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AN INTRODUCTION TO THE VALUATION OF COMMODITY OPTIONS

BY
JAMES W. HOAG

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AN INTRODUCTION TO THE VALUATION OF COMMODITY OPTIONS

by

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This paper summarizes my work on commodity options from 1973 through 1978 and is based on my dissertation and subsequent papers presented at various meetings and seminars at Stanford, University of British Columbia and London Graduate School of Business. Comments made during those presentations have improved this work, but results beyond 1978 are not contained herein. This work has also benefited from discussions with colleagues at Stanford, and especially with the late Paul Cootner. A comprehensive transaction data base for commodity option prices has been assembled since then and results of tests using that data will soon be available.
AN INTRODUCTION TO THE VALUATION OF COMMODITY OPTIONS

§ 1 Introduction

This paper presents the preliminary results of research conducted on the nature of options on commodities. In recent years, much has been written on option pricing. However, the basic setting has always been in terms of financial assets: stocks and options on those stocks (warrants, Over the Counter (OTC) and Chicago Board Option Exchange (CBOE) options). Fortunately, these techniques are generally applicable to the study of commodity options.

There exists an extensive literature on the motivation and interaction of participants (hedgers and speculators) in commodity futures markets -- some notable examples being Brennan [1958], Cootner [1967] and Working [1948, 1949]. Partially due to lack of availability and intermittent illegality, the use of commodity options in hedging policy has not been examined in the academic (or popular) literature. One aspect of the difficulty of examining commodity options is the lack of theory for pricing of commodity options and for the entire area of commodity pricing and inventory storage policy. In recent research, Cootner [1977] related spot commodity prices to the optimal inventory storage pattern, and produced a model with future and spot prices endogenously computed. The value of a commodity stored by inventory holders included a premium for the inventory holder's option to sell to current consumers at the spot price or to sell to future consumers at the future spot price. Thus, commodity options are really compound options (that is, options on
other options). Throughout this work, arbitrage relationships among prices of spot, futures and options are considered to be implicit functions of aggregate supply (represented by inventory holdings). Understanding the underlying inventory dependence will be useful when examining the approximations which are presented later in this paper.

Recently, hedging practice has come to include the purchase or sale of call or put options. These efforts attempt to hedge price risks connected with uncertain supply and demand for raw materials used in production.\(^2\) Hence, the pricing of these options is an important aspect in hedging policy. Although hedging motives and practice provide incentives to create option markets in these commodities, these markets are also investment or speculative vehicles for individuals and corporations. As with futures markets, the delicate interplay between hedgers and speculators serve to equilibrate commodity option prices.

In this paper, a summary of the theory concerning the pricing of commodity options is examined in a consistent framework. Empirical results on the pricing of options on futures (London Commodity options on cocoa, coffee and sugar) and options on metals (dealer silver options) are presented. Evidence of efficient market pricing of commodity options is necessary to rationally discuss the implications of regulatory policy on social and individual welfare. Although tentative in many aspects, this paper links the appropriate areas of commodity theory with the basic option pricing model to provide a consistent introductory background for research in this area.
§ 2 Brief Economic History of Commodity Options

Commodity options (known then as privileges, indemnities, bids, offers or guaranties) developed simultaneously with commodity futures markets for agricultural commodities (wheat, corn, oats and rye) in the mid 1800s. Both options and futures along with the commodity exchanges were viewed with suspicion by farmers during the early period of trading. Legends of massive price manipulation by "speculators" fill the popular literature of that period. Many efforts were initiated by farmers to have state legislatures and the Federal Congress ban futures and options during the period.

In 1885, the Illinois Supreme Court in the case of Pearce vs. Foote declared option transactions illegal based on an 1874 Illinois statute. During this period the Chicago Board of Trade was quite ambivalent in their position, banning option transactions and subsequently ignoring and not enforcing their own ban.

By the 1892-93 session of Congress, both the House and the Senate came within a technicality of passing bills levying a large tax on futures trading. Thereafter, the sentiment against futures trading subsided until the early 1920s.

In 1900, the court, in the case of Booth vs. People (57 NE 798), held with respect to options that (see Mehl, p. 7):

The prohibition of the right to enter into contracts which do not contemplate the creation of an obligation on the part of one of the contracting parties to accept and pay for the commodity which is the purported subject matter of the contract, but only to invest him with the option or privilege to demand the other contracting party shall deliver him the grain, if he desires to purchase it, tends materially to the suppression of
the very evil of gambling in grain options which it was the legislative intent to extirpate, for the reason such evil injuriously affected the welfare and safety of the public.

Although futures received a reprieve, options received continuing attention from both government and the commodity exchanges throughout the early 1900s. In 1920, the Future Trading Act imposed a prohibitive tax on options which essentially halted trading in options.

The next year, the Grain Futures Act provided the government with fact finding capability and elemental tools to deal with the commodity exchanges. In early 1926, the United States Supreme Court held that the tax on options was unconstitutional, and option trading recommenced immediately. The 1922 Act permitted the collection of data on option trading during August and September 1926. It is this data which Mehl analyzes in his 1934 study of grain options. In that study, commodity options on grain futures represented on the order of 10 to 15 percent of the volume (in bushels) of grain traded in futures contracts. Mehl presents little evidence on price manipulation with options. His summary is relatively inconsistent and unsupported by his own empirical evidence:

Privilege trading is considered useful by many members of the grain trade in that it affords protection against price changes, makes possible the financing of speculative transactions on a small capital, ...

Its unfavorable aspects are the following: The small amount of capital required to trade in privileges encourages speculation by traders of limited financial resources. The practice of trading against privileges bought and protecting those sold causes artificial price movements.

In July 1933, options trading on the Chicago Board of Trade was suspended following a three year decline in grain prices. After extensive hearings before Congress, the Commodity Exchange Act of 1936 banned
trading in commodity options on regulated commodities. During the hearings (see To Amend the Grain Futures Act, Hearings before the Committee on Agriculture and Forestry, United States Senate), Mehl filed very ambiguous "evidence" with these sentences:

I think I shall have very little to say on the subject of privilege trading. The record of the exchanges themselves on that question over a period of years shows conclusively that the trade itself has never had any unanimity of opinion regarding the desirability of trading in privileges.

Mehl did not mention the sizable volume of options in grains shown by his study. Although the Congress did ban commodity options, there seems to be very little evidence presented either for or against option trading in the hearings.

The relevant portions of the 1936 Commodity Exchange Act are given in §2(a) which lists the commodities regulated by the Act and §4(c):

It shall be unlawful for any person to offer to enter into, or confirm the execution of, any transaction involving any commodity ...

(B) if such transaction is, or is the character of, is of commonly known to the trade as, a "privilege," "indemnity," "bid," "offer," "put," "call," "advance guaranty," or "decline guaranty," or ...

The only minor modifications of this Act, as related to commodity options, changed and added to the list of regulated commodities during the period from 1936 to 1974.

During the period before 1974, there was no restriction on commodity option transactions in unregulated commodities (those not explicitly stipulated in the Commodity Exchange Act). Commodity options have been bought and sold by reputable metals dealers. These dealer options were written on unregulated metals, such as silver, platinum, palladium
and, later, copper. Commodity options, traded on the London Commodity Exchanges (now the International Commodities Clearing House (ICCH)) and the London Metal Exchange, were also available in the United States.

Few statistics exist on the volume of usage of these options, but the existence of the instruments does indicate at least sufficient interest to cover the transaction costs of establishing the market (which, of course, could be quite low). The volume of options compared to futures transactions for soft commodities (coffee, sugar and cocoa) traded on the ICCH is presented for the most recent years in Table 1.

In the early 1970s, dealers came into existence to market options to the general public for investment (as opposed to hedging) purposes. These dealers attempted to satisfy a market need in terms of selling options. Some dealers not only acted as brokers or clearing houses for options, they took a net (or naked) position in options without hedging that risk (see, for instance, the SEC vs. Goldstein, Samuelson, Inc.). These dealers subsequently discovered the hard lessons of other merchandising firms by carrying the full price risk of their product — commodity options. The demise of commodity option dealers inspired renewed regulatory and public interest in commodity options.

In 1974, Congress modified the Commodity Exchange Act to create the Commodity Futures Trading Commission (CFTC). In addition, the Act was changed in two important respects. Section 2(a) was modified to include:

... All other goods and articles, except onions as provided in Public Law 85-839, and all services, rights, and interests in which contracts for future delivery are presently or in the future dealt in.
<table>
<thead>
<tr>
<th>Year</th>
<th>Coffee Futures</th>
<th>Coffee Options</th>
<th>Sugar Futures</th>
<th>Sugar Options</th>
<th>Cocoa Futures</th>
<th>Cocoa Options</th>
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<td>824263</td>
<td>6436</td>
<td>742557</td>
<td>1538</td>
</tr>
</tbody>
</table>

*Volume for both options and futures is measured in number of contracts traded. Futures are bought and sold easily, but the options are essentially non-tradeable. Contract size and description for futures and option contracts are presented later.


#All maturities.

†Calls, puts, and doubles in all maturities.
Section 4c(B) continued the ban on options on the commodities which had previously been banned in the 1936 Commodity Exchange Act and its subsequent amendments. A new section 4c(b) regulates option transactions in the newly regulated commodities which are:

... contrary to any rule, regulation, or order of the Commission prohibiting any such transaction or allowing such transaction under such terms and conditions as the Commission shall prescribe ...

The Report of the Advisory Committee on Definition and Regulation of Market Instruments to the Commodity Futures Trading Commission entitled "Recommended Policies on Commodity Option Transactions" suggests rules under which option transactions would be allowed. The CFTC allowed transactions in these commodity options through April 1978, when they again prohibited all commodity option transactions in the United States and forhsde marketing of London commodity options. At the time of the ban, several exchanges had submitted proposals for exchange-traded, CFTC-regulated commodity options.

In view of the concern about the viability of commodity options, a thorough examination of the evidence on the pricing of commodity options follows.

§ 3 Commodity Options as Contingent Claims

A commodity option is a contract conveying the right to buy or sell a specific amount of the spot commodity or future contract at a given price on (or before) a given date. The value of the commodity is directly contingent upon the price of the underlying commodity contract. Through the commodity price, the option is also contingent upon supply,
demand, current inventory and expected future harvests of the underlying commodity.

Commodity options can be written on either the spot commodity (e.g., dealer precious metals options) or on a futures contract (e.g., London Soft Commodity Options). Thus it is necessary when discussing commodity options to distinguish the underlying variable upon which the option is written. An option on spot commodity requires delivery of the physical commodity, if exercised. An option on a futures contract requires delivery of a futures contract (not the spot commodity).

The given price of purchase is typically called the exercise price (E, or basis for London options). The last exercise date is the maturity of the option (T). The price of the underlying commodity will be designated S, if a spot price, or F, if a futures price. The exercise price will always be in the same units as the spot or futures price. An option to buy (sell) a unit of a commodity is a call (put) and its price or premium will be designated as C(S,T;E) (P(S,T;E)) or for options on futures C(F,T;E) (P(F,T;E)). If the call (put) can be exercised at any time during the period to maturity it is an American call (C, or American put, P). Calls (puts) which can only be exercised at maturity are European calls (c, or European put, p). Many relationships between puts and calls of both the American and European varieties are developed in Kruizenga (in Cootner [1964]), Samuelson and Merton [1969], Stoll [1969], and Merton [1973].

Commodity options (puts or calls on either spot commodities or futures contracts) are limited liability contracts. Option contracts are generally assumed to satisfy the so-called "bucket shop" assumption
concerning zero aggregate supply (see Merton [1973], footnote 6, for a brief history of this nomenclature). In essence the buyer and seller create a side bet on the price of the commodity. Unless the introduction of option trading changes the state space spanned by the extant securities, the distribution of commodity prices in the future will be unaffected (Ross [1976]). Unfortunately, without a full set of state contingent claims (only unconditional futures contracts currently exist), this assumption may not be realistic for commodity options.

The commodity option provides one form of contingent claim on a commodity. For instance, a European call on a commodity offers to the owner one unit of the commodity on a specific, single day at a fixed price, regardless of the then current price of that commodity. Of course, the call need not be exercised if the commodity price is below the exercise price — the commodity can be purchased in the market. At maturity, when the price of the option is below the exercise price, the premium (original price of the option) paid to insure purchase at or below the exercise price was, ex post, paid unnecessarily.

A futures contract provides a different contingent claim on a unit of spot commodity. Specifically, a futures contract purchases one unit of commodity on a specific date at a fixed price, regardless of the then prevailing price. There is, however, no option. The agreed-upon price is due on the delivery date in payment for the goods which are also deliverable at that time.

Both a commodity option written on a futures contract and a commodity option written on the spot commodity are contingent claims on the spot commodity. Potentially, both option and futures markets provide
instruments for suppliers or purchasers of inventories of the commodity to hedge against some of the relevant price risks. The hedge created with futures contracts exchanges price risk for basis risk. The basis risk (fluctuations in the difference between futures and spot price) typically has smaller magnitude than the risk of price fluctuations. Hedges created with commodity options (at least, American options) also partially hedge the basis risk. Holders of physical inventories of a commodity can conveniently hedge inventory price risk by writing options against inventories. Since both options and futures hedge changes in price of the spot commodity, both contingent claims depend upon the spot price in some as yet undetermined fashion.

Additionally, the inventory of the commodity involved will also affect all prices of claims for that asset (spot, future, and option). Inventory carrying theory is used to estimate the carrying charges for moving the spot commodity through time. These charges are necessary to distinguish the differential benefits (costs) of holding the spot commodity, a futures contract or a commodity option.

§ 4 Payouts Against the Commodity: Carrying Charges and the Basis

The calculation of profitability for holding inventories of the spot commodity (either hedged or unhedged) rests on the determination of the influence of inventories, production and consumption on the expected spot price. In the commodity literature, supply of storage theory describes this relationship.
The direct costs of holding inventory include warehouse rental and insurance. These costs are thought to be relatively constant over a wide range of inventory levels. Holding inventory ties up capital. An implicit interest charge is usually included in the direct costs of carrying inventory. The difference between futures price (a surrogate for expected spot price) and spot price is termed the basis. When the direct costs of carrying inventory are netted out of the basis, there is an empirically substantiated residual component termed the marginal convenience yield. This relationship (the "supply of storage" curve) is depicted in Figure 1.

FIGURE 1
"SUPPLY OF STORAGE" CURVE
The carrying charges per unit of spot commodity \( c(t) \) for the period from now until time \( t \) consist of interest plus other costs and benefits \( q \):

\[
c = rS + w + i + y = rS + q
\]

where \( r \) = current period interest rate

\( w \) = warehouse rent for \( t \) periods

\( i \) = insurance for \( t \) periods

\( y \) = convenience yield

The classic explanation of the marginal convenience yield phenomenon is that convenient inventories provide benefits to holders of inventory by reducing stockout costs. The theory contends that there is a reduction of plant down time and start-up costs which reduces the costs of idle capital. Also, convenient inventories reduce the chance of turning away good customers (or increase the chance of adding new customers) when inventories are scarce. Thus this theory holds that consumers will pay inventory holders for reducing their costs to locate supplies in times of scarcity. While there has been no completely satisfactory explanation of the supply of storage phenomenon, the empirical effect exists for many commodities (e.g., Brennan [1958], Working [1948, 1949]).

Supply of storage phenomena are relevant to all markets where there exists some possible option value to holding inventory. Mining and smelting capacity and harvest size and timing provide supply uncertainty. Uncertain demand for commodities adds to the uncertainty about expected spot price. If the price can be rationally expected to decline in the future, Cootner [1977] has shown that inventories should be held in
positive quantities even if the price rises at less than the rate of interest (plus direct storage costs). The difference between the going rate of interest (plus direct storage costs) and the rate of interest reflected in the difference between futures and spot prices can be interpreted as an option granted by consumers to inventory holders. Future consumers grant inventory holders the current use of stocks, if it is profitable to use them now, but if the stocks are stored into the future, consumers will buy them then. Inventory holders pay for this put (option to sell in the future) by selling forward (for future delivery) to consumers at a price lower than that necessary to store inventories into the future. This option value is the difference known heretofore as the marginal convenience yield.

The relevant question herein is not the explanation of the "convenience yield." The empirical existence of payouts (in the form of direct and indirect carrying charges) against the commodity requires different theoretical analysis for options written on the commodity. The values of a nonpayout protected option may differ from the value of a payout protected option (Merton [1973]). The ability of the (American) option owner to capture this value is at issue.

§ 5 Payout Effects on Commodity Option Valuation

Carrying the physical commodity and carrying the hedged commodity provide different expected rates of return. To see this, consider the strategies available for acquiring the commodity at some future time.
(1) The spot commodity can be bought and held at a direct known cost by purchasing now and storing until needed. Empirically, there are some implicit benefits to this strategy (the convenience yield). The expected return is a function of the known costs (warehouse rental, insurance and interest) and unknown benefits.

(2) A futures contract with sufficiently long maturity can be entered into at a known price. When the need for the commodity arises, the futures contract is sold and the spot commodity is bought in the market at that time. The expected return of this strategy is dependent upon the movement of the spot price with respect to the futures price.

(3) A commodity option with sufficiently long maturity can be purchased at a known premium. This option can be written on either a futures contract or the spot commodity.

(3a) When the need arises for the commodity, a European option (which cannot be exercised until maturity) is sold and the commodity is purchased in the spot market at that time. European options on spot and futures contracts may have different expected returns, but the mechanics of acquiring the spot commodity are identical. The expected return depends upon the comovement of the option and spot prices for the commodity.

(3b) With an American option two strategies are possible for obtaining the commodity. The American option can be sold and the spot commodity purchased. In addition, an American option can be exercised, thus making available at the known, previously established exercise price either the spot commodity, or a futures contract. If a futures contract is obtained by exercise, it is then sold, and the spot commodity is purchased. The expected return on these strategies depends on the comovement of the spot (and futures) and the option price.

(4) Finally, the spot commodity can be bought in the market only when needed. Carrying costs are avoided, but the risk of spot price change is borne over that period of time.

Each of these strategies has a different rate of return contingent upon the realized state of the world (as characterized, for instance, by the estimated or realized size of the next harvest). Each strategy would have different risks, and dominance or arbitrage will determine the
equilibrium relationship between expected return and risk for the previous strategies.

What is not obvious from the scenarios above is whether an option holder can earn the convenience yield which accrues to the holder of the spot commodity. If this is possible, it should be reflected in the premium paid for the option. This is extremely important in terms of the option pricing models, since early developments in security option models assume that there are no payouts against the security and no unfavorable exercise price changes.\textsuperscript{9} Under that assumption, the value of a European and American call is equal \((C = c)\).

The following digression on options on stock with payouts will assist in the analysis of options on commodities with payouts. If an option is not protected against unfavorable exercise price changes or dividend payouts on the underlying stock, the value of the option (Merton [1973]) is reduced. As the dividend increases (ceteris paribus), the current value of a nonpayout protected call decreases. If the exercise price per unit is increased without corresponding change in the number of units to be purchased, the current value of a call decreases. Lack of contractual arrangements (in the option) for payout protection requires that all anticipated payouts against the stock be discounted in the value of the stock option.\textsuperscript{10}

A commodity option is analogous to an option on a stock which pays dividends. The expected return available to the holder of a (non-payout protected) commodity option is different from the expected return on the spot commodity. The different rates of return arise due to different risks assumed and potential payouts (in the form of carrying
charges) to the holder of the spot commodity. Thus, it is necessary to consider option pricing models which incorporate the value of protection against payouts. 11

The owner of the European commodity option cannot benefit from the convenience of having the commodity currently available at a known cost (the direct costs of storage). A holder of an American option may partially benefit from the convenience yield by exercising the option prematurely to earn any payouts on the spot commodity. Of course, the holder of an American commodity option may pay for the right to exercise at any time during the life of the option. An American call will always be worth as much as a European call, and perhaps more (Merton [1973]). The differential premium on American and European commodity call options must be related to the carrying charges for holding the physical asset (the basis).

Analysis of the relationship between the payouts on commodity options (in the form of carrying charges as a function of total inventory) and the overall profitability of carrying inventory leads to a simplified formula for the commodity option price. However, it is necessary to go beyond a simple arbitrage relationship between the commodity option and spot commodity to include some overall inventory carrying theory. In later sections, payouts on the spot commodity are incorporated into the analysis of options on the spot commodity in a simple and useful fashion.
§ 6 Dynamics of Spot Prices

Before continuing on to the evaluation of commodity option prices, the dynamics of the spot price (the price for current delivery of the commodity) will be specified.

For stock option pricing, Samuelson [1965], Samuelson and Merton [1969], Black and Scholes [1972] and Merton [1973] assume that prices follow a relative (geometric) diffusion

\[ dS = \mu S dt + \sigma S dz \]

In this formulation, \( \mu \) and \( \sigma \) are assumed to be nonstochastic time-invariant constants. Cox [1975] uses models of stock dynamics with constant elasticity of variance diffusions \( (0 \leq \beta \leq 2) \)

\[ dS = \mu(S,t) dt + \sigma S^{(\beta/2)} dz \]

Cox and Ross [1976] and Merton [1976] use Poisson processes and/or mixes of the above diffusions for the underlying stock processes. The form of their respective option pricing models follows from the assumptions made about stock price dynamics. 12

An adequate theory of commodity option pricing requires a description of the underlying process (either spot or futures) price dynamics. In the following paragraphs, aspects of the underlying problem of specifying the spot and future price dynamics are examined. Although a full solution of the entire formulation is not presented here, various important features will be noted for use in approximations of commodity option values.
Cootner [1977] specified the relationship between spot price $(S(x,t))$ and inventory $(x)$ in a stochastic form which differs significantly from the original work by Samuelson [1957 to 1971]. Inventory $(x)$ is assumed to follow a locally Brownian\textsuperscript{13} process:

$$dx = \mu^X(x,t)dt + \sigma^X(x,t)dz$$

where $dz$ is a normal Wiener process with zero drift and unit instantaneous variance

$\mu^X(x,t)$ is the drift of inventory, and

$\sigma^X(x,t)$ is the instantaneous standard deviation of inventory.\textsuperscript{14}

Using Ito's Lemma,\textsuperscript{15} the spot price dynamics are:

$$dS = \left(S_t + \mu^X_S S_x + 1/2[\sigma^X]^2 S_{xx}\right)dt + (S_x \sigma^X)dz$$

Without pursuing the solution of the optimal inventory/spot price policy, it is apparent that the spot price dynamics will be dependent on inventory levels and can be specified generally as:

$$dS = \mu^S(x,S,t)dt + \sigma^S(x,S,t)dz$$

This model is sufficiently flexible to admit a basis (future-spot price) versus inventory relationship that is consistent with the general form of the (empirical) supply of storage curve (Figure 1). A complete solution of that problem is useful, but it is not required. The spot process form, dynamics and parameters are specified in sufficient generality to accommodate a reasonable approximate solution to the option pricing problem.
Several simplifying assumptions are made. The spot price dynamics are assumed to follow the locally Brownian motion:

\begin{equation}
\frac{dS}{S} = \mu(S,t)dt + \sigma(S,t)dz
\end{equation}

\begin{align*}
\mu(S,t) &= S_t + \mu^x S_x + \frac{1}{2} \sigma^x S_{xx} = \mu^S(x,S,t) \\
\sigma(S,t) &= S_x \sigma^x = \sigma^S(x,S,t)
\end{align*}

The dependence on inventory is suppressed, but not forgotten \textit{(especially in the empirical estimation)}. Starting with a simple diffusion

\begin{equation}
\frac{dS}{S} = \mu S dt + \sigma S dz
\end{equation}

mismodels the process by oversimplification. In some respects, the degree of nonlinearity and nonstationarity of \( \mu(x,S,t) \) and \( \sigma(x,S,t) \) are at question. Some evidence on this empirical question exists which indicates substantial nonlinearity and nonstationarity.\(^{16}\) For the dynamics of spot price, a nonspecific locally Brownian process\(^ {17}\) is assumed. By adaptation of the parameters, equation (1) models many possible spot commodity stochastic processes.

\section*{§ 7 Dynamics of Futures Prices}

In addition to specifying the parametric form and dynamics of the spot price, the futures price process must also be specified. Futures contracts, which are defined in § 10, and commodity options written on futures contracts are both contingent upon the futures price.
The futures price is the price agreed upon now for future delivery of the commodity. When this price is paid (by the buyer to the seller) is specified in the futures contract. The current conventions in futures markets will be described with the definition of futures contracts. Consistency with the specification of the spot price and with empirical observation are both necessary in the dynamic description of futures prices.

Whether or not the futures price is equal to the expected spot price is an unresolved issue in the literature. Cootner [1967] concluded that hedgers in futures markets (either long or short, whichever predominate) pay speculators a risk premium which is implicit in the futures price. Thus he claims the futures price should not equal the expected spot price. Miller [1971] in a capital asset pricing model application suggests that her evidence allows for no risk premium. Unfortunately, the key issue of the Cootner [1967] paper was ignored. If hedgers sometimes pay a premium to long speculators, and at other times pay a premium to short speculators, then the optimal holding strategy will not be a buy and hold strategy (which is what Miller [1971] assumes implicitly by calculating average holding period returns to a long position in futures contracts). Thus, the evidence of that study cannot be relied upon without further empirical work. Further evidence on the variance of future prices is presented in §11.

Without loss of generality, assume that the futures price for \( T \) periods hence is denoted (without an indication of the maturity which will be obvious in all contexts) as \( F(t) \). The future price dynamics
follow the locally Brownian motion (the superscripts on $\mu$ and $\sigma$ will be omitted when there is no chance for confusion):

\begin{equation}
\begin{aligned}
\text{(3)} \quad dF &= \mu(F,t)dt + \sigma(F,t)dz \\
\mu(F,t) &= \mu^F(x,S,F,t) \\
\sigma(F,t) &= \sigma^F(x,S,F,t)
\end{aligned}
\end{equation}

The drift and instantaneous standard deviation are both suppressed functions of inventory and spot price as well as time and the futures price. The dependence on inventory and spot price is suppressed, but not forgotten, especially in the empirical estimation.

Again, a broadly specified model (which can be simplified as the evidence dictates) is assumed. Thus, if no mean drift for the futures price process is observed, $\mu(F,t)$ can be set to zero. Or, if the evidence dictates, a relative diffusion for futures prices can be assumed.

\section*{§ 8 Valuation of Options on the Spot Commodity}

The spot price dynamics from § 6 is used below to evaluate an option on the spot commodity. The commodity option is a security, $C$, the value of which depends on the spot price and time $C(S,t)$. The spot commodity price $S(t)$ has dynamics given by equation (1) previously

\begin{equation}
\begin{aligned}
dS &= \mu(S,t)dt + \sigma(S,t)dz.
\end{aligned}
\end{equation}

However, the maintenance of the spot value requires the payment of the direct carrying charges (insurance and warehouse costs) and the potential
receipt of the convenience yield. The instantaneous payout function \( q(S,t) \) summarizes these costs and benefits.

Consider the portfolio consisting of a mixture of the spot commodity and the commodity option

\[
P = n_S S + n_C C(S,t)
\]

where \( n_S \) is the number of contracts of spot

\( n_C \) is the number of commodity options

\( P \) is the value of the portfolio.

The change in the value of the portfolio (including the payouts) is

\[
dP = n_S dS + n_C dC - n_S q(S,t)dt
\]

The return on the portfolio is given by

\[
\frac{dP}{P} = \alpha_S \left[ \frac{dS - q(S,t)dt}{S} \right] + \alpha_C \frac{dC}{C}
\]

where \( \alpha_S = \frac{n_S S}{P} \) is the percentage of the portfolio invested in spot

\( \alpha_C = \frac{n_C C}{P} \) is the percentage of the portfolio invested in commodity options

\( \alpha_S + \alpha_C = 1 \)

Assume that the option can be represented in differential form as

\[
dC = \mu_C dt + \sigma_C dz
\]
Substituting the dynamics of spot \((dS)\) and option \((dC)\) into the portfolio return equation and rearranging

\[
\frac{dp}{p} = \left\{ \frac{\alpha_s (\mu(S, t) - q(S, t))}{S} + \frac{\alpha_c \mu_c}{C} \right\} dt + \left\{ \frac{\alpha_s \sigma(S, t)}{S} + \frac{\alpha_c \sigma_c}{C} \right\} dz
\]

By choosing the portfolio weights properly, the portfolio can be made riskless.

\[
\frac{\alpha_s \sigma(S, t)}{S} + \frac{\alpha_c \sigma_c}{C} = 0
\]

This implies that the portfolio must earn the riskless rate of return

\[
\frac{dp}{p} = \left\{ \frac{\alpha_s (\mu(S, t) - q(S, t))}{S} + \frac{\alpha_c \mu_c}{C} \right\} dt = \mu_c dt
\]

Rearranging gives the fundamental option valuation relationship

\[
\frac{\mu(S, t) - q(S, t) - rS}{\sigma(S, t)} = \frac{\mu_c - r_c}{\sigma_c}
\]

Ito's Lemma is applied to \(C(S, t)\) to specify the dynamic motion of the commodity option

\[
dC = \{C_c + \mu(S, t)c_S + b\sigma^2(S, t)c_{SS}\} dt + \{\sigma(S, t)c_S\} dz \equiv \mu_c dt + \sigma_c dz
\]

Substituting the parameters \(\mu_c\) and \(\sigma_c\) from (5) into (4) and rearranging gives the partial differential equation for the value of the commodity option \(C(S, t)\).
\[ \frac{1}{2} \sigma^2(S,t)C_{SS} + (rS + q(S,t))C_S - rC + C_T = 0. \]

The option differential equation is not analytically tractable, except for very special cases of the payout function \( q(S,t) \).\(^{21}\) Fortunately the market provides empirical estimates of \( q(S,t) \) as the difference between futures and spot prices net of interest costs. These yield naturally to solutions using numerical methods such as finite differences.

Some minor changes facilitate solution of the equation. If time is measured from the expiration date, then \( dt = -dT \). The terminal condition for the option and a boundary condition\(^{22}\) at \( S = 0 \) completes the specification of the partial differential equation for the spot commodity option value.

\[ \frac{1}{2} \sigma^2(S,T)C_{SS} + (rS + q(S,t))C_S - rC - C_T = 0 \]

(6) \[ C(S,0) = \max(0,S-E) \]

\[ C(0,T) = 0. \]

\[ \S \ 9 \ Evidence \ on \ the \ Pricing \ of \ Spot \ Commodity \ Options \]

The model for pricing options on spot commodities will be tested on a sample of 388 call options written on silver.\(^{23}\) The options were offered with fixed maturity dates (four series a year) and with various exercise prices near the then current price of silver.

The spot price of silver was extrapolated from the Commodity Exchange, Inc. futures market prices. The daily spot and futures price
series extend from January 1967 to June 1974. During the period of the study (May–September 1973) the price of silver ranged from $2.3430/oz. to $3.0100/oz. without any discernible trend. The carrying charges for silver were also estimated from the current day's future price term structure.

The riskless rate for the maturity of the contract was estimated from the prevailing government bond structure in much the same way that the carrying charges were estimated from the silver term structure.

Estimation of the instantaneous variance is a serious problem which affects evaluation of any option pricing model. Black and Scholes [1972] found that estimating variance from past data caused their option pricing formula for stocks to overprice options on high variance stocks and underprice options on low variance stocks. They suggested that this was evidence of measurement error or possibly nonstationarity in the variance.24

Obtaining a useful estimate of the instantaneous variance is difficult, but the problem is not insoluble. Accurate utilization of prior information in modeling the stochastic process of commodity future returns is necessary to obtain unbiased estimates of the process parameters. A technique which samples the generating stochastic differential equation for price is used. An adaptive Kalman filter then computes the updated variance parameter estimate.25 The actual change in the variance parameter of the spot price process over time was substantive. The estimated daily variance parameter ranged from 2.5¢/oz. to 3.5¢/oz. over the period of the study. Any single estimate would have provided a poor approximation to the variance parameter even during this short period.
A model option price is calculated with the same exercise price, maturity and spot price as each of the actual option prices. The interest rate and variance were estimated from historic information as indicated above. For each set of parameters corresponding to a sample price, the option valuation partial differential equation (Equation (6)) was solved.

An example of the output of the numerical solution to the partial differential equation is shown in Table 2. It is necessary to compute a range of current spot prices and times to maturity to obtain the single necessary data point. On May 11, 1973, with silver selling at $2.43/oz., a 294 day option with exercise price $2.40 was priced by the model at $1896. The option was actually selling at $2950.

<table>
<thead>
<tr>
<th>Current Spot Price</th>
<th>Days to Exercise</th>
</tr>
</thead>
<tbody>
<tr>
<td>24000</td>
<td>0</td>
</tr>
<tr>
<td>24750</td>
<td>750</td>
</tr>
<tr>
<td>25500</td>
<td>1500</td>
</tr>
<tr>
<td>26250</td>
<td>2250</td>
</tr>
<tr>
<td>27000</td>
<td>3000</td>
</tr>
<tr>
<td>27750</td>
<td>3750</td>
</tr>
<tr>
<td>28500</td>
<td>4500</td>
</tr>
<tr>
<td>29250</td>
<td>5250</td>
</tr>
<tr>
<td>30000</td>
<td>6000</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Exercise Price</th>
<th>C(S,T) Silver Option</th>
<th>May 11, 1973</th>
</tr>
</thead>
<tbody>
<tr>
<td>24000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>24750</td>
<td>479</td>
<td>517</td>
</tr>
<tr>
<td>25500</td>
<td>641</td>
<td>687</td>
</tr>
<tr>
<td>26250</td>
<td>846</td>
<td>896</td>
</tr>
<tr>
<td>27000</td>
<td>1036</td>
<td>1151</td>
</tr>
<tr>
<td>27750</td>
<td>1396</td>
<td>1453</td>
</tr>
<tr>
<td>28500</td>
<td>1747</td>
<td>1806</td>
</tr>
<tr>
<td>29250</td>
<td>2152</td>
<td>2210</td>
</tr>
<tr>
<td>30000</td>
<td>2608</td>
<td>2665</td>
</tr>
</tbody>
</table>
The model prices were compared to the prevailing market prices with OLS regression for descriptive purposes. Although the model explained 82 percent of the variation in market prices, the model prices were downward biased estimates of market price.26

One test for market efficiency is to purchase "undervalued" options and sell "overvalued" options at market prices. Black and Scholes [1972] use this technique for testing their stock option valuation formula. These strategies follow individual commodity options from issuance to expiration. The options are hedged with the spot commodity at the going market price to balance the portfolio position using the appropriate weights given in the theoretical valuation formula. If the model of commodity option pricing is valid, the riskless rate of interest would be earned on these positions. If this strategy gives significant returns over the risk free rate, then a hypothesis of market inefficiency (before transactions costs) might be entertained. In practical terms, the positions called for must be feasible. Execution of orders at market prices such as the open and close may not be feasible due to price limits in most commodity markets. Care is taken to eliminate situations that require trading when none occurred. The transactions costs of continuously adjusting these positions are large. Anyone attempting to emulate this strategy would look beyond individual positions to the entire commodity portfolio (options plus spot) and balance the costs of transacting against the potential losses of holding unbalanced positions.

A market consensus ex ante estimate of the variance should be necessary for the option pricing formula to give correct values. If both the market and the model fully use the same information sources
then the strategy would earn the riskless return. If the market uses information that the model does not, the strategy would earn no more than the riskless rate, and possibly less. If the model uses information more fully than the market, then the strategy would earn in excess of the riskless rate. This test sheds some light on the relative efficiency of the models' (vis a vis the market's) use of market information to estimate the instantaneous variance for the commodity.

The result of pursuing the arbitrage portfolio during the time period of the sample is given in Table 3.

TABLE 3

<table>
<thead>
<tr>
<th>Excess Dollar Returns from Riskless Arbitrage Portfolios</th>
<th>(Per Contract Per Day)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average Excess $ Return</td>
<td>Standard Error of Excess $ Return</td>
</tr>
<tr>
<td>- $16</td>
<td>$17.78</td>
</tr>
</tbody>
</table>

Interpretation of these results should be approached with care. A null hypothesis that the average excess dollar return is zero cannot be rejected.

The model of § 8 was tested against a sample of 388 options available during 1973. Regression analysis indicates that the model provided biased, but useful estimates of the market price. An attempt to exploit differences between model and market prices and to earn riskless arbitrage profits showed results insignificantly different from zero. Thus, a hypothesis of market efficiency could not be rejected.
§ 10 Valuation of Options on Futures Contracts

A futures contract is an unconditional promise to deliver a certain amount (one contract) of the commodity on the delivery date \((T)\) for payment of a price \(F\) (known as the futures price for that delivery date). The price is to be paid on the delivery date. Thus, if \(F(s)\) is the current futures price, then the value of a futures contract at time \(t, s < t < T\) is

\[
Z(t) = (F(t) - F(s))B(t)
\]

where \(B(t) = e^{\tau(t-T)}\) is the value of a riskless discount bond.

Two problems plague the evaluation of an option on this futures contract. First, an option on this futures contract written at price \(F(s)\) will be traded only randomly in the period until maturity of the option. The London Metal Exchange futures and options market closely conform to this particular theoretical construct. Second, most traded futures contracts (The International Commodity Clearing House in London and all American exchanges) required daily settlement by each side of the trade to mark to the current market price. That is, a loser must pay losses to the winner at the close of each day. At any time within the period from initiation to delivery, the actual settlement futures contract value is a complicated function of the discounted disbursements from the settlement process. What started out to be the simple evaluation of an American commodity option without payouts is actually a problem with daily payouts and only a numerical solution.
In what follows, assume that the future price sets the value of a pure futures contract to zero at initiation time \( s \) where \( f = F(s) \) is the futures price at option initiation

\[
Z(s) = (F(s) - f)B(s) = 0.
\]

The futures price \( F(t) \) follows an absolute diffusion with instantaneous drift and variance which do not depend on the level of prices

\[
dF = \mu dt + \sigma dz.
\]

By Itô's Lemma, the dynamics of the future contract are

\[
dZ = \frac{\partial Z}{\partial t} dt + \frac{\partial Z}{\partial F} dF + \frac{1}{2} \frac{\partial^2 Z}{\partial F^2} (dF)^2 \equiv \mu_Z dt + \sigma_Z dz.
\]

There are no intermediate payouts from the option or the futures contract, the maturity of the option and future coincide and the exercise price for this option is zero. Once the premium is paid for this option, nothing else is due to complete this option contract! Upon delivery of the futures contract, the liability associated with the futures contract is due, but this is not part of the option contract.

Using arguments similar to those in §8, the partial differential equation for the option price \( c(Z,t) \) including the terminal condition is

\[
\frac{1}{2} \sigma^2 \frac{\partial^2 c}{\partial Z^2} + rcZ - rc + c_r = 0
\]

\[
c(Z(T),T) = Z(T)^+.\]
Performing standard change of variables to the futures price $F = B(t)^{-1}Z + f$, the partial differential equation for the value of the option $C(F,t)$ is:

\begin{equation}
\frac{1}{2}\sigma^2 C_{FF} - rC + C_t = 0
\end{equation}

\[ C(F(T),T) = (F(T)-f)^+ \]

Using a technique due to Feller [1951], the solution for the value of an option on a pure futures contract is \(^2\)

\begin{equation}
C(F,t) = B(t)\{F(t)(N(Y_1) + N(Y_2)) - F(s)(N(Y_1) - N(Y_2))\}
+ \sigma\sqrt{T-t}(n(Y_1) - n(Y_2))
\end{equation}

where

\[ Y_1 = \frac{(F(t)-F(s))B(t)}{\sigma\sqrt{T-t}} \]

\[ Y_2 = \frac{-(F(t)+F(s))B(t)}{\sigma\sqrt{T-t}} \]

and

\[ N(x) = \int_{-\infty}^{x} n(y)dy \]

\[ n(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \]

§ 11 Evidence on the Pricing of Commodity Options on Futures Contracts

Although the proposed commodity option valuation equation does not conform exactly to the settlement convention on the International Commodities Clearing House, it is used as a close surrogate of the
correct equation. Data on 155 options traded on seven contracts for three different commodities is presented for the period January 12, 1977 through March 31, 1977. This data represents approximately 1413 contracts of the various commodities. A description of the London Commodity Options appears in Table 4, the observation period is described in Table 5 at the back of this section, and the distribution of maturities in the sample is displayed in Figure 2.

London Commodity Options are (almost) always written with the exercise price equal to the then current futures price, thus the contract trades initially at the money. Since the exercise price is not one of a small subset of predetermined prices, it is rare for a London option to trade in the secondary market. London options are effectively sold and held by the purchaser.

For each of the 155 option observations, the actual futures price and the maturity were used along with the other parameters to compute a model price for each option. London short term interest rates for various maturities were obtained from the Financial Times. The variance was estimated for both absolute and relative diffusions using various periods of data and techniques. Evidence on how these estimates affect the model option prices is presented later in this section.

The model prices were estimated using an absolute diffusion option model with an adaptive variance estimate. These model prices are compared to actual market prices in Table 6 using simple regression techniques. Overall, the model price explained 84 percent of the variance in market price but this figure is deceptive for reasons which will become apparent shortly. The regression statistics for each of the three
### TABLE 4

LONDON COMMODITY OPTIONS ON SOFT COMMODITY FUTURES CONTRACTS

<table>
<thead>
<tr>
<th>Commodity</th>
<th>Contract Size</th>
<th>Premium and Basis</th>
<th>Quoted</th>
<th>Commissions***</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coffee</td>
<td>5 metric tons</td>
<td>£ per metric tons</td>
<td></td>
<td>£ 32</td>
</tr>
<tr>
<td>Sugar</td>
<td>50 metric tons</td>
<td>£ per metric tons</td>
<td></td>
<td>£ 36</td>
</tr>
<tr>
<td>Cocoa</td>
<td>10 metric tons</td>
<td>£ per metric tons</td>
<td></td>
<td>£ 32</td>
</tr>
</tbody>
</table>

*Premium is the price of the option.

**Basis is the exercise price for the option.

***1/2 listed commission charged at origination, 1/2 charged at declaration (no declaration charge, if option abandoned).

### FIGURE 2

LONDON COMMODITY OPTIONS: DISTRIBUTION OF MATURITIES
commodities and all seven contracts are presented in Table 6. The intercepts which are all expected to be zero are significantly nonzero. The slopes for Coffee and Cocoa options are insignificantly different from one, but Sugar options are mispriced by the model compared to current market price. The residuals for all of the regressions appear to be normal.

A similar regression utilizing prices of options divided by the exercise price is computed to look at the relative pricing ability of the model. Overall the relative model price only explains 34 percent of the variance in relative market price, but the slope is insignificantly different from one for the whole sample. The regression statistics for the relative pricing are given in Table 7.

To check for major differences using absolute or relative diffusion option pricing models, the model prices for each diffusion and all variance estimating techniques were computed. Given the relative homogeneity of the parameters, there was no essential difference based on the type of diffusion model used. Nor is a difference expected. If these options were not all at the money, the differences would be much larger.

The effect of various variance estimators and diffusion processes on the overall fit of the model versus the market is estimated. The model fit as measured by $R^2$ is virtually identical for both processes and all variance techniques. The slope and intercept for the regressions utilizing the adaptive variance estimate are demonstrably better than the others, with the intercept insignificantly different from zero and the slope nearer to one (although still significantly different than one).
To check the variance estimators another way, the implied standard deviations using an absolute diffusion option model were calculated and compared to the various standard deviation estimates. As Table 9 shows, the adaptive estimates had the lowest mean square and mean absolute errors of the four variance techniques.

TABLE 9

LONDON COMMODITY OPTIONS: DIRECT COMPARISON OF IMPLIED STANDARD DEVIATION ESTIMATES WITH A PRIORI ESTIMATED STANDARD DEVIATIONS

<table>
<thead>
<tr>
<th>Variance Technique</th>
<th>Mean Square Error</th>
<th>Mean Absolute Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>All Previous Obs.</td>
<td>32.5681</td>
<td>4.86966</td>
</tr>
<tr>
<td>100 Day M.A.</td>
<td>21.3111</td>
<td>4.09137</td>
</tr>
<tr>
<td>30 Day M.A.</td>
<td>14.4184</td>
<td>3.06333</td>
</tr>
<tr>
<td>Adaptive</td>
<td>12.3335</td>
<td>2.82644</td>
</tr>
</tbody>
</table>

As with the spot commodity options, an arbitrage portfolio was constructed consisting of the correct theoretical portions of the commodity option, futures contract and riskless security. The results of this arbitrage portfolio are presented in Table 10. Without including transaction costs, the initial average discrepancy between the market and model price was 436 pounds. The cumulative average hedging loss was 331 pounds, and the total return from the whole arbitrage operation was 104 pounds. While all of these estimates are statistically significant, none
include any transaction costs. The inclusion of only the transaction
cost of the initial purchase or sale of the option would make the whole
arbitrage operation have profits insignificantly different from zero.

To further analyze the discrepancies of the model, and point the
way to further improvement, the various market to model performance meas-
ures were compared to the starting date, maturity, variance estimate, and
exercise prices in Tables 11 through 14. Neither the starting date nor
the maturity seem relative specially to the various performance measures.
Although the initial guess of the model price is negatively related to
the variance estimate, the cumulative hedging returns are positively re-
lated to the variance estimate. The total return from the option pur-
chase and hedging operation is not significantly related to the variance
estimate. The model performance measures seem to be minimally related
to the exercise price.

Overall, the model seems to provide estimates of commodity op-
tion prices on commodity futures contracts which are generally in line
with those observed in the market. There does not appear to be signifi-
cantly positive arbitrage opportunities in the London soft commodity op-
tion market.

§ 12. Summary and Conclusions

In this paper, some introductory theoretical aspects of the pric-
ing of commodity options on spot commodities and futures contracts are
developed. A detailed discussion of the relevant inventory carrying
theory precedes the development of the models for spot and futures prices.
The stochastic processes provided here determine the nature of the solutions for commodity option prices.

Empirical evidence on the efficiency of the market is considered by comparing the theoretical model prices to the actual market prices, and by looking for riskless arbitrage profits. A sample of traded commodity options on London futures contracts (sugar, coffee, cocoa) is analyzed. Differences from market values are within the bounds of transaction costs. Another sample of options on spot commodities (silver) also demonstrates and confirms the general accuracy of the models previously proposed. Empirically, the option pricing model provides estimates that are in agreement with the prices of traded options. A hypothesis of commodity option market efficiency cannot be rejected.
<table>
<thead>
<tr>
<th>Contract</th>
<th>Observation Period</th>
<th># of Trading Days</th>
<th># Observations Available</th>
<th>Premium: ( # of Contracts) During Observation Period</th>
<th>Volume* ( # of Contracts) During Observation Period</th>
<th>Average Premium During Observation Period (Per Contract)</th>
<th>Expiration Date</th>
</tr>
</thead>
<tbody>
<tr>
<td>March Coffee</td>
<td>Jan 12, 1977 - Jan 20, 1977</td>
<td>7</td>
<td>4</td>
<td>18</td>
<td>£1375</td>
<td>Feb 28, 1977</td>
<td></td>
</tr>
</tbody>
</table>

*Volume reported is morning plus afternoon transactions where available. Actual volume exceeds this due to missing volume data in early part of period.
TABLE 6

LONDON COMMODITY OPTIONS: COMPARISON OF MARKET TO MODEL* PRICE AND RESIDUAL ANALYSIS

(Market Price) = α + β(Model Price)

<table>
<thead>
<tr>
<th>Analysis Variable</th>
<th>$H_0: \beta=0$</th>
<th>$t_\beta$</th>
<th>α</th>
<th>$t_\alpha$</th>
<th>$R^2_{ADJ}$</th>
<th>$\sigma$(Resid)</th>
<th>$\sigma(\beta)$</th>
<th>$t_\beta$</th>
<th>Number Observations</th>
<th>Studentized Range of Residuals</th>
</tr>
</thead>
<tbody>
<tr>
<td>Overall</td>
<td>1.271</td>
<td>27.976</td>
<td>36</td>
<td>.550</td>
<td>.8354</td>
<td>364.211</td>
<td>.045</td>
<td>5.969</td>
<td>155</td>
<td>5.131</td>
</tr>
<tr>
<td>Coffee</td>
<td>.835</td>
<td>8.006</td>
<td>809</td>
<td>5.098</td>
<td>.4670</td>
<td>381.178</td>
<td>.104</td>
<td>1.581</td>
<td>73</td>
<td>3.979</td>
</tr>
<tr>
<td>Sugar</td>
<td>1.313</td>
<td>11.666</td>
<td>-205</td>
<td>-2.821</td>
<td>.7299</td>
<td>134.178</td>
<td>.113</td>
<td>2.779</td>
<td>51</td>
<td>5.264</td>
</tr>
<tr>
<td>Cocoa</td>
<td>.888</td>
<td>13.481</td>
<td>825</td>
<td>6.082</td>
<td>.8576</td>
<td>211.110</td>
<td>.066</td>
<td>-1.699</td>
<td>31</td>
<td>3.871</td>
</tr>
<tr>
<td>May Coffee</td>
<td>-.236</td>
<td>-.507</td>
<td>1958</td>
<td>3.585</td>
<td>.0000</td>
<td>342.629</td>
<td>.047</td>
<td>-2.658</td>
<td>27</td>
<td>3.499</td>
</tr>
<tr>
<td>July Coffee</td>
<td>.708</td>
<td>3.714</td>
<td>1084</td>
<td>3.228</td>
<td>.2378</td>
<td>389.965</td>
<td>.191</td>
<td>-1.531</td>
<td>42</td>
<td>3.409*</td>
</tr>
<tr>
<td>May Sugar</td>
<td>1.831</td>
<td>4.864</td>
<td>-352</td>
<td>-2.119</td>
<td>.6538</td>
<td>157.290</td>
<td>.376</td>
<td>2.207</td>
<td>13</td>
<td>3.368</td>
</tr>
<tr>
<td>May Cocoa</td>
<td>1.374</td>
<td>12.449</td>
<td>110</td>
<td>.629</td>
<td>.9112</td>
<td>98.610</td>
<td>.110</td>
<td>3.391</td>
<td>16</td>
<td>3.129</td>
</tr>
<tr>
<td>July Cocoa</td>
<td>.959</td>
<td>7.640</td>
<td>600</td>
<td>1.942</td>
<td>.8038</td>
<td>256.945</td>
<td>.125</td>
<td>-.323</td>
<td>15</td>
<td>3.497</td>
</tr>
</tbody>
</table>

*Model using absolute diffusion option model with adaptive variance estimate.

* The probability that the estimated statistic (given the sample size) is from a normal distribution is less than .01.
<table>
<thead>
<tr>
<th>Analysis Variable</th>
<th>$\beta$</th>
<th>$t_\beta$</th>
<th>$\alpha$</th>
<th>$t_\alpha$</th>
<th>$R^2_{\text{adj}}$</th>
<th>$\sigma_{\text{(Resid)}}$</th>
<th>$\beta$</th>
<th>$t_\beta$</th>
<th>$\beta=0$</th>
<th>Number Observations</th>
<th>Studentized Range of Residuals</th>
</tr>
</thead>
<tbody>
<tr>
<td>Overall</td>
<td>.825</td>
<td>9.010</td>
<td>.0359</td>
<td>4.324</td>
<td>.3424</td>
<td>.0269</td>
<td>.0916</td>
<td>1.910</td>
<td>.0916</td>
<td>155</td>
<td>4.443</td>
</tr>
<tr>
<td>Coffee</td>
<td>.600</td>
<td>4.173</td>
<td>.0692</td>
<td>5.448</td>
<td>.1857</td>
<td>.0245</td>
<td>.1438</td>
<td>-2.782</td>
<td>-2.782</td>
<td>73</td>
<td>4.149</td>
</tr>
<tr>
<td>Sugar</td>
<td>1.267</td>
<td>10.697</td>
<td>-.0268</td>
<td>-2.343</td>
<td>.6940</td>
<td>.0206</td>
<td>.1184</td>
<td>2.255</td>
<td>2.255</td>
<td>51</td>
<td>5.171</td>
</tr>
<tr>
<td>Cocoa</td>
<td>.907</td>
<td>16.512</td>
<td>.0317</td>
<td>6.761</td>
<td>.9005</td>
<td>.0084</td>
<td>.0549</td>
<td>-1.694</td>
<td>-1.694</td>
<td>31</td>
<td>4.049</td>
</tr>
<tr>
<td>March Coffee</td>
<td>5.956</td>
<td>10.314</td>
<td>-.1501</td>
<td>-6.090</td>
<td>.9723</td>
<td>.0039</td>
<td>.5775</td>
<td>8.582</td>
<td>8.582</td>
<td>4</td>
<td>1.943</td>
</tr>
<tr>
<td>May Coffee</td>
<td>.476</td>
<td>1.071</td>
<td>.1343</td>
<td>4.289</td>
<td>.0056</td>
<td>.0203</td>
<td>.4443</td>
<td>-1.179</td>
<td>-1.179</td>
<td>27</td>
<td>3.693</td>
</tr>
<tr>
<td>May Sugar</td>
<td>1.744</td>
<td>4.300</td>
<td>-.0486</td>
<td>-1.784</td>
<td>.5931</td>
<td>.0249</td>
<td>.0406</td>
<td>18.325</td>
<td>18.325</td>
<td>13</td>
<td>3.466</td>
</tr>
<tr>
<td>August Sugar</td>
<td>1.502</td>
<td>9.458</td>
<td>-.0543</td>
<td>-3.255</td>
<td>.7051</td>
<td>.0175</td>
<td>.1502</td>
<td>3.342</td>
<td>3.342</td>
<td>38</td>
<td>3.269†</td>
</tr>
<tr>
<td>July Cocoa</td>
<td>.946</td>
<td>9.697</td>
<td>.0256</td>
<td>2.549</td>
<td>.8635</td>
<td>.0104</td>
<td>.0976</td>
<td>-0.55</td>
<td>-0.55</td>
<td>15</td>
<td>3.504</td>
</tr>
</tbody>
</table>

*Model using absolute diffusion option model with adaptive variance estimate.

†The probability that the estimated statistic (given the sample size) is from a normal distribution is less than .01. 31
<table>
<thead>
<tr>
<th>Diffusion Process</th>
<th>Variance Technique</th>
<th>$\beta$</th>
<th>$t_\beta$</th>
<th>$\alpha$</th>
<th>$t_\alpha$</th>
<th>$R^2_{\text{adj}}$</th>
<th>$\sigma(\text{Resid})$</th>
<th>$\hat{\sigma}(\beta)$</th>
<th>$t_\beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Absolute</td>
<td>All Previous Obs.</td>
<td>2.257</td>
<td>25.387</td>
<td>-219.4</td>
<td>-2.707</td>
<td>.8069</td>
<td>394.491</td>
<td>.0889</td>
<td>14.129</td>
</tr>
<tr>
<td>Absolute</td>
<td>100 Day M.A.</td>
<td>1.411</td>
<td>31.138</td>
<td>230.4</td>
<td>4.306</td>
<td>.8628</td>
<td>332.505</td>
<td>.0453</td>
<td>9.068</td>
</tr>
<tr>
<td>Absolute</td>
<td>30 Day M.A.</td>
<td>.781</td>
<td>25.624</td>
<td>598.5</td>
<td>11.409</td>
<td>.8098</td>
<td>391.536</td>
<td>.0305</td>
<td>-7.200</td>
</tr>
<tr>
<td>Absolute</td>
<td>Adaptive</td>
<td>1.271</td>
<td>27.976</td>
<td>35.9</td>
<td>.550</td>
<td>.8354</td>
<td>364.212</td>
<td>.0454</td>
<td>5.966</td>
</tr>
<tr>
<td>Relative</td>
<td>All Previous Obs.</td>
<td>2.268</td>
<td>25.103</td>
<td>-211.4</td>
<td>-2.589</td>
<td>.8034</td>
<td>398.086</td>
<td>.0904</td>
<td>14.038</td>
</tr>
<tr>
<td>Relative</td>
<td>100 Day M.A.</td>
<td>1.419</td>
<td>31.057</td>
<td>233.5</td>
<td>4.36</td>
<td>.8622</td>
<td>333.258</td>
<td>.0457</td>
<td>9.174</td>
</tr>
<tr>
<td>Relative</td>
<td>30 Day M.A.</td>
<td>.788</td>
<td>25.692</td>
<td>597.5</td>
<td>11.412</td>
<td>.8106</td>
<td>390.702</td>
<td>.0307</td>
<td>-6.93</td>
</tr>
<tr>
<td>Relative</td>
<td>Adaptive</td>
<td>1.279</td>
<td>27.934</td>
<td>37.2</td>
<td>.568</td>
<td>.8350</td>
<td>364.667</td>
<td>.0458</td>
<td>6.096</td>
</tr>
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</table>

*155 observations of price per contract in £.
<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>Standard Deviation</th>
<th>Minimum</th>
<th>Maximum</th>
<th>Skewness</th>
<th>Kurtosis</th>
<th>Studentized Range</th>
<th>Z**</th>
</tr>
</thead>
<tbody>
<tr>
<td>Market Price/Model Price</td>
<td>1.272</td>
<td>.384</td>
<td>.599</td>
<td>2.84</td>
<td>.874</td>
<td>1.317</td>
<td>5.81</td>
<td>8.790</td>
</tr>
<tr>
<td>Market Price - Model Price (£)</td>
<td>385.4</td>
<td>403.0</td>
<td>-208.9</td>
<td>1214.9</td>
<td>.204</td>
<td>-1.236</td>
<td>3.53</td>
<td>11.868</td>
</tr>
<tr>
<td></td>
<td>Market Price - Model Price</td>
<td>435.6</td>
<td>347.7</td>
<td>.4</td>
<td>1214.9</td>
<td>.440</td>
<td>-1.086</td>
<td>3.49</td>
</tr>
<tr>
<td>Cumulative Hedging £ Return</td>
<td>-331.2</td>
<td>645.7</td>
<td>-2227.9</td>
<td>1271.3</td>
<td>-.629</td>
<td>.508</td>
<td>5.42</td>
<td>-6.365</td>
</tr>
<tr>
<td>Total £ Return</td>
<td>104.4</td>
<td>615.9</td>
<td>-1815.1</td>
<td>2169.7</td>
<td>-.428</td>
<td>.959</td>
<td>6.47</td>
<td>2.103</td>
</tr>
<tr>
<td>Final Day Hedge £ Return</td>
<td>5.8</td>
<td>42.3</td>
<td>-197.1</td>
<td>371.6</td>
<td>4.936</td>
<td>44.451</td>
<td>13.45</td>
<td>1.702</td>
</tr>
</tbody>
</table>

*Model price for this comparison is calculated with variances modeled by an absolute diffusion estimated with an adaptive filter technique.

**H₀: Mean = 0, except Market Price/Model Price H₀: Mean = 1; 155 observations.

#The probability that the estimated statistic (given the sample size = 155) is from a normal distribution is less than .01.
TABLE 11

LONDON COMMODITY OPTIONS: PERFORMANCE OF MODEL VERSUS MARKET PRICE --
RELATIONSHIP WITH STARTING DATE

\[ \text{Performance Measure} = \alpha + \beta \text{(Starting Date)} \]

<table>
<thead>
<tr>
<th>Market to Model Performance Measure</th>
<th>( \beta )</th>
<th>( t_{\beta} )</th>
<th>( \alpha )</th>
<th>( t_{\alpha} )</th>
<th>( R_{ADJ}^2 )</th>
<th>( \sigma_{(\text{Resid})} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Market Price/Model Price</td>
<td>-.0064</td>
<td>-5.28</td>
<td>1.55</td>
<td>26.1</td>
<td>.149</td>
<td>.3551</td>
</tr>
<tr>
<td>Market Price - Model Price (f)</td>
<td>-2.16</td>
<td>-1.58</td>
<td>478.2</td>
<td>7.1</td>
<td>.010</td>
<td>401.1</td>
</tr>
<tr>
<td></td>
<td>Market Price - Model Price (f)</td>
<td>-.34</td>
<td>-1.13</td>
<td>493.0</td>
<td>8.5</td>
<td>.002</td>
</tr>
<tr>
<td>Cumulative Hedging &amp; Return</td>
<td>3.08</td>
<td>1.41</td>
<td>-463.5</td>
<td>-4.3</td>
<td>.006</td>
<td>643.7</td>
</tr>
<tr>
<td>Total f Return</td>
<td>1.74</td>
<td>.83</td>
<td>29.55</td>
<td>.3</td>
<td>.000</td>
<td>616.5</td>
</tr>
</tbody>
</table>

TABLE 12

LONDON COMMODITY OPTIONS: PERFORMANCE OF MODEL VERSUS MARKET PRICE --
RELATIONSHIP WITH MATURITY

\[ \text{Performance Measure} = \alpha + \beta \text{(Maturity)} \]

<table>
<thead>
<tr>
<th>Market to Model Performance Measure</th>
<th>( \beta )</th>
<th>( t_{\beta} )</th>
<th>( \alpha )</th>
<th>( t_{\alpha} )</th>
<th>( R_{ADJ}^2 )</th>
<th>( \sigma_{(\text{Resid})} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Market Price/Model Price</td>
<td>-.0006</td>
<td>-.82</td>
<td>1.34</td>
<td>16.0</td>
<td>.000</td>
<td>.3853</td>
</tr>
<tr>
<td>Market Price - Model Price (f)</td>
<td>-.49</td>
<td>-.60</td>
<td>434.4</td>
<td>4.7</td>
<td>.000</td>
<td>403.8</td>
</tr>
<tr>
<td></td>
<td>Market Price - Model Price (f)</td>
<td>-.66</td>
<td>-.93</td>
<td>500.8</td>
<td>6.7</td>
<td>.000</td>
</tr>
<tr>
<td>Cumulative Hedging &amp; Return</td>
<td>-1.05</td>
<td>-.81</td>
<td>-226.3</td>
<td>-1.6</td>
<td>.000</td>
<td>646.5</td>
</tr>
<tr>
<td>Total f Return</td>
<td>-1.71</td>
<td>-1.377</td>
<td>274.5</td>
<td>2.1</td>
<td>.006</td>
<td>614.1</td>
</tr>
</tbody>
</table>
TABLE 13

LONDON COMMODITY OPTIONS: PERFORMANCE OF MODEL VERSUS MARKET PRICE -- RELATIONSHIP WITH VARIANCE ESTIMATE

\[ \text{Performance Measure} = \alpha + \beta \text{[Variance Estimate]} \]

<table>
<thead>
<tr>
<th>Market to Model Performance Measure</th>
<th>(\beta)</th>
<th>(t_\beta)</th>
<th>(\alpha)</th>
<th>(t_\alpha)</th>
<th>(R^2)</th>
<th>(\sigma(\text{Resid}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Market Price/Model Price</td>
<td>-.0018</td>
<td>-8.83</td>
<td>1.49</td>
<td>42.2</td>
<td>.333</td>
<td>.3143</td>
</tr>
<tr>
<td>Market Price - Model Price (€)</td>
<td>-1.44</td>
<td>-11.43</td>
<td>651.6</td>
<td>19.5</td>
<td>.457</td>
<td>297.0</td>
</tr>
<tr>
<td>(</td>
<td>\text{Market Price - Model Price}</td>
<td>(€))</td>
<td>-1.14</td>
<td>-9.75</td>
<td>645.1</td>
<td>20.9</td>
</tr>
<tr>
<td>Cumulative Hedging &amp; Return</td>
<td>1.31</td>
<td>5.13</td>
<td>-572.1</td>
<td>-8.5</td>
<td>.141</td>
<td>598.4</td>
</tr>
<tr>
<td>Total &amp; Return</td>
<td>.17</td>
<td>.65</td>
<td>73.0</td>
<td>1.1</td>
<td>.000</td>
<td>617.1</td>
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</tbody>
</table>

TABLE 14

LONDON COMMODITY OPTIONS: PERFORMANCE OF MODEL VERSUS MARKET PRICE -- RELATIONSHIP WITH EXERCISE PRICE

\[ \text{Performance Measure} = \alpha + \beta \text{[Exercise Price]} \]

<table>
<thead>
<tr>
<th>Market to Model Performance Measure</th>
<th>(\beta)</th>
<th>(t_\beta)</th>
<th>(\alpha)</th>
<th>(t_\alpha)</th>
<th>(R^2)</th>
<th>(\sigma(\text{Resid}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Market Price/Model Price</td>
<td>-.00002</td>
<td>4.99</td>
<td>.96</td>
<td>13.9</td>
<td>.135</td>
<td>.3580</td>
</tr>
<tr>
<td>Market Price - Model Price (€)</td>
<td>.035</td>
<td>9.64</td>
<td>-150.5</td>
<td>-2.5</td>
<td>.374</td>
<td>318.9</td>
</tr>
<tr>
<td>(</td>
<td>\text{Market Price - Model Price}</td>
<td>(€))</td>
<td>.028</td>
<td>8.43</td>
<td>12.4</td>
<td>.2</td>
</tr>
<tr>
<td>Cumulative Hedging &amp; Return</td>
<td>.038</td>
<td>-5.69</td>
<td>249.5</td>
<td>2.2</td>
<td>.168</td>
<td>589.1</td>
</tr>
<tr>
<td>Total &amp; Return</td>
<td>-.010</td>
<td>-1.47</td>
<td>262.9</td>
<td>2.23</td>
<td>.008</td>
<td>613.6</td>
</tr>
</tbody>
</table>
FOOTNOTES

1. The seminal piece of work on option pricing is Samuelson's [1965] paper on rational warrant pricing. In that article, and Samuelson and Merton's [1969] article, an equation for the warrant price is developed, based on relations among the stock, the warrant and the riskless asset. This research has had several important extensions (Merton [1973], Black and Scholes [1972, 1973]) in theory and empirical testing. For recent reviews of stock option pricing, see Smith [1976] and Cox and Ross [1976].

2. For example, metal dealers make option markets (primarily selling calls) and mortgage bankers buy puts (standbys) from packagers (and others).

Although no direct evidence exists, Mehl [1934] supports the contention that the trade (large holders) sells options to smaller traders. Throughout that study, Mehl contends (with very little empirical justification) that writers profited far more than did purchasers of privileges (options). (This contention is also popular in the stock option literature.) The question of whether holders of inventories have a bias toward selling options is, as yet, unanswered.

3. Some demand is obvious to the extent that firms like Goldstein, Samuelson, Inc. grew from nothing to an estimated monthly premium volume of $25 million in 20 months. To what extent this is due to fraudulent advertising and intense marketing is unknown. It is also difficult to infer demand from their growth, since there is evidence of serious underpricing of their commodity options.

4. In the 1977 case of Lloyd, Carr, Inc., there is evidence of serious overpricing of commodity options. Since deceptive sales techniques have been alleged, it is difficult to infer demand, but annual sales volume when Lloyd, Carr, Inc. closed down was about $50 million. Again, some demand for these options must have existed.

5. The issue of whether a risk premium is part of the price paid by the hedger in the futures case is a controversial topic with much academic research (see Cootner [1967], Johnson [1960], Schrock [1971], and Miller [1971]). The payment of a premium for bearing risk is not directly at issue here.

However, it is possible to have one market (either option or futures) providing price insurance to hedgers at better rates, with lower transactions costs, or with increased flexibility in terms of hedging their supplies or needs. Essentially, this is an empirical question requiring investigation of the efficacy and cost of establishing various positions in each market.
One conjecture about the relationship between futures and options states that if there were no transaction costs, a future contract could be made into a European type option by placing an instantaneous buy and sell stop loss order on the future contract at the chosen exercise price.

Practically, completion of the instantaneous orders to buy and sell at the exercise price will not always be feasible due to possible jump discontinuities in price and maximum price fluctuations allowable in each day's trading (limits).

With continuous trading, assuming that no limit prices exist and price discontinuities did not occur, it seems that the conversion of the future contract utilizing the trading scheme described above yields an option at zero premium. Perhaps then, the actual premiums observed (which are non-zero) imply something about the costs of establishing option and futures markets or the magnitude of risk premia in futures markets.

Unfortunately, even with the convenient assumption of continuous trading with diffusion price processes and no price limits, a portfolio policy of holding the future contract only when its current price exceeds some exercise price is an anticipating function of current price. Since futures prices are random, no such portfolio policy can exist.

This theory was developed by Holbrook Working [1948, 1949], expository by Brennan [1958] and Cootner [1964], and extended in Weymar [1968]. An excellent summary of these concepts appears in Cootner [1967].

Cootner [1977] has extended the literature in this area by providing a theoretical justification for the supply of storage phenomenon.

Although convenience yield is generally thought to be evident only in soft commodities such as wheat and cocoa, Hoag [1971] demonstrates the effect for at least one precious metal -- silver.

Exercise price change is always present, because the exercise price is in nominal terms. The option contract is not protected against inflation. The issue is an index problem, since the value of the exercise price in terms of a basic bundle of commodities changes through time in a stochastic fashion. With short term options the magnitude of this effect should not be large, but with perpetual options, this effect could predominate.

Merton [1973] demonstrates that the necessary call option protection for a dividend payout would be to increase the number of shares to be purchased for the exercise price by a percentage given by the dividend divided by the expayout price.
For short maturity stock options, with known dividends, reducing the stock value by the present value of dividend payouts before the maturity of the option is a good approximation (Scholes [1973]). Models developed by Samuelson [1965] and Samuelson and Merton [1969] consider the general case when returns on the option and its underlying stock differ. Cox [1975] and Cox and Ross [1975] consider the general case where payouts against the underlying return stream are possible.

The original work on option pricing by Bachelier [1900] utilized an absolute (arithmetic) diffusion. This process is criticized in contemporary literature as an unrealistic representation of stock prices due to a positive probability of negative prices. Since stock is a limited liability asset, this can be an undesirable feature. An appropriate solution to the limited liability problem adds a constraint which converts $S=0$ to an absorbing or reflecting barrier depending upon the magnitude of bankruptcy costs.

In the case of commodities, rational economics will dictate that negative inventory will be an inaccessible (natural) boundary due to the price mechanism. Thus, an underlying absolute diffusion process for inventory would be controlled by the problem to generate positive prices and have positive inventories held.


In the following developments, subscripts will indicate partial derivatives with respect to the indicated variable; superscripts will designate which stochastic process inventory ($x$), spot price ($S$), future price ($F$), payout stream ($Q$) or commodity option ($c,C$) is currently being considered; and exponents will be used exclusively in conjunction with square brackets where confusion may arise [for example, $[S]^{B/2}$].


There is some empirical evidence of nonconstant variance for commodity prices (see Rocca [1969] and Mann and Heifner [1976]). These studies ignore the inventory dependency and obtain results consistent with a general specification of $\mu(S,t)$ and $\sigma(S,t)$, but not with (2). Developments by Samuelson [1965] and Miller [1971] assume that the variance is nonconstant. Evidence in Hoag [1978] supports a model consistent with equation (1), but not the relative diffusion (2).
17 It is interesting to note that Cootner's [1977] characterization of the spot price consists of a component which values the inventory through time at the discounted future price plus an option to sell the spot inventory at any intermediate time, if the price then seems reasonable compared to the discounted future price. This formulation is consistent with the specification of the spot price dynamics as a locally Brownian motion.

18 The carrying charge function q(S,t) is assumed to be known with certainty. The estimation of the carrying charge function from the current futures term structure is discussed in Hoag [1978].

19 The derivation of the option pricing formulae follows the heuristic technique used in previous literature (see Cox [1975], Cox and Ross [1975]). For a detailed proof of the formula including proof of the existence of the portfolio weights using stochastic control theory, see Hoag [1978].

20 The value of the riskless portfolio weights given \( \sigma = C_s \sigma(S,t) \) is \( q_s = -\alpha C s / C \) or \( n_s = -n C s \).

The option pricing formula is strictly valid only for European options.


If the carrying charges \( q(S,t) \) are a power function of the spot price with an exponent one more than the exponent of a constant elasticity of variable diffusion, an analytical solution (involving Bessel functions) can be derived.

22 Although other specifications are possible, the absorbing boundary at \( S = 0 \) is the simplest condition to impose. The numerical solution does not actually require any specification other than the terminal condition.

23 These options for 10,000 troy ounces of spot silver were bought and sold by Mocatta Metals. Data on exercise price, time to maturity and premiums came from the Mocatta Metals Daily Bulletin.

24 However, Samuelson [1965] demonstrates that many possible stationary stochastic processes are consistent with observed returns on both stocks (with a mean drift) and commodities (with seasonals). This theoretical suggestion implies that the pattern of process parameter variation over time should be subject to estimation.
25. The procedure and intermediate results are presented in detail in Hoag [1978].

26. The OLS regression equation was market price = 600 + 1.45 model price. OLS estimates of variances are inappropriate in this case since option prices are affected by the same underlying process and hence any errors would not be expected to be independent.

27. The distribution of the errors is nonnormal, and the errors are not independent. Thus, the conclusion should be interpreted carefully until further analysis on the larger data base is available.

28. For further valuation results on commodity options for the two types of futures contracts actually traded in London, see Hoag [1978].


The solution (10) assumes (a second order approximation) that

\[ \sigma \left[ \frac{1 - e^{2r(T-t)}}{2r} \right]^{\frac{1}{2}} \approx \sigma \sqrt{T-t} \]

30. The techniques and evidence on which price process best fits the futures price data is presented in Hoag [1978].

31. Since the errors in measurement of option value are not independent, the tests of significance are suspect. With data from the large transaction base, modifications will be made to perform the correct significance tests.
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