Option Pricing with Different Interest Rates for Borrowing and for Lending

by

Yaacov Z. Bergman
RESEARCH PROGRAM IN FINANCE AT THE
WALTER A. HAAS SCHOOL OF BUSINESS,
UNIVERSITY OF CALIFORNIA, BERKELEY

The Research Program in Finance in the Walter A. Haas School of Business at the University of California has as its purpose the conduct and encouragement of research in finance, investments, banking, securities markets, and financial institutions. The present reprint and working paper series were established in 1971 in conjunction with a grant from the Dean Witter Foundation.

INSTITUTE OF BUSINESS AND ECONOMIC RESEARCH
Carl Shapiro, Director

The Institute of Business and Economic Research is an organized research unit at the University of California, Berkeley, whose mission is to promote research by faculty and graduate students in the fields of business and economics. The Institute carries out its mission by organizing programs and activities that enrich the research environment, administering extramural research awards, publishing working papers, and making direct grants for research.
OPTION PRICING WITH DIFFERENT INTEREST RATES FOR BORROWING AND FOR LENDING

by

Yaacov Z. Bergman

March 1981

Working Paper No. 109
1. Summary and Conclusions

While the body of knowledge on option pricing has very rapidly expanded in the last few years, it was predominantly confined to perfect-market environments. The subject of option pricing under imperfect market conditions has hardly been touched. This paper deals with the effects of a particular market imperfection upon option pricing, namely, a market with different interest rates for borrowing and for lending. The results herein may also shed some light on what is to be expected under other imperfect market conditions.

It is common knowledge within the profession that, even though a simple option (stock call or put option) has a unique price under perfect market and no arbitrage conditions, this no longer holds when interest rates for lending and for borrowing differ. Rather, the price of a call option can be bounded within what we call here a contingent band. The upper bound of this price is given as the maximum amount a buyer is willing to pay for the call option and is computed by solving the Black and Scholes (1973) partial differential equation with the interest rate for borrowing; the lower bound of which is given by the minimum price a call writer is willing to accept and is computed using the lending interest rate. This is so because a buyer of a call option can exactly duplicate the position by holding a portfolio of the underlying stocks long and borrowing, while the writer of a call
can exactly duplicate his position by a portfolio of the underlying stocks held short and by lending.

Turning now to portfolios of simple options, one can easily prove that in a perfect market the unique price of such a portfolio can be computed from the prices of the individual options in the portfolio. One of the main results of this paper is that this no longer holds in a market with different interest rates for borrowing and for lending. While one can still use the contingent bands for the individual options to derive a contingent band for the portfolio, the result would be a highly exaggerated band width. It is shown that the band can be drastically narrowed by directly solving for the portfolio of stocks and bonds which duplicates the portfolio of options. The reason is that the individual options may entail substantial borrowing or lending in their corresponding duplicating portfolios; when combining them in a portfolio and forming the corresponding duplicating portfolio for the combination, borrowing and lending should be offset against each other, and the resultant portfolio would cost much less than the simple combination of the components. This is a direct result of two different interest rates, since with a unique interest rate there is no need for offsetting borrowing against lending.

The implications of this argument reach beyond the effects on the contingent band of a portfolio of options, since a narrower band for the portfolio forces the individual options to bound each other in bands still narrower than the original ones. Moreover, it is shown
from the above argument that the narrowest band for a given option
depends on the observed prices for all other options available in the
market.

An additional result concerns pricing of portfolios of options,
each with a different underlying asset. Although correlations between
the underlying assets' prices do not play any role in pricing the
portfolio in a perfect market environment, this is no longer true when
interest rates for borrowing and for lending differ. This market im-
perfection is shown to bring forth the cross correlations that appear
explicitly in the pricing formulae.

The outline of the paper is as follows: sections 2 and 3 dis-
cuss the derivation of the duplicating portfolio. The duplicating
portfolio for a simple call option in a perfect market, and a Geometric
Brownian Motion for the underlying asset's price, is solved for in
section 4. Section 5 introduces different interest rates for lending
and for borrowing for this simple case; section 6 does the same for a
money spread of call options using it to motivate the general treatment
which follows. Section 7 treats the generalized option on one under-
lying asset with two different interest rates. A partial differential
equation (PDE) with an interest rate "switching" function is derived
for the value of the duplicating portfolio. The function switches be-
tween the two rates according to whether borrowing or lending is implied
by the duplicating portfolio. Section 8 provides a numerical solution
to this PDE as an example that serves to illustrate the points made
in this paper and to demonstrate the significant implications to
options investment strategies. Section 9 concludes with the discussion of pricing of a portfolio of options on different underlying assets and different expiration dates; it derives the appropriate PDE and provides a solution procedure for this general case.

2. The Traditional Derivation of the Black and Scholes Equation

But these two errors—it is like a miracle—cancel out in the most precise manner, as I shall prove further down. [From Johannes Kepler's account of discovery of his Second Law in *Astronomia Nova*, 1609.]

Some of the most important scientific discoveries were not hampered by technical errors done en route, and the Black and Scholes seminal Option Pricing Model may be counted among this honorable number.

To point out the mathematical error in the traditional derivation of the option pricing partial differential equation, use will be made of C. W. Smith's (1975) exposition of it in his excellent review paper.

A hedge portfolio is formed by holding one share of the underlying stock long and writing \( \frac{1}{\partial C / \partial S} \) calls, it is then claimed to be riskless. It will now be shown that this claim is incorrect.

The value of this hedge portfolio is given by

\[
V_H = S - \left( \frac{1}{C_S} \right) C
\]

where \( C_S \) denotes partial derivative w.r.t. \( S \).

By Ito's lemma:

\[
dV_H = dS - \left( \frac{1}{C_S} \right) dC - Cd \left( \frac{1}{C_S} \right) - d \left( \frac{1}{C_S} \right) dC. \]
But by the same lemma:

\[ \frac{dC}{C_s} = \frac{\partial C}{\partial S} dS + O(dt) \]  \hspace{1cm} (3)

where \( O(dt) \) means: of the order of \( dt \).

And:

\[ d \left( \frac{1}{C_s} \right) = -\frac{C_{SS}}{(C_s)^2} dS + O(dt) . \]  \hspace{1cm} (4)

Substituting (3) and (4) into (2) gives:

\[ dV_H = \frac{C^*C_{SS}}{(C_s)^2} dS + O(dt) . \]  \hspace{1cm} (5)

Since \( C^*C_{SS} \) is not identically zero for the whole lifetime of the option, it obviously follows from (5) that the return to the hedge portfolio is indeed stochastic and not riskless as has been claimed.

The origin of this error stems from ignoring the last two terms on the RHS of equation (2). In trying to justify this, a heuristic argument is called upon to the effect that in maintaining the hedge portfolio one chooses to hold \(-\frac{1}{C_s}\) calls at the beginning of each instant and then skillfully manages to hold this number constant over the extent of the whole instant, and, hence, \( d \left( \frac{1}{C_s} \right) = 0 \). This heuristic argument is, unfortunately, incorrect since Ito's lemma tells us otherwise, as calculated in (4) above. It is also reassuring to note that we
are really not in need of the above incorrect argument since the derivation of the Black and Scholes equation goes through even with strict adherence to the principles of mathematics as is shown in the following section.

3. The Duplicating Portfolio

The approach taken in the present section parallels that of Merton (1977) but adds some novel aspects. The interested reader is also referred to a rigorous treatment of this and related subjects by J. M. Harrison and S. R. Pliska (1981). The idea is to construct a portfolio of securities whose prices are known and to identify a feasible trading strategy for this portfolio that will cause it to generate exactly the same returns as the option to be priced. Then, if arbitrage opportunities are not to be present, the price of the option should equal that of the duplicating portfolio throughout the lifetime of the former. Specifically, the duplicating portfolio will be constructed of positions in the underlying asset (stock) and riskless bonds. The assumptions employed are the standard:

(A1) Perfect capital markets: no transaction costs or taxes. Borrowing and shortselling are allowed.

(A2) Borrowing and lending are carried out at a unique constant interest rate \( r \) by holding short or long positions in bonds. Without loss of generality, the price of one unit of bonds is normalized to equal 1 at a chosen time origin \( t=0 \). Hence, the price dynamics for a bond unit is given by
\[ B(t) = e^{rt} \]  

(A3) There is a risky asset (stock) whose value \( S(t) \) follows a diffusion process formally described by the stochastic differential equation:

\[ dS = \mu_S S dt + \sigma_S S dZ \]

where \( \mu_S, \sigma_S^2 \) are the expectation and variance per unit time of the rate of return, both may be functions of \( S \) and \( t \); \( dZ \) is a standard Wiener process. Although not essential, it is assumed for simplicity that the stock does not pay dividends.

(A4) There is another asset (an option) which promises to pay at a specified future date \( T \) (expiration) an amount \( h(S(T)) \), which is totally determined by the value of the underlying asset (stock) at that date. Although not essential, it is assumed that the option does not pay dividends.

(A5) Trade in the three assets—stocks, bonds, and options—can take place continuously in time.

(A6) Investors prefer more to less. They agree on \( \sigma_S^2 \) but not necessarily on \( \mu_S \).

In order to concentrate on the issues taken up in this article, only European options will be discussed.

Our aim is now to form a duplicating portfolio consisting of \( \alpha \) shares of stock and \( \beta \) units of bonds. We will try to vary \( \alpha \) and \( \beta \) with
the stock price and advancing time in such a way so that the portfolio will exactly duplicate the returns to the option. This amounts to a two-fold requirement. First, the value of the duplicating portfolio must end up equal to the prespecified function of the stock price at expiration. Second, since the option does not generate any returns prior to expiration nor does it require any payments, the same should be the case with the duplicating portfolio. By construction, the value of the duplicating portfolio must then be equal to the price of the option which is duplicated, if arbitrage opportunities are not allowed.

Formally, if we denote the value of the portfolio at time $t$ by:  

$$V(S,t) = \alpha(S,t)S + \beta(S,t)B(t)$$  

Then the first requirement is: $V(T) = h(S(T))$.

For the second, using Ito's lemma, the change in the portfolio's value is given by:

$$dV = \alpha dS + \beta dB + [(d\alpha)S + (d\beta)B + (d\alpha)(dS) + (d\beta)(dB)]$$

Along the argument by Merton (1971), one can identify the first two terms on the RHS as the change in the portfolio's value arising only from changes in prices of the stock and bond, while the other terms are identified with changes arising from withdrawals from and external additions to the portfolio, which, in our case, must be zero.
This gives:

\[(d\alpha)S + (d\beta)B + (d\alpha)(dS) + (d\beta)(dB) = 0\]  

(9)

and from (8):

\[dV = \alpha dS + \beta dB\]  

(10)

These two equations are the so-called "budget constraint" where no withdrawals or additions are made.

Note that equation (10) arises not from the incorrect assumption of \(d\alpha = d\beta = 0\) mentioned at the end of section 2, but from the requirement that the duplicating portfolio should obey the budget constraint.

It will now be shown that equation (9) gives rise to a system of two partial differential equations in the unknown function \(\alpha, \beta\) with arguments \(S, t\). By Ito's lemma,

\[d\alpha = \mu_{\alpha} dt + \sigma_{\alpha} dZ\]

\[d\beta = \mu_{\beta} dt + \sigma_{\beta} dZ\]

where

\[\mu_{\alpha} = \frac{\partial}{\partial t} + \frac{\partial}{\partial S} \mu_{\alpha} S + \frac{1}{2} \frac{\partial^2}{\partial S^2} \sigma_{\alpha} S^2\]

\[\sigma_{\alpha} = \frac{\partial}{\partial S} \sigma_{\alpha} S\]

with analogous expressions for \(\mu_{\beta}, \sigma_{\beta}\). From (6): \(dB = rB dt\).
Substituting for \(da, dB, dS, d\beta\) in (10) using the multiplication rule \(dz^2 = dt\) and ignoring \(o(dt)\) terms, we get:

\[
(m_\alpha S + m_\beta B + \sigma_\alpha S)dt + (m_\alpha S + \sigma_\beta B)dz = 0
\]  
(11)

In order to maintain this identity to zero for all \(t\) and \(S\) we must equate the deterministic and the stochastic parts of (11) each to zero:

\[
\sigma_\alpha S + \sigma_\beta B = 0 \tag{12a}
\]

\[
m_\alpha S + m_\beta B + \sigma_\alpha S = 0 \tag{12b}
\]

Substituting in (12) for \(m_\alpha, m_\beta, \sigma_\alpha, \sigma_\beta\) and for \(B\) we get a system of two parabolic PDEs.

\[
S \frac{\partial S}{\partial S} + e^{rt} \frac{\partial B}{\partial S} = 0 \tag{13a}
\]

\[
S \left[ \frac{\partial S}{\partial t} + m_\beta S \frac{\partial S}{\partial S} + \frac{1}{2} \sigma_\beta^2 S^2 \frac{\partial^2 S}{\partial S^2} \right] + \sigma_\beta^2 S^2 \frac{\partial S}{\partial S} +
+ e^{rt} \left[ \frac{\partial B}{\partial t} + m_\beta S \frac{\partial B}{\partial S} + \frac{1}{2} \sigma_\beta^2 S^2 \frac{\partial^2 B}{\partial S^2} \right] = 0 \tag{13b}
\]

in the domain \(0 < t < T, 0 < S\) and with boundary condition:

\[
V(S(T), T) = \alpha(S(T), T)S(T) + \beta(S(T), T)e^{rt} = h(S(T))
\]

where \(h(S(T))\) is the contracted dollar return to the option at expiration \(T\). Fortunately, system (13) can be condensed into a single PDE in the value of the duplicating portfolio. Let us make an educated substitution:
\[ \alpha(S, T) = \frac{\partial V}{\partial S} \quad (14) \]

Then from (7):

\[ \beta(S, t) = e^{-rt} \left[ V - S \frac{\partial V}{\partial S} \right] \quad (15) \]

Substituting these \( \alpha, \beta \) in (13a) results in \( \theta = 0 \), but substituting in (13b) and after some algebra we happily get:

\[ \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = r \left[ V - S \frac{\partial V}{\partial S} \right] \quad 0 < t < T, \quad 0 < S \quad (16) \]

which is the celebrated Black-Scholes PDE for the value of the option and which together with the boundary condition

\[ V(S(T), T) = h(S(T), T) \]

has a unique solution. Substituting this solution back into (14) (15) yields the unique solutions for \( \alpha \) and \( \beta \).

From (15) \( \left[ V - S \frac{\partial V}{\partial S} \right] \) is the dollar amount invested in bonds in the duplicating portfolio. Hence, the R.H.S. of (16), after being formally multiplied by \( dt \), may be interpreted as the instantaneous pure interest return to the bond component. Note, however, that the instantaneous change in the dollar amount invested in bonds is not equal to the instantaneous pure interest payment and, in fact, using Ito's lemma, is given by:

\[ d \left( V - S \frac{\partial V}{\partial S} \right) = dV - \frac{\partial V}{\partial S} dS - \sigma \sqrt{S} dW \left( \frac{\partial V}{\partial S} \right) - \left[ \sigma \sqrt{S} dW \left( \frac{\partial V}{\partial S} \right) \right] \]

\[ = \left[ \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right] dt + \left[ \sigma \sqrt{S} dW \left( \frac{\partial V}{\partial S} \right) \right] \quad (17) \]
The change thus consists of two parts. The first bracketed expression on the R.H.S. of (17) can be identified by the help of (16) as the instantaneous pure interest payment to the bonds, whereas the second bracketed expression is the change in the dollar amount held in bonds as the consequence of the decision to follow the active investment strategy as prescribed by (14), (15).

Note also that the portfolio which holds one option long and \( \frac{3V}{2S} \) stocks short does not obey the budget constraint. This can be seen from the fact that the second bracketed expression in (17) is not identically zero. This implies external stochastic nonprofit payments in and out of this portfolio which certainly cannot be interpreted in the traditional spirit as simulating a risk free return.

4. A Particular Explicit Solution for the Duplicating Portfolio

It is instructive to consider an explicit solution for a particular case. Specifically, assume in addition to assumptions (A1) through (A6) of section 3 that the stock price follows a Geometric Brownian Motion and that the contractual agreement for the option is that of a simple call option, namely:

\[
h(S(T)) = \max[S(T) - K, 0]
\]

where \( K \) is the exercise price.

The solution to the Black-Scholes PDE (16) under the above assumption is well known and is given by the Black and Scholes call option pricing formula.
\[ V(S,t) = S \cdot N(d_1) - e^{-r(T-t)} K \cdot N(d_2) \]

where \( N(*) \) is the cumulative normal distribution function, 

\[ d_1 = \frac{\ln S/K + \left( r + \frac{1}{2} \sigma_S^2 \right) (T-t)}{\sigma_S \sqrt{T-t}} \]  \hspace{1cm} (20) 

\[ d_2 = \frac{\ln S/K + \left( r - \frac{1}{2} \sigma_S^2 \right) (T-t)}{\sigma_S \sqrt{T-t}} \]  \hspace{1cm} (21) 

Substituting this solution for \( V(S,t) \) in (14), (15), we get the solution for the number of shares of stocks and of bonds held in the duplicating portfolio at each point in time and for any stock price level:

\[ \alpha(S,t) = N(d_1) \]

\[ \beta(S,t) = - (e^{-rT} \cdot K \cdot N(d_2)) \]  \hspace{1cm} (22) 

It is easy to see that at expiration time \( (t=T) \) the duplicating portfolio satisfies the necessary boundary condition (18) since

\[ N(d_1) = N(d_2) = 1 \quad \text{if} \quad S \geq K \]

\[ 0 \quad \text{if} \quad S < K. \]

It is also obvious from this example and, more generally, from (14), (15) that one cannot arbitrarily set to 1 or any other constant the number of stocks
or bonds in the duplicating portfolio as has been done in some previous works. Both parts of the portfolio must vary stochastically.

5. Relaxing the Unique Interest Rate Assumption—Simple Call Option

In this section the assumption of a unique interest rate is relaxed; it is assumed there are two interest rates, one for borrowing \( r_B \) and one for lending \( r_L \), where \( r_B > r_L > 0 \). All other assumptions necessary for the Black-Scholes option pricing formula are maintained.

Consider the simple call option. It will now be necessary to distinguish between a long and a short position in the call. From (22) it follows that in duplicating a long position in a call option one has to hold a nonpositive position in bonds irrespective of time and stock price level since \( \beta \leq 0 \) vs. \( t \), with equality occurring only at expiration and only when \( S(T) \leq K \), i.e., in maintaining the portfolio duplicating the long position in the call, only borrowing takes place. Now, since the interest rate enters only through the price of the bonds held in the duplicating portfolio, it is obvious that the interest rate for borrowing \( r_B \) is to be used in the Black-Scholes pricing formula (19) for a long position in a call.

\[
B(S,t) = SN(d_1) - e^{-r_B(T-t)}K\cdot N(d_2)
\]

where \( d_1, d_2 \) are those of (20) (21) with \( r \) replaced by \( r_B \). \( B(S,t) \) is an upper bound for the price an investor is willing to pay for a call.
option contract. The investor will not pay any higher price since he can always duplicate a long position in a call option with the duplicating portfolio priced at \( B(S,t) \). For convenient reference only, we will call this the option "bid" price.

By an analogous argument the portfolio duplicating a written call involves lending only, and the lower bound for the price which an investor is willing to accept for writing a call—the call option "ask" price—is given by:

\[
A(S,t) = SN(d_1) - e^{-r_L(T-t)} K \cdot N(d_2)
\]

where \( d_1, d_2 \) are those of (20) (21) with \( r \) replaced by \( r_L \).

Writing the interest rate as an explicit argument in the Black–Scholes option pricing formula, we have:

\[
V(S,t,r_B) = B(S,t)
\]
\[
V(S,t,r_L) = A(S,t).
\]

Since \( \frac{\partial V}{\partial r} > 0 \) for \( r \geq 0 \), then from \( r_B > r_L > 0 \) it follows that:

\[
B(S,t) > A(S,t) \quad 0 \leq t \leq T, \quad 0 \leq S
\]

i.e., the call option bid price is higher than the ask price, and together they determine a range of prices all of which are consistent with the assumptions of the present section. We will call this range of prices "the contingent band." The actual price will be called "the equilibrium price"
and will be denoted by $C(S,t)$; this will be determined in the full equilibrium setting by taking into account preferences, beliefs, endowments, and availability of other assets. Formally put:

$$A(S,t) \leq C(S,t) \leq B(S,t).$$  \hspace{1cm} (23)

6. **Different Interest Rates for Borrowing and for Lending—Money Spread of Call Options**

It is well known that in a perfect market setting the price of a generalized option whose payoffs can be written as a linear combination of the payoffs to a set of simple call options is equal to the same linear combination of the prices of the simple options. The reason is that the portfolio of stock and bonds that duplicates the generalized option is simply the linear combination of portfolios that separately duplicate the simple call options. This situation does not carry over to the case where different interest rates for borrowing and for lending prevail.

To understand this point, consider a particular generalized option, namely, a money spread of call options. A money spread involves the purchase of one option with exercise price $K_1$ and the sale of another with exercise price $K_2$ written on the same stock with the same expiration date. As a matter of convention, when $K_2 > K_1$ we shall call the position "long in the spread," and the opposite position "short in the spread." Now consider calculating the contingent band for the spread. At first it seems tempting to suggest that the duplicating portfolio for the long (short) position in the spread is the simple combination of the portfolio
duplicating the long (short) position in the first call \( (K_1) \) and the portfolio duplicating the short (long) position in the second call \( (K_2) \). Now, the first call is bounded in its contingent band.

Rewriting (23) we have:

\[
A(K_1) \leq C(K_1) \leq B(K_1)
\]

where the dependence on \( S, t \) is suppressed and the dependence on the exercise price made explicit. For the second call we similarly have:

\[
B(K_2) \geq C(K_2) \geq A(K_2)
\]

A simple combination of the duplicating portfolios would now entail subtraction of (25) from (24) to get the contingent band for the spread:

\[
A(K_1) - B(K_2) \leq C(K_1) - C(K_2) \leq B(K_1) - A(K_2)
\]

where \( (C(K_1) - C(K_2)) \) is the spread's equilibrium price.

Although the contingent band for the spread given in (26) is correct, it can be greatly narrowed. A closer examination reveals that the portfolio suggested above cannot be the one duplicating the spread. This is so because the portion duplicating the long (short) position in the \( K_1 \) call entails nonzero borrowing (lending) and the portion duplicating the short (long) position in the \( K_2 \) call entails nonzero lending (borrowing) as was shown in the previous section. Together this means that nonzero borrowing and lending are simultaneously taking place in the duplicating portfolio proposed above. But this is in conflict with the budget
constraint on the duplicating portfolio which requires that no funds should be wasted out of it (see section 2), and since the borrowing interest rate is assumed strictly higher than the lending rate, simultaneous borrowing and lending is a strategy that wastes funds unnecessarily. To stop this leakage, the true duplicating portfolio for the spread involves offsetting the borrowing and lending against each other; either lending or borrowing will take place at any one time. This means that the true duplicating portfolio will cost less than the one proposed above as a first guess. This means that the bid price for the long position in the spread is strictly less than \( B(S,t,K_1) - A(S,t,K_2) \), and that the ask price for a short position in the spread is strictly higher than \( A(S,t,K_1) - B(S,t,K_2) \).

A special and somewhat trivial case may prove instructive. Consider a money "spread" in two identical calls, \((K_1 = K_2)\). When calculating the contingent band using (26) one gets a contingent band of finite width, but when offsetting of the borrowing and lending is affected, then this money spread can be identified as an identically zero position, and the contingent band is clearly narrowed to zero width.

The implications of the argument in the present section reaches beyond the effects on the contingent band of the money spread, since a narrower band for the spread forces the individual calls to bound each other in bands still narrower than the originals. To see this, let us denote the strictly narrower band for the spread by \((AN, BN)\) where:

\[
A(K_1) - B(K_2) < AN \leq C(K_1) - C(K_2) \leq BN < B(K_1) - A(K_2)
\]
from which it follows that

\[ [A(K_1) - B(K_2)] + C(K_2) < AN + C(K_2) \leq C(K_1) \leq BN + C(K_2) < [B(K_1) - A(K_2)] + C(K_2). \]

(27)

Now suppose that \( C(K_2) \) is observed in the economy to be equal to its lower bound \( A(K_2) \), then substituting for \( C(K_2) \) in the right part of (27):

\[ C(K_1) \leq BN + A(K_2) < [B(K_1) - A(K_2)] + A(K_2) = B(K_1) \]

i.e., \( [BN + A(K_2)] \) is better than \( B(K_1) \) as an upper bound for \( C(K_1) \), the price of the \( K_1 \) call, given that the \( K_2 \) call is observed to be priced at \( A(K_2) \). Similarly, it can be shown that \( [AN + B(K_2)] \) is better than \( A(K_1) \) as a lower bound for the price of the \( K_1 \) call, given that the \( K_2 \) call is observed to be priced at \( B(K_2) \). Usually what will happen is that for each observed value of \( C(K_2) \) at least one bound for \( C(K_1) \) and most probably both will be improved. A striking example for this is displayed in figure 1, section 8, where a numerical solution is presented. The above can, of course, be reversed to bound the \( K_2 \) call by the observed prices of the \( K_1 \) call.

The money spread of two calls has served us well in the present section. The time has come now to generalize to portfolios of generalized options. But before we do so, let us draw one more conclusion. What has been shown above implies that the very possibility of combining options in a portfolio causes the no-arbitrage-consistent price of any one option or of a subportfolio of options to be bounded more narrowly (than when
considered in isolation) by the observed prices of the rest of the options in the portfolio. Moreover, since portfolios including all options in the economy can be formed, it follows that in looking for the narrowest contingent band for one option or a portfolio of options (including our money spread as a special case), one must consider the observed prices of all other options in the economy and all possible economy-wide options portfolio combinations. The bottom line for this section becomes: life in an imperfect market may be hard but also interesting.

7. Different Interest Rates for Borrowing and for Lending—Generalized Option on One Underlying Asset

In this section we maintain assumptions (A1) and (A3) through (A6) of section 3. Assumption (A2) is replaced by:

(A2') Borrowing is carried out at a constant interest rate \( r_B \) by holding a short position in bonds whose unit price develops according to:

\[
B(t) = e^{r_B t}.
\]

Lending is carried out at a constant interest rate \( r_L \) \( (r_B > r_L) \) by holding a long position in bonds whose unit price develops according to:

\[
L(t) = e^{r_L t}.
\]

We shall now form the hedge portfolio that duplicates the returns to a general option the features of which are described in (A4) above.
In addition to $\alpha$ shares of the underlying stock, our duplicating portfolio will contain a short position of $\beta$ units of borrowing bonds ($\beta \leq 0$), and a long position of $\lambda$ units of lending bonds ($\lambda \geq 0$). The value of the duplicating portfolio is given by:

$$V(S,t) = \alpha(S,t)S + \beta(S,t)B(t) + \lambda(S,t)L(t).$$  \hfill (28)

By Ito's lemma:

$$dV = \alpha dS + \beta dB + \lambda dL + (Sd\alpha + Bd\beta + Ld\lambda + dS\alpha + dB\beta + dL\lambda).$$  \hfill (29)

Imposing the budget constraint on this duplicating portfolio and using the same argument as in section 3, we conclude that the bracketed expression in (23) is equal to zero, i.e.:

$$Sd\alpha + Bd\beta + Ld\lambda + dS\alpha + dB\beta + dL\lambda = 0$$  \hfill (30)

$$dV = \alpha dS + \beta dB + \lambda dL.$$  

Applying Ito's lemma to $\alpha$, $\beta$, $\lambda$, $B$, and $L$, substituting the differentials in equation (30), and setting the deterministic and stochastic parts of the resulting equation both to zero exactly in the same way as in section 3, one arrives at the two PDEs:

$$S \frac{\partial \alpha}{\partial S} + e^{B_t} \frac{\partial B}{\partial S} + e^{L_t} \frac{\partial \lambda}{\partial S} = 0$$  \hfill (31a)

$$S \mu_\alpha + \sigma^2 S \frac{\partial \mu_\alpha}{\partial S} + e^{B_t} (\mu_\beta) + e^{L_t} (\mu_\lambda) = 0$$  \hfill (31b)
in the domain $0 \leq t \leq T$, $0 \leq S$, where $\mu_\alpha$ is the drift in the number of shares of stocks given by

$$
\mu_\alpha = \mu_S S \frac{\partial \alpha}{\partial S} + \frac{\partial \alpha}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 \alpha}{\partial S^2}
$$

(32)

and $\mu_\beta, \mu_\lambda$ are the drifts in $\beta$ and in $\lambda$ respectively, given by analogous expressions (replace $\alpha$ by $\beta$ and by $\lambda$ in (32)).

As was discussed in section 6, when $r_B > r_L$ the duplicating portfolio should involve either borrowing or lending but not simultaneously both in order to meet the budget constraint. This means that at any one time at least one out of the pair $\beta, \lambda$ should be equal to zero, or formally put:

$$
\beta(S,t) \cdot \lambda(S,t) = 0 \quad 0 \leq t \leq T, \quad S \geq 0.
$$

(31c)

The boundary condition for (31) is given by:

$$
V(S(T),T) = \alpha(S(T),T) S(T) + \beta(S(T),T) e^{r_B T} + \lambda(S(T),T) e^{r_L T} = h(S(T))
$$

where $h(S(T))$ is the contracted dollar return to the option at expiration date $T$.

For the unique interest rate case it was shown in section 3 that

$$
(V - S \frac{\partial V}{\partial S})
$$

is the dollar amount lent in the duplicating portfolio. Motivated by this fact, one can make the following somewhat involved substitution:

$$
\alpha = \frac{\partial V}{\partial S}.
$$
If \( (V - S \frac{\partial V}{\partial S}) > 0 \) (lending), then let \( \beta = 0 \), \( \lambda = e^{-r_L t} (V - S \frac{\partial V}{\partial S}) \).

If \( (V - S \frac{\partial V}{\partial S}) < 0 \) (borrowing), then let \( \lambda = 0 \), \( \beta = e^{-r_B t} (V - S \frac{\partial V}{\partial S}) \).

Equation (31c) is now satisfied by construction, (31a) yields \( 0 = 0 \), and (31b) gives:

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma_S^2 S^2 \frac{\partial^2 V}{\partial S^2} = r_L (V - S \frac{\partial V}{\partial S}) \tag{33}
\]

for \( 0 \leq S \) and \( 0 \leq t \leq T \) for which \( (V - S \frac{\partial V}{\partial S}) > 0 \)

and

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma_S^2 S^2 \frac{\partial^2 V}{\partial t^2} = r_B (V - S \frac{\partial V}{\partial S}) \tag{34}
\]

for \( 0 \leq S \) and \( 0 \leq t \leq T \) for which \( (V - S \frac{\partial V}{\partial S}) < 0 \).

Define a step function \( f(\cdot) \) by:

\[
f(X) = r_L \quad \text{if} \quad X > 0
\]

\[
= r_B \quad \text{if} \quad X < 0.
\]

Using the function \( f(\cdot) \) we can now combine (33) and (34) into one PDE in the value of the duplicating portfolio:

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma_S^2 S^2 \frac{\partial^2 V}{\partial S^2} = \left[ \frac{\partial (f(V - S \frac{\partial V}{\partial S}))}{\partial S} \right] (V - S \frac{\partial V}{\partial S}) \tag{35}
\]

for \( 0 \leq S, \ 0 \leq t \leq T \).
Thus, the system (31) of three PDEs in $\alpha$, $\beta$, $\lambda$ is condensed into a single PDE in $V$. The function $f(\cdot)$ serves in (35) as a switch for the interest rate. When $\left( V - S \frac{\partial V}{\partial S} \right) \geq 0$ and lending takes place in the duplicating portfolio, $f(\cdot)$ switches to the lending rate, and when $\left( V - S \frac{\partial V}{\partial S} \right) < 0$ and borrowing takes place, $f(\cdot)$ switches to the borrowing rate.

The boundary condition for (35) is:

$$V(S(T), T) = h(S(T)). \quad (36)$$

Note that if, by convention, we choose to call the contractual agreement which promises to pay on expiration $h(S(T))$ dollars "long in the option," then the short position in the option is the contract which promises to pay $-h(S(T))$ dollars on expiration. Thus, on one hand, the unique solution to (35) with the boundary condition (36) yields the long side boundary of the contingent band within which the equilibrium price of the option must lie to be consistent with the no-arbitrage condition, and, on the other hand, the unique solution to (35) together with the short position boundary condition

$$V(S(T), T) = -h(S(T))$$

gives the short side boundary of the contingent band. Which boundary is the option ask price and which is the option bid price would depend on the specific form of $h(\cdot)$ as well as on $S$ and $t$. Note also that in general either boundary may depend on both interest rates for lending and for borrowing.
Substituting the unique solution \( V(S,t) \) back into \( \alpha, \beta, \lambda \) above gives the unique trading strategy for the duplicating portfolio for either the long position or the short position in the option.

8. A Numerical Solution

A numerical solution\(^{12}\) will be used to illustrate the points made this far and to demonstrate the significant implications to options investment strategies. A stock, currently selling for $30 and whose price is assumed to follow a Geometric Brownian Motion with volatility 0.3/year, was chosen as the underlying asset. The generalized option was taken as a money spread of two call options, one bought and one written, with exercise prices $20 and $30 and with the same expiration date of one year from now. The interest rates of \( r_B = 13\% / \text{year} \), \( r_L = 14\% / \text{year} \) were taken to reflect levels prevailing at the time of writing this article. Table 1 presents the contingent bands for the separate calls calculated by the Black-Scholes pricing formula as in section 5. Note that the implied borrowing and lending in the duplicating portfolio for both calls are relatively large, about $16,\(^{13}\) hence, one would expect a band width of about \((16 \times 0.19 - 16 \times 0.14) = 0.80. This, in fact, is the case: the actual band widths are $0.70 and $0.71 for the \( K = 20 \) and \( K = 30 \) calls, respectively.

Table 2 presents two contingent bands for the money spread calculated by two methods respectively. First, a band calculated from the bands for the separate calls according to (26) resulting in the value of
TABLE 1
CONTINGENT BANDS FOR THE SEPARATE CALLS

<table>
<thead>
<tr>
<th></th>
<th>Call 1</th>
<th>Call 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$S = 30$</td>
<td>$S = 30$</td>
</tr>
<tr>
<td></td>
<td>$\sigma = 0.3/\text{year}$</td>
<td>$\sigma = 0.3/\text{year}$</td>
</tr>
<tr>
<td></td>
<td>$T = 1 \text{ year}$</td>
<td>$T = 1 \text{ year}$</td>
</tr>
<tr>
<td></td>
<td>$r_L = 14%/\text{year}$</td>
<td>$r_L = 14%/\text{year}$</td>
</tr>
<tr>
<td></td>
<td>$r_B = 19%/\text{year}$</td>
<td>$r_B = 19%/\text{year}$</td>
</tr>
<tr>
<td></td>
<td>$K_1 = 20$</td>
<td>$K_2 = 30$</td>
</tr>
<tr>
<td></td>
<td>$K_1 = 20$</td>
<td>$K_2 = 30$</td>
</tr>
<tr>
<td></td>
<td>Price</td>
<td>Price</td>
</tr>
<tr>
<td></td>
<td>Implied</td>
<td>Implied</td>
</tr>
<tr>
<td></td>
<td>Lending*</td>
<td>Lending*</td>
</tr>
<tr>
<td>Bid</td>
<td>$13.26$</td>
<td>$6.22$</td>
</tr>
<tr>
<td></td>
<td>$-16.18$</td>
<td>$-16.80$</td>
</tr>
<tr>
<td>Ask</td>
<td>$12.56$</td>
<td>$5.51$</td>
</tr>
<tr>
<td></td>
<td>$16.65$</td>
<td>$16.13$</td>
</tr>
<tr>
<td>Contingent</td>
<td>$13.26 - 12.56 = 0.70$</td>
<td>$6.22 - 5.51 = 0.71$</td>
</tr>
<tr>
<td>Band-Width</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

* Part of the duplicating portfolio initially invested in bonds (lent).

Equals \( (V - \frac{\sigma_v}{\sigma_s} - S) \) |

\( t = 0, S(0) \)

minus for borrowing.
### Table 2

Continent Band for a Money Spread of Two Calls, One Long and One Short with \( k_1 = 20 \) and \( k_2 = 30 \), Respectively

<table>
<thead>
<tr>
<th>Contingent Band of Spread, Calculated from bands for Separate Calls</th>
<th>Price</th>
<th>Implied Lending*</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Bid</strong></td>
<td>$13.26 - 5.51 = 7.75</td>
<td>$16.03 - 16.18 = -0.05</td>
</tr>
<tr>
<td><strong>Ask</strong></td>
<td>$12.56 - 6.22 = 6.34</td>
<td>$16.65 - 16.80 = -0.15</td>
</tr>
<tr>
<td><strong>Band Width</strong></td>
<td>$7.75 - 6.34 = 1.41$</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Contingent Band of Spread, from Numerically Solving Equation (35)</th>
<th>Price</th>
<th>Implied Lending*</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Bid</strong></td>
<td>$7.18</td>
<td>$0.19</td>
</tr>
<tr>
<td><strong>Ask</strong></td>
<td>$6.95</td>
<td>-$0.11</td>
</tr>
<tr>
<td><strong>Band Width</strong></td>
<td>$7.18 - 6.95 = 0.23$</td>
<td></td>
</tr>
</tbody>
</table>

* See Table 1.

† Width shrinks sixfold, from $1.41$ to $0.23$. 
$1.41. Second, directly (numerically) solving PDE (35) for the band of the spread, results in the value of $0.21, a remarkable sixfold narrowing of the band width. This drastic narrowing of the spread's contingent band is due to the fact that the second method allows for offsetting borrowing against lending in the duplicating portfolio for the spread that results in net borrowing or lending of only about $0.15 one year from expiration compared with about $16.00 borrowing and lending in the separate calls.\textsuperscript{14}

Figure 1 depicts contingent bands for the call with exercise price $K_1 = $20. The wide horizontal band is the one computed for this call in isolation as in section 5. The narrower band (rising from left to right) is the one computed for the first call, given observed prices for the second call ($K_2 = $30) using (27). The resulting contingent band is the striped area, which is the intersection between the two bands mentioned above. The narrowing of the contingent band is also remarkable; from the original width of $0.70 to $0.23 for $5.61 < C(K_2) < 6.09$ dropping to a mere $0.09 at $C(K_2) = 6.22$.

It is instructive to note the following. First, the $0.23$ figure may not be the narrowest obtainable band for $C(K_1)$ given observed $C(K_2)$, since only one particular portfolio has been considered in the current example, namely, the spread. But a continuum of portfolios of these two calls can be formed\textsuperscript{15} and the narrowest band searched for. Second, if prices of options other than $C(K_2)$ are also observed, then the band for $C(K_1)$ given all these observed prices may be still narrower.
1) Lines a and b give the upper and lower bounds for \( C(K_1) \) considered in isolation.

2) Lines c and d are the upper and lower bounds for \( C(K_1) \) given the observed price for \( C(K_2) \).

3) The striped area is the resulting contingent band for \( C(K_1) \) given observed \( C(K_2) \); the intersection of the above two bands.
The same applies to the contingent band of the spread itself, given the observed prices of all other options.

As illustrated by the numerical example presented in this section, the implications to options investment strategies are quite significant.

9. Different Interest Rates for Borrowing and for Lending—Generalized Option on Many Underlying Assets

It was shown earlier that the contingent band for a portfolio of options on one underlying asset cannot be deduced from the contingent bands for the individual options, rather the problem had to be solved in the options portfolio context directly. The same is true for the more general case of a portfolio of options on a number of different underlying assets. The interesting aspect of this case is the following: While in the case of a unique interest rate for lending and borrowing the price of a portfolio of options, each having a different underlying asset, can be deduced from the individual options prices and, thus, correlations among the underlying assets do not play any role in pricing of the portfolio. This is no longer the case when different interest rates for borrowing and for lending prevail; cross-correlations between the underlying assets do appear explicitly in the pricing formulas. Thus, a market imperfection brings forth a correlation effect which was absent in the perfect market environment.

A. Many Underlying Assets, One Expiration Date

The approach in treating this case is a generalization of that in section 7 above.
Let $S_i (i = 1, \ldots, N)$ denote values of $N$ underlying assets whose dynamics is given by the set of stochastic differential equations:

$$dS_i = \mu_i dt + \sum_{r=1}^{N} \nu_{ir} dW_r \quad (i = 1, \ldots, N)$$

where $W_r(t)(r = 1, \ldots, N)$ are $N$ independent Wiener-processes obeying the formal multiplication rule:

$$dW_i dW_j = dt \quad \text{if} \quad i = j$$

$$= 0 \quad \text{if} \quad i \neq j$$

from which it follows that

$$dS_i dS_j = (\sigma_{ik} S_k S_j) dt,$$

where

$$\sigma_{ik} S_k S_j = \sum_{u=1}^{N} \nu_{ku} \nu_{ju}$$

also: 16

$$\nu_{i} = \nu_{i}(S_1, \ldots, S_N, t)$$

$$(i, r = 1, \ldots, N)$$

$$\nu_{ir} = \nu_{ir}(S_1, \ldots, S_N, t).$$

For simplicity of exposition it will be assumed at this stage that all options in the portfolio expire at the same date $T$. This assumption
will later be relaxed. The derivation follows the same lines as in section 7; the duplicating portfolio will now contain \( \alpha_i \) units of underlying asset \( S_i \) \((i = 1, \ldots, N)\) and borrowing or lending. In particular, equations (28) through (30) will remain the same except that \( S \) is now to be interpreted as the vector of asset prices \( (S_1, \ldots, S_N) \), \( \alpha \) becomes the vector \( (\alpha_1, \ldots, \alpha_N) \) and \( \alpha \cdot S, \alpha \cdot dS, S \cdot d\alpha \) become scalar products. The rest remains the same.

A system of \( N + 2 \) PDEs follows now from equation (30) exactly the same as in section 7.

\[
\sum_{m=1}^{N} S_m \frac{\partial \alpha_m}{\partial t} + \sum_{m=1}^{N} \frac{\partial \alpha_m}{\partial S_j} \frac{\partial S_j}{\partial t} + e^{r_B t} (\mu_\beta) + e^{r_L t} (\mu_\lambda) = 0 \tag{37a}
\]

\[
\sum_{j=1}^{N} \left[ \sum_{m=1}^{N} S_m \frac{\partial \alpha_m}{\partial S_j} + e^{r_B t} \frac{\partial \beta}{\partial S_j} + e^{r_L t} \frac{\partial \lambda}{\partial S_j} \right] v_{j\tau} = 0 \quad (r = 1, \ldots, N) \tag{37b}
\]

\[
\beta \lambda = 0 \tag{37c}
\]

where

\[
\mu_\alpha = \frac{\partial \alpha_m}{\partial t} + \sum_j \frac{\partial \alpha_m}{\partial S_j} \mu_j + \frac{1}{2} \sum_k \sum_l \frac{\partial^2 \alpha_m}{\partial S_k \partial S_l} \sigma_k l \sigma_k S_k S_l
\]

\[
\mu_\beta = \frac{\partial \beta}{\partial t} + \sum_j \frac{\partial \beta}{\partial S_j} \mu_j + \frac{1}{2} \sum_k \sum_l \frac{\partial^2 \beta}{\partial S_k \partial S_l} \sigma_k l \sigma_k S_k S_l
\]

and analogously for \( \mu_\lambda \).
The boundary condition for (37) is given by

\[
\sum_{i=1}^{N} a_i(S(T), T) S_i(T) + \beta(S(T), T) e^{-r_g T} + \lambda(S(T), T) e^{-r_L T} = h(S(T))
\]

where \( h(S(T)) \) is the sum of the contracted dollar returns to all the options in the portfolio.

Substituting as in section 7,

\[
a_i = \frac{\partial V}{\partial S_i} \quad (i = 1, \ldots, N).
\]

If \( \left( V - \sum_{i=1}^{N} S_i \frac{\partial V}{\partial S_i} \right) \geq 0 \) (lending), then let \( \beta = 0, \lambda = e^{-r_L t} \left( V - \sum_{i=1}^{N} S_i \frac{\partial V}{\partial S_i} \right) \).

If \( \left( V - \sum_{i=1}^{N} S_i \frac{\partial V}{\partial S_i} \right) < 0 \) (borrowing), then let \( \lambda = 0, \beta = e^{-r_B t} \left( V - \sum_{i=1}^{N} S_i \frac{\partial V}{\partial S_i} \right) \).

Substituting in (37) and using the function \( f(\ast) \) as defined in section 7, we finally get:

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\partial^2 V}{\partial S_i \partial S_j} S_i S_j \sigma_{ij} = \left[ f \left( V - \sum_{i=1}^{N} S_i \frac{\partial V}{\partial S_i} \right) \right] \left( V - \sum_{i=1}^{N} S_i \frac{\partial V}{\partial S_i} \right) \quad (38)
\]

which is the generalization of (35). The boundary condition for (38) is

\[
V(S(T), T) = h(S(T))
\]

for the long position in the portfolio of options, and

\[
V(S(T), T) = - h(S(T))
\]

for the short position in this portfolio.
It is important to note that if the portfolio of options had consisted of options each with one underlying asset only, and if there had been a unique constant interest rate for borrowing and for lending, i.e., \( f(\cdot) = r \), then equation (42) would have been separable into \( N \) PDEs of the form of the Black-Scholes equation (16), each having the value of the individual option as the unknown variable and its underlying asset price and time as its independent variables; then the cross-correlation terms would have disappeared. But it is sufficient that interest rates for borrowing and for lending be different in order to prevent this separability from going through. In this case one has to solve (38) directly, and, eventually, the cross-correlation terms in (38) will appear in the solution for the contingency band for the portfolio of options as was claimed in the beginning of this section. The underlying reason for this inseparability is the same as for a portfolio of options on a single underlying asset: it is necessary to offset lending against borrowing in the duplicating portfolio.

B. Many Underlying Assets, Many Expiration Dates

Consider a portfolio of \( M \) options, each written on the \( N \) underlying assets (or a subset of them), and each option \( k \) contracted to pay to the holder \( h_k(S(T_k)) \) dollars on expiration date \( T_k (k = 1, \ldots, M) \). This portfolio can be treated as the one with a unique expiration date \( T_M \) (in this case), the only difference being that now the portfolio of options can be viewed as being paid a "dividend" at the rate of

\[
D(t, S(t)) = \sum_{k=1}^{M-1} \delta(T_k - t)h_k(S(T_k)),
\]  

(39)
where $\delta(\cdot)$ is the Dirac-delta function with the defining properties:

$$\delta(x) = 0 \quad \text{if} \quad x \neq 0$$

$$\int_{-\infty}^{\infty} \delta(x) dx = 1.$$ 

The last payment to the option portfolio $h_M(S(T_M))$ is excluded from the "dividend" stream (39) and is reserved to serve as the boundary condition.

The introduction of a dividend stream $D(t, S(t))$ which is paid to the portfolio of options leads to the introduction of an identical dividend stream paid to the duplicating portfolio. This, in turn, has the effect of altering the budget constraint (9) which now will indicate that the change in value of the duplicating portfolio, which does not arise from changes in prices of its components, is equal to the dividend stream. Formally, this has the effect of replacing the zero on the RHS of (9) by $D(t, S(t)) dt$. This hardly affects the derivation of the PDE in the value of the duplicating portfolio which now is:

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\partial^2 V}{\partial S_i \partial S_j} S_i S_j \sigma_{ij} = \left[ f \left( V - \sum_{i=1}^{N} S_i \frac{\partial V}{\partial S_i} \right) \right] \left( V - \sum_{i=1}^{N} S_i \frac{\partial V}{\partial S_i} \right) +$$

$$\sum_{k=1}^{M-1} \delta(T_k - t) h_k (S(T_k))$$

in the domain $0 \leq S$, $0 \leq t \leq T_M$ and boundary condition:

(40)
\[ V(T_M) = h_M(S(T_M)). \]  \hspace{1cm} (41)

For the short position in the portfolio of options, \( h_k(*) \) should be replaced by \( -h_k(*) (k = 1, \ldots, M) \).

Although (40) is a very compact formal summary of a necessary condition for the dynamics of the value of the duplicating portfolio, it is of little practical help in actually solving the problem. The following suggests a practical iterative procedure for solution. Start from the boundary condition (41) at \( t = T_M \) and integrate (40), backward in time up to \( t = T_{M-1} \) noting that the last term on the RHS of (40) vanishes for \( T_{M-1} < t \leq T_M \). At \( t = T_{M-1} \) add \( h_{M-1}(S(T_{M-1})) \) to the result obtained from the above integration. The sum thus obtained serves as a new boundary condition at \( t = T_{M-1} \). Now iterate the procedure from one expiration date to the one immediately preceding it all the way back to \( t = 0 \), where the final solution is arrived at.
FOOTNOTES

1 Note that the dependence of α and β on S and t exclusively is not an assumption, but only a recipe for cooking up the duplicating portfolio. As is usually the case with recipes, its success will be judged by the results.

2 Especially equations (11), (12), (13) with imposition of no withdrawals and no external additions to the portfolio, i.e., C(t)dt = dy = 0 in Merton's notation.

3 The domain is an integral part of the specification of a differential equation. Section 7 will illustrate that sometimes this is not a trivial matter.

4 This number of bonds in the duplicating portfolio is similar to that obtained in Cox-Ross-Rubinstein's discrete time framework (1979) except for the exp(-rt) factor which does not appear there because CRR renormalize the value of the bond to 1 at each instant of time.

5 See, for example, theorem 2 in Cox, Ingersoll, and Ross (1980), where the number of bonds in the duplicating portfolio is arbitrarily set to 1.

6 This might have discriminating implications. Suppose, for example, that transacting stocks were costless but transacting bonds were not, then one could evade transaction costs on the duplicating portfolio by maintaining a constant number of bonds. But this should be recognized as an impossibility since the number of bonds must be varied as well as the number of stocks.

7 For example, a no-coupon junior bond can be viewed as such a generalized option on the value of the firm as the underlying asset. The junior bond holders receive h(F(T)) dollars upon maturity date T depending on the value of the firm then F(T), where

\[ h(F(T)) = 0 \quad \text{if} \quad F(T) < K \]
\[ = F(T) - K \quad \text{if} \quad K \leq F(T) < J \]
\[ = J \quad \text{if} \quad J \leq F(T) \]

where J is total face value of the junior debt and K is the same for the senior debt. This option can be viewed as a money spread in call options, see below.
For expository simplicity it is assumed that $A(S,t,K_1) > B(S,t,K_2)$, but this is not essential.

Options on one underlying asset. This is relaxed later resulting in similar conclusions.

See footnote 1.

Dependence of the interest rate on the amount borrowed or lent can be easily incorporated into the analysis via an appropriate definition of the function $f(*)$.

I wish to thank Professor M. E. Rubinstein for letting me use one of his computer programs modified to accommodate different interest rates for borrowing and for lending.

Further calculations indicate that this figure changes only about $2 either way when the stock price develops in time.

This figure can change about $2 either way when time advances towards expiration, which accounts for the seemingly exaggerated figure of $0.23$ for the directly computed band width.

Positive weights for both calls may be immediately excluded, since the effect of offsetting borrowing against lending is then absent.

Expanding the dependence of the $u_i$'s on additional state variables would hardly affect the treatment.

Since then $V = \sum_{i=1}^{N} u_i(S_i,t)$, hence $\frac{\partial^2 V}{\partial S_i \partial S_j} = 0$ for $i \neq j$. 
REFERENCES


