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A CHARACTERIZATION OF SELF-FINANCING PORTFOLIO STRATEGIES

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A CHARACTERIZATION OF SELF-FINANCING PORTFOLIO STRATEGIES

By

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Abstract

A self-financing portfolio is one that funds are neither added to nor withdrawn from, even when its value is reallocated amongst its constituent assets. The main result in this study is that Merton's (1971) self-financing condition is mathematically equivalent to the Black-Scholes (1973) partial differential equation, and that this equivalence is not at all predicated on the existence of options or on the absence of arbitrage opportunities. This result also helps uncover errors in the traditional derivation of the Option Pricing Model, and paves the way for a natural and clear derivation that is also amenable to extension to option pricing in imperfect markets.

1. Introduction

Suppose that you live in a perfect market environment, which does not admit arbitrage opportunites. Suppose further that you ask your broker to buy and manage[1] a portfolio of Q_S shares of stocks and Q_C call options on the stock, where

$$Q_S = \frac{e^{At}}{2S} \tag{1}$$

$$Q_C = \frac{e^{At}}{2C} \tag{2}$$

where C is the price of the option written on the stock[2] with price S. Time t is measured from some fixed origin, and A is an arbitrary positive number.

The value of this portfolio is:

$$V_{GE} = \left(\frac{e^{At}}{2S}\right)\cdot S + \left(\frac{e^{At}}{2C}\right)\cdot C \tag{3}$$

To your delightful surprise you discover that you have struck a rich gold vein, since quick calculation reveals:

$$V_{GE} = e^{At} \tag{4}$$

which means that your carefully designed portfolio returns at the constant rate A. But of course you can choose A as large as you wish; larger than the interest rate, even larger than the expected rate of return on either the stock or the option. This is clearly a paradox!

The resolution of the paradox in section 2 will serve a number of purposes. It will help identify (section 3) shortcomings in the traditional derivation of the Option Pricing Model (OPM). It will also help shed new light on the role played by Merton's (1971) budget equation in the OPM. I will show that Merton's budget equation is mathematically equivalent to Black and Scholes' (1973) partial differential equation and, moreover, that this mathematical equivalence is, surprisingly, neither predicated on the impossibility of arbitrage opportunities nor on the existence of options. This Equivalence Theorem, which is the main result of this paper, is proved in section 4.

In section 5, options are introduced into an economy which does not allow arbitrage opportunities. The Option Pricing Model then follows as a simple corollary from the Equivalence Theorem. This derivation has three advantages. First, it avoids the difficulties which beset the traditional derivation. Second, it makes clear exactly where the "no arbitrage" assumption is required in the derivation process -- surprisingly, it is not at all required for the derivation of the Black-Scholes partial differential equation. Third, our derivation, unlike its predecessors, lends itself to a natural extension of the Option Pricing Model to imperfect and incomplete capital markets. But this is the subject of a separate paper.

2. Paradox Resolved

Let us first consider a stronger version of the paradox; its resolution will then be obvious.

Consider forming the following hedge portfolio:

$$Q_S = \frac{e^{At}}{S} \tag{5}$$

$$Q_C = 0 \text{ (zero)} \tag{6}$$

Where S, A, and t are the same as above. Note that this is a simpler portfolio strategy in the sense that it consists exclusively of stock; it does not involve options at all. The value of this new hedge is:

$$V_{SGE} = (e^{At}/S) \cdot S + 0 \cdot C = e^{At} \tag{7}$$

Once again it seems we have paradoxically succeeded in simulating a riskless discount bond with an arbitrary large rate of return A, but this time using only stock.

Not Quite!

One of the defining properties of a riskless discount bond is that it is self-financing,[3] i.e., it does not require any external net inflow or net outflow of funds. This condition is obviously violated by the above hedge portfolio, since it requires maintaining a stochastically changing number of shares of stock, which necessarily implies that, when the strategy calls for an increase in the number of stocks, a free external net flow of funds is injected into the portfolio in order to finance the purchase of stock, and conversely with a decrease in the number of stocks. Clearly, if an unspecified free external net inflow of funds is allowed, there is no longer a paradox in obtaining an arbitrarily high riskless rate of change in its value, even if arbitrage opportunities are excluded. The paradox is thus resolved.

The above discussion conveys the correct impression that one should be very careful when dealing with hedge portfolios that are not self-financing, lest paradoxes arise. The following is a more specific conclusion:

Proposition 1. If a nonself-financing hedge portfolio exhibits a riskless rate of change in its value, then from the absence of arbitrage opportunities it does not necessarily follow that this rate of change in value is equal to the riskless interest rate.

Proof. A clear counterexample is our second paradoxical portfolio consisting of (e^{At}/S) shares of stock, the value of which changes at the riskless rate A, which may differ from the riskless interest rate. Obviously an external infusion of funds may well result in a rate of change which is strictly higher than the interest rate, and conversely with a net outflow. Q.E.D.

In section 3 I will show that the most celebrated hedge portfolio, namely, the one which is used by Black and Scholes (1973) to derive the Option Pricing Model is not self-financing. Hence, even if the change in its value is riskless, it follows from Proposition 1 that the rate of value change is not necessarily equal to the riskless interest rate. Thus a vital step in the Black-Scholes derivation is missing.

One more point should be made. While there is no harm in interchanging "change in value" and "return" as far as self-financing portfolios are concerned, these two terms must not be confused when a non-self-financing portfolio is considered. In fact, the rate of return should be defined by the following accounting identity:

$$\text{Rate of dollar return} \equiv \text{Rate of change in value} - \text{Rate of free external net inflow of funds} \qquad (8)$$

This point is worth emphasizing, since the essential distiction between "return" and "change in value" has often been overlooked in the literature.

Before concluding this section, let us resolve our original paradox using the ideas discussed above. The key lies in recognizing that the prescribed portfolio [eq. (1), (2)] is infeasible, unless a continuous net flow of external funds is injected into the portfolio. Our broker will call us instantly back to require this inflow of funds, without which he would not be able to execute our portfolio strategy[4]. If we do provide this inflow of funds, then obviously there is no paradox in having the portfolio change in value at an arbitrary large rate A.

In section 6 I will prove that a self-financing hedge portfolio in stock and options does exist, and that it is unique. This hedge is given by:

$$Q_S = \frac{(-\partial C/\partial S)e^{rt}}{(C - \frac{\partial C}{\partial S} \cdot S)} \tag{9}$$

$$Q_C = \frac{e^{rt}}{(C - \frac{\partial C}{\partial S} \cdot S)} \tag{10}$$

where r is the constant interest rate. No other hedge can possibly be self-financing and hence its return cannot be deduced from the rate of its value change unless the external cash flows are explicitly specified.

Also note that from the uniqueness of (9), (10), it follows that setting $A = r$ in (1), (2) will not help to make that strategy self-financing.

3. Errors in the Traditional Derivation of the Option Pricing Model

> But these two errors--it is like a miracle--cancel out in the most precise manner, as I shall prove further down. [From Johannes Kepler's account of discovery of his Second Law in Astronomia Nova, 1609.]

Some of the most important scientific discoveries were not hampered by technical errors done en route, and the Black and Scholes (1973) seminal Option Pricing Model may be counted among this honorable number.

For simplicity of exposition, I will use, in this section only, the same notation as in Black-Scholes (1973). Let w be the price of an option on a share of stock with price x. Also, let w_1 denote $\partial w/\partial x$, $w_{11} \equiv \partial^2 w/\partial x^2$ and $w_2 \equiv \partial w/\partial t$.

For convenient reference let us quote the following from Black-Scholes (1973, p. 642):

> "In general, since the hedged position contains one share of stock long and $1/w_1$ options short, the value of the equity in the position is:
>
> $$x - w/w_1 \qquad \langle 2\rangle$$
>
> The change in the value of the equity in a short interval Δt is:
>
> $$\Delta x - \Delta w/w_1 \qquad \langle 3\rangle$$
>
> Assuming that the short position is changed continuously, we can use stochastic calculus to expand Δw , which is $w(x+\Delta/x,t+\Delta t)-w(x,t)$ as follows:
>
> $$\Delta w = w_1 \Delta x + \frac{1}{2} w_{11} v^2 x^2 \Delta t + w_2 \Delta t \qquad \langle 4\rangle"$$

Proposition 2. The Black and Scholes (1973) hedge portfolio is not self-financing.

Proof. The Black and Scholes hedge portfolio consists of maintaining one share of stock long and $1/w_1$ options short. Suppose that

$$\left(\frac{1}{w_1}\right)_{t_1} = 5$$

and

$$\left(\frac{1}{w_1}\right)_{t_2} = 2 ,$$

i.e., at time t_1 the hedge portfolio consists of one share long and five options short, and at time t_2 the same hedge consists of (again) one share long and two options short. Where did the funds required for repurchase of three options come from? In other words, since one share of stock must be maintained throughout, and no fraction of it can be sold in order to finance the repurchase of the three options, it obviously follows that the repurchase of the options must be financed by some free external injection of funds.[5] Q.E.D.

The following is a corollary of Propositions 1 and 2: Even if we accept the claim by Black and Scholes, that the change in the value of equity in their hedge portfolio in a short interval is riskless, since by Proposition 2 this hedge portfolio is not self-financing, hence by Proposition 1, we cannot be sure that the rate of change in the hedge value is equal to the interest rate; the external inflow or outflow of funds may have distorted the change rate and it might be equal to some unknown constant A . Thus, a central step in the Black-Scholes derivation is missing.

Proposition 3. Contrary to the claim of Black and Scholes, the change in value of the hedge portfolio in a short (infinitesimal) interval Δt is not riskless.[6]

Proof. Denote by $V(x,t)$ the value of the equity in the Black-Scholes hedge at time t when the stock price is x.

$$V(x,t) = x - w/w_1 \tag{11}$$

This is Black-Scholes equation <2>. To write the change in the value of equity in a short interval Δt, let us use a Taylor expansion exactly as Black-Scholes do to derive their equation <4>.

$$\Delta V = V(x+\Delta x, t+\Delta t) - V(x,t) = \frac{\partial V}{\partial x}\Delta x + \frac{\partial V}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}(\Delta x)^2 + o(\Delta t) \tag{12}$$

All partial derivatives are taken at the point (x,t). Now, $(\Delta x)^2$ is of the order of Δt and from (11) we get

$$\frac{\partial V}{\partial x} = \frac{w \cdot w_{11}}{(w_1)^2} \neq 0.$$

It follows that there is only one stochastic term on the RHS of (12), namely the term in Δx; all the other terms are of the order Δt or higher, hence there is no other term to cancel out the stochastic term in Δx, i.e., ΔV is stochastic.

Q.E.D.

Quite apart from the proof of Proposition 3 it may be useful to trace the origin of the erroneous claim of Black and Scholes, since the same type of error recurrs in the literature.

Black-Scholes correctly apply Ito's lemma in their equation <4> to compute the change in the option price, but they should have also applied the same lemma to calculate the change in the value of their hedge portfolio. The result would have been then[7];

$$\Delta V = \Delta x - \Delta w/w_1 - w\cdot\Delta(1/w_1) - \Delta w\cdot\Delta(1/w_1) \tag{13}$$

Instead, they write their equation <3>:

$$\Delta V = \Delta x - \Delta w/w_1 \tag{14}$$

i.e., two terms on the RHS are simply missing. A short calculation also reveals that (13) is in complete agreement with (12) as expected.

To justify (14) a heuristic argument is often called upon to the effect that the short position in the options is being kept constant across each interval Δt, and hence $\Delta(1/w_1) = 0$. Hard as it is to accept[8], this argument is in direct conflict with what Black-Scholes assume in the line following their equation <3>:

> "Assuming that the short position is changed continuously, we can use stochastic calculus to expand Δw, which is $w(x+\Delta x, t+\Delta t) - w(x,t)$, ..."[9]

How can the short position be changed continuously for the purpose of calculating Δw, but at the same time be kept constant over Δt for the purpose of computing ΔV? Note also, that Δw appears in the expression for ΔV, hence whatever is assumed for Δw should also be assumed for ΔV.

The difficulty pointed out in Proposition 3 overshadows the difficulty which was shown to follow from Propositions 1 and 2; if the change in the value of Black and Scholes' hedge portfolio is stochastic, it cannot possibly be equal to the riskless interest rate.

It is worthwhile to identify the same difficulty, which was pointed out in proposition 3, also in Smith's (1976) exposition of the OPM, because it is widely known and referred to. For this purpose, the following excerpt from Smith's review will be quoted:

> "The value of the hedge portfolio, V_H, can be expressed as the stock price times the number of shares of stock plus the call price times the number of calls in the hedge,
>
> $$V_H \equiv SQ_S + CQ_C, \qquad \langle 36 \rangle$$
>
> where V_H is the value of the hedge portfolio, Q_S the quantity of stock and Q_C the quantity of calls (for one share each). The change in the value of the hedge, dV_H, is the total derivative of <36>
>
> $$dV_H = Q_S dS + Q_C dC. \qquad \langle 37 \rangle$$
>
> Black and Scholes use stochastic calculus to express dC, the change in the call price. Itô's lemma provides a technique by which certain functions of Wiener processes may be differentiated. If it is assumed that the stock price, S, follows Geometric Brownian Motion, then Itô's lemma can be employed to express dC."
>
> Smith (1976, p. 20).

Since, by Smith's equation <36>, V_H is also a smooth function of a Wiener process, namely the stock price S, Ito's lemma should also have been applied to V_H in order to calculate its total derivative, for the same reason that Smith applies it to calculate the change in the option price. This would have correctly resulted in:

$$dV_H = Q_S \cdot dS + Q_C \cdot dC + dQ_S \cdot S + dQ_C \cdot C + dQ_S \cdot dS + dQ_C \cdot dC \qquad (15)$$

Instead, Smith writes his equation <37>

$$dV_H = Q_S \cdot dS + Q_C \cdot dC \tag{<37>}$$

i.e., four terms are completely missing on the RHS.

In a recent review, Smith (1979, p. 80) provides the following rationale for the above missapplication of Ito's lemma.

> "Since at a point in time the quantities of options and stock are given, the change in the value of the hedge results from the change in the prices of the assets."

But what conceivable quantities are not "given at a point in time"? What about the prices of the stock and the option--are they changing "at a point in time"?

To highlight the error in Smith's argument from still another angle, consider forming the following hedge:

$$Q_S = \frac{e^{rt}}{2S}, \quad Q_C = \frac{e^{rt}}{2C}.$$

The value of this hedge is as in Smith's <36>:

$$V_H = SQ_S + CQ_C = S(e^{rt}/2S) + C(e^{rt}/2C) = e^{rt}$$

and the change in its value over an infinitesimal time interval is:

$$dV_H = d(e^{rt}) = re^{rt}dt \tag{16a}$$

which is clearly (and correctly) riskless.

But now let us use the Black-Scholes-Smith argument to compute the very same change in value. Using Smith's <37> we get:

$$dV_H = Q_S dS + Q_C dC = (e^{rt}/2S)dS + (e^{rt}/2C)dC \qquad (16b)$$

which is risky since $Q_S/Q_C \neq -\partial C/\partial S$. But this is obviously a contradiction. What is wrong in (16b) is again the unjustified omission of the four terms on the RHS as advocated by Black-Scholes-Smith.

The only possible way to reconcile (15) with <37> is to set the sum of the four missing terms equal to zero.

$$dQ_S \cdot S + dQ_C \cdot C + dQ_S \cdot dS + dQ_C \cdot dC = 0 \qquad (17)$$

However, this remedy does not come free. Every choice of Q_S, Q_C will now have to satisfy (17). Unfortunately, the Black-Scholes-Smith arbitrary choice

$$Q_S = 1 \quad , \quad Q_C = -1/(\partial C/\partial S) \qquad (18)$$

violates equation (17) and the suggested remedy fails for this choice. However, the choice of Q_S, Q_C as in (9) (10), is consistent with (17) and, as mentioned in section 2, is the unique hedge portfolio which returns at the riskless interest rate.

A simple substitution of the arbitrary choice (18) into Smith's equation <36> and application of Ito's lemma thereof, will prove that the change in V_H is indeed stochastic and not riskless as claimed. On the other hand, substitution of the choice (9) (10) in Smith's <36> and application of Ito's lemma will yield a riskless (deterministic) rate of value change, simply because the value of the portfolio is e^{rt}.

To conclude this section it would be interesting to see how a correct result, namely the Black - Scholes partial differential equation, could be derived despite the errors discussed above. It turns out that the two errors acted to cancel each other out precisely.

It has been shown in this section that the two central steps in the traditional derivation of the OPM are incorrect. One step was to show that the change in the hedge value is riskless:

$$\Delta V_H = -\left(\frac{1}{2} w_{11} v^2 x^2 + w_2\right)\Delta t / w_1 \tag{19}$$

This was shown to be incorrect; rather,the change in value is stochastic. The second step was to show that since the change is riskless, it must be equal to the riskless interest rate:

$$\Delta V_H = (x - w/w_1) r \Delta t \tag{20}$$

This was also shown to be incorrect; r might really be an arbitrary constant $\bar{A}$, since the hedge portfolio is not self-financing. But, although (19) and (20) are both incorrect, by equating $\Delta V_H = \Delta V_H$ in (19) and (20), the correct Black - Scholes partial differential equation follows. Thus, the Black - Scholes derivation is an example of two wrongs which do make a (most important) right. This is not the first event of such nature in the annals of scientific discovery as evidenced by the quotation from Johannes Kepler in the beginning of the section.

Although the Black-Scholes partial differential equation has been derived in the literature in ways which do not face the same difficulties as does the traditional derivation, e.g. Merton (1974), Garman (1976), these derivations, unlike ours, which is given in the next section, do assume existence of options and absence of arbitrage opportunites.[10] Thus, these derivations do not reveal the full generality of the Black-Scholes PDE, which will be shown to play an important role even in economies without options or with arbitrage opportunities.

4. An Equivalence Theorem

The discussion in the previous sections suggests that self-financing portfolios play a significant role in the theory of option pricing. To find out the nature of this role, we need a workable formal definition of a self-financing portfolio. Fortunately, we do not have to search far, since the required definition is provided by a special case of Merton's (1971) budget equation [eq. (12), p. 379]. This equation describes a condition on a portfolio in which wage income is used to finance new purchases of securities, and from which assets are liquidated to finance consumption. In order to apply the budget equation to the case of a self-financing portfolio, both the income and the consumption flows will be set to zero and, I will call the resulting equation "Merton's self-financing condition."

Although Merton (1974) uses this condition implicitly in his derivation of the Black-Scholes partial differential equation, he also makes use of options and of the no-arbitrage condition. My aim in this section is to prove a much stronger result; that Merton's self-financing condition and Black-Scholes partial differential equation are mathematically equivalent to each other, and furthermore, that this equivalence is independent of the existence of options or of the absence of arbitrage opportunities.

Throughout this study attention is concentrated exclusively on stochastic processes of the Ito type. Generalization to other types of processes, although possible, are not attempted herein.

In order to fix ideas it will be useful to restate Merton's self-financing condition. Let us start with a general portoflio strategy, which calls for holding $N_i(t)$ shares of asset i that does not pay dividends and that is priced at $P_i(t)(i = 1, \dots n)$. The value of this portfolio strategy at time t is given by:

$$V = \sum_{i=1}^{n} N_i P_i \tag{21}$$

It is assumed that the prices P_i are generated by Ito's processes, and that all assets are continuously and costlessly traded in perfect markets. We will focus on portfolio strategies such that the $N_i(t)$ $(i = 1,\dots n)$ are functions of the assets price levels and time only, and also are regular enough so that Ito's lemma can be applied to (21) to give the change in the value of the portfolio:

$$dV = \sum_i N_i(dP_i) + \sum_i (dN_i)(dP_i) + \sum_i (dN_i)P_i \tag{22}$$

Intuitively, one can identify the first term on the RHS of (22) as a contribution to change in value of the portfolio arising from changes in prices only, and the third term as a value contribution arising from changes in the number of shares held through influx of external funds. However, as Merton points out, intuition per se does not suffice to identify the middle term, which might be associated either way. Using a very delicate limiting argument, Merton identifies the middle term as belonging together with the third, i.e., the middle and the third term constitute the contribution to the change in value coming from external sources, while the first term is the contribution arising solely from changes in prices. Later on we will see the result of ignoring Merton's argument and incorrectly associating the middle term with the first, rather than with the third term.

Now, a self-financing portfolio strategy is, by definition, a strategy in which all of the change in value arises from changes in prices exclusively, with no external value contributions. This amounts to imposing the condition

$$\sum_i (dN_i)(dP_i) + \sum_i (dN_i)P_i = 0 \tag{23}$$

from which it also follows that

$$dV = \sum_i N_i(dP_i) \tag{24}$$

Equation (23) is what I term "Merton's self-financing condition."

It should be emphasized that this condition merely characterizes a particular portfolio strategy as being self-financing. It does not exclude the possibility of other nonself-financing strategies in the same set of assets, or even other strategies which earn arbitrage profits.[11]

It is noteworthy that Harrison and Kreps (1979) concentrate on condition (24) as the defining property of a self-financing portfolio strategy. I focus, instead, on condition (23) to derive the new results reported herein.

To prove the Equivalence Theorem, we need to assume that one of the assets is a bond returning at a constant interest rate. For simplicity of exposition I will also assume that, in addition to the bond, there is only one risky asset. Both assumptions will be relaxed later.

Specifically, borrowing and lending are carried out at a unique constant interest rate r by holding short or long positions in bonds. Without loss of generality, the price of one unit of bonds is normalized to equal 1 dollar at a chosen time origin $t = 0$. Hence, the price dynamics for a bond unit is given by

$$B(t) = e^{rt} \tag{25}$$

The risky asset (stock) has price S which follows an Ito process formally described by the stochastic differential equation:

$$dS = \mu_S S dt + \sigma_S S dZ \tag{26}$$

where μ_S, $\sigma_S^2 > 0$ are the expectation and variance per unit time of the rate of return, both may be functions of S and t; dZ is a standard Wiener process. It is assumed for simplicity that the stock does not pay dividends.

Let us define a portfolio strategy, in this context, as an ordered pair (α,β) where $\alpha = \alpha(S,t)$ is the number of shares of stock held in the portfolio at time t when the stock price is S, and where $\beta = \beta(S,t)$ is the number of bond units held. The value of the portfolio strategy at time t and stock price S is then given by:

$$V(S,t) = \alpha \cdot S + \beta \cdot B \tag{27}$$

I will occasionally refer to $V(S,t)$ as the value of the strategy (α,β), and to (α,β) as the strategy which sustains the value function $V(S,t)$. Note that the correspondence between value functions and sustaining strategies is one to many, since $(\alpha, e^{-rt}(V-\alpha S))$ sustains $V(S,t)$ for any α .

The setup is now complete and we are ready to propose the following:

Theorem 1. (Equivalence Theorem)

$V(S,t)$ is the value of a portfolio strategy of stock and bonds, which obeys Merton's (1971) self-financing condition, if and only if $V(S,t)$ satisfies Black-Scholes (1973) partial differential equation, in which case the unique self-financing strategy is maintaining $(\partial V/\partial S)$ shares of stock and $e^{-rt}[V-(\partial V/\partial S)S]$ bonds.

Before engaging in the proof, it is important to emphasize that the proposed equivalence between Merton's self-financing condition and Black-Scholes partial differential equation is of pure mathematical nature; no economic reasoning is necessary for its proof. In particular, the equivalence theorem is not predicated on the absence of arbitrage opportunities. In fact, there may well exist arbitrage profits generating strategies involving the stock, the bond, and additional assets, e.g., an option. But once a particular strategy is identified as obeying Merton's self-financing condition, then the Equivalence Theorem guarantees that its value function $V(\bar{S},t)$ satisfies the Black-Scholes partial differential equation, and that the given strategy is the unique self-financing strategy that sustains $V(S,t)$ as its value. The converse is also true. If a function $V(S,t)$ solves the Black-Scholes partial differential equation then the unique portfolio strategy that obeys Merton's self-financing condition can be identified, the value of which is the same $V(S,t)$.

Note also that options are not mentioned at all in the Equivalence Theorem, i.e., contrary to common wisdom the Black-Scholes partial differential equation is not at all an exclusive property of options. In fact, the book value of a closed-end fund invested in the stock and the bond, which is managed continuously, costlessly and smoothly according to price levels, must obey the Black-Scholes partial differential equation, even though option type boundary conditions do not apply in this case. (The boundary condition will be the fund's initial equity value.) An example is a strategy which prescribes holding all the time half the equity in the stock and half in bonds. This general applicability of the Black-Scholes PDE is now clear in light of our theorem, which characterizes it as the mathematical equivalent of Merton's self-financing condition.

Proof of the Equivalence Theorem

Necessity (Merton → Black-Scholes)

Let $V(S,t)$ be the value of a portfolio strategy (α,β), i.e.

$$V(S,t) = \alpha(S,t)\cdot S + \beta(S,t)\cdot B(t) \tag{28}$$

The change in the value of the portfolio is calculated by Ito's lemma:

$$dV = \alpha dS + \beta dB + [(d\alpha)S + (d\beta)B + (d\alpha)(dS) + (d\beta)(dB)] \tag{29}$$

Now require that this portfolio strategy (α,β) satisfies Merton's self-financing condition:

$$(d\alpha)S + (d\beta)B + (d\alpha)(dS) + (d\beta)(dB) = 0 \tag{30}$$

It will now be shown that the self-financing condition on α and β which takes the form of a single stochastic differential equation (30) can be transformed into an equivalent condition on α and β, that takes the form of a system of two nonstochastic partial differential equations. By Ito's lemma,

$$d\alpha = \mu_\alpha dt + \sigma_\alpha dZ$$

$$d\beta = \mu_\beta dt + \sigma_\beta dZ$$

where

$$\mu_\alpha = \frac{\partial\alpha}{\partial t} + \frac{\partial\alpha}{\partial S}\mu_S S + \frac{1}{2}\frac{\partial^2\alpha}{\partial S^2}\sigma_S{}^2 S^2$$

$$\sigma_\alpha = \frac{\partial\alpha}{\partial S}\sigma_S S$$

with analogous expressions for μ_β, σ_β.

Recall: $$dS = \mu_S S\ dt + \sigma_S S\ dZ$$

and from $B = e^{rt}$: $$dB = re^{rt}dt$$

Substituting for all the differentials in (30), using the formal multiplication rule $dZ^2 = dt$, ignoring o(dt) terms and collecting, results in

$$(\mu_\alpha S + \mu_\beta B + \sigma_\alpha\sigma_S S)dt + (\sigma_\alpha S + \sigma_\beta B)dZ = 0 \tag{31}$$

In order to maintain this identity at zero for all t and S we must equate the deterministic and the stochastic parts of (31) each to zero:

$$\sigma_\alpha S + \sigma_\beta B = 0 \tag{32a}$$

$$\mu_\alpha S + \mu_\beta B + \sigma_\alpha \sigma_S S = 0 \tag{32b}$$

Substituting in (32) for μ_α, μ_β, σ_α, σ_β and for B we get a system of two parabolic PDEs.

$$S \frac{\partial \alpha}{\partial S} + e^{rt} \frac{\partial \beta}{\partial S} = 0 \tag{33a}$$

$$S\left(\frac{\partial \alpha}{\partial t} + \frac{1}{2} \sigma_S^2 S^2 \frac{\partial^2 \alpha}{\partial S^2}\right) + \sigma_S^2 S^2 \frac{\partial \alpha}{\partial S} + e^{rt}\left(\frac{\partial \beta}{\partial t} + \frac{1}{2} \sigma_S^2 S^2 \frac{\partial^2 \beta}{\partial S^2}\right) = 0 \tag{33b}$$

Note that the drift in the stock price has dropped out.

From (28):

$$\beta = e^{-rt}(V - \alpha S) \tag{34}$$

Substitution for β from (34) into (33a) results in

$$\alpha = \frac{\partial V}{\partial S} \tag{35}$$

and for α in (34):

$$\beta = e^{-rt}\left(V - S \frac{\partial V}{\partial S}\right) \tag{36}$$

Note that (35) and (36) uniquely determine the self-financing strategy (α, β) in terms of the value function $V(S,t)$ of the portfolio. This proves the uniqueness assertion in the theorem in a constructive way.

Substituting for α, β from (35), (36) into (33b) yields

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_S^2 S^2 \frac{\partial^2 V}{\partial S^2} = r\left(V - S\frac{\partial V}{\partial S}\right) \tag{37}$$

which is the celebrated Black-Scholes partial differential equation. This completes the proof of the necessity assertion.

Note that the Black-Scholes PDE has been derived from Merton's self-financing condition in a purely mathematical sequence of steps, as promised.

We also have a nice interpretation of the Black-Scholes partial differential equation. To see this, first note that from (35):

$$\alpha S = S\frac{\partial V}{\partial S} \tag{38}$$

is the dollar amount invested in the stock component of our self-financing portfolio. Hence $(S\frac{\partial V}{\partial S})\cdot\mu_S$ is the non-stochastic contribution to the rate of dollar return on the stock component in the portfolio. Similarly from (36):

$$\beta B = V - S\frac{\partial V}{\partial S} \tag{39}$$

is the dollar amount invested in bonds, hence the rate of dollar return on the bond component is given by:

$$r\left(V - S\frac{\partial V}{\partial S}\right) \tag{40}$$

which is precisely the RHS of the Black-Scholes partial differential equation. Now let us look at the non-stochastic part of the rate of dollar change of the portfolio as a whole, i.e. the drift in the value of the portfolio, which by Ito's lemma is:

$$\text{drift } (V) = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_S^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) + \mu_S S \frac{\partial V}{\partial S} \tag{41}$$

The first two terms on the RHS of (41) can now be identified by the help of the Black-Scholes partial differential equation as the pure interest contribution to the drift of the portfolio, while the third term is the contribution from the drift of the stock component.

Sufficiency (Black-Scholes → Merton)

Assume that a function $V(S,t)$ satisfies the Black-Scholes partial differential equation. Let us now choose the particular portfolio strategy $(\alpha_{SF}, \beta_{SF})$ as prescribed by (35) and (36). A quick calculation shows that this strategy sustains $V(S,t)$ as its value function (as do many other strategies). Now, substitution and using the Black-Scholes partial differential equation (37) shows that these α_{SF}, β_{SF} satisfy (30), i.e. $(\alpha_{SF}, \beta_{SF})$ obeys Merton's self-financing condition. Any choice of a strategy which is different from $(\alpha_{SF}, \beta_{SF})$ will violate (30), i.e., will not be self-financing. Thus uniqueness is also proved. Q.E.D.

To recapitulate, the Equivalence Theorem characterizes the set of all functions $V(S,t)$, to which Ito's lemma can be applied, in the following way. Recall that any one function $V(S,t)$ can be sustained as a value by infinitely many portfolio strategies of stocks and bonds. Now, the set of all functions $V(S,t)$ can be divided into two disjoint subsets, one containing all functions $V(S,t)$ that satisfy the Black-Scholes partial differential equation, and the other containing all functions which do not. The Equivalence Theorem then tells us that among all portfolio strategies which sustain any given function in the first subset (satisfying the Black-Scholes PDE) there exists exactly one strategy which is self-financing. The theorem also prescribes a way to identify it; it is the strategy which calls for holding $\partial V/\partial S$ shares of stock and the rest of the funds are invested in bonds. On the other hand, for any given function in the second subset (not satisfying Black-Scholes PDE), our theorem informs, that none of the strategies, that sustain this function, is self-financing.

The Equivalence Theorem can be put to immediate use.

Proposition 4. $V(t) = \exp(rt)$ is the unique value-function of time only (with normalized value of unity at time zero) that can be sustained by a self-financing portfolio strategy.

Proof. Consider a general function of time only $V(t)$ with $V(0) = 1$. by the Eqivalence Theorem, $V(t)$ must satisfy the Black-Scholes partial differential equation in order to be sustained by a self-financing portfolio strategy. Substition of $V(t)$ in Black-Scholes partial differential equation yields

$$\frac{\partial V(t)}{\partial t} = rV(t) \tag{42}$$

which together with $V(0) = 1$ has a unique solution $V(t) = e^{rt}$. Q.E.D.

It follows from Proposition 4 that if a portfolio strategy is observed to have a value $\exp(At)$ with $A \neq r$, then it is certainly a nonself-financing strategy. This, recall, is related to the paradox resolution in section 2.

As another example consider $V(S,t) = S^2$. Since, excluding the peculiar case $r = -\sigma^2$, S^2 does not satisfy Black-Scholes partial differential equation, there is no self-financing strategy which can sustain this value function. As a matter of fact, the only function of stock price alone that can be sustained by a self-financing strategy, is $V(S) = aS$ where a is a constant, i.e., a buy-and-hold strategy.

Note also that it follows from the Equivalence Theorem that there is no self-financing portfolio strategy in which the number of units of one asset is kept constant while the other is varied. This is, of course, intuitively plausible.

I will now bring forth another manifestation of the equivalence between Merton's self-financing condition and the Black-Scholes partial differential equation. Suppose that we did not follow Merton's (1971) limiting argument and instead of associating the middle term on the RHS of (22) with the third, we erroneously associated it with the first term. In that case the self-financing condition would have taken the form:

$$\sum_i (dN_i)P_i = 0 \tag{43}$$

In our framework of portfolios of a stock and bonds, (43) would have taken the form

$$(d\alpha)S + (d\beta)B = 0 \tag{44}$$

Then it can be shown that the Equivalence Theorem would go through with one major change; the Black-Scholes partial differential equation is replaced by a different one:

$$\frac{\partial V}{\partial t} - \frac{1}{2}\sigma_S^2 \cdot S^2 \frac{\partial^2 V}{\partial S^2} = r(V - S\frac{\partial V}{\partial S}) \qquad (45)$$

Compare to (37). (The proof follows exactly the same lines as that of the Equivalence Theorem.) In other words, if the self-financing condition does not take Merton's particular form, the Black-Scholes partial differential equation does not follow.

To conclude this section let us state the following: assume that $S_i(i=1,\ldots,n)$ are the Ito price processes of n risky assets with a nonsingular returns covariance matrix with elements $\sigma_{ij} = \sigma_{ij}(S_1,\ldots,S_n,t)$, $(i,j=1,\ldots,n)$.

Equivalence Theorem (Generalization).

(a) $V(S_1, \ldots, S_n,t)$ is the value of a self-financing portfolio strategy in the n risky assets and a riskless bond, if and only if $V(S_1, \ldots, S_n,t)$ satisfies

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}S_iS_j\frac{\partial^2 V}{\partial S_i \partial S_j} = r(V - \sum_{i=1}^{n}\frac{\partial V}{\partial S_i}\cdot S_i) \qquad (46)$$

In this case, the self-financing strategy is unique; the number of units of asset i is $\partial V/\partial S_i(i = 1,\ldots n)$ and the number of bonds is

$$e^{-rt}(V - \sum_{i=1}^{n}\frac{\partial V}{\partial S_i}S_i).$$

(b) $V(S_1,\ldots,S_n,t)$ is the value of a self-financing portfolio strategy in the n risky assets[12], if and only if $V(S_1,\ldots,S_n,t)$ satisfies:

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} S_i S_j \frac{\partial^2 V}{\partial S_i \partial S_j} = 0 \qquad (47a)$$

$$V = \sum_{i=1}^{n} \frac{\partial V}{\partial S_i} S_i \qquad (47b)$$

The unique self-financing strategy again holds $\partial V/\partial S_i$ units of asset i and, of course, no bonds.

The proof for (a) is given elsewhere.[13] The reader is invited to provide the proof for (b).

5. Application to Option Pricing Theory

Options and the "no-arbitrage" condition are now first introduced into the analysis, and the Option Pricing Model is derived as a corollary from the Equivalence Theorem. To bring out the main points, only European options that do not pay dividends will be considered.

A generalized European option is a security which entitles the holder to a payoff of $h(S(T))$ dollars at expiration time T contingent upon the value $S(T)$ of the underlying asset (stock).

Corollary 1. (Option Pricing)

If arbitrage profits generating strategies in the stock, the riskless bond and the option are ruled out, then the price of the option is a function of the stock price and time alone, i.e., $C = C(S,t)$, and it must satisfy the Black-Scholes partial differential equation with boundary condition:

$$C(S(T),T) = h(S(T)) \qquad (48)$$

Proof. The idea is to construct a portfolio of stock and bonds, that will exactly duplicate the contracted cash flows to the option. Then, if arbitrage opportunities are ruled out, the price of the option must equal the value of the portfolio strategy, which will be termed the "duplicating portfolio", at any time prior to or at expiration.

The proof rolls as follows. Since the option is obviously self-financing, it is required that the duplicating portfolio also has the same property. Then by the necessity part of the Equivalence Theorem the value $V(S,t)$ of the self-financing duplicating portfolio must satisfy the Black-Scholes partial differential equation. (Note, because of self-financing, not because of "no-arbitrage".) Now, in order to duplicate the contracted payoff to the option at expiration, we must require that the duplicating portfolio winds up at the value $h(S(T))$ at expiration T. This has the effect of appending the boundary condition $V(S(T),T) = h(S(T))$ to the Black-Scholes partial differential equation, which jointly produce (under appropriate conditions on $h(\cdot)$) a unique solution $V(S,t)$ $0 \leq t \leq T$ for the value of the duplicating portfolio. Note that even though we have identified the value function $V(S,t)$, we have not yet identified its sustaining self-financing strategy. Now we summon the sufficiency part of the Equivalence Theorem to identify the unique self-financing strategy that sustains the solution $V(S,t)$. This strategy is given by equations (35) and (36).

Now that we have completed the construction of a portfolio strategy that exactly duplicates the returns to the option, it is at this point that the "no arbitrage" condition is introduced[14] to conclude that the option price must be equal to the value of the duplicating strategy throughout the option's lifetime. Formally:

$$C = V(S,t) \qquad 0 \leq t \leq T \tag{49}$$

hence, C inherits all the properties of $V(S,t)$, i.e., the option price is a function of S and t only, and it solves the Black-Scholes partial differential equation with boundary condition (48). Q.E.D.

Note that the dependence of the option price on S and t alone was not assumed apriori, but was, indeed, proved. Note also that both directions of the Equivalence Theorem were used to prove the corollary and that the duplicating strategy was naturally identified. It is also worth emphasizing that allowing arbitrage opportunities to exist would only prevent the corollary from going through, i.e., the Option Pricing Model would not hold, but the Equivalence Theorem will remain unharmed.

For a moment, let us modify again the form of the self-financing condition as was done above in equation (44). It was shown there that if this is done, the partial differential equation (45), which is different from that of Black-Scholes, would follow. Given that, we could prove again a result similar to corollary 1, with one major difference, this time the option price would satisfy PDE (45) instead of the Black-Scholes PDE.

This poses still another difficulty to the traditional derivation of the Option Pricing Model by Black-Scholes, since their derivation is, evidently, independent of the particular form taken by the self-financing condition, a fact, as we have just shown, that cannot possibly be true. It should again be realized that the correct form of the self-financing condition is that of Merton [equation (23)], and it is for this reason that the option price satisfies Black-Scholes partial differential equation and not equation (45).

To conclude this section it will be useful to examine a European call option which is defined by the boundary condition:

$$h(S(T)) = \max[S(T) - K, 0] \tag{50}$$

where K is the exercise price. Black-Scholes (1973) derive the explicit solution to the price of the call when σ is a constant. This will be written in the following format:

$$C(S,t) = [N(d_1)] \cdot S + [-e^{-rT} \cdot K \cdot N(d_2)] \cdot e^{rt} \tag{51}$$

where $N(\cdot)$ is the cumulative normal distribution function, and

$$d_1 = \frac{\ln(S/K) + \left(r + \frac{1}{2}\sigma_S^2\right)(T-t)}{\sigma_S\sqrt{(T-t)}}$$

$$d_2 = \frac{\ln(S/K) + \left(r - \frac{1}{2}\sigma_S^2\right)(T-t)}{\sigma_S\sqrt{(T-t)}}$$

In order to identify the unique self-financing portfolio strategy which has (51) as a value, use will be made of the Equivalence Theorem by substituting $C(S,t)$ from (51) into (35) and (36) to get that the number of shares of stock in the duplicating portfolio is

$$\alpha(S,t) = N(d_1) \tag{52}$$

and the number of bonds is

$$\beta(S,t) = -e^{-rT} \cdot K \cdot N(d_2) \tag{53}$$

This clearly explains the format in which (51) was written. It also follows from (52) and (53) that in order to duplicate a call option, one is always long in the stock and borrowing.

6. Characterization of Self-financing Portfolios in Stock and Options.

Since hedge portfolios of stock and options were extensively used in the literature, it is interesting to see what are the conditions that allow these portfolios to be self-financing in Merton's sense. Another purpose of this section is to make good on promises given in sections 2 and 3.

To simplify matters, I will embed the present discussion in an economy with a riskless asset returning at constant rate r, and I will futher assume that arbitrage opportunities are ruled out. This is necessary because in such an economy, as was shown in section 5, an option price depends only on time and price of the underlying asset.

Theorem 2.

$V(S,t)$ is the value of a self-financing portfolio strategy involving an option with price $C(S,t)$ and its underlying asset with price S, if and only if $V(S,t)$ satisfies the Black-Scholes partial differential equation. In this case, the unique[15] self-financing strategy is maintaining $Q_C(S,t)$ options and $Q_S(S,t)$ units of the underlying asset, where

$$Q_S = \frac{\left(C\frac{\partial V}{\partial S} - V\frac{\partial C}{\partial S}\right)}{\left(C - \frac{\partial C}{\partial S}\cdot S\right)} \tag{54}$$

$$Q_C = \frac{\left(V - \frac{\partial V}{\partial S}\cdot S\right)}{\left(C - \frac{\partial C}{\partial S}\cdot S\right)} \tag{55}$$

Note that (54), (55) are the analogs of (35), (36). The proof follows similar lines to that of the Equivalence Theorem, a sketch is provided.

Necessity

Let us start with a general portfolio strategy, which maintains $Q_S(S,t)$ shares of stock and $Q_C(S,t)$ options. Its value is given by:

$$V(S,t) = Q_S \cdot S + Q_C \cdot C(S,t) \tag{56}$$

The change in value is by Ito's lemma:

$$dV = Q_S \cdot dS + Q_C \cdot dC + dQ_S \cdot S + dQ_C \cdot C + dQ_S \cdot dS + dQ_C \cdot dC \tag{57}$$

Now impose Merton's self-financing condition

$$dQ_S \cdot S + dQ_C \cdot C + dQ_S \cdot dS + dQ_C \cdot dC = 0 \tag{58}$$

From which it follows that:

$$dV = Q_S \cdot dS + Q_C \cdot dC \tag{59}$$

This is the remedy to Smith's (1976, 1979) error discussed[16] in section 3. Now we will find the cost of this remedy.

As above S follows the Ito process:

$$dS = \mu(S,t)dt + \sigma(S,t)dZ \tag{60}$$

Using Ito's lemma, substituting for all differentials in (58), separating into deterministic and stochastic parts and equating each to zero, we get a system of two partial differential equations in Q_S and Q_C:

$$S \frac{\partial Q_S}{\partial S} + C \frac{\partial Q_C}{\partial S} = 0 \qquad (61a)$$

$$S \frac{\partial Q_S}{\partial t} + C \frac{\partial Q_C}{\partial t} + \frac{1}{2} \sigma^2 \Big(2 \frac{\partial Q_S}{\partial S} + 2 \frac{\partial C}{\partial S} \frac{\partial Q_C}{\partial S} + S \frac{\partial^2 Q_S}{\partial S^2} + C \frac{\partial^2 Q_C}{\partial S^2}\Big) = 0 \qquad (61b)$$

This strict condition on Q_S, Q_C is the cost of our remedy. It can immediately be verified that the Black-Scholes-Smith arbitrary choice

$$Q_S = 1 \quad , \quad Q_C = -1/ \frac{\partial C}{\partial S} \qquad (18)$$

violates Merton's self-financing condition (58) and its equivalent system (61). Thus this hedge defeats its remedy. Note also that merely setting $(Q_S/Q_C) = -(\partial C/\partial S)$ is not sufficient to satisfy (58) or (61).

Now, from (56) and (61a), using some algebra, we get (54) and (55). Substituting (54) and (55) back into (61b) and using the fact that the option price must satisfy the Black-Scholes partial differential equation in our economy, it follows that V must also satisfy the same partial differential equation. This completes the proof of necessity.

Sufficiency

Using the fact that $V(S,t)$ satisfies the Black-Scholes partial differential equation it can be verified that Q_S, Q_C as in (54) and (55) are the only choice which satisfies Merton's self-financing condition (58). Q.E.D.

An immediate consequence of this theorem is the following:

Corollary 2. $V(t) = \exp(rt)$ is the unique value-function of time alone, that can be sustained by a self-financing strategy in stock and options. The unique self-financing sustaining strategy is given by:

$$0_S = \frac{(-\partial C/\partial S)e^{rt}}{(C - \frac{\partial C}{\partial S} \cdot S)} \tag{9}$$

$$0_C = \frac{e^{rt}}{(C - \frac{\partial C}{\partial S} \cdot S)} \tag{10}$$

Proof. By theorem 2, any function $V(t)$ that can be sustained by a self-financing strategy must satisfy the Black-Scholes partial differential equation. Hence, exactly as in Proposition 4, $\exp(rt)$ is the only solution. Substituing $V = \exp(rt)$ in (54) and (55) yields the desired strategy (9), (10) which is unique. Q.E.D.

Theorem 2 can also be employed to provide an alternative one-line proof for Proposition 2. Simply note that the value of the Black-Scholes (1973) hedge portfolio does not satisfy the Black-Scholes partial differential equation, hence by theorem 2 it cannot possibly be self-financing.

7. Conclusions

Our main result in this paper is the Equivalence Theorem which shows that imposing Merton's self-financing condition on a portfolio strategy is equivalent to imposing Black-Scholes partial differential equation on its value even in economies with arbitrage or without options.

This result brought into light the central role played by Merton's self-financing condition in the Option Pricing Model. In particular it helped point out serious difficulties in the traditional derivation of the Option Pricing Model and paved the way for a natural derivation that clarifies the issues and avoids the difficulties. It also lends itself to extension of the Option Pricing Model into imperfect and incomplete capital markets, but this is the subject of a subsequent paper.

FOOTNOTES

1. Manage until one day before expiration to avoid zeros in the denominator.
2. For simplicity, the stock is assumed throughout not to pay dividends.
3. This term was coined by Harrison and Kreps (1979).
4. When $A < r$ our broker will find out that he can withdraw funds and still maintain a value dynamics identical to the one requested.
5. This proof would not follow if the option price were a linear function of the stock price, since then $1/w_1$ = constant, but aposteriori we know that this is seldom the case.
6. See also Kwon (1979).
7. By Ito's lemma $\Delta(f \cdot g) = f \cdot \Delta g + g \cdot \Delta f + \Delta f \cdot \Delta g$ on very short intervals Δt.
8. This argument, if correct, could be used to show that $\Delta(f \cdot g) = f \cdot \Delta g$, even in classical calculus; just keep f constant across the interval Δt.
9. I have used precisely the same procedure in proving Proposition 3.
10. Note that I refer to the Black-Scholes partial differential equation only, not to option pricing, which follows from the former and which obviuosly must deal with options.
11. We may even have self-financing portfolio strategies that produce arbitrage profits. As a concrete example consider the two assets defined by $P_1 = e^{At}$, $P_2 = e^{rt}$ with $A > r$, and form the portfolio: $N_1 = 1$, $N_2 = -1$. On one hand, the value of this portfolio is zero at $t = 0$ and strictly positive for all $t > 0$. On the other hand, Merton's self-financing condition (23) is clearly satisfied.

12. A riskless asset does not necessarily exist in the economy in this case.

13. Bergman (1981).

14. A logical gap, that has often been left unnoticed in the literature, must be bridged at this point. The "no arbitrage" condition cannot be freely imposed on the set of three assets--bond, stock, and option--without some preparation, since the assumed price process for the bond and for the stock might have already admitted arbitrage opportunities involving these two assets alone. As an example for such a case take $\sigma_S = 0$ and $\mu_S = A \neq r$, then see footnote 11. The proof that arbitrage opportunities in the bond and stock alone are impossible, follows from the Equivalence Theorem together with Theorem 5.2 in Friedman (1975, p. 147).

15. The case $C(S,t) = f(t) \cdot S$ must be treated separately, since then Q_S, Q_C as in (54) (55) are not defined. To qualify as an option $C(S,t)$ must solve B-S PDE, which can only be true if $f(t)$ = constant. In this case a self-financing strategy is not unique, but is also indistinguishable from "buy and hold a constant number of stocks."

16. Note that (56), (57), (58), (59) are the same as <36>, (15), (17), <37>, respectively, i.e., the essential remedy discussed in section 3 is really imposing Merton's self-financing condition.

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