OPTIMAL DURATION OF GROWTH INVESTMENTS--
A BAYESIAN APPROACH*

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ABSTRACT

The optimal policy of terminating growth investments is analyzed for the case where the firm may revise its beliefs about the mean return and mean growth of its investment (the Bayesian case). This policy, consisting of a set of reservation prices, is compared with the policy obtained when no revision of beliefs takes place (the non-Bayesian case). In both cases a positive correspondence exists between the reservation prices, the value of the investment, and the variance of the proceeds. Unlike the non-Bayesian case, the reservation prices in our case need not be finite. It is shown that the higher the growth rate and the variances of the proceeds, and the lower the discount rates, the more likely that the Bayesian firm has higher reservation prices than the non-Bayesian firm.
1. Introduction

A classical problem in finance is the following: a firm has a forest that grows with time. The forest will be worth $x_t$ if cut at time $t$. When should the forest be cut down? and what is its present value? The solution of this problem is straightforward when the function $x_t$ is known with certainty (see, e.g., Bierman [2], Brealey and Myers [4], Hartman [11], Hirshleifer [12]): the forest should be cut down when the proportionate increase in the forest's sale value equals the interest rate. If the optimal cutting time is $T^*$, then the present value of the forest is $\rho^{T^*}x_{T^*}$, where $\rho$ is the appropriate discount factor.

This problem is more difficult when the proceeds $x_t$ from cutting and selling the forest are unknown (i.e., random). The optimal strategy of selling the asset (henceforth we shall refer to the forest as "the asset") is then defined by a set of reservation prices which are functions of the age of the asset. At each period $t$, the firm compares $x_t$ with the appropriate reservation price. If $x_t$ is higher the firm sells the asset, otherwise it holds it for one more period. The reservation prices must be determined using dynamic programming techniques since the decision whether or not to sell the asset affects the firm's opportunity set (e.g., if the firm sells, it will not have the asset next period). Venezia and Brenner [17], and Brock, Rothschild and Stiglitz [6] analyzed the firm's optimal strategy, and investigated the effect of increased dispersion of the proceeds $X_t$ on the reservation prices and on the value of the asset. They show that increased dispersion increases the reservation prices and the value of the asset. In these studies, however, it has been assumed that the distributions of future proceeds from selling the asset are known at the start and will not be reassessed. This assumption is disturbing when the duration of
growth investments is long. The firm may then obtain information which changes its prior beliefs about the future prospects of the asset; it may observe a streak of extremely high or low offers for the asset.

The main objectives of this article are therefore to (1) determine the optimal selling strategy of the firm, assuming that it reassesses its beliefs about future prospects, using Bayes' rule, whenever new data is observed, and (2) compare the results obtained in the present model (the Bayesian case) with those of the non-Bayesian case.²

In the Bayesian case, as in the non-Bayesian case, each period the firm compares the current offer with the value of the asset if it is kept. If the former is higher, the firm sells the asset, otherwise it keeps it. In the Bayesian case however, the value of the asset if it is kept depends on the current offer because the current offer affects the firm's beliefs about the future prospects of the asset. We demonstrate how the optimal selling strategy is determined, and prove that as in the non-Bayesian case, the reservation prices and value of the asset increase with increased dispersion. We show that in the Bayesian case the reservation prices may be infinite (this cannot happen in the non-Bayesian case), and we present and interpret conditions which allow this. It is also shown that the lower the discount rate, the higher the growth rate, and the higher the variance, the more likely that the Bayesian firm has higher reservation prices than the non-Bayesian firm.

The paper is organized as follows. In Section II we present the basic model and the statement of the objective function of the firm. Applications of our model are also discussed. In Section III, as a preliminary to the analysis of subsequent sections, we analyze the case of stationary investments (by "stationary" we mean that time per-se does not affect the ex-
pected proceeds from selling the asset). In Section IV we consider the case of growth investments, and in Section V we compare the Bayesian and non-Bayesian decision policies.

II. The Model

Consider a firm whose shares are traded in a perfect securities market and which holds an asset (say, a forest) whose value increases stochastically with time.\(^3\) Suppose that the firm is small enough so that at any period \(t\), its actions do not affect the market price of lumber \(\pi_t\). If the volume of lumber contained in the forest at \(t\) is \(k_t\), then by cutting the forest and selling its lumber, the firm may receive the cash flow \(x_t = \pi_t k_t\), and we say that the firm obtained for its asset at period \(t\) an offer of \(x_t\). During any period, the future offers the firm will obtain for its asset are uncertain because future prices of lumber are unknown, and since the physical growth pattern of the asset is uncertain. These two sources of uncertainty are represented by saying that prior to \(t\), \(\tilde{x}_t\) is some random variable. It is assumed that the firm does not know with certainty the mean of \(\tilde{x}_t\) and its mean growth rate. The firm, however, can learn about these parameters from previous observations of offers. We address the questions: when should the firm terminate its investment and what is its value?\(^4\)

Consider another example: Suppose the asset the firm holds is a vintage wine whose quality is an increasing function \(Q(t)\) of its age \(t\). The market price of wine is some increasing function \(\pi(Q)\) of its quality \(Q\). Assuming that our firm's supply does not affect the market's prices, the firm can sell its wine for \(\tilde{x}_t = k\tilde{\pi}_t(Q(t))\) where \(k\) denotes the volume of wine. Uncertainty in this case stems from the unknown evolution of the wine's quality, \(\tilde{Q}(t)\), and uncertainty about future market conditions, \(\tilde{\pi}_t(Q)\).
How long should the firm age the wine? What is its value?

If the firm must terminate the investment at some specified period in the future $T$, and if it does not revise its estimates of the relevant parameters, then the value of the asset is:

$$W_0 = E_0(\tilde{x}_T) \prod_{t=1}^{T} 1/[1 + E(R_t)] \tag{1}$$

where $E_0(\tilde{x}_T)$ is the expected cash flow, as seen at the start, from selling the asset at $T$, and $E(R_t)$ is an appropriate risk adjusted discount rate. \(^5\) This rate depends on the value of $\text{Cov}(\tilde{x}_t, \tilde{R}_{Mt})/E_t(\tilde{x}_T)$, where $\tilde{R}_{Mt}$ and $E_t(\tilde{x}_T)$ denote the market rate of return and expectation of $\tilde{x}_T$ at time $t$, respectively.

In our case, the parameters of the distribution of $\tilde{x}_T$ will be revised during the interval $(0, T)$ since the firm learns about the expected offers from experience. However, since Fama [8] has shown that (1) is the proper measure of the asset's value if the only reassessments of parameters are reassessments of means, and since it is shown below that only such reassessments are needed in our model, (1) is indeed the appropriate measure of the asset's value.

The appropriate measure for $E(R_t)$ is the risk free interest rate, $r_{Ft}$, in the case where $\text{Cov}(\tilde{x}_t, \tilde{R}_{Mt}) = 0$. This is the case when all uncertainty stems from imperfect knowledge of the pattern of physical growth of the asset or, more generally, of idiosyncratic fluctuations of $\tilde{x}_t$. In this case (1) can be rewritten:

$$W_0 = E_0(\tilde{x}_T) \prod_{t=1}^{T} \rho_t \tag{2}$$

where

$$\rho_t \equiv 1/(1 + r_{Ft}). \tag{3}$$

When $\text{Cov}(\tilde{x}_t, \tilde{R}_{Mt}) \neq 0$, $E(R_t) = r_{Ft} + \lambda \text{Cov}(\tilde{x}_t, \tilde{R}_{Mt})/W_{t-1}$.\(^5\)
where $\lambda$ is the market's price of risk, and $W_{t-1}$ is the value of the asset at $(t-1)$. In this case (1) no longer is an appropriate measure of the asset's value, since $W_{t-1}$ may be revised during the interval $(0, t-1)$. For the preceding examples, however, it seems plausible to assume that $\text{Cov} (\tilde{x}_t, \tilde{r}_{Mt})$ is zero or close enough to zero (the correlation between the prices of lumber and wine, and the market rate of return on securities seems to us to be small). For simplification, and since we show that the same results are also obtained when $\text{Cov}(\tilde{x}_t, \tilde{r}_{Mt}) \neq 0$ (see footnote 11 and Appendix II), we shall henceforth use $r_{Ft}$ for $E(R_t)$.

Suppose now that the firm does not have to sell the asset at $T$ but can sell it at any period prior to $T$. In this case the firm will base its decision whether or not to sell the asset at any period $t$ $(t < T)$, on the current offer it has for the asset, and on its expectations about future offers. The sale time $\bar{S}$ is not known at the start since the decision to sell depends on future offers. The firm has then to determine a sales strategy that will maximize

$$W_0 = E_0(\tilde{x}_{\bar{S}}) \prod_{t=1}^{\bar{S}} \rho_t.$$  \hspace{1cm} (4)

In the next section we analyze the learning process of the firm and its optimal sale strategy for the case where there is no growth trend in the evolution of $\tilde{x}_t$. This is a necessary preliminary to the analysis of subsequent sections.

III. The Case of Stationary Investments

Suppose that the firm obtains at each period $t$ one offer, $\tilde{x}_t$, which is normally distributed with an unknown mean $\tilde{\mu}_t$ and a known variance $\sigma_t^2$. That is

$$\tilde{x}_t = \tilde{\mu}_t + \tilde{\varepsilon}_t, \hspace{0.5cm} t = 1, 2, \ldots$$  \hspace{1cm} (5)
where the \( \tilde{\varepsilon}_t \)'s are normally distributed with mean zero and variance \( \sigma_t^2 \).

They are independent of each other and of the \( \tilde{\mu}_t \)'s.

Since the firm does not know \( \tilde{\mu}_t \), it may regard this parameter as a random variable. The firm can learn about this random variable from past observations of offers. It is assumed that the firm uses Bayes' rule when revising its beliefs about \( \hat{\mu}_t \). The revision is done as follows. Suppose that at the beginning of period \( t \) the beliefs of the firm about \( \hat{\mu}_t \) are represented by the normal distribution with mean \( m_t \) and variance \( V_t \), that is,

\[
\hat{\mu}_t = m_t + \delta_t
\]

where \( \delta_t \) is normally distributed with zero mean and variance \( V_t \). It can be shown (see, e.g., DeGroot [7], p. 167) that if an offer \( x \) has been observed at period \( t \), then the posterior distribution of \( \hat{\mu}_t \) at the end of period \( t \) is normal with mean \( m'_t \) and variance \( V'_t \) given by

\[
m'_t = \gamma m_t + (1-\gamma)x
\]

\[
V'_t = \gamma V_t
\]

where

\[
\gamma = \sigma_t^2 / (\sigma_t^2 + V_t)
\]

The weights \( \gamma \) and \( 1-\gamma \) of \( m \) and \( x \) in the future mean reflect the relative information content of prior beliefs and observations of offers. The higher \( V_t \) the more uncertain the firm is of its prior, hence the lower the weight, \( \gamma \), of \( m \). The higher \( \sigma_t^2 \) the less informative are observations of \( x \), therefore the lower the weight \( (1-\gamma) \) of \( x \).

Since we are considering the case of stationary investments ("stationary" here means that time per-se does not affect the value of the asset), it follows that the prior distribution \( \hat{\mu}_{t+1} \) at the beginning of period \( (t+1) \) is normal with mean \( m_{t+1}(x) \) and variance \( V_{t+1}(x) \) given by

\[
m_{t+1}(x) = m'_t = \gamma m_t + (1-\gamma)x
\]

\[
V_{t+1} = V'_t = \gamma V_t
\]
where \( \gamma \) is given by (8).

Having described the learning process of the firm, we now turn to describe its decision process. One observes from (9) that the whole sequence of variances, \( V_t \), is known at the start since revisions of \( V_t \) do not depend on \( x \), and that the only parameters reassessed between the origin and period \( t \) are the means \( m_1, m_2, \ldots, m_t \). It thus follows, as claimed in Section II, that (4) is, indeed, the appropriate measure of the asset's value. The objective of the firm is therefore to determine a selling strategy that will maximize (4). For this the following definition is needed.

Let \( W_t(x, m, V) \) denote the maximal obtainable expected discounted proceeds from the asset (i.e. the asset's value), given that at the beginning of period \( t \) the parameters of the distribution of \( \tilde{\mu}_t \) are \( m \) and \( V \), and that during the period an offer of \( x \) has been made for the asset.

Suppose that the firm has received an offer \( x \) at period \( t \). The firm can either accept it, and obtain the proceeds \( x \), or reject it, and then move to period \( (t+1) \) where the decision process continues. In the latter case the parameters of the distribution of \( \tilde{\mu}_{t+1} \) at the beginning of period \( (t+1) \) will be \( m_{t+1}(x) \) and \( V_{t+1} \) given by (9). At period \( (t+1) \) an offer \( \tilde{x}_{t+1} \) will be made, distributed according to the parameters relevant at period \( (t+1) \). It hence follows that the expected maximal obtainable proceeds from \( (t+1) \) on, when discounted to period \( t \), are given by

\[
p_t E[W_{t+1}(\tilde{x}_{t+1}, m_{t+1}(x), V_{t+1})].
\]

The expectation \( E \) is taken with respect to the marginal distribution of \( \tilde{x}_{t+1} \) which is normal with mean \( m_{t+1}(x) \) and variance \( (V_{t+1} + \sigma^2_{t+1}) \).

Since \( W_t(x, m, V) \) represents the outcome of a maximizing behavior:

\[
W_t(x, m, V) = \text{Max} \{ x, p_t E[W_{t+1}(\tilde{x}_{t+1}, m_{t+1}(x), V_{t+1})] \}.
\]

The firm will thus sell the asset only if
\[ x \geq \rho_t E[W_{t+1}(\hat{x}_{t+1}, m_{t+1}(x), V_{t+1})] \]  

In order to determine the optimal strategy of the firm, and the value of the asset, the functions \( W_t(x, m, V) \) should be computed. This is straightforward in the case where the planning horizon is some finite \( T \).\(^7\)

In this case \( W_T(x, m, V) = x \), and hence\(^8\)

\[ W_{T-1}(x, m, V) = \text{Max} \{ x, \rho E[\hat{x}_T | m_T(x), V_T] \} \]

\[ = \text{Max} \{ x, \rho[\gamma m + (1-\gamma)x] \} . \]

Based on \( W_{T-1}(x, m, V) \) one can compute \( W_{T-2}(x, m, V) \), and similarly all other \( W_t(x, m, V) \) can be computed.

From (12) it follows that at period \((T-1)\) the firm will sell the asset if and only if the offer \( x \) satisfies

\[ x \geq x^*_T \leq m_T(1-\rho) + \rho \gamma . \]  

For periods \( t < T-1 \), it is also true that there exist "reservation prices" \( x^*_t \) such that the asset will be sold if and only if \( x \geq x^*_t \). These reservation prices can be obtained by solving (after replacing the inequality sign by equality sign) the implicit equation (11). It is shown in Proposition A.1 (in Appendix I) that for any \( t, m, V \), the reservation price \( x^*_t \) is unique and finite (this is not the case, as we shall later see, with growth investments).

We now investigate the effect of increased dispersion on the reservation prices. The function \( W_t(x, m, V) \) represents the value of an offer \( x \), in the sense that it denotes the maximal expected discount proceeds, given that an offer \( x \) has been received. Proposition A.2 of Appendix I shows that the functions \( W_t(x, m, V) \) are convex functions of \( x \) for all \( t \).

\[ E[W_{t+1}(\hat{x}_{t+1}, m_{t+1}(x), V_{t+1})], \]  
is the expectation of a convex value function, and therefore increases as dispersion \( \sigma^2_{t+1} \) increases. It follows from (10) and (11) that so do the value of the asset, \( W_t(x, m, V) \),
and the reservation price \( x^*_t \).

The result that increased dispersion increases the reservation prices and the value of the asset has also been obtained in non-Bayesian models by Venezia and Brenner [17], and Brock, Rothschild and Stiglitz [6].

In the non-Bayesian case \( V \) equals to zero and the function \( \tilde{W}_t(x, m) \) defined analogously to \( W_t(x, m, V) \) satisfies

\[
\tilde{W}_t(x, m) = \max \{ x, \rho E[\tilde{W}_{t+1}(\tilde{x}, m)] \} = \max \{ x, \phi_t \}. \tag{14}
\]

In this case convexity of \( \tilde{W}_t(x, m) \) is straightforward since \( \tilde{W}_t(x, m) \) is the maximum between \( x \) and a constant. This is not so in our case where \( E[W_{t+1} (\tilde{x}_{t+1}, m_{t+1}(x), V_{t+1})] \) is also a function of \( x \).

In the non-Bayesian case the economic intuition for the above result is the following. Increased dispersion increases the probability of both high and low offers. If the investor observes a low offer he may wait until a "better" one comes along. The larger the dispersion, the larger the probability of obtaining high offers.

In the Bayesian case a low offer reduces the probability of obtaining high offers in the future. However, the future mean of the offers is a convex combination of the offer \( x \) and the prior \( m \), thus the "damage" in observing a low \( x \) is partially offset by the prior.

An alternative explanation for the positive correspondence between value and variance in the non-Bayesian case is the following. From (14) it follows that

\[
\tilde{W}_t(x, m) = \phi_t + \max \{ x - \phi_t, 0 \}. \tag{15}
\]

The second term in the right hand side of (15) can be considered as the value of an option with exercise price \( \phi_t \), where the price of the underlying stock is \( x \). Since it is well known (e.g., Merton [14]) that the value
of an option rises with increased dispersion of the underlying stock, so should \( \tilde{W}_t(x, m) \). In the Bayesian case \( \phi_t \) is a function of \( x \) and the reasoning does not go through. However, in light of the above results one might conjecture that the price of options with state-dependent exercise price, where the exercise price is a convex function of the price of the stock, should be an increasing function of dispersion. This result has been obtained by Goldman, Sosin, and Gatto [10], p. 1119, for "sell at the high" options.

IV. The Case of Growth Investments

We first consider the case where the pattern of growth is known. As in Section III, it is assumed that offers \( \tilde{x}_t \) are drawn from a normal distribution with uncertain mean and known variance. The mean, \( \tilde{\mu}_t \), of \( \tilde{x}_t \) is assumed to grow following the process

\[
\tilde{\mu}_t = \tilde{\mu}_{t-1} + \alpha_t \quad t = 1, 2, \ldots
\]

where the \( \alpha_t \)'s are known constants.

As in the former case, the only unknown parameter is \( \tilde{\mu}_t \). The learning process of the firm is similar to that described in Section III; however, (9) is replaced by

\[
m_{t+1}(x) = m_t' + \alpha_t = \gamma m_t + (1-\gamma)x + \alpha_t
\]

where \( \gamma \) is given by (8).

A more interesting case is the one where the pattern of growth is unknown and the investor learns about this pattern from past observations. In this case we assume that the increase \( \alpha_t \) in the mean of the distribution is unknown, and (16) is replaced by

\[
\tilde{\mu}_t = \tilde{\mu}_{t-1} + \tilde{\alpha}_t, \quad t = 1, 2, \ldots
\]

where the tilde over \( \alpha_t \) signifies that now \( \alpha_t \) is considered a random var-
liable. It is also assumed that the beliefs of the firm about $\tilde{a}_t$ are represented by a normal distribution with mean $\eta_t$. That is

$$\tilde{a}_t = \eta_t + \tilde{u}_t$$

where the $\tilde{u}_t$'s are normally distributed random variables with zero mean and known variance $U_t$.

Here the firm must update each period its beliefs both about $\tilde{a}_t$ and $\tilde{u}_t$. The firm does so using two pieces of data: observations of past offers provide knowledge about $\tilde{u}_{t-1}^t$, and observations of past differences in offers provide information about $\tilde{a}_t$. We shall first show how beliefs about $\tilde{a}_t$ are revised.

From (5) and (18) it follows that

$$\tilde{D}_t = \tilde{x}_t - \tilde{x}_{t-1}^t = \tilde{u}_t^t - \tilde{u}_{t-1}^t + (\tilde{e}_t - \tilde{e}_{t-1}^t) = \tilde{a}_t + (\tilde{e}_t - \tilde{e}_{t-1}^t)$$

It hence follows from Bayes' rule that if at the beginning of period $t$ the prior distribution of $\tilde{a}_t$ has been normal with mean $\eta_t$ and variance $U_t$, and if at period $t$ an offer $x$ is received yielding $D = x - x_{t-1}$, then the posterior distribution of $\tilde{a}_t$ is normal with mean $\eta_t$ and variance $U_t$ given by

$$\eta_t = \gamma_2 \eta_t + (1 - \gamma_2)D$$

$$= \gamma_2 \eta_t - (1 - \gamma_2)x_{t-1} + (1 - \gamma_2)x$$

$$U_t = \gamma_2 U_t$$

where

$$\gamma_2 = 2\sigma_t^2/(2\sigma_t^2 + U_t)$$

Since no trend with time is assumed about the evolution of $\tilde{a}_t$, the prior distribution of $\tilde{a}_{t+1}$ is the same as the posterior distribution of $\tilde{a}_t$, that is

$$\eta_{t+1} = \eta_t^t, \quad U_t = U_t$$

Consider now the revision of beliefs about $\tilde{u}_t$. The posterior of $\tilde{u}_t$
given the offer $x$ is the same as that presented in Section III. Here, however, since the mean offers grow with time, the prior distribution of $\bar{\mu}_{t+1}$ is not the same as the posterior of $\bar{\mu}_t$. Actually, since $\bar{\mu}_{t+1} = \bar{\mu}_t + \bar{a}_t$, it follows that the mean of $\bar{\mu}_{t+1}$ equals the mean of the posterior of $\bar{\mu}_t$ plus the mean of the posterior of $\bar{a}_t$. Thus, $\bar{\mu}_{t+1}$ is normally distributed with mean and variance given by

$$m_{t+1}(x) = \gamma_1 m_t + (1 - \gamma_1) x + \gamma_2 n_t + (1 - \gamma_2)d$$

$$= [\gamma_1 m_t + \gamma_2 n_t - (1 - \gamma_2)x_{t-1}] + [(1 - \gamma_1) + (1 - \gamma_2)]x$$

$$V_{t+1} = \gamma_1 V_t + \gamma_2 U_t$$

where $\gamma_2$ is defined in (22) and $\gamma_1$ is the same as $\gamma$ defined in (8).

In its decision process the firm must now consider four parameters: $m, \eta, V, U$. Its selling strategy can be computed as in Section III, except that now the function $W_t(\cdot)$ is defined over the above mentioned four parameters. Arguing as before, one can readily observe that $W_t(\cdot)$ satisfies:

$$W_t(x, m, \eta, V, U) = \max\{x, pE[W_{t+1}(x_{t+1}, m_{t+1}(x), n_{t+1}(x), V_{t+1}, U_{t+1})]\}$$

The functions $W_t(\cdot)$ are computed recursively using the boundary condition $W_T(x, m, \eta, V, U) = x$.

Since $W_T(x, m, \eta, V, U)$ is a convex function of $x$, one can show, reasoning as in Section III, that $W_t(x, m, \eta, V, U)$ is also a convex function of $x$ for all $m, \eta, V$ and $U$. From this follows the result that reservation prices (if they exist) and the value of the asset are increasing functions of the dispersion of offers in the future periods.

V. Comparison of Bayesian and Non-Bayesian Decisions

In the present model, unlike the cases of stationary investments and non-Bayesian decision makers, the reservation prices need not be finite. This can be seen from analyzing $W_{T-1}(\cdot)$. Since $W_T(x, \cdot) = x$, from (24)
and (25):

\[ W_{T-1}(x, m, \eta, V, U) = \max \{ x, \rho \mathbb{E}[\tilde{x}_T \mid m_T(x), \eta_T(x), V_T, U_T] \} \]

\[ = \max \{ x, \rho m_T(x) \} \]

\[ = \max \{ x, \rho[\gamma_1 m + \gamma_2 \eta - (1 - \gamma_2)x_{T-1}] + \rho[(1 - \gamma_1) + (1 - \gamma_2)]x \} \]

Now if,

\[ \rho[(1 - \gamma_1) + (1 - \gamma_2)] > 1 \] and \[ \rho[\gamma_1 m + \gamma_2 \eta - (1 - \gamma_2)x_{T-1}] > 0 \]

then

\[ x < \rho[\gamma_1 m + \gamma_2 \eta - (1 - \gamma_2)x_{T-1}] + \rho[(1 - \gamma_1) + (1 - \gamma_2)]x \]

for all \( x \geq 0 \). This means that no matter how high the offer is, the firm should not sell the asset. Constraint (27) will hold if the \( \gamma \)'s are small, i.e. if the prior information is not reliable. In this case most of the information about \( \tilde{\mu}_t \) and \( \tilde{\alpha}_t \) comes from observing \( x \), and if a high offer is made, it raises even higher the expectations of future offers. For (27) to hold the discount rate should be small, since if the discount rate were high (e.g., \( \rho < \frac{1}{2} \)), the losses from waiting would always be smaller than the gains from the increased expected growth.

In the case where

\[ \rho[(1 - \gamma_1) + (1 - \gamma_2)] > 1 \] and \[ \rho[\gamma_1 m + \gamma_2 \eta - (1 - \gamma_2)x_{T-1}] < 0 \]

the interpretation of the reservation prices is reversed. In this case, if \( x < x^*_{T-1} \) the firm should sell the asset and otherwise it should wait. This occurs because low offers decrease the firm's expectations of future proceeds (the firm may even expect that its asset is deteriorating), hence it is better off by selling the asset.

The non-Bayesian case can be considered as a special case of the Bayesian case where \( \gamma_2 = \gamma_1 = 1 \), that is, the firm does not revise its beliefs when observing new information. We consider, for simplicity, the case where the \( \alpha_t \)'s are known. The function \( W_t(\cdot) \) for \( t = T-1 \) in the Bayesian
case is
\[ W_{T-1}(x, m, V) = \max \{ x, \rho [\gamma_1 m + \eta + (1 - \gamma_1) x] \}, \]  
\[ (29) \]
and its non-Bayesian counterpart is given by
\[ \tilde{W}_{T-1}(x, m) = W_{T-1}(x, m, 0) = \max \{ x, \rho [m + \eta] \} . \]
\[ (30) \]
These two functions intersect when \( x = m \). One can easily observe from
Figures 1 and 2 that if \( \tilde{x}_{T-1} > m \), then \( x_{T-1} > \tilde{x}_{T-1} \) (where \( \tilde{x}_{T-1} \) is the
reservation price in the non-Bayesian case).

For the case \( \tilde{x}_{T-1} > m \), by the definition of \( \tilde{x}_{T-1} \), a non-Bayesian firm
receiving an offer \( \tilde{x}_{T-1} \) is indifferent between selling the asset and keeping
it (i.e. moving to period \( T \) possessing an asset whose mean offer is \( m + \eta \)).
If a Bayesian firm receives an offer of \( \tilde{x}_{T-1} \) and rejects it, it will possess
at \( T \) an asset whose mean offer is larger than \( m + \eta \) (since \( \tilde{x}_{T-1} > m \)).
Thus, the Bayesian firm has a better incentive (than the non-Bayesian) to
keep the asset. It hence follows that if the non-Bayesian firm is indifferent
between keeping and selling the asset, the Bayesian firm strictly prefers keeping
the asset. This implies that the Bayesian firm needs a higher offer in order to sell the asset, i.e. its reservation price is higher than
that of the Bayesian firm. Likewise, one can argue that the opposite is
true when \( \tilde{x}_{T-1} < m \).

The functions \( \tilde{W}_t(x, m) \) and \( W_t(x, m, V) \), \( t < T - 1 \) are depicted
in Figure 3. In this case the functions \( \rho E[W_{t+1}(\tilde{x}_{t+1}, m_{t+1}(x), V)] \) and
\( \rho E[\tilde{W}_{t+1}(\tilde{x}_{t+1}, m+\eta)] \) depend, in addition to the means, also on the variances,
and hence need not intersect at \( x = m \). However, since the expected value
of \( \tilde{x}_{t+1} \) is a major determinant of the values of these functions and since
\( m_{t+1}(x) = m + \eta \) when \( x = m \), they should intersect "close" to \( m \). Thus,
the higher \( \tilde{x}_t \) relative to \( m \) the more likely that \( \rho E[W_{t+1}(\tilde{x}_{t+1}, m_{t+1}(x), V)] \)
will intersect \( \rho E[\tilde{W}_{t+1}(\tilde{x}_{t+1}, m)] \) before it intersects the 45° line, i.e.
the more likely that $x_t^* > x_t$. Since $x_t$ is positively related to the variance of returns and the rate of growth, and negatively related to the discount rates, it follows that the higher the variances and the growth rates, and the lower the discount rates, the more likely that a Bayesian firm has higher reservation prices than a non-Bayesian firm.

Conclusion

We have extended and modified results obtained in previous models of growth investments, to the case where the firm reassesses its beliefs whenever confronted with new information. We have described the learning process and the optimal policies for terminating the investment when there is uncertainty about the mean return and mean growth rate of the investment assuming that the underlying distributions are normal. A comparison of the Bayesian and non-Bayesian optimal policies has shown that (1) the Bayesian reservation prices may be infinite (this cannot happen in the non-Bayesian case) if uncertainty about the unknown parameters is considerable and if discount rates are low, (2) as in the non-Bayesian case there is a positive correspondence between reservation prices, the value of the asset, and the variance of the proceeds from selling the asset; and (3) the higher the variances and the rate of growth, and the lower the discount rates, the more likely that a Bayesian firm has higher reservation prices than a non-Bayesian firm.
FOOTNOTES

1. By "selling the asset," we mean terminating the physical investment and, in the forest's case, cutting and selling the forest for lumber.

2. The strategy of selling an asset when the parameters of the distribution are unknown has also been investigated by DeGroot [7], pp. 136-141, and Ross [16], pp. 139-140. They do not consider, however, the case where there is a growth trend in the offers for the asset, do not compare the Bayesian solution to the non-Bayesian one, and do not investigate the effect of increased dispersion on the results. Furthermore, DeGroot assumes no discounting, and Ross only considers inference between two different distributions.

3. Our results also apply (with appropriate modifications) to the case of a firm selling an asset whose value decreases stochastically with time.

4. One may wonder why the firm is not always indifferent between selling or keeping the asset, i.e. why the value of the asset at t is not just \( x_t \). The answer is that whereas lumber (of a given quality) is a homogeneous commodity, forests are not since they may differ in age, and thus in growth potential. It may be profitable to cut and sell for lumber an older forest, at the going price of lumber but not a younger one. Actually, owning a forest is the same as having an option to sell it for lumber at the market price for lumber. If there were a perfect market for un-cut forests, the firm would be indifferent between selling the un-cut forest and keeping it. This does not eliminate the problem of determining when to cut the forest.

5. See Bogue and Roll [3], Brennan [5], Fama [8], Myers and Turnbull [15].

6. There is an unfortunate feature to our model since (5) permits negative prices to occur with positive probabilities. This is justified if there are transaction costs (e.g., costs of cutting the forest). We assume, however, that in our case these probabilities are negligible. The same results also hold when \( \tilde{X}_t \) is lognormally distributed. (5) can then be replaced by: \( \tilde{x}_t' = \tilde{\mu}_t e^{\tilde{\sigma}_t} t \), and by taking logarithms we obtain \( \tilde{x}_t = \tilde{\mu}_t + \tilde{\sigma}_t \) where \( \tilde{x}_t' = \ln \tilde{x}_t' \) and \( \tilde{\mu}_t = \ln \tilde{\mu}_t \). In this case \( \tilde{x}_t' \) should be interpreted as the yield on the investment at time t.

7. Since the case where the planning horizon is infinite does not provide any further economic insights, we shall not consider it here.

8. To simplify notation we drop the index t from \( p_t \) whenever this does not lead to confusion.

9. A similar effect of increased dispersion has been obtained in other types of economic activities: e.g., job search models (Kohn and
Shavell [13]), and liquidity models (Baldwin and Meyer [1]). All these models, however, are non-Bayesian.

10. Merton proved this property by showing that the value of the option is a convex function of the price of the underlying stock (ibid., p. 148).

11. Alternative, more complicated, assumptions can be analyzed using the same techniques we use here. For example, the case of a linear trend in growth can be studied by replacing (18) by: \( \tilde{\mu}_t = \tilde{\mu}_{t-1} + a_0 - t\tilde{\alpha}_t \), where \( a_0 \) is some known constant and \( \tilde{\alpha}_t \) is defined as above. The case where \( (\tilde{x}_t, \tilde{R}_{Mt}) \neq 0 \), can be analyzed by replacing (5) with:
\[
\tilde{x}_t = \tilde{\mu}_t + b\tilde{R}_{Mt} + \tilde{\varepsilon}_t,
\]
where \( b \) is some known constant. Since the results of this and the following sections are preserved also under these alternative assumptions (see Appendix II), we preferred using the simpler assumptions in order to facilitate the exposition.

12. This result is similar to the one obtained in the "secretary problem" (see, e.g. Gilbert and Mosteller [9]). In this problem a decision maker samples from a distribution he knows nothing about. His objective is to find the maximal element of a finite sequence. The optimal strategy of the decision maker is to reject the first \( s \) observations and then accept the largest one thus far. We note that he rejects the first \( s \) observations (where \( s \) depends on the length of the sequence he samples from) no matter how large they are.
REFERENCES


APPENDIX I

Proposition A.1. For all \( t, m, V \), there exist unique and finite reservation prices \( x^*_t \) such that the asset will be sold if and only if \( x > x^*_t \).

Proof: From Lemma A.1 it follows that

\[
\psi_t(x) = \mathbb{E}[W_{t+1}(\tilde{x}_{t+1}, \gamma m + (1-\gamma)x, V)] \leq \gamma m + (1-\gamma)x + C \text{ where } C \text{ is independent of } x.
\]

It hence follows from (11) that if \( x > p(\gamma m + C)/(1-p(1-\gamma)) \), then the firm should sell the asset. This implies that the reservation price must be finite.

When proving Proposition A.2 it has been shown that \( \psi_t(x) \) is convex. It is, of course, also increasing. It therefore follows that \( \psi_t(x) \) cannot cross the function \( f(x) = x \) more than twice (see Figure A.1). If these two functions cross twice, this implies that the reservation price is infinite, but this has already been ruled out. Hence these two functions can cross at most once, and since \( \psi_t(0) > 0 \), they cross exactly once. This implies the proposition.

Q.E.D.

In what follows the parameter \( V \) will be omitted since it plays no role in the analysis. It will also be assumed, without loss of generality, that the variance of the marginal distribution of \( \tilde{x}_{t+1} \) is 1. To economize space, we shall omit \( 1/\sqrt{2\pi} \) whenever the density of the standard normal distribution appears.

Lemma A.1 \[ \mathbb{E}[W_{t+1}(\tilde{x}, \gamma m + (1-\gamma)x)] \leq \gamma m + (1-\gamma)x + C \]

where \( C \) is independent of \( x \).

Proof: By definition

\[
\psi_t(x) = \mathbb{E}[W_{t+1}(\tilde{x}, \gamma m+(1-\gamma)x)] = \int_{-\infty}^{\infty} W_{t+1}(z, \gamma m+(1-\gamma)x) \exp \left[-1/2(z-\gamma m-(1-\gamma)x)^2\right] dz
\]

From Lemma A.2 it follows that
\[ \psi_t(x) \leq \gamma m + (1-\gamma)x + \int_{-\infty}^{\infty} W_{t+1}(z-\gamma m - (1-\gamma)x, 0) \exp \left[-\frac{1}{2}(z-\gamma m - (1-\gamma)x)^2\right] dz \]
\[ = \gamma m + (1-\gamma)x + \int_{-\infty}^{\infty} W_{t+1}(r, 0) \exp \left(-\frac{1}{2} r^2\right) dr \]
\[ = \gamma m + (1-\gamma)x + C. \]

The last equality defines C which is clearly independent of x.

Q.E.D.

**Lemma A.2.** For all k, \( W_t(x, m) \leq k + W_t(x-k, m-k) \)

**Proof:** By induction. The statement is clearly true for \( t = T \). Suppose it is true for \( t+1 \). One can verify that

\[ W_t(x, m) = \max \{ x, pE[W_{t+1}(x, \gamma m + (1-\gamma)x)] \} \]
\[ \leq \max \{ x, p[k + EW_{t+1}(x-k, \gamma m + (1-\gamma)x-k)] \} \]
\[ \leq \max \{ x, k + p \int_{-\infty}^{\infty} W_{t+1}(z-k, \gamma m + (1-\gamma)x-k) \exp \left[-\frac{1}{2}(z-\gamma m - (1-\gamma)x)^2\right] dz \}
\[ = \max \{ x, k + p \int_{-\infty}^{\infty} W_{t+1}(r, \gamma m + (1-\gamma)x-k) \exp \left[-\frac{1}{2} (r-\gamma m - (1-\gamma)x+k)^2\right] dr \}
\[ = k + W_t(x-k, m-k). \]

The first inequality stems from the induction assumption. The second inequality holds true since \( pk < k \), and the following equality is obtained by a simple change of variables. The equation before last holds since for any numbers a, b, k, \( \max (a, b) = k + \max (a-k, b-k) \). The last equality follows from the definition of \( W_t(x-k, m-k) \) and using the fact that

\[ \gamma(m-k) + (1-\gamma)(x-k) = \gamma m + (1-\gamma)x - k. \]

Q.E.D.

**Proposition A.2.** \( W_t(x, m) \) is a convex function of \( x \) for all \( t \leq T \).

**Proof:** We shall actually show that \( W_t(\cdot) \) is a convex function of \( x, m \).

The proof is by induction. The statement is true for \( t = T \), since \( W_T(x, m) = x \). Suppose the proposition is true, for \( (t+1) \). From the definition of \( W_{t+1}(\cdot) \), and using standard integration rules one obtains

\[ \psi_t(x, m) = E[W_{t+1}(x, \gamma m + (1-\gamma)x)] \]
It hence follows from the convexity of $W_{t+1}(.)$ that $\psi_t(x, m)$ is convex too, since convexity is preserved under linear transformations and the integral operation. Since $W_t(x, m) = \text{Max} \{x, \psi_t(x, m)\}$, it is convex, being the maximum between two convex functions.

Q.E.D.
APPENDIX II

The Case of a Linear Trend in Growth

In this case (18) is replaced by

\[ \tilde{\mu}_t = \tilde{\mu}_{t-1} + a_0 - \tilde{a}_t \cdot t \]

where \( a_0 \) is some known constant, and the unknown parameter \( \tilde{a}_t \) is defined as in Section IV. In this case, as in Section IV the firm considers the same four parameters of interest: \( m_t, \ n_t, \ V_t, \ U_t \) which are updated each period. Here, however, the updating rule is somewhat different and is given by:

\[
\begin{align*}
    m_{t+1}(x) &= \gamma_1 m_t + (1 - \gamma_1)x + a_0 - \frac{(t+1)n_{t+1}(x)}{t} \\
    n_{t+1}(x) &= \gamma_3 n_t + (1 - \gamma_3)[a_0 - (x - x_{t-1})]/t \\
    U_{t+1} &= \gamma_3 U_t \\
    V_{t+1} &= \gamma_1 V_t
\end{align*}
\]

where

\[ \gamma_3 = 1 / [V_t(t^2/2\sigma^2) + 1] \]

One may easily verify that the economic implications of this model are quite similar to those of the one discussed in Section IV.

The Case of Correlation Between \( \tilde{x}_t \) and \( \tilde{R}_{Mt} \)

Suppose that the process generating the cash flows is given by

\[ \tilde{x}_t = \tilde{\mu}_t + b\tilde{R}_{Mt} + \tilde{\varepsilon}_t \quad , \quad t = 1, 2, \ldots \]

where \( b \) is some known constant, where the \( \tilde{\varepsilon}_t \)'s are independent of \( \tilde{R}_{Mt} \) where \( \tilde{R}_{Mt} \) is normally distributed, and where the same assumptions as in section IV are made about the other parameters.

The revision of beliefs about \( \tilde{\mu}_t \) and \( \tilde{a}_t \) is done as before except that one must consider the observations \( (x - R_{Mt}) \) instead of \( x \).
Defining \( \tilde{d}_t = (\tilde{x}_t - b\tilde{R}_{Mt}) - (\tilde{x}_{t-1} - b\tilde{R}_{Mt-1}) \), one can show that: if at the beginning of period \( t \) the priors of \( \tilde{a}_t \) and \( \tilde{a}_T \) have been normal with means \( \eta_t \) and \( m_t' \), and variances \( U_t \) and \( V_t \), respectively, and if during that period an offer \( x \) has been made, and the market return has been \( R_{Mt} \), then the distributions of \( \tilde{a}_{t+1} \) and \( \tilde{a}_{t+1} \) at the beginning of period \( (t+1) \) are normal with parameters:

\[
\eta_{t+1}(x) = \gamma_2 \eta_t + (1 - \gamma_2) d_t
\]

\[
= [\gamma_2 \eta_t - (1 - \gamma_2)(x_{t-1} - bR_{Mt-1}) - (1 - \gamma_2)bR_{Mt}] + (1 - \gamma_2)x
\]

\[
U_{t+1} = \gamma_2 U_t
\]

\[
m_{t+1}(x) = \gamma_1 m_t + (1 - \gamma_1)(x - bR_{Mt}) + \eta_{t+1}(x)
\]

\[
V_{t+1} = \gamma_1 V_t + \gamma_2 U_t
\]

where \( \gamma_1 \) and \( \gamma_2 \) are defined in Section IV. The marginal expectation of \( \tilde{x}_{t+1} \),

\[
E_{t+1}(\tilde{x}_{t+1} \mid x), \text{ is } m_{t+1}(x) + bE(R_{M_{t+1}}).
\]

In the present case (4) no longer is the value of the asset since in this case \( E(R_t) \) equals \( r_{Ft} + b\lambda/W_{t-1}(x, m, V) \) and hence it is revised between the origin and period \( t \). The value of the asset, however, can be computed recursively as follows. Suppose that at the beginning of period \( T-1 \), the firm received an offer \( x \) and has rejected it. Assuming that no more offers will be received during the period, the value of the asset at that time, \( W_{T-1}(x, m, V) \), is the risk adjusted present value of the proceeds at \( T \), i.e.*

\[
W_{T-1}(x) = E_{T-1}(\tilde{x}_T \mid x) / [1 + r_{Ft} + b\lambda/W_{T-1}(x)]
\]

where \( E_{T-1}(\tilde{x}_T \mid x) \) denotes the marginal expectation of \( \tilde{x}_T \) at \( (T-1) \), given the offer \( x \). Rearranging terms one obtains

\[
W_{T-1}(x) = [E_{T-1}(\tilde{x}_T \mid x) - \lambda b] / (1 + r_{Ft})
\]  

(A.2)

From the maximizing behavior of the firm it follows that

* We henceforth suppress the parameters \( m, \eta, V, U \).
\[ W_{T-1}(x) = \text{Max} \{ x, W^l_{T-1}(x) \} = \text{Max} \{ x, \rho_T [E_{T-1}(\tilde{x}_T | x) - \lambda b] \} \quad (A.3) \]

and hence the reservation price, \( x^*_T \) is the minimal price satisfying

\[ x \geq \rho_T [E_{T-1}(\tilde{x}_T | x) - \lambda b] \]

Similarly one can define

\[ W^l_{t-1}(x) = \{ E_{t-1}[W_t(\tilde{x}_t | x)] - \lambda b \} / (1 + r_{F_t}), \quad t = 1, 2, \ldots, T-1 \quad (A.4) \]

\[ W_{t-1}(x) = \text{Max} \{ x, W^l_{t-1}(x) \} \]

and the reservation price \( x^*_t \) is the minimal price satisfying

\[ x \geq \rho_t [E_{t-1}(W_t(\tilde{x}_t) | x)] \]

From (A.1), (A.2), (A.3) and (A.4) it can be verified that the same qualitative conclusions obtained in Sections IV and V hold true in this case as well.
Figure 2
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