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OF A MODERN MARKET CLEARING SYSTEM

BY
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ABSTRACT

Modern technological developments have made it possible to create security markets that could only be dreamed of a century ago when our current system of trading was adopted. In this paper we examine one type of architecture that can be used for a security market, namely the periodic call system. We develop an analytical model for comparing and ranking several different designs for the market clearing system and present some criteria which can be used in their evaluation. We find that two design parameters in particular are crucial to the quality of the market's performance. They are the resolution of the price grid and the slope of the aggregate excess demand function for the security. Of these two, only the resolution of the price grid can be controlled. We discuss the costs and benefits of doing so.
I. INTRODUCTION

Starting in the 1960's academic interest in the functioning of security markets and the actions of people operating in them has grown steadily. This has been partly due to the stand taken by the Securities and Exchange Commission (1972) on the future of security markets, partly because of the emerging institutional power in the market, and of course, partly due to the tremendous information communication and processing capability that has become available. In recent years many academicians have devoted their efforts to developing models that explain the actions of market agents such as the specialist in the market environment, or explain the bid-ask spread; among them Demsetz (1968), Tinic (1972), Tinic and West (1972, 1974), Benston and Hagerman (1974), Branch and Freed (1977), Cohen, Maier, Ness, Okuda, Schwartz and Whitcomb (1977), and Stoll (1978a, 1978b). Other contributions to the market structure literature have come from Garman (1976), Beja and Hakansson (1976), Mendelson, Peake and Williams (1977), Goldman and Sosin (1977), Garbade and Silber (1979), Goldman and Beja (1979), and Hakansson, Beja and Kale (HKB, 1980).

Following Garman's work (1976) most of the research in this area has been of the "market microstructure" variety. It is concerned with the order arrival process in today's
security market, the execution of those orders and the resulting price series. In this paper we take a wider perspective. Broadly speaking we address two types of issues: (1) given the technology we have today how would we design a security market, and (2) given a set of alternative designs, how do we evaluate them. Of course, this is based on the presumption that a security market is necessary, in that securities do play an important role in risk-sharing in the economy (Arrow(1971)). These issues are different from the ones that have been discussed traditionally within the area of "market microstructure." Indeed they fall more naturally into an area that may be described as "Security Market Architecture."

We have chosen to restrict this study to several variations of a single fundamental design for the Market Clearing System (MCS), namely a "call market." We develop an analytical model that is used to demonstrate the importance of certain parameters in this architecture. We discuss criteria for evaluating different designs of the MCS and use them to rank the alternatives presented. HBK (1980) have also studied some of the issues pertinent to the ranking procedure, but their study is different in that they use a computer simulation model (with different stochastic properties), and their goal is to demonstrate the feasibility of running a securities market where investors submit sets of limit ord-
ers which are cleared in a call market in accordance with a
prespecified rule.

To facilitate appreciation of the relevant issues a
brief discussion of call markets and their relation to con-
tinuous markets follows. The current system of trading is
termed "continuous," in that anyone wishing to trade can do
so without a break between 10 a.m. and 4 p.m., Monday thru
Friday (excluding certain holidays). A call market is dif-
f erent; a trader can trade only when the market is called,
an event which takes place periodically during the day.
Call markets are not new; most markets start out as call
markets. The New York Stock Exchange was a call market
until 1871, and currently there are several functioning call
markets around the world (Spray (1964)). Even today each
market opening at the N.Y.S.E. operates like a market call;
the specialist aggregates (more or less) the set of buy ord-
ers and sell orders to decide an opening price.

A modern call market would consist of an electronic
network which would channel information from individuals to
a central location. Each individual would communicate the
price-quantity combination (excess demand function) accept-
able to him for trading. The market would be called period-
ically, at which time all individual demand would be aggre-
gated at the center and then the market would be cleared by
a Market Clearing System (MCS). The MCS would pick the
price at which all trades are made, as well as take the necessary position in the security in order to clear the market and thus avoid rationing of trades. The system described here falls into the category of one-step trading mechanisms. There is no iterative bidding unlike most auctions or Walras' (1876) system which may need an infinite number of iterations to converge to a market clearing price.

The frequency of market calls is an important parameter in the design of a call market. There are two traditional methods for handling this problem. The first is to call the market for each stock once during the day. The second is to start at the top of the list of stocks, call the market for each stock in turn all the way to the bottom of the list, and then repeat the process until the end of the day. The optimal frequency of market calls should depend on the attributes of the stock in question and the pattern of information dissemination in the market. Some work has been done on the impact on trading, of certain patterns of communication and inference of information. Copeland (1976) has modeled the effect of sequential arrival of information on asset trading. Goldman and Sosin (1977) have also developed an information-dissemination model, but with the specific purpose of contrasting continuous markets with call markets. They conclude that there is an optimal positive time interval between market calls that maximizes market
efficiency. Garbade and Silber (1979) come to a similar conclusion by using a model that minimizes an individual's exposure to the risk of his order trading at a price that is different from his estimate of the equilibrium price at a specified time before his order is traded. However, the time interval between calls need not be fixed, it could be drawn randomly from an appropriate probability distribution. Yet another scheme would trigger a trade each time any individual changed his excess demand function, which would in effect mimic a "continuous" market. Thus the "continuous" market is contained in the set of designs that can be implemented with a call market architecture. Although this particular issue is important, it is separate from the actual clearing of the market when it is called. In this paper we concentrate on the latter.

Section II lays the framework for the analysis; Section III puts forth possible designs for an MCS and discusses their properties; Section IV explores the sensitivity of the system to changes in the design parameters; Section V concludes the paper.
II. THE MODEL

We assume that the excess demand function for each trader is continuous and has a negative slope. This assumption is consistent with risk-averse behaviour and that the value of the security at any time cannot be less than zero. In the set up of a call market this assumption implies that an equilibrium price and allocation exists at each call (Debreu (1959)), which means that it is always possible to clear the market. In our analysis we go one step further and assume for simplicity that the demand functions are not only downward sloping, but also linear. Given that the number of shares outstanding is fixed, this additional assumption results in a unique equilibrium price and allocation which can be calculated readily if every trader's excess demand function is known.

In the case described above, the MCS is nothing more than a device that aggregates excess demand functions and identifies the exact price corresponding to zero aggregate excess demand, namely the equilibrium price. In this case there is only one design for the MCS, so the problem of design has a straightforward solution.

We take the model a step closer to reality by assuming that trading prices are discrete. This assumption reflects
current practice in the stock market where prices are expressed as multiples of $1/8, $1/16, etc. Let $D$ be the size of the interval between any two adjacent prices on the price grid. Thus $D$ is a measure of the resolution of the price grid. Currently the resolution is finer for low priced stocks and coarser for stocks with higher prices. The structure of the underlying demand is still assumed to be described by continuous linear downward sloping individual excess demand functions. Since the resulting aggregate excess demand function will also be continuous, linear and downward sloping, the equilibrium price at each call will lie in one of the intervals between two adjacent prices on the price grid. We arbitrarily assign a price of 0 to the lower bound of the interval containing the equilibrium price; it follows that the upper bound of the interval is at price $D$. Thus the $(0, D)$ interval moves along with the equilibrium price, as the equilibrium price falls into different intervals on the price grid at successive calls of the market. We will use the $(0, D)$ interval as the frame of reference for the remainder of the study. At the $T$'th call of the market, let $P_T$ be the equilibrium price relative to the interval $(0, D)$, i.e., $0 < P_T < D$. Our final assumption is that $P_T$ is a random variable such that, given $P_T$ is contained in $(0, D)$, $P_T$ has a uniform probability distribution in the interval $(0, D)$ and is independent of $P_{T-1}$. This assumption may not be satisfactory for every situation that
may be encountered during trading. Nonetheless when D is "small," i.e., when price changes from call to call are of larger magnitude than D, it would be hard to find a better assumption. Later on we present several reasons for keeping D small, which can be done since the value of D is after all under the control of the designer.

Once the aggregate excess demand function is available, the task of identifying the (0,D) interval is easy. The problem then is choosing one of 0, or D as the clearing price, which we label $P_C$. Our primary criterion for this choice is the magnitude of the difference between the equilibrium price and $P_C$, the smaller this value, the better the performance of the MCS. This criterion is in keeping with the prevailing literature in finance. Its justification is that a smaller error in the price signal reduces the misallocation of assets involved in the current exchange along with providing a more accurate signal for the allocation of resources in the economy at large.

In order to rank different designs of the MCS we need criteria that can be used directly. For the sake of consistency these should be derived from the primary criterion set forth above. In the multiperiod environment of the periodic call market, the steady state performance of the MCS is our main concern. In this context the process by which the stock inventory of the MCS changes at each call
determines the level of that inventory at any time. If the stock inventory becomes large relative to the stock outstanding, then it introduces a significant external factor in the determination of the clearing price. Consequently the error in the price signal is larger, the larger the level of the MCS' stock inventory, other things remaining the same. In an extreme situation the MCS might hold all of the stock outstanding, in which case the clearing price can hardly be regarded as a surrogate for the equilibrium price. The level of the stock inventory held by the MCS is thus a natural candidate for a secondary criterion; the lower this level the better the performance of the MCS. Let $I_T$ be the stock inventory of the MCS after the $T$'th call. To minimize external interference from the very beginning, the MCS starts with a stock inventory of 0, i.e., $I_0=0$. Let $-b$ be the slope of the aggregate excess demand function (Fig. 1), then the value of $I_T$ depends upon $D$, $-b$, $p_T$, and of course the design of the MCS.
Fig. 1. The aggregate excess demand function.
III. ALTERNATIVE DESIGNS FOR A MARKET CLEARING SYSTEM

Six different designs of an MCS are presented below, starting with the simplest and progressing to the most sophisticated. Beginning with the simplest designs allows a step by step approach to the development of the analytics of the more sophisticated designs; it also helps to highlight the superior properties of the better designs.

Design 1

$P_C=D$ (buy). The MCS picks the upper bound of the price interval as the clearing price at each call. Let $dI_t$ be the increase in the stock inventory at the $t$'th call. For all values of $P_T$ contained in $(0,D)$ the aggregate excess demand is negative for $P_C=D$ (see Fig. 1), thus $dI_t=(D-P_T)/b$ is greater than 0 at each call. Given that $I_T$ is defined as,

$$I_T = \sum_{t=1}^{T} dI_t,$$

with the passage of time $I_T$ grows without bound. This design continually increases the external interference in the market and is therefore unacceptable. The next design is equally simple, but requires the MCS to take the opposite position.
Design 2

$P_C = 0$ (sell). The MCS picks the lower bound of the price interval at each call. For all values of $P_T$ contained in $(0,D)$ the aggregate excess demand is positive for $P_C = 0$ (see Fig. 1), thus $dI_T = -P_T/b$ is less than 0 at each call. As a result, the MCS has a short position whose magnitude grows without bound. Hence this design is also unacceptable.

Design 3

Select randomly between $P_C = 0$ (sell), and $P_C = D$ (buy), the probability of each outcome being 1/2. Such a scheme has the advantage that it is impossible for any trader to game against it. The probability distribution of $dI_T$ can be derived from the probability distribution of $P_T$ and the relation between $dI_T$ and $P_T$, deduced in Designs 1 and 2 above.

If the outcome of the random selection is $P_C = D$ (buy), then $dI_T = (D - P_T)/b$. Since $P_T$ is uniformly distributed over $(0,D)$, $dI_T$ is uniformly distributed over $(0,D/b)$. If on the other hand, the outcome of the random selection is $P_C = 0$ (sell), then $dI_T = -P_T/b$, is uniformly distributed over $(-D/b, 0)$. Since each outcome has a probability of occurrence of 1/2, $dI_T$ is uniformly distributed over $(-D/b, D/b)$. Therefore, $E[dI_T] = 0$, and $\text{var}(dI_T) = D^2/3b^2$. At each call the
expected value of \( dI_T \) is 0 and the magnitude of the maximum change possible in the inventory is \( D/b \).

Since we have assumed that \( P_T \) is independent of \( P_{T-1} \), given the random selection of \( P_C \), it follows that \( dI_T \) is independent of \( dI_{T-1} \). In Appendix A the distributions of \( I_1, I_2 \) and \( I_3 \) are derived. \( I_1 \) is uniformly distributed over \((-D/b, D/b)\), \( I_2 \) has a symmetric triangular probability density function (pdf) over \((-2D/b, 2D/b)\), while \( I_3 \) has a pdf given by,

\[
f(i_3) = \frac{(i_3)^2}{16(D/b)^3} + \frac{3i_3}{8(D/b)^2} + \frac{9}{16(D/b)}, \quad -3D/b < i_3 < -D/b
\]

\[
= -\frac{(i_3)^2}{8(D/b)^3} + \frac{3}{8(D/b)}, \quad -D/b < i_3 < D/b
\]

\[
= \frac{(i_3)^2}{16(D/b)^3} - \frac{3i_3}{8(D/b)^2} + \frac{9}{16(D/b)}, \quad D/b < i_3 < 3D/b
\]

\[
= 0, \quad \text{elsewhere.}
\]

The pdf's of the three distributions are illustrated in Fig. A.1 in Appendix A. Extending this result to time \( T \), the pdf of \( I_T \) is composed of arcs of \( T \) different polynomials of degree \( T-1 \), over the interval \((-TD/b, TD/b)\), and a value of 0
elsewhere. While the distribution appears to approach the familiar bell shape, by applying the Central Limit Theorem we see that it is actually the distribution of \( \frac{I_T}{(T)^{1/2}} \) that approaches the normal distribution with mean 0 and variance \( D^2/3b^2 \), as \( T \) approaches infinity.

Although the distribution of \( I_T \) is very complicated, the independence of the \( dI_T \) for all \( T \) makes it easy to calculate its first two moments.

\[
E[I_T] = \sum_{t=1}^{T} E[dI_t] = 0
\]

\[
\text{var}(I_T) = \sum_{t=1}^{T} \text{var}(dI_t) = \frac{TD^2}{3b^2}
\]

With this design the drift in \( I_T \) is zero, a desirable property. However as \( T \), the number of calls, approaches infinity, so does the variance of \( I_T \). As Garman (1975) pointed out, this property implies that in the long run, the probability that at some time the MCS will hold all the outstanding stock or be short a similarly large amount is 1. The only condition for which this situation is avoided is if \( D \), the resolution of the price grid is zero, which means prices would have to be continuous. Although this design does better than the two previous ones in the short run in that it has a 0 drift, it will also fail in the long run.
Design 4

Minimize the magnitude of $dI_T$. This design minimizes external interference in the market for any given call. Fig. 2 shows the relation between $dI_T$ and $P_T$. The MCS picks $P_C=0$ (sell) when $0<P_T<D/2$, and $P_C=D$ (buy) when $D/2<P_T<D$. Since $P_T$ is uniformly distributed over $(0,D)$, $dI_T$ is uniformly distributed over $(-D/2b,D/2b)$. Thus $E[dI_T]=0$, and $\text{var}(dI_T)=D^2/12b^2$. Since $P_T$ is independent of $P_{T-1}$, $dI_T$ is independent of $dI_{T-1}$. The distribution of $I_T$ is the same as in Design 3, except for a difference in the value of the parameter, which in this case is $D/2b$ instead of $D/b$. The first two moments of the distribution are,

$$E[I_T] = 0, \quad \text{var}(I_T) = \frac{TD^2}{12b^2}$$

The drift in $I_T$, the stock inventory is zero, the same as for Design 3. The variance of the inventory value is one fourth of that obtained for Design 3, but it approaches infinity as $T$ approaches infinity. Once again, the perpetually increasing variance of the stock inventory makes this design unacceptable.

Design 5

$P_C=0$ (sell) if $I_{T-1}>0$, and $P_C=D$ (buy) if $I_{T-1}<0$. If
Fig. 2. $dI_T$ for Design 4.
\( I_{T-1} = 0 \), which is true at least at \( T=0 \), then at \( T \) a random choice is made between 0 and \( D \) with the probability of each outcome being \( 1/2 \). Unlike any of the preceding designs, this design incorporates the overall position of the MCS, i.e., the value of \( I_T \), in an attempt to reduce the total level of external interference in the market.

At \( T=1 \), this design is the same as Design 3, so that \( dI_1 \) and \( I_1 \) are both uniformly distributed over \((-D/b,D/b)\). Consequently the probability that \( I_1 > 0 \) is equal to \( 1/2 \), as is the probability that \( I_1 < 0 \).

After the first call the distribution of \( I_T \) becomes more complex. In Appendix B, the distribution of \( I_2 \) (Fig. B.1) and \( I_3 \) are derived. It turns out that \( I_2 \) and \( I_3 \) have identical distributions, with a pdf that is symmetric triangular over \((-D/b,D/b)\). By induction it follows that that for \( T>1 \), the distribution of \( I_T \) is stationary, so that the pdf of \( I_T \) for all \( T>1 \) is also symmetric triangular over \((-D/b,D/b)\). The first two moments of this distribution are,

\[
E[I_T] = 0, \quad \text{var}(I_T) = \frac{(D/b)^2}{6}, \quad T=2,3,\ldots
\]

As in the two previous designs the drift in \( I_T \) is zero, but now for the first time the value of \( I_T \) is bounded. The importance of this result is that this is the first design where the extent of external interference in the market has
a finite bound which can be controlled by controlling the resolution of the price grid (D). The maximum magnitude of the stock inventory at any time will be less than D/b. This design is very simple, yet its properties make it acceptable as an MCS.

**Design 6**

Minimize the magnitude of $I_T$. This design reduces the overall external interference in the market more systematically than does Design 5. At $T=1$, this design has the same effect as Design 4, so $I_1$ is uniformly distributed over $(-D/2b, D/2b)$.

Consider an arbitrary value of $I_1=D/4b$. Fig. 3 shows the relation between $I_2$ and $P_2$. The two lines in the lower left of the figure represent $P_C=0$ (sell), while the two lines in the upper right of the figure represent $P_C=D$ (buy). If $P_2=0$ and if the MCS chooses $P_C=0$ (sell), then $dI_2=0$ and $I_2=D/4b$; but if on the other hand the MCS chooses $P_C=D$ (buy), then $dI_2=D/b$ and $I_2=5D/4b$. Clearly, when $P_2=0$, to minimize $|I_2|$, the clearing rule will pick $P_C=0$. The rule will continue to pick $P_C=0$ for all values of $P_2$ such that $0<P_2<3D/4$. For larger values of $P_2$, the MCS will pick $P_C=D$. $I_2$ is therefore a piecewise linear function of $I_1$ and $P_2$. 
Fig. 3. $I_2$ and $dI_2$ for Design 6, given $I_1 = \frac{D}{4b}$. 
\[ I_2 = \frac{D}{4b} - \frac{P_2}{b}, \quad 0 < \frac{P_2}{b} < \frac{3D}{4b} \]

\[ = \frac{5D}{4b} - \frac{P_2}{b}, \quad \frac{3D}{4} < \frac{P_2}{b} < D \]

Since \( P_2 \) is uniformly distributed over \((0,D)\), \( I_2 \) is uniformly distributed over \((-D/2b,D/2b)\).

\( I_1 = D/4b \) was an arbitrary choice for the value of \( I_1 \), so the distribution of \( I_2 \) derived above remains unchanged for other values of \( I_1 \) contained in the interval \((-D/2b,D/2b)\). Since both \( I_1 \) and \( I_2 \) have the same probability distribution, by induction, \( I_T \) has a stationary uniform distribution over \((-D/2b,D/2b)\). The first two moments of this distribution are.

\[ E[I_T] = 0, \quad \text{var}(I_T) = \frac{(D/b)^2}{12}, \quad T=1,2,3,\ldots \]

Like Designs 3-5, this rule results in a zero drift in the stock inventory. Like Design 5 the value of the inventory is bounded, but with the additional advantage that the bounds and the variance of the inventory in this case are half the size of the Design 5 bounds and variance respectively. For Designs 5 and 6 the pdf's of \( I_T \) are compared in Fig. 4. Ironically, the stock inventory for the most sophisticated design has the simplest probability distribution.
Fig. 4. The probability density functions of $I_T$ for Designs 5 and 6.
IV. EVALUATION OF THE MARKET CLEARING SYSTEM

In Section III we demonstrated that a very simple algorithm can be used to clear the market once the right inputs, in the form of individual excess demand functions, are available to the MCS. Among the designs analysed, Design 6 was shown to be the best for an MCS, so it will be used for the following discussion, which examines the relation between the quality of the MCS and its design parameters.

As before, the quality of the MCS is determined by the magnitude of values that $I_T$ can attain. This value has an upper bound of $D/2b$. To improve the quality of the MCS this upper bound has to be lowered. This can only be accomplished by decreasing $D$, the resolution of the price grid, because $-b$, the slope of the aggregate excess demand function, cannot be controlled. If the value of $b$ is known, and if $N$ is the desired upper bound for $I_T$, then it is a simple matter to calculate the appropriate value of $D$. Assuming that smaller values of $D$ are costlier, $D$ is the largest feasible value $< 2bN$. When the value of $b$ is not known, it has to be estimated. A direct way to estimate $b$ is to run an experiment in which different values of $D$ are used in a call market and the corresponding distributions of $I_T$ are observed. Together they can be used to estimate $b$. Obviously this is a simplistic representation of the larger
problem, namely optimizing D. There are several other types of costs and benefits associated with smaller values of D, some of which we discuss below. Undoubtedly this is a rich area for further research.

Even though we cannot control it, the parameter b is an important determinant of the quality of the MCS, so a few comments about its properties are in order. It depends on the slopes of the demand functions of individual traders whose demands are aggregated, as well as the number of those individuals. For example, if there are P individuals, each having a linear demand function with slope \(-1\), then the slope of the aggregate demand function is \(-1/P\), i.e., \(b = 1/P\). Thus a market with many traders will have a flatter aggregated excess demand function and is therefore "deep." In contrast, a market with few traders will have a steeper aggregate excess demand function and is therefore "thin." The number of traders in the market varies considerably between times of intense interest in the market and times of relative indifference to the market. If a procedure can be set up for estimating and reporting b, say on a daily basis, then it will act as yet another source of information to investors. It will certainly be more interesting and informative than the daily volume which is currently reported.

For a fixed value of D, our model suggests that on
average, the MCS will have to take a larger position for a flatter aggregate excess demand function than for a steeper one, because when b is smaller $E[|I_T|] = D/4b$ is larger. The value of b was shown to decrease as the number of traders is increased, consequently in a call market, the larger the number of traders, the larger the positions the MCS has to take. This is contrary to the finding made by HBK. Their simulation indicates that increasing the number of traders decreases the specialist's position. Their result can be explained by examining the process they used to generate the aggregate excess demand function. HBK use a sparse individual demand function created by using five or less limit orders per trader, and the total number of their traders is small compared to the number of available prices on the price grid at which limit orders can be placed. Thus initially as the number of traders grows, the aggregate excess demand function becomes denser, which produces more opportunities for the MCS to take a smaller position, which is what HBK observed. Nevertheless, even in the HBK simulation once the aggregate excess demand function becomes sufficiently dense, adding more traders to the market will increase the size of the position that the MCS has to take.

Turning now to the cash required to operate the system, we will localize our discussion to the $(0,D)$ interval. Cash flow is generated by the MCS as it buys or sells stock to
clear the market. The probability distribution of this cash flow is derived below.

$P_T$, the equilibrium price, is uniformly distributed over $(0,D)$. $dI_T$ and $I_T$ are both uniformly distributed over $(-D/2b,D/2b)$ for each $T$ (Design 6). In the context of this model, the buy decision is the important one, because the MCS buys at price $D$. When it sells at price $0$ the cash position is unaffected. At time $T$, given $P_T$, the conditional probability that the MCS will pick $P_C=D$ (buy) is (see Fig. 3),

$$Pr[I_{T-1} < \frac{P_T}{b} - \frac{D}{2b}] = \frac{(P_T/b-D/2b)-(-D/2b)}{D/b} = \frac{P_T}{D}$$

Since $P_T$ is uniformly distributed over $(0,D)$, the probability that the MCS will pick $P_C=D$ (buy) for any $T$ is

$$\int_0^D \frac{P_T}{D} \, dP_T = \frac{1}{2}$$

Given that the MCS picks $P_C=D$ (buy), the increase in the stock inventory is $dI_T=D/b-P_T/b$. Therefore the increase in the cash inventory is,

$$dC_T = -D\left[\frac{D}{b} - \frac{P_T}{b}\right] = -\frac{D^2}{b} + \frac{DP_T}{b}$$

Since $P_T$ is uniformly distributed over $(0,D)$, $dC_T$ is uniformly distributed over $(-D^2/b,0)$. The buy and sell decisions are equally probable, thus $dC_T=0$ with probability $1/2$. 

and $dC_T$ is uniformly distributed over $(-D^2/b, 0)$ with probability 1/2.

$$E[dC_T] = \frac{1}{2} E[\text{Buy decision}| P_C = D] + \frac{1}{2} E[\text{Sell decision}| P_C = 0]$$

$$= \frac{1}{2} \cdot \frac{-D^2}{2b} + \frac{1}{2} \cdot 0$$

$$= \frac{-D^2}{4b}$$

The calculation for the variance is somewhat longer, the result is

$$\text{var}(dC_T) = \frac{5D^4}{48b^2}$$

The expected value of $dC_T$ is negative, indicating a downward drift in the cash inventory. This result is expected since the MCS buys at price $D$ and sells at price $0$. The expected value of $dC_T$ suggests that the cash outflow with this design will average about $D^2/4b$ per call. It can be covered by setting up a commission structure for trades. This aspect of market clearing is similar to the commissions and premiums charged by brokers and specialists in today's exchanges. The difference is that although the investors pay commissions, the MCS buys high ($D$) and sells low ($0$) in return, with the result that the commissions paid do not represent a real cost to the investors. The reverse is true in today's markets where the specialist buys low and sells high to "make a market," thereby adding to the the cost of commissions already charged by the brokers. We do not mean
to suggest that the total cost of such a system is zero. However it must be emphasized that the real costs will come from the maintenance of the plant, equipment and personnel needed to operate the system, and not from the market clearing activity.

There is one type of cost which such a system will impose. It is a type of cost that exists in today's market as well. This cost stems from the redistribution of wealth caused by trading at a price other than the equilibrium price, and also from the collection of commissions even if eventually they are partly offset by the MCS. The magnitude of this redistribution can only be minimized by minimizing D.

In any system where the market is cleared by an automated procedure there is always the danger that certain investors or groups of them can game against the system and beat it. If a coalition of investors does form in the proposed system, then it will be gaming against the remaining investors as well, and not against the MCS alone. That fact reduces the vulnerability of the MCS. If a situation develops where the coalition consists of all investors, then by gaming against the MCS they are really gaming against themselves, because the MCS can offset the cash outflow caused by buying high and selling low by charging commissions from the same group of investors. Furthermore the
charges to investors will also include the real costs of maintaining the plant, equipment and personnel needed to operate the MCS. Consequently a coalition of all investors will definitely lose by trying to game against the MCS. Finally the benefits from gaming by any group of investors can be reduced by reducing D to the point that the gain from being able to predict whether the market clears at 0 or D is too small compared to the aforementioned transactions costs.
V. CONCLUSION

The model developed in the paper demonstrates the importance of the resolution of the price grid (D), as a parameter in the design of a modern Market Clearing System (MCS) that is called periodically. The other important parameter is the slope of the aggregate excess demand function (-b). The two together determine the quality of the MCS. Of these two parameters, only D can be controlled, while b depends on the attributes of the community of investors and their level of interest in the market. The model suggests that smaller values of D improve the quality and robustness of the MCS. On the other hand, we expect that decreasing the value of D will increase the cost of the MCS, because the system will have to handle larger amounts of information. It is likely that D will also depend on individual stock characteristics, e.g., the value of b for that stock. Hopefully, progress in this area of research will contribute to the estimation of an optimal value of D.

A question that has not yet been addressed, is the manner in which traders will convey their excess demand functions to the MCS. One solution would require that each investor supply the two parameters necessary for constructing a linear approximation to his excess demand function, namely the slope and the intercept. In this way an investor
can stay in the market at all times. Of course, if he wishes to withdraw from the market, he can do so by not submitting an excess demand function, which is the same as submitting a vertical excess demand function at 0 shares. If he wishes to submit a market order for \( N \) shares then he simply submits a vertical excess demand function at \( N \) shares for the next call. There are several ways of handling non-linear excess demand functions, e.g., piecewise linear approximations, restricting price movements from call to call, etc., but that is a separate topic in itself.

The discussion of suitable criteria for evaluating the MCS used the price signal as the point of departure. A more satisfactory criterion would also account for the allocation resulting from the exchange taking place at the clearing price. This particular issue has hardly been touched in the finance literature; it is yet another promising venue for further research.

The advantages of an MCS such as Design 6 are numerous. The MCS treats all investors equally. The market does not have to be maintained by one person whose capacity to handle heavy trading activity is limited. The "crowd" of traders does not have to be physically present on the Exchange floor, it can be spread all over the country, or for that matter all over the world. If more investors choose to stay in the market at all times, not only will the depth of the
market increase, but in addition, the price signal will reflect a truer aggregation of individual demand. With this type of design it is also possible to control the performance of the MCS by adjusting D.

Besides the designs presented here, there are several other designs that can be implemented within the call market architecture. Our purpose here has been to highlight the resolution of the price grid (D), and the slope of the aggregate excess demand function (-b) as important parameters in a call market architecture. Research into designs other than those presented here and also different types of market architecture should prove to be very fruitful.
APPENDIX A

Design 3:

The changes in inventory $dI_T$, $T=1,2,3,...$ are independent and identically distributed with a uniform distribution over $(-D/b, D/b)$. Since $I_0$ is equal to 0, $I_1$ has the same distribution as $dI_1$, i.e., uniform over $(-D/b, D/b)$. The probability density function (pdf) of $I_1=dI_1$ is shown in Fig. A.1.

Let $X_1=dI_1$, $X_2=dI_2$, $X_3=dI_3$, and $a=D/b$. Then $X_1$, $X_2$ and $X_3$ are independent random variables, each with a uniform distribution over the interval $(-a,a)$. Also let $Y=X_1+X_2=I_2$ and $Z=X_1+X_2+X_3=I_3$. First we derive the distribution of $Y$ and then use that to derive the distribution of $Z$.

The pdf of $X_j$, $j=1,2,3$ is,

$$g(x_j) = \frac{1}{2a}, \quad -a < x_j < a$$

$$= 0, \quad \text{elsewhere.}$$

The distribution function of $Y$ is,

$$F(y) = 0, \quad y < -2a$$
Fig. A.1. The probability density functions of $I_1$, $I_2$, and $I_3$ for Design 3.
\[ y + a \frac{y - x_2}{4a^2} \]  
\[ = \int_{-a}^{y} \int_{-a}^{-a} \frac{1}{4a^2} \, dx_1 \, dx_2, \quad -2a < y < 0 \]

Solving the integral, differentiating \( F \) wrt to \( y \), replacing \( y \) with \( i_2 \) and \( a \) with \( D/b \), and using the symmetry of \( f(i_2) \), the pdf of \( I_2 \) is (Fig. A.1),

\[ f(i_2) = \frac{i_2}{4(D/b)^2} + \frac{1}{2(D/b)}, \quad -2(D/b) < i_2 < 0 \]

\[ = - \frac{i_2}{4(D/b)^2} + \frac{1}{2(D/b)}, \quad 0 < i_2 < 2(D/b) \]

\[ = 0, \quad \text{elsewhere.} \]

To get the pdf of \( Z \), we divide the half space \((-\infty, 0)\) into three segments and calculate the distribution function of \( Z \) separately for each. For the interval \((-\infty, -3a)\), \( F(z) = 0 \). Over the interval \((-3a, -a)\),

\[ F(z) = \int_{-2a}^{z} \int_{-a}^{-y} \left[ \frac{y}{8a^3} + \frac{1}{4a^2} \right] \, dx_3 \, dy, \]

while over the interval \((-a, 0)\),

\[ F(z) = \frac{1}{6} + \int_{-2a}^{z-a} \int_{-a}^{-a-y} \left[ \frac{y}{8a^3} + \frac{1}{4a^2} \right] \, dx_3 \, dy \]
\[ + \int_{z-a}^{z+y} \int_{a-y}^{a+y} \left[ \frac{y}{8a^3} + \frac{1}{4a^2} \right] \, dx_3 \, dy \]

\[ + \int_{0}^{z+a} \int_{-a}^{a} \left[ -\frac{y}{8a^3} + \frac{1}{4a^2} \right] \, dx_3 \, dy \]

Solving the integrals, differentiating wrt \( y \), replacing \( z \) with \( i_3 \) and \( a \) with \( D/b \), and using the symmetry of \( f(i_3) \), the pdf of \( I_3 \) is (Fig. A.1),

\[ f(i_3) = \frac{(i_3)^2}{16(D/b)^3} + \frac{3i_3}{8(D/b)^2} + \frac{9}{16(D/b)}, \quad -3D/b < i_3 < -D/b \]

\[ = -\frac{(i_3)^2}{8(D/b)^3} + \frac{3}{8(D/b)}, \quad -D/b < i_3 < D/b \]

\[ = \frac{(i_3)^2}{16(D/b)^3} - \frac{3i_3}{8(D/b)^2} + \frac{9}{16(D/b)}, \quad D/b < i_3 < 3D/b \]

\[ = 0, \quad \text{elsewhere}. \]
Design 5: 

$I_1$ is uniformly distributed over $(-D/b, D/b)$. At time $T=2$, if $I_1 < 0$, then $P_c = D$ (buy), so that $dI_2$ is independently and uniformly distributed over $(0, D/b)$. The probability that $I_1 < 0$ is $1/2$; the conditional pdf of $I_1$ given $I_1 < 0$, is $(b/2D)/(1/2) = b/D$.

Let $X_1 = (I_1 | I_1 < 0)$, $X_2 = (dI_2 | I_1 < 0)$, and $a = D/b$. Define $Y = X_1 + X_2$. The pdf of $Y$ is symmetric about 0. For $y < 0$, the distribution function of $Y$ is,

$$F(y) = 0, \quad y < -a$$

$$= \int_{-a}^{y} \int_{0}^{y-x_1} \frac{1}{a^2} \, dx_2 \, dx_1, \quad -a < y < 0$$

Working through the derivation of the distribution function of $I_2$ given $I_1 > 0$ leads to the same distribution. Therefore, differentiating $F$ wrt $y$, using the symmetry of the pdf of $y$, and replacing $y$ with $i_2$ and $a$ with $D/b$, the the pdf of $I_2$ is (Fig. B.1),

$$f(i_2) = \frac{i_2}{(D/b)^2} + \frac{1}{(D/b)}, \quad -D/b < i_2 < 0$$
Fig. B.1. The probability density function of $I_2$ for Design 5.
\[- \frac{i_2}{(D/b)^2} + \frac{1}{(D/b)}, \quad 0 < i_2 < D/b\]

\[= 0, \quad \text{elsewhere.}\]

At time \(T=3\), consider the case \(I_2 < 0\). Let \(P_C = D\) (buy) so that \(dI_3\) is independently and uniformly distributed over \((0, D/b)\). The probability that \(I_2 < 0\) is \(1/2\); the conditional pdf of \(I_2\) given \(I_2 < 0\) is \(f(i_2)/(1/2)\) for \(-D/b < i_2 < 0\), and 0 elsewhere.

As before set \(a = D/b\). Let \(Y = (I_2 | I_2 < 0)\) and \(X_3 = (dI_3 | I_2 < 0)\). Define \(Z_1 = Y + X_3\). The pdf of \(Z_1\) is not symmetric. This makes it necessary to break down the derivation of the distribution of \(I_3\) into two parts. In this first part we calculate the distribution function of \(Z_1\) for \(Z_1 < 0\).

\[G(z_1) = 0, \quad z_1 < -a\]

\[= \int_{-a}^{z_1} \int_{0}^{z_1-y} \left[ \frac{2y}{a^3} + \frac{2}{a^2} \right] dx_3 \, dy, \quad -a < z_1 < 0\]

Now consider the case \(I_2 > 0\) at \(T=3\). The probability that \(I_2 > 0\) is \(1/2\); the conditional pdf of \(I_2\) given \(I_2 > 0\) is \(f(i_2)/(1/2)\) for \(0 < i_2 < D/b\), and 0 elsewhere. Define,
\[ Z_2 = (I_2 | I_2 > 0) + (dI_3 | I_2 > 0) \]

The distribution function of \( Z_2 \) is obtained by using the knowledge that its pdf is symmetric to the pdf of \( Z_1 \).

\[ \Pr(Z_2 < z_2) = \Pr(Z_1 > z_1), \quad \text{for } z_1 = -z_2 \]

From the joint pdf of \( y \) and \( x_3 \),

\[ \Pr(Z_1 > z_1) = \int_{z_1 - a}^{a} \int_{z_1 - y}^{a} \left( \frac{2y}{a^3} + \frac{2}{a^2} \right) \, dx_3 \, dy, \quad 0 < z_1 < a \]

\[ = \frac{(z_1)^3}{3a^3} - \frac{z_1}{a} + \frac{2}{3} \]

Therefore, the distribution function for \( Z_2 \) is,

\[ H(z_2) = -\frac{(z_2)^3}{3a^3} + \frac{z_2}{a} + \frac{2}{3}, \quad -a < z_2 < 0 \]

The probability that \( I_3 < i_3 \) is,

\[ \frac{1}{2} \Pr(Z_2 < i_3) + \frac{1}{2} \Pr(Z_1 < i_3) \]

More generally, the distribution function of \( I_3 \) is the weighted sum of the distribution functions of \( Z_1 \) and \( Z_2 \).
Weighting each distribution with its probability of realisation, setting $z_1 = z_2 = i_3$, and replacing $a$ with $D/b$, the distribution function of $I_3$ is,

$$F(i_3) = \frac{1}{2} G + \frac{1}{2} H$$

$$= \frac{(i_3)^2}{2(D/b)^2} + \frac{i_3}{(D/b)} + \frac{1}{2}, \quad -D/b < i_3 < 0$$

Differentiating $F$ wrt $i_3$, and making use of the symmetry of $f(i_3)$, the pdf of $I_3$ is,

$$f(i_3) = \frac{i_3}{(D/b)^2} + \frac{1}{(D/b)}, \quad -D/b < i_3 < 0$$

$$= -\frac{i_3}{(D/b)^2} + \frac{1}{(D/b)}, \quad 0 < i_3 < D/b$$

$$= 0, \quad \text{elsewhere.}$$

It turns out that the distribution of $I_3$ is identical to the distribution of $I_2$. Hence, by induction, $I_T$ has the same symmetric triangular distribution (Fig. B.1) for all $T$. 
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