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RISK–ADJUSTED DISCOUNTING

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RISK-ADJUSTED DISCOUNTING

Risk-adjusted discount rates (RADM) are widely used by practitioners in evaluating investment projects. Therefore the question of whether such an approach is consistent with modern asset valuation methods must be addressed.\(^1\)

Perhaps the most ambitious effort in this direction has been made by Fama [1977].\(^2\) Extending the single-period Sharpe-Lintner capital asset pricing model (CAPM) into a multiperiod context, he shows that a risk-adjusted discounting approach is valid.

Unfortunately, the use of the multiperiod CAPM requires mutually contradictory assumptions. Fama stresses the fact that multiperiod CAPM valuation is correct only when the source of uncertainty-resolution over time is reassessment of expected future cash flows. Specifically, this requires that the market price of risk is nonstochastic. But as Rubinstein [1973] has shown, a nonstochastic market price of risk required investors' utility functions to exhibit constant proportional risk aversion (CPRA). These utility functions will be consistent with the CAPM only if asset returns are normally distributed. But normally distributed returns are not integrable (i.e., consistent) with CPRA utility functions.\(^3\)

Since formally the multiperiod CAPM is inconsistent, we must reject risk-adjusted discounting based on such an approach. The question remains whether risk-adjusted discounting is consistent with a reasonable alternative economic environment. We show that if investors
exhibit CPRA tastes, and asset cash flows follow a Fama-type evolution of expectations, then risk-adjusted discount rates may be used regardless of the distribution of asset returns, so long as expected utilities are defined.

We examine the determinants of the risk-adjusted discount rate in these environments, and show they bear similarity to the single-period CAPM discount rate. Thus, with a relatively minor change in the economic environment assumed, we can formally justify the use of risk adjusted discount rates in a multiperiod context.

I. The Model

Determining the desirability of an investment project requires a valuation of the project's cash flow stream. The valuation technique most commonly used by practitioners discounts the sequence of expected cash flows from the project, $E(\tilde{x}_t)$ (t=1,...), by a risk-adjusted discount rate $k$, i.e.

$$ P = \sum_{t} \frac{E(\tilde{x}_t)}{k^t} $$

where $K = 1 + k$.

Models of market equilibrium pricing suggest that the value $P$ should reflect the certainty equivalent cash flows at each period, discounted by the risk-free interest rate $r$. Assuming the CAPM, the value of the risky payoff to be received at $t=1$, $\tilde{x}_1$, would be
\[ P_1 = \left[ E_1(\tilde{x}_1) - \lambda_1 \text{cov}(\tilde{x}_1, \tilde{r}_m) \right]/R_1 \]  

(2)

where \( \lambda, r_m \) and \( R \) are the market price of risk, the return on the market portfolio and unity plus the risk-free interest rate \( r \), respectively.

Since the price determined by equation (2) must agree with the price determined by the RADR technique, it follows that

\[ K_1 = R_1 \left[ 1 - \lambda_1 \text{cov}\left[ \frac{\tilde{x}_1}{E_1(\tilde{x}_1)}, \tilde{r}_m \right] \right]^{-1}. \]  

(3)

With the additional assumptions of Fama, i.e., if \( \tilde{x}_t \) follow a stationary random walk and if \( R, \lambda, \) and \( \text{cov}\left[ \frac{\tilde{x}_t}{E(\tilde{x}_t)}, \tilde{r}_m \right] \) are known and constants, the discount rate in equation (3) can be used for all future expected cash flows. Fama obtains his result by assuming that the CAPM holds for every period, pricing the cash flow backwards, thereby producing a multiperiod framework. However, as discussed in the introduction, the assumptions required to implement this result are mutually contradictory. It follows, therefore, that the use of risk-adjusted discounting techniques cannot be rigorously justified by the CAPM.

Nonetheless, by adopting an alternative valuation model to the CAPM, the RADR approach may be justified.

The following discussion makes explicit an environment consistent with risk-adjusted discounting and examines the nature of risk adjustment as stated in the following theorem.
Theorem:

Assume - (a) the existence of a consensus investor with stationary CPRA utility, additively separable over time, i.e., \( U = \left( \frac{1}{\gamma} \right) \sum_{t} \beta_t C_t^\gamma \) where \( \beta_t \) is the subjective time discount factor for consumption at time \( t \), and \( \gamma < 1 \).

(b) Market value and the evolution of each expected future cash flow follow multiplicative random walks.

Then, the value of any asset with payouts \( \tilde{x}_t \) can be expressed as:

\[
P = \sum_{t} \frac{E(\tilde{x}_t)}{\Pi_{\tau=1} K_{t,\tau}} \tag{4}
\]

where

\[
K_{t,\tau} = R_t \left( 1 + \frac{\text{cov}[(1+\tilde{r}_{m,\tau})^{-\alpha}, (1+\tilde{e}_{t,\tau})]/[E(1+\tilde{r}_{m,\tau})^{-\alpha}]} \right)^{-1}
\]

where \( \alpha = 1 - \gamma \) and \( \tilde{e}_{t,\tau} = [E_t(\tilde{X}_t)/E_{t-1}(\tilde{X}_t)] - 1 \), i.e.,

\[
E_t(\tilde{X}_t) = E_{t-1}(\tilde{X}_t)(1+\tilde{e}_{t,\tau}), \text{ with } E(\tilde{e}_{t,\tau}) = 0 \text{ and } \tau < t.
\]

Corollary I: If the evolution of expected cash flows follows an inter-temporally independent process which is identical for each cash flow, then \( K_{t,\tau} \) is identical across time \( t \) and
\[ P = \sum_{t} \frac{E(\tilde{X}_t)}{t} \cdot \prod_{\tau=1}^{t} K_{t}^{-\gamma} \]

**Corollary II:** If the evolution of expected cash flows follows an inter-temporally independent process which is identical for each future cash flow and for each future period, and if the interest rate is constant for each future period, then

\[ P = \sum_{t} \frac{E(\tilde{X}_t)}{t} \cdot K_{t}^{-\gamma} \]

where the risk-adjusted discount rate is given by

\[ K = R \{ 1 + \frac{\text{cov}[(1+r_{m})^{-\gamma}, (1+\tilde{e})]/[E(1+r_{m})^{-\gamma}]} \}^{-1} \]

**Proof:** The investor's strategy is to maximize expected utility of future consumption,

\[ \max E[\sum_{t} \beta_{t} U(\tilde{c}_t)] , \]

where \( \tilde{c}_t \) is the (aggregate) consumption at \( t \) and where the marginal utility of consumption \( U'(\tilde{c}_t) = \tilde{c}_t^{-\alpha} \) with \( \alpha > 0 \). Therefore, an asset yielding random payoffs \( \tilde{x}_t \) through time will be priced such that its value will be equal the value of the (normalized) expected marginal utility it provides:

\[ P = E[\sum_{t} \beta_{t} U'(\tilde{c}_t)\tilde{x}_t] . \quad (5a) \]
However, since CPRA implies \( U'(\tilde{c}_t) = \tilde{c}_t^{-\alpha} \), valuation (5a) may be written as:

\[
P = E[\sum_t \beta_t \tilde{c}_t^{-\alpha} \tilde{x}_t].
\] (5b)

It should be noted that the consumption at \( t \) depends on (aggregate) wealth at that time. Because we assume CPRA, the consumption at \( t \), \( \tilde{c}_t \), is related to that period's wealth, \( \tilde{w}_t \), via a constant \( h_t \), i.e.

\[
\tilde{c}_t = h_t \tilde{w}_t
\]

with \( h_t \) being independent of \( t \) as long as returns are identically distributed through time.\(^9\)

Substituting for \( \tilde{c}_t \) yields

\[
P = E[\sum_t \beta_t h_t \tilde{w}_t^{-\alpha} \tilde{w}_t^{-\alpha} \tilde{x}_t].
\] (5c)

To specify the price function in greater detail, we use the assumption that the market's and the asset's returns are intertemporally independent, i.e., the aggregate wealth and the expected cash flows both follow a multiplicative random walk such that

\[
\tilde{w}_t = w_0 \prod_{t=1}^{T} (1 + \tilde{r}_{mt}),
\]
and
\[ \tilde{x}_t = E_0(\tilde{x}_t) \prod_{\tau=1}^{t} (1+\tilde{\varepsilon}_{t\tau}). \]

Given this assumption, equation (5c) may be written as
\[ P = E_0 \left[ \sum_{t} \beta_t h_t \tilde{w}_t \prod_{\tau=1}^{t} (1+\tilde{r}_{t\tau}) \right] \prod_{\tau=1}^{t} \left[ E_0(\tilde{x}_t) \prod_{\tau=1}^{t} (1+\tilde{\varepsilon}_{t\tau}) \right]. \] (5d)

Because we assume that returns are intertemporally independent, equation (5d) may be stated as:
\[ P = \sum_{t} \beta_t h_t \tilde{w}_t \prod_{\tau=1}^{t} E_0(\tilde{x}_t) \prod_{\tau=1}^{t} \left[ E_t \left( (1+\tilde{r}_{t\tau})^{-\alpha}(1+\tilde{\varepsilon}_{t\tau}) \right) \right]. \] (6)

Equation (6) suggests that, given our assumptions on the economy, asset prices will be inversely related to current wealth and positively related to the expected value of the asset as seen at \( t=0 \).

As equation (6) represents the correct valuation of every asset, then it must hold true also for an investment in the riskless asset of one dollar. Applying equation (6) to this investment yields
\[ 1 = \beta_t h_t \tilde{w}_t \prod_{\tau=1}^{t} \left[ \prod_{\tau=1}^{t} E_{t} \left( (1+\tilde{r}_{t\tau})^{-\alpha} \right) \right] \prod_{\tau=1}^{t} R_{t}. \]

or,
\[ \beta_t h_t \tilde{w}_t = \left\{ \prod_{\tau=1}^{t} E_{t} \left( (1+\tilde{r}_{t\tau})^{-\alpha} \right) \prod_{\tau=1}^{t} R_{t} \right\}^{-1}. \] (7)

Equation (7) represents the marginal utility of future consumption of
one dollar invested at \( t=0 \). Substituting equation (7) into equation (6) and rearranging terms, we obtain:

\[
P = \sum_{t} E_{0}(\tilde{x}_{t}) \prod_{\tau=1}^{t} \left\{ \frac{E_{\tau}((1+\tilde{r}_{m\tau})^{-\alpha}(1+\tilde{e}_{t\tau}))/R_{\tau}[E_{\tau}(1+\tilde{r}_{m\tau})^{-\alpha}]}{1 + \text{cov}_{\tau}[(1+\tilde{r}_{m\tau})^{-\alpha}, (1+\tilde{e}_{t\tau})]/E_{\tau}(1+\tilde{r}_{m\tau})^{-\alpha}} \right\}
\]  

Equation (8) may be further simplified by noting that the expected value of a product of two random variables is the product of their expected values plus their covariance.

Equation (8) can therefore take the form:

\[
P = \sum_{t} E_{0}(\tilde{x}_{t}) \prod_{\tau=1}^{t} \left\{ 1 + \text{cov}_{\tau}[(1+\tilde{r}_{m\tau})^{-\alpha}, (1+\tilde{e}_{t\tau})]/E_{\tau}(1+\tilde{r}_{m\tau})^{-\alpha} \right\}
\]  

From (9), the RADR now may be stated as:

\[
K_{t\tau} = \prod_{\tau=1}^{t} \left\{ 1 + \text{cov}_{\tau}[(1+\tilde{r}_{m\tau})^{-\alpha}, (1+\tilde{e}_{t\tau})]/E_{\tau}(1+\tilde{r}_{m\tau})^{-\alpha} \right\}^{-1}
\]  

QED

Using Fama's assumption that the evolution of expected cash flow follows an identical random walk for every future period, then:

\[
\tilde{e}_{t\tau} = \tilde{e}_{t
}
\]

implying

\[
k_{t\tau} = k_{t}
\]

for all \( t \),

thereby proving Corollary I.
When, in addition we assume that the project's expected cash flows are identical random walks, then

\[ \tilde{e}_t = \tilde{e} \quad \text{for all } t. \]

If \( r \) is constant, then

\[ k_t = k \]

and Corollary II is proved.

We therefore demonstrated that the traditional RADR used in the capital budgeting process is obtained under the above assumptions which are very similar to some of the assumptions used by Fama. However, our results do not require the CAPM assumptions.\textsuperscript{11,12,13}

The result shown in equation (10) deserves a clarification. Recall that for most cases the RADR is larger than \( r \) (the required risk premium). Therefore, the argument in the denominator of equation (10) should be smaller than unity. This is obtained by noting that the positive covariance between \( (1 + \tilde{r}_{mt}) \) and \( (1 + \tilde{e}_{tt}) \) usually implies negative covariance between \( (1 + \tilde{r}_{mt})^{-\alpha} \) and \( (1 + \tilde{e}_{tt}) \). This is always the case under the following circumstances:

(i) \( \tilde{r}_{mt} \) and \( \tilde{e}_{tt} \) are jointly normally distributed (see footnote 11). In this case the

\[ \text{cov}[g(\tilde{y}), \tilde{x}] = E[g'(\tilde{y})]\text{cov}(\tilde{y}, \tilde{x}), \]

which shows the result, or

(ii) \( \text{When market risks are small, in which case a Taylor series approximation of } (1 + \tilde{r}_{mt})^{-\alpha} \text{ about } [1 + E(\tilde{r}_{mt})] \text{ can be used to show the result.} \]
II. Concluding Observations

We have shown that risk-adjusted discounting is quite appropriate, given the assumptions of CPRA investors and intertemporal independence of the evolution of expected cash flows.

Examination of formula (10), describing the discount rate, shows that RADR, \( k \), will be higher as the covariance between the asset's return and the market's return raised to the power \(-\alpha\) (the marginal utility of consumption) becomes negative. If market and asset returns are positively correlated, this generally will imply a negative covariance between the asset's return and the market return raised to a negative power. Thus, greater correlations between asset and market returns generally imply higher discount rates, as we would expect.

Our examination also reveals the case where risk discounting is inappropriate. Most notably this is when the evolution of expected cash flows is not intertemporally independent. While market efficiency may suggest that the rates of return to marketed assets are intertemporally independent, where rate of return includes price appreciation (or depreciation), unmarketed assets (such as investment projects chosen by firms) need not exhibit intertemporally independent expected returns. Values could not be described by risk-discounted returns in such cases.

Finally, we note that the continuous time valuation model of Breeden and Litzenberger [1978] also suggests that risk discounting such as shown in equation (10) is appropriate for assets whose returns follow a logarithmic diffusion process (implying rates of return that are intertemporally independent). Breeden and Litzenberger's study, unlike
this study, assumes that the market follows a logarithmic diffusion
process. Their assumptions also imply CPRA tastes (Theorem 3, p. 647).
Footnotes

1 For survey reports, see Klammer [1972] and Gitman and Forrester [1977].

2 Other studies which examine the applicability of the RADR include Robichek and Myers [1966]; Bar-Yosef and Mesznik [1977]; Bogue and Roll [1974]; and Myers and Turnbull [1977].

3 Rubinstein [1973] argues that "... it is perhaps worthwhile noting the formal inconsistency between constant proportional risk aversion and the assumptions of the mean-variance model. ... In the absence of quadratic utility, the mean-variance model, if (1) it is to be consistent with measurable utility and (2) at least two sufficiently different risky securities exist, requires normality. Therefore, any proposed utility function must be well defined over all real values of \( \tilde{w} \). ... Proportional risk aversion ... functions ... are either imaginary over some value of \( \tilde{w} \) or the assumptions of \( u' > 0 \) and \( u'' < 0 \) are violated over some real values of \( \tilde{w} \)." (p. 608, fn. 10).

4 See Fama equation (38) [note that the exponent \( t \) has been omitted, as can be seen from equation (37)]. Fama's

\[
E(\tilde{R}) = R[1 - \lambda \operatorname{cov} (X, \tilde{R})]^{-1}
\]

from his equation (3).

5 It is only fair to suggest that Fama is not completely comfortable with his conclusions when he states: "One can reasonably find this result unpalatable, but the basic quarrel must be with the multi-period version of the [CAPM] model" (p. 16).

6 See Rubinstein [1974, 1976].
Note that this becomes log utility as $\gamma \rightarrow 0$.

By normalized, we mean that the original marginal utilities are divided by the marginal utility of current wealth. Thus, they may be interpreted as marginal willingness to pay for $1$ delivered at time $t$ in state $s$.

See Hakansson [1970].

Note the equation (5c) may be written as

$$ P = \sum_s \sum_t \rho_{st} h^{-\alpha} w^{-\alpha}(s)x_t(s) $$

where $\rho_{st}$ designates the probability of the occurrence of the state $s$ at $t$. Consequently, we get the familiar contingent claim pricing.

$$ P = \sum_s \sum_t q_t(s)x_t(s) $$

where the contingent claim $q_t(s) = \rho_{st} h^{-\alpha} w^{-\alpha}(s)$.

If we make the contradictory assumption of normality, applying Rubinstein's [1976] result that

$$ \text{cov}[g(\tilde{y}), \tilde{x}] = E[g'(y)] \text{cov}(\tilde{y}, \tilde{x}) $$

to the last term of equation (10) to obtain:

$$ \text{cov}[(1+\tilde{r}_m)_t^{-\alpha}, (1+\tilde{e}_it)]/E(1+\tilde{r}_m)_t^{-\alpha}E(1+\tilde{e}_it) = -\frac{\alpha E(1+\tilde{r}_mt)^{-\alpha-1}}{E(1+\tilde{r}_mt)^{-\alpha}} \cdot \text{cov}(\tilde{r}_mt, \tilde{e}_it) \cdot \alpha E(1+\tilde{r}_mt)^{-\alpha-1} $$

Note that the argument $\frac{\alpha E(1+\tilde{r}_mt)^{-\alpha-1}}{E(1+\tilde{r}_mt)^{-\alpha}} \equiv \lambda_t$ (the market price of risk). Assuming constant $\lambda$ and $r$ recalling that $(1+\tilde{e}) = \tilde{x}/E(\tilde{x})$, equation (10) may be written as:
\[ K = R \{ 1 - \lambda \text{cov}(\tilde{\mu}/E(\tilde{\mu}), \tilde{r}_m) \}^{-1} \]

which is equation (3) applied to \( t = 1 \ldots \).

\footnote{If investors are risk neutral (\( \alpha = 0 \)), or asset returns \( \tilde{e}_{it} \) are independent of the market returns \( \tilde{r}_{mt} \), then \( K = R \).}

\footnote{These results are consistent with those reported by Breeden and Litzenberger [1978] and Rubinstein [1976].}
References


