COMMENTS ON THE VALUATION OF DERIVATIVE ASSETS

by

Avi Bick*

School of Business Administration
University of California
Berkeley, California 94720

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This paper presents an alternative approach to derive the Breeden-Litzenberger valuation formula, which expresses the price of an arbitrary derivative security in terms of call options' prices. This valuation formula follows from the observation that a continuous derivative security can be replicated by a portfolio including a bond and call options with all possible exercise prices. Discrete terms are added to the original Breeden-Litzenberger formula to reflect possible discontinuities of the call option price's derivative with respect to the exercise price. These discontinuities are subsequently shown to correspond to mass points of the probability distribution of the stock price. Several applications of the Breeden-Litzengberger valuation formula are demonstrated.
1. INTRODUCTION AND SUMMARY

Breeden and Litzenberger (1978, henceforth BL) show that
\[ c_{xx} = \frac{\partial^2 c}{\partial x^2}, \]
where \( c(T,x) \) is the price of a European call option on
a given underlying security with exercise price \( x \) and maturity date
\( T \), is a pricing density function for the valuation of general derivative
assets on the same underlying security. Thus, the current price of a
derivative security that pays \( q(M_T) \) at time \( T \), where \( M_T \) is the
value of the underlying security at time \( T \), and zero at any other time
will be:

\[ V(q) = \int_0^\infty q(y)c_{xx}(T,y)dy. \]  

(1)

In the case of a derivative cash stream over time with density \( q(T,M) \)
(i.e., \( q(T,M)dT \) is paid in the time interval \( [T,T + dT] \)), (1) is
generalized (denoting the present by \( T = 0 \)) to

\[ V(q) = \int_0^\infty \int_0^\infty q(T,y)c_{xx}(T,y)dy \ dT. \]  

(1')

In section 2, an alternative proof for the valuation formula (1)
is presented. The reason for this is twofold: first, while BL implic-
itly assume the existence of \( c_{xx} \) at each point, \(^1\) this paper will
generalize the formula for the case in which \( \partial c/\partial x \) may not exist in
some points. (The economic interpretation of this case will be dis-
cussed in section 3.) For example, (1) is obviously not true if \( c \) is
piecewise linear in the exercise price [see, e.g., Cox, Ross, and Rubinstein (1979, eq. (6)) or Cox and Ross (1976, eq. (29)) for such cases]. Even if $\frac{\partial^2 c}{\partial x^2}$ is continuous, (1) implies that the current price of a bond which pays one dollar at time $T$ is $-\frac{\partial c}{\partial x}(T,0^+)$, which is not necessarily true (see section 3).

Second, while BL's proof is intuitively appealing, it is not complete from a formal point of view. They implicitly use an approximation theorem which states that the given "continuous-state" economy (in which $M_T$ can obtain any nonnegative value) can be approximated by a "discrete-state" economy (in which $M_T$ can obtain only integer non-negative multiples of some $\Delta M$) in the sense that price systems will converge to the price system in the given economy as $\Delta M \to 0$. While it is plausible that such a theorem can be proved under certain conditions, this problem can be circumvented altogether by "remaining" in the given economy and approximating "on portfolios." The method employed here is to construct the derivative security explicitly from the call options, thus generalizing the known piecewise-linear case [see Garman (1976), Cox and Rubinstein (1982)], and then the implied valuation formula will turn out to be mathematically equivalent to the generalized BL's formula.

Section 3 deals with the relation between investors' beliefs and the structure of call options' prices (as a function of the exercise price). In section 4, the valuation formula will be written for the Black-Scholes case, and it will be shown that this is consistent with the continuous-trading-strategy approach. The usefulness of this
formula will be demonstrated in sections 5 and 6, where it will be employed to price compound options and call options on a stock with constant dividend yield, respectively.

2. DERIVATION OF THE PRICING FORMULA

Consider a future time $T$ which will be held fixed throughout this section (and hence omitted from the notation). The following securities, among other securities, are assumed to be traded in a perfect financial market:

(i) A given "underlying" stock $^3$ whose price at time $T$ is denoted $M$ (a random variable).

(ii) For each $x \geq 0$, a European call option $^4$ on the above stock with maturity at $T$ and exercise price $x$. (In subsequent sections we shall call it "the $(T,x)$-call option.") Its price $c(x)$ at $t = 0$ is assumed to be continuous in $x$.

(iii) A bond which yields one dollar at time $T$. Its price at $t = 0$ will be denoted $B = B_T$.

The market is assumed to be open at times $t = 0$ and $t = T$, but no assumptions are made regarding whether or not there is trade in between or regarding the dividend pattern of the underlying stock.

**Proposition 1:** Let $q:[0,\infty) \rightarrow R$ be a function with a continuous second derivative except in a finite set of points in which the left-hand and right-hand derivatives $q'_-', q'_+, q''_-, q''_+$ exist and are finite.$^5,^5a$ Then, (a) The cost of the derivative asset which pays $q(M)$ at time $T$ and zero at any other time is given by$^6$
\[ V(q) = q(0)B - \sum_{a \in D(q_+)} (q_+(a) - q(a))c^+(a) - \sum_{a \in D(q_-)} (q(a) - q_-(a))c_-(a) \]
\[ + q_+(0)c(0) + \int_{a \in D(q')} (q'_+(a) - q'_-(a))c(a) + \int_0^\infty q''(y)c(y)dy \quad (2) \]

where \( D(q') \) is the set of discontinuity points of \( q' \), \( D(q_+) \) is the set of points \( a \) in which \( q_+(a) \equiv \lim_{x \to a^+} q(x) \neq q(a) \) and \( D(q_-) \) is defined similarly.

(b) Assume that \( c \) has a continuous second derivative with respect to the exercise price except in a finite set of points in which the left- and right-hand derivatives \( c'_-, c'_+ \) and \( c''_+ \) exist and are finite. Then,

\[ V(q) = \int_0^\infty q(y)c''(y)dy + q(0)(c_+(0) + B) + \sum_{a \in D(c')} q(a)(c'_+(a) - c'_-(a)) \quad (3) \]

where \( D(c') \) is the set of discontinuity points of \( c' \).

**Proof:** (a) Suppose first that \( q \) is continuous. Consider a portfolio of all the call options with weight function \( q'' \).

The portfolio's cost will be \( \int_0^\infty q''(y)c(y)dy \) and its payout at time \( T \), given a realization of \( M \), will be

\[ \int_0^\infty q''(y)\max(M-y,0)dy = \int_0^M q''(y)(M-y)dy. \] It can be viewed as a limit of
a sequence of portfolios, each one with a finite number of options, which corresponds to the Riemann's sums in the definition of the integral. Suppose, for the sake of simplicity, that there is only one positive point \( a \) in which \( q \) does not necessarily have a continuous second derivative. Integration by parts gives

\[
\int_0^M q''(y)(M-y)dy = q(M) - q(0) - q'_+(0)M \\
\text{if } M \leq a
\]

\[
\int_0^M q''(y)(M-y)dy = \int_0^a q''(y)(M-y)dy + \int_a^M q''(y)(M-y)dy = \\
q(M) - q(0) - q'_+(0)M - (q'_+(a) - q'_-(a))(M-a) \\
\text{if } a < M.
\]

Thus \( q(M) \) can be produced by holding \( q(0) \) bonds, \( q'_+(0) \) call options with zero exercise price (or \( q'_+(0) \) shares of the underlying stock, if it is certain that the stock does not pay dividends in the time interval \([0,T]\)) , \( q'_+(a) - q'_-(a) \) call options with exercise price \( a \) and all call options weighted by \( q'' \). The cost of this portfolio is given by (2) (where in this case \( D(q') \) is one-point and the second and third terms are zero).

Next, if \( q \) is not continuous, it can be written in the form

\[
q(M) = Q(M) + \sum_{a \in D(q_+)} [q_+(a) - q(a)]x_{(a,\infty)}(M) + \sum_{a \in D(q_-)} [q(a) - q_-(a)]x_{(a,\infty)}(M)
\]

where
\[ \chi_{(x, \pm)}^{(M)} = \begin{cases} 1 & \text{if } x < M \\ 0 & \text{otherwise} \end{cases} = \lim_{n \to \infty} n[\max(M-x, 0) - \max(M-x - 1/n, 0)] \]

\[ \chi_{[x, \pm)}^{(M)} = \begin{cases} 1 & \text{if } x \leq M \\ 0 & \text{otherwise} \end{cases} = \lim_{n \to \infty} n[\max(M-x, 1/n, 0) - \max(M-x, 0)] \]

and \( Q \) is a continuous function that has the same derivatives as \( q \) and such that \( Q(0) = q(0) \). It is clear that the cost of \( \chi_{(x, \pm)}^{(M)} \) is
\[ \lim_{n \to \infty} n[c(x) - c(x + 1/n)] = -c'_+(x) \] and the cost of \( \chi_{[x, \pm)}^{(M)} \) is \(- c'_-(x) \). It is left to the reader to conclude the argument.

(b) Transforming equation (2) into the form (3) is only a matter of calculus. Again, suppose for the sake of simplicity that there is only one positive point \( a \) in which \( c \) or \( q \) do not necessarily have a continuous second derivative and in which \( q \) is continuous on the left but not necessarily on the right. The origin may possibly be another discontinuity point of \( q \). The reader can verify that integration by parts twice gives
\[ \int_{0}^{a} q''(y)c(y)dy = -q_+'(0)c(0) + q_+(0)c'_+(0) - (q_+(a) - q_-'(a))c(a) \]
\[ + q_+(a)c'_+(a) - q(a)c'_-(a) + \int_{0}^{a} q(y)c''(y)dy \]
\[ + \lim_{b \to \infty}[q'(b)c(b) - q(b)c'(b)]. \] (4)

It is assumed that the last term is zero \( \frac{9}{9} \) (a more rigorous approach is presented in appendix 1). Now Substitute (4) in (2) (modified to our
simple case) and note, in particular, that if \( c' \) is continuous at \( a \), all the discrete terms which correspond to \( a \) are cancelled. This will give the desired result. \( \text{Q.E.D.} \)

The general valuation formula (3) is a combination of equation (2) in BL's paper, which is correct when \( c_{xx} \) is continuous, and their discrete distribution formula (their equation (1)) which correctly values payoffs of mass points. The above proof follows the economic interpretation of equation (2) of this paper, and (3) follows from it mathematically. It is also interesting to prove proposition 1 by applying BL's method, which provides insights for equation (3). This will be done in appendix 2.

The reader can verify that (3) holds in particular for the case of a bond, a call option, or the stock itself.\(^{10} \) Applying the formula for the case \( q(M_T) = \max(0, x - M_T) \), one obtains \( V(q) = c(x) - c(0) + xB \) i.e., the put-call parity.

Note that the valuation functional \( V(\cdot) \) is linear and satisfies \( V(q) \geq 0 \) for \( q \geq 0 \) [as \( c'' \geq 0, c'_b(0) + B \geq 0 \) (see Smith (1976)) and \( c'_b(a) - c'_b(a) \geq 0 \) (see Rockafellar (1970, theorem 25.3))], in accordance with the works of Rubinstein (1976), Garman (1978), Ross (1978), and others.

3. INVESTORS' BELIEFS AND THE STRUCTURE OF CALL OPTIONS' PRICES

The price of a derivative asset is expressed in the previous section in terms of the bond's and the call options' prices. This relies on the spanning property of these securities and the lack of
arbitrage opportunities, and it is true regardless of the way in which the price system is determined. For example, the analysis is valid in a two-date, multiperiod or continuous-time framework and nothing is assumed on investors' preferences or beliefs. While $c(T,x)$ may be determined in equilibrium by all these characteristics of the economy, the functional relation between $V(q)$ and $c(T,x)$ is independent of them.

The discrete terms which were added to the original BL's formula correspond to mass points in the probability distribution of the underlying stock $M_T$. If $M_T$ has a continuous probability distribution with no mass points, then (2) and (3) reduce to the BL (continuous) formula (1). As $c_x(T,a^+)-c_x(T,a^-)$ is the implicit price of a contingent claim that pays one dollar at time $T$ if and only if $M_T = a$, it is clear that this is positive if and only if all investors attribute positive probability to this event. Similarly $c_x(T,0^+)+B_T$ is positive if and only if all investors attribute positive probability to the case of bankruptcy before or at time $T$. In the pure discrete case where $M_T$ is atomic (investors are assumed to agree on the mass points but not necessarily on their weights) it follows from arbitrage considerations that $c(T,x)$ must be piecewise linear in $x$, and thus only the discrete terms in (3) remain.

This can be demonstrated more explicitly in the context of a multiperiod (finite or infinite horizon) exchange economy where investors' beliefs are not necessarily homogenous. In addition to the given underlying stock and its call options (with notation as before), other securities and their derivative assets may be traded. Suppose at time
\( t = 0 \) the \( i \)th individual maximizes (with respect to all feasible dynamic strategies) \( \min \sum_{t} E_{i}[U_{it}(d_{it})] \) where \( U_{it} \) is his/her utility of time-\( t \) consumption, \((d_{i0}, d_{i1}, d_{i2}, \ldots)\) is the consumption stream in dollars (random variables for \( t \geq 1 \)) and \( E_{i} \) is the expectation operator.

Suppose the \( i \)th investor's subjective probability distribution of \( M_{T} \) is determined, except in a finite number of mass points, by a continuous density function \( f_{iT} \). It follows\(^{12}\) that in equilibrium, for each investor \( i \) and time \( T \geq 1 \),

\[
c(T,x) = K_{iT}E_{i}[U'_{iT}(d_{iT}) \max(M_{T}-x,0)] = K_{iT}E_{i}[\max(M_{T}-x,0)E_{i}[U'_{iT}(d_{iT})|M_{T}] ] \\
= K_{iT} \int f_{iT}(m)(m-x)E_{i}[U'_{iT}(d_{iT})|M_{T} = m]dm \\
+ K_{iT} \sum_{a > x} \text{Prob}_{i}(M_{T} = a)(a-x)E_{i}[U'_{iT}(d_{iT})|M_{T} = a] \tag{5}
\]

where the summation is on the mass points of \( M_{T} \) (as viewed by the \( i \)th individual) and where \( K_{iT} = B_{i}/E_{i}[U'_{iT}(d_{iT})] \). Differentiation yields:

\[
B_{T} + c_{x}(T,0+) = \text{Prob}_{i}(M_{T} = 0)K_{iT}E_{i}[U'_{iT}(d_{iT})|M_{T} = 0] \tag{5a}
\]

\[
c_{x}(T,a+) - c_{x}(T,a-) = \text{Prob}_{i}(M_{T} = a)K_{iT}E_{i}[U'_{iT}(d_{iT})|M_{T} = a] \tag{5b}
\]

for each \( a > 0 \)

and

\[
c_{xx}(T,x) = f_{iT}(x)K_{iT}E_{i}[U'_{iT}(d_{iT})|M_{T} = x] \tag{5c}
\]

for each \( x \) which is not a mass point.
To further see the role of $c_{xx}$ as a density function and to demonstrate an application of the valuation formula, it will now be shown that $c_{xx}$ satisfies the Chapman-Kolmogorov equation [see, e.g., Schuss (1980), p. 101]. This is true under the assumption that there is a deterministic function $c(t,M,T,x)$ which gives the $(T,x)$-call option's price at each time $t$ as a function of the underlying stock's price $M = M_t$. This is the case, for example, in the Black-Scholes model (see section 4) and related models, but here continuous trading or an explicit distribution is not postulated. It is assumed, however, that $c$ is suitably smooth such that (3) is reduced to (1). Suppose, now, that the market is open at the three times $t < \tau < T$. Then at time $t$ one knows that the $(T,x)$-call option can be sold at time $\tau$ for $q(M_\tau) = c(\tau,M_\tau,T,x)$. This payoff can be constructed at time $t$ from call options that expire at time $\tau$, and (1) gives

$$c(t,M,T,x) = \int_0^\infty c(\tau,y,T,x) \frac{\partial^2 c}{\partial x^2} \bigg|_{(t,M,\tau,y)} dy.$$ 

Hence

$$\frac{\partial^2 c}{\partial x^2} \bigg|_{(t,M,T,x)} = \int_0^\infty \frac{\partial^2 c}{\partial x^2} \bigg|_{(\tau,y,T,x)} \frac{\partial^2 c}{\partial x^2} \bigg|_{(t,M,\tau,y)} dy$$

which is the analoge to the Chapman-Kolmogorov equation in probability theory. Garman (1978), referring to general pricing operators, termed this property "intertemporal consistency."
4. THE BLACK-SCHOLES CASE

It is interesting to write equation (3) for this important case. To recapitulate, if continuous trading is allowed and the underlying stock's price follows a geometric Brownian motion, and, for the sake of simplicity, it does not pay dividends in the relevant time interval, then the Black-Scholes (1973) formula implies that the price at time $t$ of a European call option with maturity at time $T \geq t$ and exercise price $x$ is

$$c(t,S,T,x) = SN(d_1) - e^{-r(T-t)}xN(d_2)$$  \hspace{1cm} (6)

where

$$d_2 = d_2(t,S,T,x) = \frac{\ln S/x + (r - \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}}, \quad d_1 = d_2 + \sigma \sqrt{T-t},$$

$S = S_t$ is the underlying stock's price at time $t$, $r$ is the continuously compounded interest rate (a constant), $\sigma^2$ is the instantaneous variance of the stock's rate of return (a constant), and $N(\cdot)$ is the cumulative distribution function of the standard normal variable (the density function will be denoted $n(\cdot)$). The reader can verify that in this case equation (3) reduces to the BL (continuous) formula (1). Differentiation of (6) yields

$$\frac{\partial^2 c}{\partial x^2} = e^{-r(T-t)}n(d_2).$$

Hence, as BL observe, equation (1) implies that the value at time $t$ of a derivative asset which pays $q(S_{T,t})$ at time $T$ and zero at any other
time is
\[ V_q(t,S) = \frac{e^{-r(T-t)}}{\sigma \sqrt{T-t}} \int_0^\infty \frac{q(y)}{y} n(d_2(t,S,T,y)) dy. \]

Recall that in this model a call option with any exercise price can be produced by continuously trading the stock and a bond [see Merton (1977)], and thus assumption (ii) (in section 2) is satisfied. It is interesting to note that (7) can also be proved directly without using BL's valuation formula: Merton (1977) shows that one can construct a continuous trading strategy of the stock and a bond that pays (in our notation) \( q(S_T) \) at time \( T \) and zero at any other time. Merton shows that the portfolio's value, at any time \( t \) and for any \( S = S_t \), coincides with the unique twice-continuously differentiable function \( F(t,S) \) that solves the differential equation
\[ \frac{\partial F}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} + rS \frac{\partial F}{\partial S} = rF \]
with the boundary condition \( F(T,S) = q(S) \).

Thus, it is left to prove

**Proposition 2**: \( V_q(t,S) \) as in (7) is a solution to the differential equation (8) with the above boundary condition.

**Proof (outline)**: It is known that \( c(t,S,T,x) \) as in (6) satisfies (8), and this implies that
\[ \int_0^\infty q(y) \frac{\partial^2}{\partial y^2} c(t,S,T,y) dy \]
satisfies (8). To see that the boundary
condition is satisfied, take fixed \( t, S \) and \( T \) and substitute
\[ w = \sqrt{T-t} d_2(t, S, T, y) \]
in (7) to obtain
\[ V_q(t, S) = e^{-r(T-t)} \int_{-\infty}^{\infty} q \left( S e^{-\sigma w} + (r-\sigma^2/2)(T-t) \right) \frac{e^{-\frac{w^2}{2(T-t)}}}{\sqrt{2\pi(T-t)}} dw. \tag{9} \]

As \( t \to T \), the second factor in the integrand tends to the Dirac function at \( w = 0 \) [see, e.g., Schuss (1980) p. 22] i.e., the integral will tend to \( q(S) \).

Q.E.D.

This demonstrates that the BL approach, which is based on the construction at one time \( t \) of a replicating portfolio from the bond and all the call options, gives the same result as Merton's approach, which is to construct a replicating continuous-trading strategy from the bond and the stock. Of course, this method cannot offer an alternative to Merton's approach when the derivative asset itself is a call option.

5. EXAMPLE: THE VALUATION OF COMPOUND OPTIONS

While Geske (1979) used stochastic calculus methods to price options on options, the same result can be obtained merely by a mechanical substitution in the BL valuation formula for the Black-Scholes case. The framework will be as in section 4. In order to keep the notation similar to Geske's, let \( V_t \) be the value at time \( t \) of the underlying security which is assumed to follow a geometric Brownian motion with instantaneous variance of return \( \sigma^2 \). Let \( S_t \) be the value of a
(T,m)-European call option on the above security, and let \( c_T \) be the value of a \((t^*,K)\)-European call option on the above European call option, where \( t^* < T \). [See Geske (1979) for the economic interpretation.] The payoff of the latter security at time \( t^* \) is

\[
q(v_{t^*}) = \max(S_{t^*} - K, 0) = \\
= \max(v_{t^*} N(d_1(t^*, v_{t^*}, T, m)) - e^{-r(T-t^*)} N(d_2(t^*, v_{t^*}, T, m)) - K, 0)
\]

where \( S_{t^*} = S(t^*, v_{t^*}, T, m) \) is substituted from (6). Let \( \bar{v} \) be the value of \( v_{t^*} \) for which \( S_{t^*} - K = 0 \). Then [see Smith (1976, eq. (45))] \( S_{t^*} - K > 0 \) if and only if \( v_{t^*} > \bar{v} \). Equation (7) now gives for \( t < t^* \):

\[
c_t = v q(t, v) = e^{-r(t^*-t)} \int_{-\infty}^{\infty} \frac{1}{\sigma \sqrt{2\pi}} y N(d_1(t^*, y, T, m)} - me^{-r(T-t^*)} N(d_2(t^*, y, T, m)) - K) n(d_2(t, v, t^*, y)) dy
\]

where \( v = v_t \). Breaking this into three integrals, then substituting \( x = d_1(t, v, t^*, y) \) in the first one and \( x = d_2(t, v, t^*, y) \) in the other two, results in:

\[
c_t = v \int_{-\infty}^{d_1(t, v, t^*, \bar{v})} N \left( \frac{d_1(t, v, T, m) - \sqrt{1/\tau_1} x}{\sqrt{(T-t^*)/\tau_2}} \right) n(x) dx \\
- m e^{-r_{\tau_2}} \int_{-\infty}^{d_2(t, v, t^*, \bar{v})} N \left( \frac{d_2(t, v, T, m) - \sqrt{1/\tau_1} x}{\sqrt{(T-t^*)/\tau_2}} \right) n(x) dx \\
- Ke^{-r_{\tau_1}} N(d_2(t, v, t^*, \bar{v})) \quad (10)
\]
where \( \tau_1 = t^*-t \), \( \tau_2 = T-t \). By using the last equation in Geske (1979, p. 80) it follows without difficulty that eq. (10) is identical to eq. (4) in Geske (1979).

6. EXAMPLE: THE MERTON PROPORTIONAL DIVIDEND MODEL

Merton (1973, p. 171) [see also Smith (1976) p. 26] modified the Black-Scholes model for the case of a constant dividend yield. It will now be demonstrated that this result can also be obtained from Black-Scholes' formula (with no dividends) and BL's valuation formula.

The continuous dividend stream is assumed to have a density of \( \delta S_t \), where \( \delta \) is a positive constant and \( S_t \) is the stock's price. This can be used to purchase \( \delta \) new shares per each share held. Fix a current time \( t_0 \) and a future time \( T \). Then reinvestment of dividends will transform one share at time \( t_0 \) to \( \delta(t-t_0) \) shares at time \( t > t_0 \) (as \( da/\alpha = \delta dt \)), regardless of the path of \( S \).

\[
\hat{S}_t = e^{\delta(t-t_0)} S_t,
\]

the value of a continuously reinvested position, is assumed to follow a geometric Brownian motion with instantaneous variance of return \( \sigma^2 \).

Now, the Black-Scholes model can be applied to \( \hat{S}_t \) and call options on \( \hat{S}_t \) (which can be produced from \( \hat{S}_t \) and a bond). The value of such a \((T,x)\)-call option, \( \hat{c}(t_0,\hat{S},T,x) \), is given by (6), with the obvious modification of notation. The unprotected \((T,x)\)-call option on \( S_t \) can be viewed as a derivative asset of \( \hat{S}_t \) with payoff at time \( T \)

\[
\max(S_T-x,0) = \max(e^{-\delta \tau}S_T-x,0), \quad \text{where} \quad \tau = T-t_0.
\]
It follows from (7)
that its current price is

\[ c(t_0, S, T, x) = \frac{e^{-\delta T}}{\sigma \sqrt{T}} \int_{x/\delta}^{\infty} \frac{e^{-\delta y - x}}{y} n(d_2(t_0, S, T, y)) dy. \]

Breaking this into two terms, substituting \( w = d_1(t_0, S, T, y) \) in the first and \( w = d_2(t_0, S, T, y) \) in the second, results in

\[ c(t_0, S, T, x) = e^{-\delta T} SN(d_1(t_0, S, T, xe^{5T})) - e^{-\delta T} xN(d_2(t_0, S, T, xe^{5T})), \]

which is identical to eq. (53) in Smith (1976). In the case where \( \delta = f(t) \) depends on the time (but not on the stock's price) one obtains

the same formula only with \( \int_{t_0}^{T} f(t) dt \) instead of \( \delta T \).
APPENDIX I

In the proof of part (b) of proposition 1 it was assumed, for simplicity, that \( \lim_{b \to \pm \infty} [q'(b)c(b) - q(b)c'(b)] = 0 \). In fact, this is necessarily the case if \( V(q) \) is finite. More specifically:

\[
\int_0^\infty q''(y)c(y)dy \quad \text{converges (i.e., \( V(q) \) is defined and finite) if and only if} \quad \int_0^\infty q(y)c''(y)dy \quad \text{converges and in this case,}
\]

\[
\lim_{b \to \pm \infty} [q'(b)c(b) - q(b)c'(b)] = 0.
\]

Proof: Applying the fact\(^{17} \) that \( \lim_{x \to \pm \infty} c(x) = 0 \) and \( \lim_{x \to \pm \infty} xc'(x) = 0 \), integration by parts gives \( \int_x^\infty (z-x)c''(z)dz = c(x) \) for \( x \geq a \). Hence

\[
\int_0^\infty q''(y)c(y)dy = \int_0^\infty (z-y)c''(z)q(y)dzdy - \int_0^\infty (z-y)c''(z)q''(y)dzdy
\]

\[
= \int_0^\infty (z-y)c''(z)q''(y)dzdy = \int_0^\infty c''(z)[\int_0^z (z-y)q''(y)dy]dz
\]

\[
= \int_0^\infty c''(z)[-q'_+(a)(z-a) + q(z) - q'_+(a)]dz = -\int_0^\infty c''(z)[q'_+(a)(z-a) + q'_+(a)]dz
\]

\[
+ \int_0^\infty c''(z)q(z)dz = c'_+(a)q_+(a) - c(a)q'_+(a) + \int_0^\infty c''(z)q(z)dz \quad (11)
\]
where \( A = \{(y,z) \in \mathbb{R}^2; \ a \leq y < \alpha, \ y \leq z < \alpha\} = \{(y,z) \in \mathbb{R}^2; \ a \leq z < \alpha, \ a \leq y < z\} \) and the technique of changing the order of integration in a double integral is used. In the fifth and seventh equalities, integration by parts and the above limits are applied.

This implies the "if and only if" part. Integration by part twice of the LHS in (11) gives (as in section 2) the same expression as on the RHS, only with the addition of \( L \). This implies that \( L = 0 \).
APPENDIX 2

An alternative proof\textsuperscript{18} for proposition 1, using BL's framework:

The notation will be as in section 2 of this paper. Consider the discrete case as described by BL, and let \( P(M) \) be the current price of one dollar contingent upon level \( M \) of the underlying stock at time \( T \). By arbitrage consideration, the value of the payoff \( q(M) \) is \( V(q) = \sum_M q(M)P(M) \). Substituting BL's formula (their equation (1)) for \( P(M) \) gives

\[
V(q) = \sum_M q(M) \left\{ \left[ \frac{c(M+\Delta M) - c(M)}{\Delta M} \right] - \left[ \frac{c(M) - c(M-\Delta M)}{\Delta M} \right] \right\} \tag{12}
\]

To prove (b), let \( \Delta M \to 0 \) in (12). The discontinuity points of \( c' \) will give \( \sum_{a \in D(c')} q(a)(c'_+(a) - c'_-(a)) \). In particular, the term which corresponds to \( M = 0 \), if it is a mass point, will be\textsuperscript{19} \( q(0)(c'_+(0) - c'_-(0)) = q(0)(c'_+(0) + B) \). Wherever \( c'' \) exists, the limit as \( \Delta M \to 0 \) of the bracketed expression in (12) may be represented as \( c''(M)dM \). Since there are only a finite number of discontinuities, the limit as \( \Delta M \to 0 \) of the remaining infinite summation equals \( \int_0^\infty q(y)c''(y)dy \). Combined with the above discrete terms, it gives equation (3).

To prove (a), it is assumed that \( q \) is continuous (the discontinuous case can then be obtained as in section 2). Let \( M \) take on discrete values \( (m\Delta) \) for \( n = 0, 1, 2 \ldots \). It follows from (12)\textsuperscript{20} that
\[ V(q) = \sum_{n=0}^{\infty} q(n\Delta)P(n\Delta) = q(0)B + q(0)c(\Delta)/\Delta - q(0)c(0)/\Delta \]

\[ + \frac{1}{\Delta} \left\{ \sum_{n=1}^{\infty} q(n\Delta)c((n-1)\Delta) - 2 \sum_{n=1}^{\infty} q(n\Delta)c(n\Delta) + \sum_{n=1}^{\infty} q(n\Delta)c((n+1)\Delta) \right\} \]

\[ = q(0)B + q(\Delta)c(0)/\Delta - q(0)c(0)/\Delta \]

\[ + \frac{1}{\Delta} \left\{ \sum_{n=2}^{\infty} q(n\Delta)c((n-1)\Delta) - 2 \sum_{n=1}^{\infty} q(n\Delta)c(n\Delta) + \sum_{n=0}^{\infty} q(n\Delta)c((n+1)\Delta) \right\} \]

\[ = q(0)B + \frac{q(\Delta)-q(0)}{\Delta} \cdot c(0) \]

\[ + \sum_{k=0}^{\infty} c((k+1)\Delta) \left[ \frac{q((k+2)\Delta)-q((k+1)\Delta)}{\Delta} - \frac{q((k+1)\Delta)-q(k\Delta)}{\Delta} \right] \]

assuming convergence of the infinite summations. Taking limits as \( \Delta \to 0 \) where possible (recognizing only a finite number of discontinuities) gives the valuation formula (2) for the case where \( q \) is continuous.
FOOTNOTES

1 They assume (p. 622) that the underlying security's value at time T has a continuous probability distribution, but they do not establish the relation between these two properties.

2 For example, Cox, Ross, and Rubinstein (1979) apply the central limit theorem in a special case.

3 In BL's derivation the underlying asset is a portfolio, but here the analysis is restricted to the case of a stock. In the case of a portfolio that can obtain only nonnegative values (as is the case for the market portfolio, which is used by BL) the analysis will be identical.

4 In the sequel, "call option" will always mean European.

5 Here \( q'(a) \) means \( \lim_{x \to a} q'(x) \), etc.

6a If this set is allowed to be discrete (possibly infinite), the analysis will be similar with the addition of some further technical points. A similar remark would apply to \( D(c') \) in part (b).

6 This derivative asset need not be traded in the market and its cost \( V(q) \) (which is not necessarily finite) is to be understood as the limit of the costs of the approximating portfolios (in a sense to be explained below). Of course, if it is traded, then by arbitrage considerations \( V(q) \) must be its current market price.

7 Notice that the second term in (3) can be considered as a part of the summation (the third term) as \( B = -c'(0) \) in the following sense: for \( x < 0 \), a portfolio of one call option with exercise price
zero and \(-x\) bonds can be considered as a call option with exercise price \(x\), as it will pay \(M - x = \max(M-x,0)\). Its price is \(c(x) = c(0) - xB\), hence \(c'(0) = -B\).

Integrate \(\int_a^b\) and \(\int_a^b\) separately and then add and take the limit as \(b \to \infty\).

This limit is zero for a large set of functions \(q\), as:

(a) \(\lim_{x \to \infty} c(x) = 0\) is a very plausible assumption because
\(\lim_{x \to \infty} \max(M-x,0) = 0\) for each \(M\). (b) \(\lim_{x \to \infty} xc'(x) = 0\) —to prove this, recall that \(c\) is convex [see Smith (1976)] and see Taylor (1955, problem 10, p. 642), or Polya and Szegö (1972, problem 113).

The latter case requires that the stock does not pay dividends, so that it can be considered as a security which pays \(M_T\) at time \(T\) and zero at any other time. Again, this condition is not required for the formula to be true for other derivative assets which pay only at time \(T\).

More specifically, \(c(T,x) = c(T,a_i) + (a_i - x)A_i\) for \(a_i - 1 < x \leq a_i\), where \(a_i - 1\) and \(a_i\) are two adjacent mass points of \(M_T\) and \(A_i\) is the price of the derivative asset \(X(a_i, \infty)(M_T)\) (see section 2).

Equation (5) is similar to equation (5) in Brennan (1979) which is derived for a two-date model with a finite number of securities. This can be generalized to our case by applying a variational method: let \(d_{i0}, d_{i1}, d_{i2}, \ldots\) be the cash stream, as viewed by investor \(i\) at time \(t = 0\), resulting from his optimal strategy (which is assumed to exist). Construct a new feasible strategy by combining this optimal
strategy with $\varepsilon$ times the following zero-investment strategy: Invest $\$1$ in purchasing $1/c(T,x)$ units of the $(T,x)$-call option and $-\$1$ in selling short $1/B_T$ units of the $T$-bond. The zero-investment strategy pays $D_T(M_T,x) = \max(M_T-x,0)/c(T,x) - 1/B_T$ at time $T$. Hence, because of optimality,

\[
0 = \left. \frac{d}{dx} \right|_{\varepsilon=0} \left\{ \sum_{t \in T} E_t [U_{it}(d_{it})] + E_t \left[ U_{it}(d_{it} + \varepsilon D_T(M_T,x)) \right] \right\}
= E_t \left[ U_{iT}(d_{iT}) D_T(M_T,x) \right].
\]

Substituting $D_T(M_T,x)$ from above and rearranging will yield equation (5).

The more conventional notation $S$ for the stock's price is now used, instead of $M$ which was used before in order to keep notation similar to BL's. Note that $S_T$ and $e^{-r(T-t)}$ correspond to $M$ and $B$, respectively, in proposition 1, and the time of evaluation is $t$ instead of 0.

See also Bergman (1981).

After having written this paper it was brought to the author's attention that this proposition was also proved in an unpublished paper by Black (1974), who also recognized (7) as the value of a general derivative asset in the Black-Scholes case.

The following identities, which are readily verified from the definitions, are used in the transformation:
\( \exp(\sigma^2 \tau_1 / 2 - x_0 \sqrt{\tau_1}) n(x - \sigma \sqrt{\tau_1}) = n(x) \)

\[
d_1(t^*, \nu^* \exp((r+\sigma^2/2)\tau_1 - x_0 \sqrt{\tau_1}), T, m) = \frac{d_1(t, \nu, T, m) - \sqrt[2]{\tau_1 / \tau_2} x}{\sqrt{\tau_2}}
\]

and the same formula is true with \( r - \sigma^2 / 2 \) on the LHS and \( d_2 \) instead of \( d_1 \) in both sides.

17 See footnote 9.

18 This proof was provided by Douglas T. Breeden.

19 See footnote 7 for the interpretation of negative exercise price. It can also be verified directly from arbitrage considerations that \( P(0) = [c(\Delta M) - c(0)] / \Delta M + B. \)

20 See also footnote 19.
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Black, F., 1974, The pricing of complex options and corporate liabilities, mimeographed (University of Chicago).


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