

**ON THE DOUBLING STRATEGY PARADOX AND THE
DEFINITION OF ARBITRAGE**

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ABSTRACT

The Harrison-Kreps doubling strategy paradox is implicitly based on the assumption that, roughly speaking, $\infty \cdot 0 = 0$.

Roughly speaking, if you believe that the doubling strategy yields arbitrage profits, then you believe that $\infty \cdot 0 = 0$.²

2. A paradox and its resolution

The final payout of the above doubling strategy is almost surely above 1, but its exact distribution is quite complexed. Instead of examining the original Harrison-Kreps paradox, let us consider another arbitrage strategy, one that is more complicated but whose final payout has a very simple distribution and thus is easier to analyze.

To recapitulate, in the Black-Scholes model it is assumed that a stock price S_t follows a geometric Brownian motion with instantaneous variance of return σ^2 (a constant), and the continuously compounded interest rate is a constant r . Without loss of generality, a bond can be defined as a security whose price is e^{-rt} at time t . Assuming that the stock is certain not to pay dividends in the time interval $[0, T]$, it follows (see Merton [11], Bergman [1]) that a derivative security that pays $h(S_T)$ at time T (and zero at any other time) can be produced by a self-financing trading strategy in the stock and the bond. $V(t, S)$, the value of the portfolio corresponding to this strategy at time t given that $S_t = S$, satisfies³:

$$V_t + \frac{1}{2} \sigma^2 S^2 V_{SS} + rSF_S = rF \quad (1)$$

$$V(T, S) = h(S).$$

Having solved this equation for $V(t, S)$, the number of bonds which are held at time t is then given by $\theta_1(t, S) = e^{-r(T-t)}(V - SV_S)$ and the number of shares of the underlying stock is $\theta_2(t, S) = V_S$.

Now to our version of the paradox. Fix $x > 0$. It can be directly verified (or see Bick [2, proposition 2]) that the solution of Eq. (1) with $h(S) = \delta(S-x)$, where δ is the Dirac function,⁴ is given by

$$V(t, S) = e^{-r(T-t)} n(d_2(t, S, T, x)) / x \sigma (T-t)^{1/2}$$

where $n(\cdot)$ is the density of the standard normal variable and

$$d_2(t, S, T, x) = [\ln S/x + (r - \frac{1}{2}\sigma^2)(T-t)]/\sigma(T-t)^{1/2}.$$

(This $V(t, S)$ is, in fact, the second derivative of the Black-Scholes formula with respect to the exercise price, in accordance with Breeden and Litzenberger [4].) Thus, by holding

$$\theta_2(t, S) = e^{-r(T-t)} n'(d_2)/xS\sigma^2(T-t)$$

shares of the underlying stock and

$$\theta_1(t, S) = e^{-rT} [n(d_2)/x\sigma(T-t)^{1/2} - n'(d_2)/x\sigma^2(T-t)]$$

bonds, one will obtain at time T a payoff of $\delta(S_T - x)$, i.e., zero if $S_T \neq x$ and ∞ if $S_T = x$. As the event $S_T = x$ is of probability zero, this means that the payoff is almost surely zero, and the fact the $V(t, S) > 0$ seems like a paradox. Reversing the strategy (replacing short by long and vice-versa), one receives a positive payment at time $t=0$ for a future liability which is zero with probability 1, i.e. arbitrage profits.

This form of the paradox indicates very clearly what the solution should be: It is economically plausible if we assume that investors do not disregard an event of probability zero, if its occurrence will yield a payment of $\pm\infty$. More formally, suppose Y is a random variable which represents a future payoff and is defined on a probability space (Ω, Σ, P) . If $P(\{\omega \in \Omega; Y(\omega) = \pm\infty\}) = 0$, but this set of states⁵ is not empty, then it should not be ignored. In our case, suppose the stochastic process S_t is defined on the probability space (Ω, Σ, P) . Let $\theta = (\theta_1, \theta_2)$ be a trading strategy in the bond and the stock,⁶ and let $V^\theta(t) = \theta_1(t)e^{rt} + \theta_2(t)S_t$ be the value of the corresponding portfolio. Then the assumption of no arbitrage opportunities (or "no free lunch," in Harrison-Kreps' terminology) is formulated by Kreps [9] as follows (in our notation):

For all self-financing strategies θ , $P(V^\theta(T) \geq 0) = 1$ and $P(V^\theta(T) > 0) > 0$ imply that $V^\theta(0) > 0$.

This paper proposes the following stricter definition:

For all self-financing strategies θ , $P(V^\theta(T) \geq 0) = 1$, $P(V^\theta(T) > 0) > 0$ and $V^\theta(T) \neq -\infty$ with certainty⁷ imply that $V^\theta(0) > 0$.

The strategy described above does not violate the second definition, and thus no contradiction arises. It seems perfectly intuitive that the prospect of a "huge" payoff contingent upon a "very small" set of states may raise above zero the value of a random payoff which is otherwise zero.

A similar argument is applicable to the Harrison-Kreps strategy. There the event of probability zero which causes the paradox is somewhat more complicated — this is the event of "losing" at each t_n . In contrast to what Harrison and Kreps say ("... you will bet finitely many times in any particular state"), this event is not empty, and the payoff under its occurrence is $-\infty$. Therefore, a value of zero for this random payoff, which is almost surely above 1, is not counterintuitive.

3. Some further remarks

It follows from the previous section that restrictions on credit or trading times are not really necessary, as the Harrison-Kreps doubling strategy (or the strategy in our example) does not yield arbitrage profits as this paper defines them. The penalty for not imposing those limitations is "merely" the existence of trading strategies that may produce infinite payments (necessarily with infinite security holdings). With probability 1, such an infinite payout will not result, but by now we distinguish between probability 1 and certainty. Thus we may decide to live with a model that can "explode" with probability zero, if certain strategies are followed, or we may choose to impose the technical limitation that strategies as above are not allowed. (One can envision a situation in which agents are requested to specify their strategies at time $t = 0$ to a market-maker who will not accept a strategy that may yield an infinite payment in some state.) This requirement is milder than Harrison-Kreps' proposed limitations.

A word of caution: Even if "potentially exploding" strategies as above are allowed, trading of securities that pay $\pm\infty$ in some states should not be permitted. Consider, for example, the

above strategy that produces $\delta(S_T - x)$ at time T , i.e., ∞ if $S_T = x$ and zero otherwise. Doubling the amount of securities in the strategy, we shall pay twice as much for starting the strategy, and we shall obtain $2\delta(S_T - x)$ at time T , which is also ∞ if $S_T = x$ and zero otherwise. As long as a security which pays ∞ if $S_T = x$ and zero otherwise is not traded in the market, this will not give rise to an inconsistency, and the value of $\delta(S_T - x)$ or $2\delta(S_T - x)$ is understood only as the cost of starting the strategy.

In texts on measure theory, it is usually implicit that the integral of an extended real (i.e., possibly infinite) function on a set of measure zero is zero. (See, e.g., Halmos [6, chapter V] or Dunford and Schwartz [5, pp. 118-119].) This is unfortunate for economic applications. Suppose, for example, that the market in section 2 is risk-neutral (the geometric Brownian motion of the stock is known to be consistent with a risk-neutral market under a certain relation between the drift parameter, the instantaneous variance, and the interest rate). Then, as the expectation of the payoff $\delta(S_T - x)$ is zero, so will be its value at time $t = 0$. Similar contradictions will occur in expected utility models, if the utility function is unbounded. This difficulty can be eliminated by regarding the integral of $\pm\infty$ on a set of probability zero as "undefined," and again, by not allowing trade in securities that are infinite in some states.

4. Conclusion

The doubling strategy paradox is implicitly based on the fact that an infinite payout contingent upon a non-empty set of states with probability zero is valued as zero. When the definition of "arbitrage profits" is corrected such that infinite payouts are not disregarded, the paradox disappears. The main restriction which is required to keep the model consistent is that securities with infinite payouts in some states are not allowed to be traded.

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FOOTNOTES

¹ The reader is referred to the above papers for formal definitions of the concepts mentioned here.

² This basic idea, in the context of a similar gambling situation, was expressed by Professor Lester E. Dubins (Department of Statistics, University of California, Berkeley) in a lecture he gave. The author is indebted to him for a discussion on this topic.

³ Subscripts denote partial differentiation.

⁴ A brief review on the Dirac function can be found in [12, section 1.5].

⁵ In this paper, "state" means an element ω of the probability space. In our case it can be viewed as a possible realization of the price process.

⁶ In the previous discussion θ was a function of (t, S_t) , but here we allow more general strategies that may depend on the previous history.

⁷ i.e., for each ω in the probability space.

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