A SIMPLE FORMULA FOR THE EXPECTED RATE OF RETURN OF AN OPTION OVER A FINITE HOLDING PERIOD

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MARK RUBINSTEIN

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Consider a call option with striking price $K$ and time to expiration $t$ on a particular underlying stock. Assume:

1. Both an investor and the "market" agree that the annualized discrete interest rate will remain constant and equal to $r - 1$, and agree that the current market price of the stock is $S$.
2. Both the investor and the market agree that the stock rate of return is lognormal over any holding period.
3. The investor's estimate of the annualized discrete expected rate of return and instantaneous volatility of the stock is $\mu - 1$ and $\sigma$, respectively, while the "markets" respective estimates are $\hat{\mu} - 1$ and $\hat{\sigma}$.
4. The market price of the call at all times through expiration is set equal to the value it would have according to the Black-Scholes formula for European options, $C(S,K,t,r,\sigma)$.

These assumptions isolate a single source of disagreement between an investor and the market about the relative mispricing of an option: the stock volatility. In practice there will be other sources of disagreement (interest rates, dividend yields, the pricing formula itself) but differences in volatility estimates are likely to prove the most important.

These assumptions also embody an hypothesis about the nature of market disequilibrium or inefficiency. If a security is believed to be over- or underpriced, to measure its expected rate of return over a given time interval, we need some hypothesis about the speed with which
this mispricing disappears. In the case of an option that is mispriced relative to its underlying stock, we know any mispricing must completely disappear on its expiration date if there are to be no general arbitrage opportunities: all option pricing models and all estimators of volatility agree about the value of an option relative to the stock on the expiration date. Therefore, simply to postulate the convergence of an option price to its equilibrium price on its expiration date is of no assistance.

One possibility is to measure the mispricing by the absolute difference between value and price and suppose that this difference dwindles as a linear function of the time remaining to expiration. A more fundamental supposition would be derived from the variables that affect the price of an option. In our case, we have assumed that the source of mispricing lies with differences in estimated volatility of the stock and that both the investor and the market persist in their opinions about the volatility through the option's expiration date. To be sure, a more sophisticated model would allow for the investor to anticipate the market to revise its estimate closer and closer to the investor's as the market accumulated more and more observations of stock price changes through the expiration date.

Under these conditions, we can derive a surprisingly simple formula for the expected future market price of the call $E(C \mid h)$ at the end of holding period $h < t$, measured as a fraction of a year:
\[ E(C|h) \] is equal to the Black-Scholes European option value \( C(S,K,t,r,\sigma) \) except that \( C \) is evaluated at

\[ C(S^h,K^h,t,\tau,\sqrt{a\sigma^2 + (1-a)\delta^2}) \]

where \( a = \frac{h}{t} \)

In other words, the expected future market price of an option at the end of holding period \( h \) is equal to the price the option would have today if the current stock price were \( S^h \), the striking price \( K^h \), the time to expiration \( t \), and the market's estimate of the volatility \( \sqrt{a\sigma^2 + (1-a)\delta^2} \) where \( a \) is the fraction of the option's time to expiration taken up by the holding period.

When the holding period is very short relative to \( t \), the volatility in this formula is almost entirely determined by the market since \( a = 0 \) and

\[ \sqrt{a\sigma^2 + (1-a)\delta^2} = \delta. \]

On the other hand, if the option is to be held to expiration, then the volatility is completely determined by the investor since \( a = 1 \) and

\[ \sqrt{a\sigma^2 + (1-a)\delta^2} = \sigma. \]

The call's annualized expected rate of return is thus

\[ \left[ \frac{E(C|h)}{C} \right]^{1/h} - 1 \]

where \( C \) is its current market price (using \( \delta \)).
The following table shows that the well-known instantaneous approximation tends to be adequate when $\sigma = \hat{\sigma}$. However, when the investor believes a call is over- or undervalued relative to its underlying stock (because $\sigma \neq \hat{\sigma}$), then the call expected return can be quite sensitive to the holding period.

Proof: Our problem is to integrate

$$E(\tilde{C}|h) = \int_{-\infty}^{\infty} \left[ Se^Y N(x) - Kr^{h-t}N(x-\hat{\sigma}\sqrt{t-h}) \right] \frac{1}{\sigma \sqrt{2\pi h}} e^{-\frac{(y-m)^2}{2\sigma^2 h}} dy$$

$N$ is the standard normal distribution function, where $y$ is the natural logarithm of the return of the stock over holding period $h$,

$$\mu = \frac{1}{h} E(\log y) = m - \frac{1}{2} \sigma^2$$

and

$$\tilde{x} = \frac{\log(Se^Y/Kr^{h-t})}{\hat{\sigma}\sqrt{t-h}} + \frac{1}{2} \sigma\sqrt{t-h}.$$ 

Observe that the expression in brackets is the Black-Scholes formula for the value of a call with time $t - h$ left to expiration, volatility $\hat{\sigma}$ and stock price $Se^Y$. Since $e^Y$ is lognormal by assumption, $y$ will be normal which explains the density function multiplying the term in brackets and the limits of the integral.

To solve this integral, first write it as the difference between two integrals, one involving $Se^YN(\tilde{x})$ and the other $N(\tilde{x} - \hat{\sigma}\sqrt{t-h})$. The
Table 1. Representative Black-Scholes Call Expected Rates of Return\(^a\)

\[ m = 1.10 \quad \sigma = .3 \quad S = 40 \quad r = 1.05 \]

| \( \hat{\sigma} \) | K | \( \text{Expiration Month} \) | \( \text{Holding Period (Days)} \) |
| --- | --- | --- | --- | --- | --- | --- | --- |
|   |   | JAN\(^b\) | APR | JUL |
|   |   | 1 | 15 | 30 | 1 | 15 | 60 | 120 | 1 | 15 | 60 | 210 |
| .25 | 35 | .63 | .64 | .64 | -- | .58 | .57 | .55 | .53 | .47 | .47 | .45 | .42 |
|   | 40 | 25.89 | 19.39 | 15.11 | -- | 1.79 | 1.72 | 1.55 | 1.38 | .95 | .93 | .89 | .77 |
|   | 45 | * | * | * | -- | 6.84 | 6.35 | 5.14 | 4.11 | 1.98 | 1.93 | 1.77 | 1.42 |
| .3 | 35 | .47 | .47 | .47 | -- | .34 | .34 | .34 | .34 | .29 | .29 | .29 | .28 |
|   | 40 | 1.08 | 1.07 | 1.06 | -- | .48 | .48 | .48 | .48 | .37 | .37 | .37 | .36 |
|   | 45 | 2.38 | 2.37 | 2.36 | -- | .66 | .66 | .65 | .65 | .45 | .45 | .45 | .44 |
| .35 | 35 | .21 | .22 | .24 | -- | .15 | .15 | .15 | .16 | .15 | .15 | .15 | .16 |
|   | 40 | -.58 | -.61 | -.64 | -- | -.01 | -.01 | -.01 | -.02 | .09 | .09 | .09 | .08 |
|   | 45 | -.98 | -.99 | -.99 | -- | -.24 | -.25 | -.27 | -.30 | -.01 | -.01 | -.02 | -.03 |

*Greater than 10,000.

\(^a\)No adjustment for dividends.

\(^b\)The January options have one month to expiration, the April's four months, and the July's seven months. \( r, m, \sigma, \hat{\sigma} \) and the option expected rates of return are expressed in annual terms.
first integral can be shown equal to

\[ S_m^h \int_{-\infty}^{\infty} N(A+Bz) \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \, dz \]

and the second integral can be shown equal to

\[ K r^{h-t} \int_{-\infty}^{\infty} N(A' + B'z) \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \, dz \]

where \( A, B, A', \) and \( B' \) are constants depending on the nonstochastic parameters in the original integral. Now, using the fact that

\[ \int_{-\infty}^{\infty} N(A+Bz) \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \, dz = N\left( \frac{A}{\sqrt{1+B^2}} \right) \]

the result follows.
FOOTNOTES

1. We speak here as though the "market" had a mind of its own. This is a fiction for deriving inferences from security market prices. For example, the "market's" estimate of stock volatility \( \hat{\sigma} \) can be inferred, given the Black-Scholes formula, from the implicit volatility contained in the current market price of an associated option.

2. A purist might bring the following objection. In the context of the Black-Scholes [1973] derivation, if two investors have different opinions about the volatility of the stock, then at least one will believe there is a riskless arbitrage opportunity. Even worse, if both investors agree the stock price follows geometric Brownian motion, they cannot for long disagree about the volatility since over any finite period an infinite number of observations will be available on a stationary stochastic process. Therefore, for this argument to be technically viable we need a context for the Black-Scholes formula which admits investor disagreement about volatility. Such a context is provided in a discrete-time development of the Black-Scholes formula in Rubinstein [1976].

3. An unstated but implicit assumption requires that the investor knows the market and he agree about \( r \), knows the market uses the Black-Scholes formula to price options, and knows the market's estimate \( \hat{\sigma} \). However, it is not required that the investor know the market's estimate \( \hat{\sigma} \).

4. See Galai and Masulis [1976], equation 13, page 60.
BIBLIOGRAPHY

