dominance does not require any quantitative information about temporal preferences for screening decision alternatives according to their net present values. A binary time dominance proposition extends recent sufficient conditions and adds necessity. The paper's main contribution is the development of set time dominance. By eliminating binary undominated projects which no one would choose, set time dominance minimizes time efficient sets without imposing further preference assumptions.
TIME DOMINANCE EFFICIENCY ANALYSIS

Stochastic dominance is by now a well established approach to choice problems under uncertainty [14]. This paper transfers the methodology to a temporal context, where the consequences of a decision alternative are distributed over time, with capital budgeting being a prime example. Whereas stochastic dominance puts successively stronger restrictions on the utility functions representing risk preferences, in time dominance the discounting functions representing temporal preferences are qualitatively restricted in a similar way. The time dominance approach provides rules for a partial ordering of temporal prospects, yielding an efficient set from which the ultimate choice will be made. Its distinct feature is that no numerical values of the discounting function are needed to identify and eliminate decision alternatives which are inferior according to the net present value (NPV) rule.

That property is especially attractive if future temporal preferences cannot be unambiguously quantified on the basis of capital market information. It may be instructive to consider some circumstances where the time dominance procedures may prove useful: a single decision maker, being uncertain about future temporal tradeoffs, may be reluctant to quantify the discounting function. A group of decision makers with possibly conflicting temporal preferences may have to work out a jointly agreed upon choice. A
syndicate organizer may attempt bringing together participants with heterogeneous evaluations of consequences across time. In agency situations the agent may have incomplete knowledge about the principal's temporal preference. The remote client problem may be even more severe in the case of public cost-benefit analysis. General principles may be shown to remain valid even when restrictive term structure of interest assumptions are somewhat relaxed.

Potential applications of the time dominance methodology include important decision-making situations, like comparing competitive technologies, ranking alternative financial management policies, selecting geographic locations, designing marketing strategies, and evaluating public projects.

As for the organization of the paper, in section I the problem is formulated in more detail. A time dominance approach links properties of higher order cumulated (or integrated) consequence values over time to superior NPVs for all decision makers whose temporal preferences satisfy certain conditions. Generalizing and extending related results of Bühren and Hansen [1] and of Pratt and Hammond [10], the binary time dominance proposition in section II depends on pairwise comparisons between individual projects. Motivated by the work of Fishburn [3] and Meyer [9] on stochastic efficiency analysis, section III uses a convex combination as a basis for comparisons. The resulting set time dominance proposition removes from the efficient set binary undominated projects which no one would choose, without imposing further preference assumptions. An extensive capital
budgeting application example in section IV demonstrates the power of the time dominance procedures in reducing the number of candidate projects in a significant way. Some final remarks in section V conclude the main body of the paper. Proofs of the propositions are outlined in an appendix, which also lists the data base for the section IV example.

I. FORMULATION

Let \( t \) denote a point in time, running from one initial point \( 0 \) onward into the future until the horizon \( H \). Suppose there is a set \( S \) of mutually exclusive decision alternatives available to the decision maker(s). Each such alternative is completely characterized by its consequences \( x = x(t) \) over time. Temporal preferences are represented by a discounting function \( v = v(t) \).

For simplicity the following exposition will be phrased in a capital budgeting framework. Then the set \( S \) is a collection of investment projects. The consequence \( x(t) \) may be interpreted as the cash flow at time \( t \). With \( R_t \) being the single-period interest rate from \((t-1)\) to \( t \), and \( r(t) \) being the instantaneous spot rate at time \( t \), the discounting function \( v(t) = \left[ (1+R_1)(1-R_2)\ldots(1+R_t) \right]^{-1} \) in discrete time, whereas \( v(t) = \exp[-\int_0^t r(s)ds] \) in continuous time.

There may be considerable flexibility as to what features are to be incorporated into the \( x \) or \( v \) functions. If \( x \) is a deterministic cash flow, then the discounting function \( v \) may represent
general equilibrium marginal rates of substitution in perfect and complete markets [2, chap. 2] or subjective marginal rates of substitution in a nonmarket context [7, chap. 9]. Under uncertainty, \( x \) may be expected values of cash flows and \( v \) implies risk-adjusted discounting, or \( x \) may represent certainty equivalents and \( v \) risk-free discounting. Under inflationary conditions, \( x \) may be expressed in nominal terms whereas \( v \) takes care of both deflating and discounting, or \( x \) may be in real terms leaving purely time preference adjustments for \( v \). In the presence of taxation, \( x \) may be before-taxes cash flow with both taxation effects and discounting embodied in \( v \), or \( x \) is an after-tax cash flow stream and \( v \) reflects only temporal tradeoffs.

The time dominance partial ordering of alternatives depends on mathematical properties of the cash flow distribution \( x \). It uses repeated summations in discrete time and repeated integrations in continuous time. Rewrite \( x(t) = X^0(t) \). Then, for \( n = 1, 2, \ldots \), the notation \( X^n(t) \) is defined as \( \sum_{s=0}^{t} X^{n-1}(s) \) in discrete time and as \( \int_0^t X^{n-1}(s) \, ds \) in continuous time. Thus, \( X^0(t) \) is the cash flow, \( X^1(t) \) is the cash balance at zero interest rate, \( X^2(t) \) is the cumulative cash balance, etc.

Such repeated cumulations of consequences correspond to repeated cumulations of probabilities in the stochastic dominance approach, although there are some important differences. Unlike probabilities, cash flows may be negative, and different projects' cash balances at the horizon do not necessarily coincide. Hence, \( X^k(t) \) may decrease in \( t \), and \( X^3(t) \neq X^1(t) \).
The major premise of this paper is that the decision maker evaluates temporal prospects by the NPV criterion. For a given discounting function \( v \), the NPV of \( x \), written \( \text{NPV}(x;v) \), is defined as

\[
\sum_{t=0}^{H} v(t)x(t) \quad \text{in the discrete time case, and as} \quad \int_{0}^{H} v(t)x(t)\,dt \quad \text{in continuous time.}
\]

When two projects with cash flows of \( x_j \) and \( x_i \) are being compared, the former project will be strictly preferred if and only if \( \text{NPV}(x_j;v) > \text{NPV}(x_i;v) \).

Similar to stochastic dominance using assumptions about utility functions, time dominance calls for curvature restrictions to classify discounting functions. Let \( v^0(t) \equiv v(t) \). Then, for any \( k = 1, 2, \ldots, n \), let

\[
v^k(t) = \begin{cases} \frac{dv^{k-1}(t)}{dt} & \text{in continuous time} \\ v^{k-1}(t+1) - v^{k-1}(t) & \text{in discrete time} \end{cases}
\]

Thus, \( v^k(t) \) is obtained by differentiating or differencing \( v(t) \) \( k \) times.

The widest class of discounting functions, denoted \( V_0 \), requires simply that at any point in time more is preferred to less. Formally, \( V_0 \equiv \{ v : v(t) > 0 \} \). By adding successive restrictions on \( v^k(t) \) as the nonnegative integer \( k \) increases, subsets of discounting functions are defined:

\[
V_k = \{ v : v \in V_{k-1}, \text{ and } (-1)^k v^k(t) > 0 \}
\]

Hence, \( v \) belongs to the class \( V_n \) if and only if \( v^k(t) \) alternates
in sign (and starting with plus), as \( k \) goes from 0 to \( n \). The domain of \( v^k(t) \) contains the interval \([0,H]\) in continuous time and \([0, H-k]\) in discrete time.

A condition such as (2) may need some interpretation. \( V_0 \) implies nonsatiation, such that \((0,1,0)\) is always preferred to \((0,0,0)\). Discounting functions representing impatience belong to \( V_1 \), implying \((0,1,-1,0)\)-necessarily being preferred to \((0,0,0,0)\). If impatience decreases over time, such that \((1,-2,1)\) is considered better than \((0,0,0)\), then \( V_2 \) contains the appropriate discounting function. Whenever the decrease in impatience increases over time, \((1,-3,3,-1)\) is considered superior to \((0,0,0,0)\), and the discounting function belongs to \( V_3 \). For higher orders of \( k \), suppose \( v \) belongs to the class \( V_{k-1} \). It can then be shown [11] that if \( v^k(t) \) does not change sign over \( t \), then \( v \) belongs to \( V_k \) as well. Alternatively, if \((-1)^{k-2} v^{k-2}(t) \) is convex, then \( V_k \) contains \( v \).

The time dominance approach is most powerful for the case of discounting functions belonging to \( V_k \) for any nonnegative integer \( k \). If the single-period interest rate \( R_t \) or the instantaneous spot rate \( r(t) \) never increase, then \( v \) is contained in \( V_k \) for all \( k \). In particular, discounting functions derived from constant single-period or instantaneous interest rates form proper subsets of \( V_k \) for any \( k \).

As convexity in the interest rate is not necessary for convexity of the discounting function, \( v \) may belong to \( V_k \) for some \( k \geq 2 \), even if the single-period (or instantaneous) interest rate at some points increases. As examples, suppose the initial single-period
interest rate is 25 percent, and then increases to a new constant value after one period. With $R_{t} = 30\%$ for $t \geq 2$, $v$ belongs to $V_2$ but not to $V_3$. If $R_{t} = 26\%$ for $t \geq 2$, the discounting function belongs to $V_1$, $V_2$, and $V_3$, but not to $V_4$. As $R_{t}$ approaches 25 percent from above for $t \geq 2$, $v$ is contained in $V_k$ for steadily increasing $k$.

Whereas a risk-preference utility function is unique up to a positive linear transformation, the temporal preference discounting function $v$ is unique only up to a positive constant. By convention, $v(t)$ will be normalized such that $v(0) = 1.0$.

Using a time dominance methodology, information about particular $X^k(t)$ values for $1 \leq k \leq n$ may suffice to conclude whether some decision alternatives are definitely inferior for all discounting functions in the class $V_n$. Projects surviving the purging process will form efficient sets, which should be as small as possible to facilitate the final choice.

II. BINARY TIME DOMINANCE

Starting out with pairwise comparisons of two projects at a time, binary time dominance may be developed.

In continuous time, a project $x_j$ dominates $x_i$ by $n$th order binary time dominance if and only if

\begin{align}
X^k_j(t) &> X^k_i(t) \quad \text{for all } k = 1,2,\ldots,n-1 \quad (3a) \\
X^n_j(t) &> X^n_i(t) \quad \text{for all } t \text{ in } [0,H] \quad (3b)
\end{align}
with (3b) holding as a strong inequality over some subinterval. This
definition resembles the corresponding \(n\)th order stochastic dominance
[6] [12]. It was introduced in a time framework by Böhren and
Hansen [1].

In discrete time a somewhat different definition is more con-
venient, building on the work of Pratt and Hammond [10]. Consider the
matrix \(\{x^k(t)\}\), where \(k = 1,2,\ldots,n\) and \(t = 0,1,\ldots,R\). An admi-
sible \(n\)th order connected path starts at the lower left corner of
\(x^n(0)\), passes through adjoining elements to the right and/or above,
and ends in the upper right corner of \(x^1(R)\). The project \(x_j\) domi-
nates \(x_i\) by \(n\)th order binary time dominance if and only if

\[
x^k_j(t) > x^k_i(t)
\]

for all \((k,t)\) pairs along an admissible \(n\)th order path, with at
least one strict inequality.

The proposition below links \(n\)th order binary time dominance to
NPV superiority for discounting functions in \(V_n\).

**PROPOSITION 1:** \(\text{NPV}(x_j:v) > \text{NPV}(x_i:v)\) for all \(v\) in \(V_n\), if
and only if \(x_j\) dominates \(x_i\) by \(n\)th order binary time dominance.

Appendix A provides an outline of a proof.

Proposition 1 extends results of Böhren and Hansen [1] and of
Pratt and Hammond [10]. Böhren and Hansen stated and proved three
propositions, which for continuous time models yield the sufficiency
parts for \(n = 1, n = 2\), and the constant interest rate subclass of
\(V_n\) for any \(n\). However, if the interest rates for two periods were
known to be different, and second-order binary time dominance does not hold, the Böhren and Hansen procedure cannot assist in ranking projects whereas Proposition 1 may still be useful. The Pratt and Hammond work on bounding the number of internal rates of return (IRR) essentially required ν to be in $V_n$ for all n. The necessity part of Proposition 1 is new, but it is consistent with the necessary conditions of stochastic dominance theorems.

Table I illustrates how Proposition 1 may be applied to the selection of a depreciation method for tax purposes. As depreciation yields a tax shield, rapid depreciation is generally desirable. Two popular methods of accelerated depreciation are SYD = "sum of the years' digits" and DDB = "double declining balance" [8, chap. 6]. The example assumes an initial outlay of 3,000 to be completely written off in five years with a tax rate of 50 percent. As DDB has the highest initial saving but SYD has the highest cash balances after three and four years, there exist different discounting functions representing impatience such that either SYD or DDB is the preferred method. However, DDB dominates SYD by second-order binary time dominance in this case. Therefore, any decision maker with decreasing impatience will select DDB under the postulated depreciation period. Note how this conclusion was obtained by a simple "back of the envelope" calculation, without using any tables of compound interest.

[Insert Table I about here]

The cash flow $x_0 = (1, -3, 2.5)$ is a prototype often used for demonstrating the nonexistence of an internal rate of return (IRR).
TABLE I
SECOND-ORDER BINARY TIME DOMINANCE

<table>
<thead>
<tr>
<th>Cumulations</th>
<th>Depreciation Method</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>DDB</td>
<td>0</td>
<td>600</td>
<td>360</td>
<td>216</td>
<td>129.6</td>
<td>194.4</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>SYD</td>
<td>0</td>
<td>500</td>
<td>400</td>
<td>300</td>
<td>200</td>
<td>100</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>DDB</td>
<td>0</td>
<td>600</td>
<td>960</td>
<td>1176</td>
<td>1305.6</td>
<td>1500</td>
<td>1500</td>
</tr>
<tr>
<td></td>
<td>SYD</td>
<td>0</td>
<td>500</td>
<td>900</td>
<td>1200</td>
<td>1400</td>
<td>1500</td>
<td>1500</td>
</tr>
<tr>
<td>2</td>
<td>DDB</td>
<td>0</td>
<td>600</td>
<td>1560</td>
<td>2736</td>
<td>4041.6</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>SYD</td>
<td>0</td>
<td>500</td>
<td>1400</td>
<td>2600</td>
<td>4000</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

TABLE II
POSTPONING THE HORIZON FROM THE PROJECT TERMINUS

<table>
<thead>
<tr>
<th>Cumulations</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>-3</td>
<td>2.5</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>-2</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>-1</td>
<td>-0.5</td>
<td>0.0</td>
<td>0.5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>0</td>
<td>-0.5</td>
<td>-0.5</td>
<td>0.0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>1</td>
<td>0.5</td>
<td>0.0</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
[5, p. 79]. Table II shows that $x_1$ dominates $x_0 = (0,0,0)$ by fourth-order binary time dominance. The sufficiency part of Proposition 1 confirms that $x_1$ has in fact no IRR. Hirshleifer [5] stated that there exist "perfectly respectable" interest patterns for which $-x_1 = (-1,3,-2.5)$ would be desirable to the null alternative. The proposition restricts these "respectable" interest rates to discounting functions which do not belong to $V_n$ for $n \geq 4$. Incidentally, Hirshleifer's successive one-period interest rates of 100 percent and 200 percent imply $v(1) = 1/2$ and $v(2) = 1/6$, such that his $v$ does not belong to $V_3$.

Table II also shows that it may be necessary to extend the horizon $H$ beyond the occurrence of the last nonzero element of $x$. As $x_1^2(2) = x_1^3(2) < 0$, no binary time dominance results could be obtained, if the project terminus was used as the horizon. Therefore, the necessity part of Proposition 1 may require the horizon to be sufficiently postponed into the future.

[Insert Table II about here]

III. SET TIME DOMINANCE

Repeated applications of Proposition 1 to weed out inferior projects from the set $S$ yield the $n^{th}$ order binary time dominance efficient set $S^*_n$. It consists of projects which are not dominated by any other project in the set according to $n^{th}$ order binary time dominance. Thus, $S^*_n$ is derived by successive pairwise comparisons. The necessity part of Proposition 1 implies that for any pair of
alternatives \( x_j \) and \( x_1 \) in \( S_n^* \), there exist both some \( v \) in \( V_n \) such that \( \text{NPV}(x_j:v) > \text{NPV}(x_1:v) \) and also different \( v \) in \( V_n \) which reverse the order of preference. It is important to note that even if \( x_j \) belongs to the \( n^{th} \) order binary efficient set \( S_n^* \), there is no presumption that there actually exists any \( v \) in \( V_n \) such that \( x_j \) is the most preferred alternative! If it can be shown that there exists no \( v \) in \( V_n \) such that \( x_j \) has the highest \( \text{NPV} \), then the project may be removed from the binary efficient set without restricting the choice opportunities of interest.

This section will show how the discriminatory power of time dominance relations may be sharpened without requiring additional assumptions. Such a further size reduction of efficient sets may be achieved by replacing a pairwise comparison between individual projects by a comparison between any project and a convex combination of other projects. This approach has been stimulated by Fishburn's [3] work on convex stochastic dominance and Meyer's [9] set stochastic dominance exposition.

As an illustration, let \( S = \{x_1, x_2, x_3\} \). Suppose no alternative can be eliminated based on pairwise comparisons, that is \( S_n^* = S \). Define \( x_4 = \lambda x_1 + (1-\lambda)x_2 \), where \( 0 < \lambda < 1 \). The fact that \( x_4 \) does not belong to \( S \) is unimportant, as it is just an auxiliary construct. Assume \( x_4 \) dominates \( x_3 \) by \( n^{th} \) order binary time dominance. Hence, \( \text{NPV}(x_4:v) > \text{NPV}(x_3:v) \) for all \( v \) belonging to \( V_n \). By the linearity of the \( \text{NPV} \) criterion,
\[ \lambda \text{NPV}(x_1:v) + (1-\lambda)\text{NPV}(x_2:v) > \text{NPV}(x_3:v) \] 

for all \( v \) in \( V_n \). But, if (5) holds, and as \( \lambda > 0 \) and \( (1-\lambda) > 0 \), then either \( \text{NPV}(x_1:v) > \text{NPV}(x_3:v) \) or \( \text{NPV}(x_2:v) > \text{NPV}(x_3:v) \) or both. In neither case will \( x_3 \) be selected by anyone whose discounting function \( v \) belongs to the class \( V_n \), despite the fact that the \( n^{th} \) order binary efficient set \( S_n^* \) contains the project \( x_3 \). Therefore, \( x_3 \) may be removed, to form an optimal \( n^{th} \) order efficient set \( S_n^{**} \) containing as few elements as possible.

To formalize these ideas, let \( S_J \subseteq S \) be a subset of feasible alternatives, and let \( J \) be the index set of alternatives in \( S_J \). Let \( \lambda(J) \) be a vector of nonnegative weights summing to unity. Set time dominance may then be defined as follows: the set \( S_J \) dominates \( x_i \) by \( n^{th} \) order set time dominance, if and only if there exists a vector \( \lambda(J) \) of nonnegative weights summing to unity, such that the convex combination \( \left( \sum_{j \in J} \lambda_j x_j \right) \) dominates \( x_i \) by \( n^{th} \) order binary time dominance.

**PROPOSITION 2:** There exists no \( v \) in \( V_n \) such that \( \text{NPV}(x_i:v) > \text{NPV}(x_j:v) \) for all \( x_j \) in \( S_J \), if and only if \( S_J \) dominates \( x_i \) by \( n^{th} \) order set time dominance.

The proof is deferred to Appendix B.

Elimination of as many alternatives as possible by Proposition 2 leaves the \( n^{th} \) order set time dominance efficient set \( S_n^{**} \). Every element in \( S_n^{**} \) has the property that there exists some \( v \) in \( V_n \) such that this project will indeed be the preferred one. As binary time dominance is a special case of set time dominance, \( S_n^{**} \subseteq S_n^* \). Thus, set time dominance yields a minimum size efficient set. The achieved
size reduction relative to binary time dominance has been obtained without imposing further restrictions.

Table III provides a numerical example with three projects. The cash flows have been selected in such a way that there cannot be any binary time dominance of any order. An inspection of the cash balances \( X^1(t) \) verifies that the \( n \)th order binary efficient set \( S^*_n = \{x_1, x_2, x_3\} \) for any \( n \geq 1 \). Now consider the convex combination \( \lambda x_1 + (1-\lambda)x_2 \). For \( 3/8 \leq \lambda \leq 1/2 \), the conditions \(-2+8\lambda \geq 1\), \( 6-8\lambda \geq 2 \), and \( 4-2\lambda \geq 3 \) are all satisfied, with at least one strong inequality. Hence, \( \{x_1, x_2\} \) dominates \( x_3 \) by first-order set time dominance. Any impatient decision maker will select either \( x_1 \) or \( x_2 \), but not the inferior \( x_3 \). Although \( x_3 \) is the only project which is guaranteed to have a positive NPV, it is also the only one which does not belong to any \( n \)th order set efficient set \( S^*_n \)!

[Insert Table III about here]

Böhren and Hansen [1] introduced a concept called normalized time dominance. It involved dividing through the cash flows by the assumed positive respective cash balances at the horizon, yielding, e.g., \( \hat{x}_j(t) = x_j(t)/X_j^1(H) \). Letting \( x_0 = (0,0,\ldots,0) \) be the null alternative, their approach is equivalent to an examination of whether the convex combination \( \{\lambda x_j + (1-\lambda)x_0\} \) dominates \( x_j \) by \( n \)th order binary time dominance, subject to

\[
X_j^1(H) \geq X_j^1(H) > 0 \quad (6a)
\]

\[
\lambda = X_j^1(H)/X_j^1(H) \quad (6b)
\]

\( x_0 \) belongs to \( S \) \quad (6c)
### TABLE III
FIRST-ORDER SET TIME DOMINANCE, NO BINARY, AND NO
NORMALIZED TIME DOMINANCE OF ANY ORDER

<table>
<thead>
<tr>
<th>Cumulations k</th>
<th>Project i</th>
<th>Time t=0</th>
<th>t=0</th>
<th>t=2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>x₁</td>
<td>6</td>
<td>-8</td>
<td>4</td>
</tr>
<tr>
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<td>x₂</td>
<td>-2</td>
<td>8</td>
<td>-2</td>
</tr>
<tr>
<td></td>
<td>x₃</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>λx₁+(1-λ)x₂</td>
<td>-2+8λ</td>
<td>8-16λ</td>
<td>-2+6λ</td>
</tr>
<tr>
<td>1</td>
<td>x₁</td>
<td>6</td>
<td>-2</td>
<td>2</td>
</tr>
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<td></td>
<td>x₂</td>
<td>-2</td>
<td>6</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>x₃</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>λx₁+(1-λ)x₂</td>
<td>-2+8λ</td>
<td>6-8λ</td>
<td>4-2λ</td>
</tr>
<tr>
<td>1</td>
<td>³x₁</td>
<td>3</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>³x₂</td>
<td>-1/2</td>
<td>3/2</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>³x₃</td>
<td>1/3</td>
<td>2/3</td>
<td>1</td>
</tr>
</tbody>
</table>
The restrictions (6a) and (6b) are redundant, and in a general setting 
(6c) does not necessarily hold.

From Table III it was seen how binary time dominance failed to 
eliminate an obviously inferior alternative, whereas first-order set 
time dominance succeeded. Inspection of the bottom part of Table III 
will show also that normalized time dominance of any order will fail 
to obtain.

If attention is limited to a proper subclass of $V_n$, then $n$th 
order set time dominance is no longer a necessary condition for infe-
iority. Table IV presents an example. Assume a constant but unspeci-
fied interest rate $R$, such that $v(t) = (1+R)^{-t}$, which is contained 
in $V_n$ for any $n$. By plotting the NPV profiles, or from inspection of 
the IRRs of the four projects themselves, as well as of the differential 
projects, it may be concluded that $x_2$ is selected if $0\% \leq R \leq 9.5\%$, 
$x_3$ is best for $9.5\% < R \leq 16.2\%$, where $x_1$ is superior for $R > 16.2\%$. 
Hence, $x_4$ is never a preferred project under the stated constant 
interest rate assumption. Still, $x_4$ cannot be eliminated on the 
basis of $n$th order set time dominance for any $n$. To see why, let 
$\lambda_1$, $\lambda_2$, and $\lambda_3$ be nonnegative weights associated with $x_1$, $x_2$, and $x_3$. 
The initial cash flow condition is

$$0.0\lambda_1 - 3.0\lambda_2 - 2.0\lambda_3 \geq -1.$$  \hfill (7a)

The final cash balance condition requires

$$0.1\lambda_1 + 1.0\lambda_2 + 0.8\lambda_3 \geq 0.5.$$  \hfill (7b)

Finally, the weights should sum to unity:
<table>
<thead>
<tr>
<th>Cumulations</th>
<th>Project</th>
<th>Time</th>
<th>IRR</th>
<th>NPV(x:V)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>t=0</td>
<td>t=1</td>
<td>t=2</td>
</tr>
<tr>
<td>0</td>
<td>x₁</td>
<td>0.0</td>
<td>0.0</td>
<td>0.1</td>
</tr>
<tr>
<td></td>
<td>x₂</td>
<td>-3.0</td>
<td>0.0</td>
<td>4.0</td>
</tr>
<tr>
<td></td>
<td>x₃</td>
<td>-2.0</td>
<td>0.0</td>
<td>2.8</td>
</tr>
<tr>
<td></td>
<td>x₄</td>
<td>-1.0</td>
<td>-4.0</td>
<td>5.5</td>
</tr>
<tr>
<td>0</td>
<td>x₂₋x₁</td>
<td>-3.0</td>
<td>0.0</td>
<td>3.9</td>
</tr>
<tr>
<td></td>
<td>x₂₋x₃</td>
<td>-1.0</td>
<td>0.0</td>
<td>1.2</td>
</tr>
<tr>
<td></td>
<td>x₂₋x₄</td>
<td>-2.0</td>
<td>4.0</td>
<td>-1.5</td>
</tr>
<tr>
<td></td>
<td>x₃₋x₁</td>
<td>-2.0</td>
<td>0.0</td>
<td>2.7</td>
</tr>
<tr>
<td>1</td>
<td>x₁</td>
<td>0.0</td>
<td>0.0</td>
<td>0.1</td>
</tr>
<tr>
<td></td>
<td>x₂</td>
<td>-3.0</td>
<td>-3.0</td>
<td>1.0</td>
</tr>
<tr>
<td></td>
<td>x₃</td>
<td>-2.0</td>
<td>-2.0</td>
<td>0.8</td>
</tr>
<tr>
<td></td>
<td>x₄</td>
<td>-1.0</td>
<td>-5.0</td>
<td>0.5</td>
</tr>
</tbody>
</table>
\[ \lambda_1 + \lambda_2 + \lambda_3 = 1.0. \] (7c)

As (7a), (7b), and (7c) cannot all be simultaneously satisfied, set time dominance of any order is precluded.

[Insert Table IV about here]

On the other hand, assume \( \bar{v}(0) = 1.000 \), \( \bar{v}(1) = 0.755 \), and \( \bar{v}(t) = 0.755(0.750/0.755)^{t-1} \) for \( t > 1 \). This discounting function belongs to \( \mathbb{V}_n \) for any \( n \), as it corresponds to a shift from a high single-period interest rate to a lower one after one period. Out of the four projects whose cash flows are given in Table IV, project \( x_4 \) does, in fact, have the highest NPV for this latter discounting function \( \bar{v} \).

IV. AN EXTENSIVE EXAMPLE

Weingartner [13, p. 180] specified the cash flows of thirty hypothetical investment projects, covering a time span of twenty-six periods. These cash flows were quite diverse: the projects' starting dates varied from year one to year twelve, the project lives ranged between six and twenty-six years, a maximum of six sign shifts occurred in the uncumulated flows, and investment costs of a large project could be about ten times as large as those of a small project. For convenience, the data base has been reproduced in Appendix C.

Table V presents the time dominance efficient sets applicable to Weingartner's comprehensive data. Postulating only impatience reduces the number of candidates to nineteen based on pairwise comparisons and
TABLE V
TIME DOMINANCE EFFICIENT SETS, WEINGARTNER'S DATA

<table>
<thead>
<tr>
<th>Order n</th>
<th>Efficient Set $S^*_n$, Binary Time Dominance</th>
<th>Efficient Set $S^{**}_n$, Set Time Dominance</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Number</td>
<td>Projects Included</td>
</tr>
<tr>
<td>1</td>
<td>19</td>
<td>{J,1,10,13,14,16,17,22,24,25,27}</td>
</tr>
<tr>
<td>2</td>
<td>18</td>
<td>{J,1,13,14,16,17,22,24,25,27}</td>
</tr>
<tr>
<td>3</td>
<td>13</td>
<td>{J,13,14,17,24}</td>
</tr>
<tr>
<td>4</td>
<td>12</td>
<td>{J,14,17,24}</td>
</tr>
<tr>
<td>5</td>
<td>12</td>
<td>{J,14,17,24}</td>
</tr>
<tr>
<td>6</td>
<td>12</td>
<td>{J,14,17,24}</td>
</tr>
<tr>
<td>7</td>
<td>12</td>
<td>{J,14,17,24}</td>
</tr>
<tr>
<td>8</td>
<td>12</td>
<td>{J,14,17,24}</td>
</tr>
</tbody>
</table>

\(J = \{2,5,6,7,8,9,15,21,23\}\) contains the indices of the core projects.
to sixteen using setwise relations. Adding a convexity assumption, the corresponding number are, respectively, eighteen and thirteen projects. As more preference restrictions are successively included, only twelve projects remain in the binary time dominance efficient set and nine projects in the set time dominance efficient set, the latter one forming a core from which the ultimate choice will be made.

[Insert Table V about here]

Note that in this example rejection of all projects was not an admissible choice. If, additionally, the null alternative \( x_0 = (0,0,\ldots,0) \) is included in the choice set \( S \), the following projects are removed from the set time dominance efficient set \( S_{n}^{**}: x_{25}^{n} \) for \( n=1 \), \( x_{27}^{n} \) for \( n=2 \), and \( x_{21}^{n} \) for \( n=8 \). One may speculate that an addition of a couple of financing projects, i.e., projects for which cash inflows are followed by cash outflows, would have reduced the size of the efficient set even further.

V. CONCLUSIONS

Building on the stochastic dominance framework, the time dominance efficiency analysis provides similar rules for a partial ordering of decision alternatives whose consequences are distributed over time. Without requiring any quantitative information about temporal preferences, qualitative restrictions on discounting functions may suffice for screening projects according to their NPVs. Two propositions permit elimination of projects which will be considered inferior by any
decision maker whose temporal preferences exhibit particular characteristics. Properties of repeatedly cumulated (or integrated) consequences over time are crucial for such purging to be feasible.

Based on pairwise comparisons between individual temporal prospects, the binary time dominance proposition extends and supplements recent related results [1] [10]. The paper's main contribution is the development of set time dominance, drawing on recent advances in set (or convex) stochastic dominance [3] [9]. Set time dominance compares individual projects to sets of alternative ones to exclude binary undominated projects which will not be the most preferred by any decision maker. It builds on the simple observation that if the NPV of a convex combination of projects exceeds the NPV of a single alternative, then at least one individual project in the combination has a higher NPV than the alternative one. The set time dominance proposition may reduce the size of the efficient set of prospects without imposing further preference restrictions. Time dominance, in general, and set time dominance, in particular, provide convenient and powerful tools for reducing the number of candidate decision alternatives requiring a detailed temporal preference evaluation, as illustrated by the application to the investment project data [13].
REFERENCES


APPENDICES

A. Outline of Proof of Proposition 1

Using cash balances $X(t) - x(t)$ rather than cash flows $x(t)$, the NPV can be written as $NPV(x;v) = \int_0^H v(t) dX^1(t)$ in continuous time. From integration by parts $n$ times, assuming $X^k(0) = 0$, the NPV difference is given by

$$NPV(x_j;v) - NPV(x_i;v) = \sum_{k=1}^n \left[ (-1)^{k-1} v^{k-1}(H) [x_j^k(H) - x_i^k(H)] - \int_0^H (-1)^k v^n(t) [x_j^n(t) - x_i^n(t)] dt \right]$$

(8)

Combining (2), (3a), and (3b) shows that the difference is positive. Hence, if $x_j$ dominates $x_i$ by $n^{th}$ order binary time dominance, then $NPV(x_j;v) > NPV(x_i;v)$ for all $v$ in $V_n$.

To show necessity, one proceeds by example, building on ideas from the stochastic dominance theorems [4, pp. 72-77].

Consider the discounting function

$$v(t) = H^{1-k}(H-t)^{k-1}[(k-1)!]^{-1} + pe^{-pt}$$

(9)

which belongs to $V_k$ for any $1 \leq k \leq n$. Hence,

$$NPV(x;v) = H^{1-k} \int_0^H (H-t)^{k-1}[(k-1)!]^{-1} dX^1(t) + p \int_0^H e^{-pt} dX^1(t).$$

As $X^k(H) = \int_0^H X^{k-1}(t) dt$, performing $(k-1)$ integrations by parts of the
left integration term shows that \( \text{NPV}(x:v) = \mathbb{H}^{1-k}x^k(H) + p\text{NPV}(x:e^{-pt}) \). Consequently,

\[
\text{NPV}(x_j:v) - \text{NPV}(x_1:v) = \mathbb{H}^{1-k}[x^k_j(H) - x^k_1(H)] + o(p) .
\] (10)

Therefore, as \( p \to 0 \), \( x^k_j(H) - x^k_1(H) < 0 \Rightarrow \text{NPV}(x_j:v) < \text{NPV}(x_1:v) \).

Hence (3a) is a necessary condition for \( \text{NPV}(x_j:v) > \text{NPV}(x_1:v) \) for all \( v \) in \( V_n \).

Proving the necessity of (3b) is done similarly. Assume \( x^n_j(\tau) < x^n_1(\tau) \) for some \( \tau \) satisfying \( 0 < \tau < H \). Let the discounting function be \( v(t) = w(t) + pe^{-pt} \), where

\[
w(t) = \begin{cases} 
\tau^{1-n}(\tau-t)^{n-1}[(n-1)!]^{-1} & \text{if } 0 \leq t < \tau \\
0 & \text{if } \tau < t \leq H 
\end{cases}
\] (11)

Thus, \( \text{NPV}(x:v) = \tau^{1-n} \int_0^\tau (\tau-t)^{n-1}[(n-1)!]^{-1}dX^1(t) + p \int_0^H e^{-pt}dX^1(t) \).

After \( (n-1) \) partial integrations of the \( w(t)dX^1(t) \) term,

\( \text{NPV}(x:v) = \tau^{1-n}x^n(\tau) + p\text{NPV}(x:e^{-pt}) \). As before, \( p\text{NPV}(x:e^{-pt}) \) approaches zero when \( p \) goes to zero, and

\[
\text{NPV}(x_j:v) - \text{NPV}(x_1:v) = \tau^{1-n}[x^n_j(\tau) - x^n_1(\tau)] + o(p)
\] (12)

For sufficiently small \( p \), \( x^n_j(\tau) < x^n_1(\tau) \Rightarrow \text{NPV}(x_j:v) < \text{NPV}(x_1:v) \), showing the necessity of (3b).

In a discrete time framework, integrations by parts are replaced by summations by parts, also known as Abel's summation identity [4, p. 85].
### TABLE VI
CASH FLOWS OF THIRTY INVESTMENT PROJECTS

| Project | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 |
|---------|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| 1 | -100 | 20 | 20 | 20 | 19 | 19 | 18 | 16 | 14 | 11 | 6  | -8 |
| 2 | -100 | 20 | 18 | 18 | 18 | 18 | 14 | 14 | 14 | 10 | 10 | 10 | 10 | 10 | 6  | 6  | 6  | 6  | 6  | 6  | 6  | 6  | 6  | 6  | 6  |
| 3 | -100 | 15 | 15 | 15 | 15 | 15 | 13 | 13 | 13 | 11 | 11 | 11 | 11 | 11 | 9  | 9  | 9  | 9  | 9  | 9  | 9  | 9  | 9  | 9  | 9  |
| 4 | -100 | 20 | 6  | 11 | 7  | 16 | 5  | 14 | 18 | 3  | 20 | 2  | 22 | 8  | 10 | 18 | 6  | 9  | 14 | 24 |
| 5 | -100 | -60 | -60 | 80 | 74 | 66 | 56 | 33 | 30 | 14 |
| 7 | -80 | 20 | 20 | 20 | 19 | 17 | 14 | 10 | 6  | 2  |
| 8 | -60 | -30 | -10 | 45 | 34 | 25 | 16 | 12 | 8  | -20 | 21 | 16 | 12 | 9  | 7  | 5  | 3  |
| 9 | -120 | 25 | 25 | 20 | 30 | 30 | 25 | 20 | 15 | 10 | 5  |
| 10 | -100 | 18 | 17 | 15 | 12 | 8  | -10 | 18 | 17 | 15 | 12 | 8  |
| 11 | -150 | 20 | 20 | 20 | 20 | 20 | 20 | 20 | 20 | 20 |
| 12 | -100 | 20 | 18 | 16 | 14 | 12 | 10 | 4  | -20 | 20 | 18 | 16 | 14 | 12 | 10 | 4  |
| 13 | -150 | -75 | -75 | 60 | 60 | 55 | 50 | 44 | 38 | 36 | 35 | 34 | 33 | 30 | 25 | 17 | 9  |
| 14 | -50 | -100 | -175 | 50 | 35 | 60 | 60 | 50 | 40 | 30 | 20 | 10 | -25 | 50 | 41 | 35 | 25 | 15 | 9  |
| 15 | -100 | -150 | -100 | 10 | 20 | 30 | 40 | 60 | 60 | 60 | 60 | 60 | 60 | 60 | 60 | 60 | 60 | 60 |
| 16 | -95 | -60 | -47 | -42 | -37 | -31 | -24 | -18 | -13 | 9  | 6  | 4  | 3  |
| 17 | -175 | 50 | 45 | 45 | 35 | 25 | 10 | -60 | 45 | 35 | 25 | 10 |
| 18 | -250 | 45 | 45 | 40 | 30 | 25 | 20 | 15 | 10 | -60 | 40 | 32 | 25 | 19 | 14 | 10 | 7  | 5  |
| 19 | -75 | -75 | -40 | 40 | 35 | 35 | 35 | 30 | 25 | 15 | 5  |
| 20 | -180 | 20 | 12 | 16 | 13 | 11 | 19 | 17 | 12 | 15 | 19 | 13 | 14 | 17 | 20 | 14 | 11 | 15 | 17 | 12 |
| 21 | -85 | 20 | 20 | 16 | 15 | 13 | 10 | 7  | 3  |
| 22 | -270 | -100 | 125 | 115 | 105 | 80 | 60 | 35 | 25 | 15 | 10 |
| 23 | -40 | 15 | 13 | 9  | 7  | 5  | 2  |
| 24 | -50 | 10 | 10 | 9  | 7  | 4  | -14 | 9  | 9  | 8  | 6  | 3  | -16 | 8  | 8  | 4  |
| 25 | -200 | 60 | 40 | 30 | 15 | -25 | -25 | 50 | 40 | 30 | 20 | 10 |
| 26 | -70 | 15 | 13 | 11 | 10 | 9  | 7  | 6  | 6  | 3  | 2  |
| 27 | -355 | 60 | 70 | 80 | 70 | 55 | 40 | 25 | 15 | 5  |
| 28 | -275 | 60 | 45 | 45 | 40 | 35 | 30 | 25 | 20 | 15 | -75 | 35 | 30 | 25 | 20 | 15 | 10 | 5  |
| 29 | -140 | 20 | 20 | 18 | 16 | 14 | 11 | 8  | -25 | 18 | 18 | 16 | 13 | 10 | 6  | -25 | 16 | 16 | 14 | 11 | 8  | 5  | 2  |

SOURCE: [13, p. 180].
In the necessity part, the function \( w(t) \) of (11) is then redefined as

\[
\begin{cases}
\left( \frac{t+n-1}{\tau} \right)^{-1} \left( \frac{\tau-t+n-1}{\tau-t} \right) & \text{if } t \leq \tau \\
0 & \text{if } \tau < t \leq H
\end{cases}
\]  

(13)

with similar changes in (9).

B. Outline of Proof of Proposition 2

If \( S_J \) dominates \( x_i \) by \( n \)th order set time dominance, then by definition there is some convex combination \( \left( \sum_{j \in J} \lambda_j x_{j} \right) \) which dominates \( x_i \) by \( n \)th order binary time dominance. Proposition 1 yields

\[
NPV(\sum_{j \in J} \lambda_j x_j; v) > NPV(x_i; v) \quad \text{for all } v \text{ in } V_n.
\]

As the NPV of a convex combination equals the convex combination of the NPVs,

\[
\sum_{j \in J} \lambda_j NPV(x_j; v) > NPV(x_i; v) \quad \text{for all } v \text{ in } V_n
\]

(14)

Because the weights \( \lambda_j \) are nonnegative and summing to unity, for any particular \( v \) in \( V_n \) there must be at least one \( x_j \) in \( S_J \) for which \( NPV(x_j; v) > NPV(x_i; v) \), which proves sufficiency. Necessity is immediate from Proposition 1.