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ARE ASSET-DEMAND FUNCTIONS
DETERMINED BY CAPM?

BY

JEFFREY A. FRANKEL
WILLIAM T. DICKENS

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ARE ASSET-DEMAND FUNCTIONS DETERMINED BY CAPM?

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Jeffrey A. Frankel
and William T. Dickens

Department of Economics
University of California
Berkeley, Ca. 94720

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ABSTRACT

The state-of-the-art tests of the Capital Asset Pricing Model (CAPM) suffer from three related drawbacks. First, they assume that expected returns are constant over time, or at best are allowed to change in a slow, ad hoc, manner. This assumption seems inconsistent with reality. Second, the tests assume that the "betas," the covariances with the market return, are also constant over time. The market return is a weighted average of returns on individual securities, where the weights are shares in the market portfolio. Since these shares in fact change over time, this assumption is inconsistent not only with reality, but also with the use of the share-weighted market return within the same model. Third, CAPM is not explicitly nested within an alternative hypothesis. We offer a technique for testing CAPM that avoids all three drawbacks. The CAPM hypothesis is shown to imply that the matrix of coefficients in a relationship between expected returns and portfolio-shares is proportional to the variance-covariance of the error term. (The constant of proportionality is the coefficient of relative risk-aversion.) When we try the technique on a portfolio of six aggregated assets, we reject the hypothesis.
1. INTRODUCTION.

The Capital Asset Pricing Model (CAPM) provides a compelling framework for modeling the asset demands of investors who care about not only expected returns but risk as well. The model has traditionally been tested empirically by looking for a significant relationship between various assets' expected returns and their risk as measured by covariances with the overall market rate of return.\footnote{Such tests were in the late 1970s subjected to some powerful methodological criticisms.} The best-known criticism may be that the results are sensitive to a failure to include all relevant assets in the portfolio. The present paper follows some others in using a portfolio that is as comprehensive as possible.\footnote{The assets are consumer durables and real estate, long-term federal debt, state and local debt, corporate debt, equities, and short-term federal securities, open market paper, and deposits. But the contribution of the paper lies elsewhere. It is an attempt to address two related problems that remain in the state-of-the-art literature.} First, the standard approach assumes that the expected returns on individual assets and on the aggregate market portfolio are constant over time. In reality it is clear that interest rates, expected inflation rates, expected rates of return on equity, etc. fluctuate over time. Most financial economists would like to be able to use models in which not only the level of expected returns, but also expected relative returns can change. For example, it is often argued that the expected return on government bonds, relative to money, goes up when the supply of government bonds goes up, or when the supply of money goes down.
Second, the standard approach assumes that the betas on individual assets—the coefficients in regressions of their returns against the market return—are constant over time. In some early studies that computed the return on the market portfolio as an arbitrary equal-weighted average of the returns on the individual assets, this second assumption did not appear to be any worse than the first assumption above; it followed from the assumption that not just the means but also the variances and covariances of the individual relative returns were constant over time. However, it is now fairly standard to compute the market return by taking the weights from the actual portfolio shares, which in practice vary over time. Thus even if the distribution of the individual asset returns were believed to be stationary, the betas would still vary to the extent that portfolio shares vary. (This is shown in equation (8) below.) Cheng and Grauer (1980, p. 661) have criticized the existing literature on these grounds.

These two problems with the standard approach are related to each other in that, given the linearity of the Security Market Line (equation (6) below), expected returns must vary over time to the extent that betas do. Studies have been done that relax the assumption of constancy. But expected returns and betas have been allowed to vary only in relatively ad hoc ways: splitting the sample period, estimating the moments from a lagged time sample of actual returns (e.g. estimating the expected return from a distributed lag or ARIMA process), or using lagged values of an arbitrarily chosen set of other variables (as in the technique of rolling regressions). The contribution of the present paper is to offer a technique for testing CAPM in which no restrictions are placed on the extent to which the expected returns (and betas) can vary.
The technique requires using data on the portfolios actually held by investors. Ours are yearly data on holdings by the aggregate U.S. household sector, though the technique could easily be applied to more disaggregated data. Many finance theorists prefer not to think about the actual portfolio behavior of investors, and rather to concentrate on the relationships that show up among returns in market equilibrium. In empirical studies they might try to justify this by a claim that the data on ex post returns are much more accurate than any possible data on portfolio holdings. Aside from the difficulty of estimating ex ante returns from ex post data, the argument falls apart as soon as one recognizes that the weights that go into the calculation of the return on the aggregate market portfolio are in any case the portfolio shares held by investors. For better or worse, we are stuck with using portfolio shares. And there will turn out to be some great advantages from focussing on them explicitly.

A third drawback of the existing CAPM tests is that they generally do not specify an explicit alternative hypothesis. In our test, the alternative hypothesis will be that investors' asset demands are some unspecified linear function of expected returns. What determines the parameters in such a function? The CAPM answer is that they are inversely related to the coefficient of risk-aversion and the variance-covariance matrix of returns. But to test CAPM we will nest it in the more general hypothesis that allows the parameters to be determined arbitrarily. It is worth recalling that the Tobin-Markowitz model was originally developed to shed light on the values of the parameters that appear in investors' asset demand functions.
2. ESTIMATION OF UNCONSTRAINED ASSET-DEMAND FUNCTIONS

We begin by specifying the general alternative hypothesis within which the CAPM hypothesis is nested. Investors choose their portfolio shares as some linear functions of expected one-period returns on the various assets, relative to the expected return on some numeraire asset:

$$x_t = A + B(E_{t+1} - \ell E_{t+1}^d)$$  \hspace{1cm} (1)

where $x_t$ is a vector of portfolio shares allocated to each of the $G-1$ assets (the $G$th asset is eliminated as redundant; in our case $G = 6$ and the redundant asset is Treasury bills and other short-term assets);

$E_{t+1}$ is a vector of the market's expected one-period real returns on each of the $G-1$ assets.

$E_{t+1}^d$ is the market's expected one-period real return on the numeraire asset (in our case, Treasury bills again);

$\ell$ is a vector of ones, of length $G-1$;

$A$ is a vector of $G-1$ constants; and

$B$ is a $(G-1) \times (G-1)$ matrix of coefficients that measures the responsiveness of asset demands to expected returns.

We invert equation (1) to express expected returns as a function of asset shares:

$$E_{t+1} - \ell E_{t+1}^d = -B^{-1}A + B^{-1}x_t.$$  \hspace{1cm} (2)

A common stumbling block is how to model expectations, which are unobservable. Usually the expected return is assumed constant, and estimated by the sample mean. At best it is allowed to change gradually over time. But
the way we have set up equation (2), all that is needed here is to assume that expectations are rational, i.e. that the ex post realized returns are given by

\[
\begin{align*}
    r_{t+1} - \text{tr}_{t+1}^d &= E_{t+1}r_{t+1} - \text{tr}_{t+1}^d + \varepsilon_{t+1},
\end{align*}
\]

where the expectational error \( \varepsilon_{t+1} \) is independent of information \( I_t \) available at time \( t \):

\[
E(\varepsilon_{t+1} | I_t) = 0.
\]

(For our purposes it is necessary only that \( I_t \) include \( x_t \). But it can contain other variables as well.)

From (2) and (3) we have

\[
\begin{align*}
    r_{t+1} - \text{tr}_{t+1}^d &= -B^{-1}A + B^{-1}x_t + \varepsilon_{t+1}.
\end{align*}
\]

Two aspects of equation (4) are noteworthy. All variables are observable. And by the rational expectations assumption, the error term is independent of \( x_t \). This means that we can use equation-by-equation Ordinary Least Squares (OLS) to estimate the constant terms and \( B^{-1} \). An attractive property of the specification of equation (4) is that it allows expected returns to vary from period to period as much as they want. Furthermore we have made no ad hoc assumptions about what determines actual returns or expected returns, other than that expectations are rational. 8

Table 1 reports the results of the OLS estimation. The estimates indicate, for example, that it would take a 31.02 per cent increase in the expected annual return on corporate debt to induce investors to accept an increase in their holdings of corporate debt equal to 1 per cent of their portfolio. This
Table 1: Unconstrained Estimation of Inverted Asset-Demand Function

Equation-by-equation OLS. Sample: 1954-1980

<table>
<thead>
<tr>
<th>Dependent variable</th>
<th>( \beta^{-1} ): coefficients on shares of portfolios allocated to:</th>
<th>D.W.</th>
<th>SSR</th>
<th>( R^2 )</th>
<th>log likelihood</th>
<th>( F(5,21) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tangible assets</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Constant</td>
<td>-.107 (.410)</td>
<td>.256 (.546)</td>
<td>.278 (.564)</td>
<td>-4.701 (2.760)</td>
<td>2.969 (2.849)</td>
<td>.007 (.446)</td>
</tr>
<tr>
<td>Long-term</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>federal debt</td>
<td>1.272 (.188)</td>
<td>-2.274 (1.726)</td>
<td>-2.284 (1.633)</td>
<td>-7.847 (7.995)</td>
<td>22.604* (8.253)</td>
<td>-1.836 (1.293)</td>
</tr>
<tr>
<td>State and local</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>debt</td>
<td>1.185 (1.701)</td>
<td>-2.467 (2.470)</td>
<td>-5.384* (2.337)</td>
<td>-17.194 (11.442)</td>
<td>41.645* (11.811)</td>
<td>-1.865 (1.850)</td>
</tr>
<tr>
<td>Corporate bonds</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Constant</td>
<td>.844 (.932)</td>
<td>-1.685 (1.354)</td>
<td>-3.136* (1.281)</td>
<td>-14.340 (6.272)</td>
<td>31.019* (6.474)</td>
<td>-1.439 (1.014)</td>
</tr>
<tr>
<td>Equities</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Constant</td>
<td>-.026 (2.413)</td>
<td>-.397 (3.504)</td>
<td>5.445 (3.316)</td>
<td>-18.233 (16.234)</td>
<td>4.301 (16.757)</td>
<td>.619 (2.625)</td>
</tr>
</tbody>
</table>

\[
\beta^{-1} - \frac{(G-1)}{2} \log 2\pi - \frac{T}{2} \log |\Omega| - \frac{T(G-1)}{2} = \log \text{likelihood}
\]

unconstrained

-124.06

448.75

-67.50

257.19*

constrained to 0

-124.06

410.35

-67.50

218.79

*Significant at the 95% level. (Standard errors in parentheses.)
assumes that the increase comes at the expense of the omitted asset, short-term bills and deposits. To calculate the effect of a 1 per cent increase in corporate debt at the expense of another asset, take the difference of the two relevant coefficients.

Only a few of the coefficients appear significantly different from zero by t-tests. But all but one of the individual equations do appear significant by F-tests. To do an overall test of the system of equations we must compare the log likelihood when the coefficients are constrained to zero, to the likelihood unconstrained. The numbers are 218.79 and 257.19, respectively. Since twice the difference is distributed $\chi^2$, the reduction in the likelihood that would result from the constraint that the coefficients are all zero is highly significant.

3. ESTIMATION OF CONSTRAINED ASSET-DEMAND FUNCTIONS

We now consider the restrictions imposed on the asset-demand function (1) by CAPM. Since the econometrics are necessarily discrete-time, we adopt a discrete-time theoretical framework. Consider four assumptions:

(A1) perfect capital markets

(A2) optimization of end-of-period expected utility

(A3) log-normal distribution of returns

(A4) constant relative risk-aversion.

As we show in Appendix 1, these assumptions imply a restriction on the asset-demand function (1) that is astonishingly simple:

$$ B = [\rho_\Omega]^{-1} $$

(5)
where $\rho$ is the constant of relative risk-aversion and $\Omega$ is the $G \times G - 1$ variance-covariance matrix of returns. Intuitively, investors will respond less to a given disparity in expected returns if the perceived uncertainty ($\Omega$) is high, or if their risk-aversion ($\rho$) is high.

In the CAPM literature this result is usually expressed in the form of the security market line:

$$E_{r_i} - r^o = \beta^i_m (E_{r^m} - r^o)$$

(6)

where $E_{r_i}$ is the expected return on asset $i$, $E_{r^m}$ is the expected return on the aggregate market portfolio, $r^o$ is the risk-free rate of return if there is one, and $\beta^i_m$ is the regression coefficient of $r^i$ against $r^m$, which captures the degree of non-diversifiable risk associated with asset $i$.

In appendix 2 we show that our formulation (5) can be easily derived from the familiar equation (6), once it is recognized that the return on the market portfolio is a weighted average of the returns on the individual securities, where the weights are the portfolio shares $x$:

$$r^m_{t+1} = x^t' r^m_{t+1} + (1 - x^t') r^d_{t+1}$$

(7)

The familiar beta turns out to be given by

$$\beta^i_m = \frac{x^t' \text{Cov}(r^i, r - r^d) + \text{Cov}(r^i, r^d)}{\text{Var}(r^m)}$$

(8)

which in turn implies equation (5).

The traditional strategy for testing CAPM is (1) to assume that $E_{r_i}$ and $E_{r^m}$ are constant and thus to estimate them by the ex post sample means, and (2) to assume that the $\beta^i_m$'s are also constant and thus to estimate them by
time series regressions. Then (3) in a "second-pass" regression, a cross-
section of estimated $\beta_i$'s are regressed against estimated $\beta_m$'s to see
if there is a statistically significant relationship. The more recent literature
has evolved in response to some of the by now well-known problems with the
traditional strategy.\(^{13}\) However, the assumption has been maintained that the
$\beta_m$'s are constant over time, even in studies where the actual numbers used
for shares in the market portfolio, $x_t$, in equation (8), vary over time. Thus
the assumption has been maintained that the expected relative returns in
equation (6) are also constant over time.

The solution recognized here, and the key insight of this paper, is that
$\Omega$ is precisely the variance-covariance matrix $\varepsilon \varepsilon'$ of the error term in
equation (4), in which portfolio shares and expected returns are free to vary,
and that the equation should be estimated subject to that constraint:

$$r_{t+1} - r^d_{t+1} = -\rho\Delta + \rho^\Omega x_t + \varepsilon_{t+1} \quad \Omega \equiv \varepsilon \varepsilon'$$  \hspace{1cm} (9)

The imposition of a constraint between the coefficient matrix and the error
variance-covariance matrix is unusual in econometrics, and requires maximum
likelihood estimation (MLE). Once we have done this estimation, we have our
test of CAPM: we compare the log likelihood at the constrained maximum, to
the log likelihood of the unconstrained version that we have already done in
Table 1. Appendix 3 shows the constrained likelihood function and its deriva-
tives, and describes the program used to maximize it.

If the aim were to assume CAPM a priori and to use the information to
get the most efficient possible estimates of the parameters, then one might
wish to impose not only the constraint that the coefficient matrix is propor-
tional to the variance-covariance matrix $\Omega$, but to impose as well an a priori value for the constant of proportionality, which is the coefficient of relative risk-aversion $\rho$. Friend and Blume (1975) offer evidence that $\rho$ may be in the neighborhood of 2.0. We report in Table 2 the parameter estimates for the case $\rho = 2.0$. The results look quite different from those in Table 1. If one believes the constraints, then the difference is simply the result of more efficient estimates. One has to invert the matrix in order to recover the original $B$ matrix and see which assets are close substitutes for which other assets. These coefficients are reported in Table 3. We can infer from the negative numbers that each of the long-term bonds, for example, is a substitute for the other two. 14

But we have chosen in this paper to emphasize the use of our technique to test the CAPM hypothesis, rather than the use of the technique to impose the hypothesis. The log likelihood for the estimates in Table 2 is 226.40, a decrease from the unconstrained log likelihood 257.19 in Table 1. In other words, the fit has worsened. Twice the difference is above the 5 per cent critical level. This constitutes a clear rejection of the CAPM hypothesis.

Perhaps the constraint that $\rho$ is 2.0 is too restrictive and accounts for the magnitude of the decline in the likelihood function. We relaxed the constraint on $\rho$ and allowed the MLE program to estimate it along with the other parameters. The likelihood function was maximized at a value of $\rho$ equal to 125.7. (The variance-covariance matrix did not look very different from the one that went into Tables 2 and 3 and so is not reported.) The log likelihood at this point is 230.76, an insignificant improvement over $\rho = 2.0$. We still reject the CAPM hypothesis. 15
Table 2: Constrained Estimation of \( \rho \Omega \), Inverted Asset-Demand Function

MLE. Sample: 1954-1980

\( \rho \) constrained to 2.0

<table>
<thead>
<tr>
<th>Dependent variable</th>
<th>Constant</th>
<th>( \beta^{-1} ): coefficients (constrained to ( \rho \Omega )) on shares of portfolios allocated to:</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Tangible assets</td>
</tr>
<tr>
<td>Real rate of return on asset relative to short-term bills:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Tangible assets</td>
<td>(-.010)</td>
<td>.00050</td>
</tr>
<tr>
<td></td>
<td>(.012)</td>
<td>(.00044)</td>
</tr>
<tr>
<td>Long-term federal debt</td>
<td>(-.030)</td>
<td>-.00034</td>
</tr>
<tr>
<td></td>
<td>(.039)</td>
<td>(.00085)</td>
</tr>
<tr>
<td>State and local debt</td>
<td>(-.030)</td>
<td>-.00014</td>
</tr>
<tr>
<td></td>
<td>(.056)</td>
<td>(.00130)</td>
</tr>
<tr>
<td>Corporate bonds</td>
<td>(-.027)</td>
<td>.00004</td>
</tr>
<tr>
<td></td>
<td>(.032)</td>
<td>(.00077)</td>
</tr>
<tr>
<td>Equities</td>
<td>(.046)</td>
<td>-.00035</td>
</tr>
<tr>
<td></td>
<td>(.041)</td>
<td>(.00189)</td>
</tr>
</tbody>
</table>

\[-(C-1) \frac{T}{2} \log 2\pi \quad - \frac{T}{2} \log |\Omega| - \frac{1}{2} \sum_{t} \hat{\epsilon}_{t} \hat{\epsilon}_{t}^{-1} \hat{\epsilon}_{t} = \log \text{likelihood}\]

\[\begin{array}{lll}
-124.06 & 350.46 & 226.40
\end{array}\]
Table 3: Constrained Estimate of $(\rho\Omega)^{-1}$, Pre-inverted Asset Demand Function

\( B^{-1} \) in Table 2 inverted. \( \rho \) constrained to 2.0

<table>
<thead>
<tr>
<th>The demand for the assets listed below</th>
<th>depends on the expected real return (relative to the real return on bills) of the following assets</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Tangible assets</td>
</tr>
<tr>
<td>Tangible Assets</td>
<td>1422.1</td>
</tr>
<tr>
<td>Long-term federal debt</td>
<td>449.9</td>
</tr>
<tr>
<td>State and local debt</td>
<td>71.7</td>
</tr>
<tr>
<td>Corporate bonds</td>
<td>-534.9</td>
</tr>
<tr>
<td>Equities</td>
<td>69.7</td>
</tr>
<tr>
<td>Short-term bills and deposits (= sum of other rows)</td>
<td>-1478.5</td>
</tr>
</tbody>
</table>
4. CONCLUSION

How could CAPM fail to hold? Do our results imply that investors are irrational? The failure of any one of the four CAPM assumptions listed above could explain the finding. Investors may be rational but may have to optimize subject to constraints such as imperfect capital markets. Or they may be maximizing an intertemporal utility function, à la Merton (1973) and Breeden (1979), that is more complicated than a function of the mean and variance of end-of-period real wealth. Or returns may not be log-normally distributed. Or investors may not have a constant coefficient of relative risk-aversion.

Our rejection of the null hypothesis could also be due to the failure of other assumptions that we have made in our model, but that are not part of CAPM most narrowly defined: homogeneous investors, a constant variance-covariance matrix of individual returns, rational expectations, the aggregation of the assets into six, and the accurate measurement of the holdings of those assets. The test could be refined with respect to most of these assumptions, especially by greater disaggregation of the assets or the holders.

The Capital Asset Pricing Model is a very attractive way to bring structure to asset-demand functions. One possibility is that true asset demands are equal to those given by the CAPM formula plus some other factors. The other factors would not necessarily have to be large for our technique to reject the null hypothesis. This is entirely appropriate. We have tested the hypothesis that CAPM holds exactly. But it does allow the possibility that CAPM may still have something to tell us about asset demands despite our statistical rejection of it.
Appendix 1

In this appendix we derive in discrete time the correct form for the asset-demands of an investor who maximizes a function of the mean and variance of his end-of-period real wealth.

Let $W_t$ be real wealth. The investor must choose the vector of portfolio shares $x_t$ that he wishes to allocate to the various assets. End-of-period real wealth will be given by:

$$W_{t+1} = W_t x_t' r_{t+1} + W_t (1-x_t') r_{t+1}^d$$

$$= W_t [x_t' z_{t+1} + 1 + r_{t+1}^d],$$

where we have defined the vector of returns on the $G - 1$ assets relative to the numeraire asset (deposits): $z_{t+1} = r_{t+1} - r_{t+1}^d$.

The expected value and variance of end-of-period wealth (5), conditional on current information, are as follows:

$$E_{t+1} W_{t+1} = W_t [x_t' E z_{t+1} + 1 + E r_{t+1}^d]$$

$$V_{t+1} W_{t+1} = W_t^2 [x_t' \Sigma x_t + V r_{t+1}^d + 2 x_t' \text{Cov}(z_{t+1}, r_{t+1}^d)],$$

where we have defined the variance-covariance matrix of relative returns:

$$\Sigma = E(z_{t+1} - E z_{t+1})(z_{t+1} - E z_{t+1})'.$$

The hypothesis is that investors maximize a function of the expected value and variance:

$$F[E(W_{t+1}), V(W_{t+1})].$$

We differentiate with respect to $x_t$:
\[
\frac{dF}{dx_t} = F_1 \frac{dEW}{dx_t} + F_2 \frac{dVW}{dx_t} = 0.
\]
\[
F_1 W_t \{EZ_{t+1}\} + F_2 W_t^2 \{2\omega x_t + 2 \text{Cov}(z_{t+1}, r_{t+1}^d)\} = 0.
\]

We define the coefficient of relative risk-aversion \( \rho \equiv -W_t F_2 / F_1 \), which is assumed constant. Then we have our result:

\[
EZ_{t+1} = \rho \text{Cov}(z_{t+1}, r_{t+1}^d) + \rho \omega x_t. \tag{A2}
\]

This is just equation (2) with \( B^{-1} \) constrained to be \( \rho \Omega \), as claimed by equation (5) in the text. Combining with the rational expectations assumption (3) is another way to get equation (9), the equation estimated in the text. (There is also a constraint imposed on the intercept term \( A \). But it is not convenient to impose this constraint in the econometrics. Nor do we need it, since the constraint on the coefficient matrix already gives us 25 overidentifying restrictions.)

For economic intuition, we can invert (A2) to solve for the portfolio shares, the form analogous to (1):

\[
x_t = -\Omega^{-1} \text{Cov}(z_{t+1}, r_{t+1}^d) + (\rho \Omega)^{-1} EZ_{t+1} \tag{A3}.
\]

The asset demands consist of two parts. The first term represents the "minimum-variance" portfolio, which the investor will hold if he is extremely risk-averse \( (\rho = \infty) \). For example, suppose he views deposits as a safe asset, which requires that the inflation rate is nonstochastic. Then his minimum-variance portfolio is entirely in deposits: the \( G - 1 \) entries in \( x_t \) are all zero because the \( \text{Cov} \) in (A3) is zero. The second term represents the "speculative" portfolio. A higher expected return on a given asset induces investors to hold more of that asset than is in the minimum-variance portfolio, to an extent limited only by the degree of risk-aversion and the uncertainty of the return.
There is one difference between equation (A2), which we have derived, and equation (2), which is estimated in the text: the former defines "percentage relative rates of return" \( \tilde{z} \) in level form whereas the latter defines them in logarithmic form. An earlier version of this paper (NBER Working Paper No. 1113) used rates of return in level form in the unconstrained estimation, with very little difference in the results. But log-normal returns are theoretically preferable to normal returns in the sense that they rule out the possibility of negative wealth. If returns follow a proportionate Ito process in continuous time, then they follow a log-normal process when observed at discrete intervals.

It has been shown that when the returns follow a proportionate Ito process in continuous time, the optimizing solution to the investor's portfolio problem gives a relationship very similar to equation (A3), i.e. that the asset shares are subject to a proportionality relationship such as equation (5). (The literature goes back to Samuelson (1970) and Merton (1969). A recent example in our notation is Friedman and Roley (1979), equation (9'). As mentioned in footnote 12, one needs the additional restrictions of logarithmic utility or temporally uncorrelated expected returns.) To go from continuous time proportionate returns to continuous time logarithmic returns it is necessary to use Ito's Lemma, which introduces a constant "convexity term" into the asset share expression. To go further from continuous time logarithmic returns to \textit{discrete time} logarithmic returns it is necessary to assume that the mean and variance of the Ito process, and therefore the asset demands, are constant at least over the unit time interval. In that case integrating is simply a matter of multiplying the mean and variance by the length of the unit
time interval (one year). The constant term in the asset demand function, $A$ in equation (1), will turn out to be something different from that derived in the discrete-time case in equation (A3) due to the convexity term. But since we do not impose any constraints on $A$ in any case, it will make no difference for our empirical test.
Appendix 2

In this appendix we rederive equation (A3), not from first principles but from the familiar "beta" formulation of CAPM. The standard equation for the expected return $E r^i$ on asset $i$ is:

$$E r^i - r^o = \beta^i_m (E r^m - r^o) \quad (A4)$$

where $\beta^i_m$ is $\text{cov}(r^i, r^m) / \text{var}(r^m)$, the regression coefficient of $r^i$ against the return on the market portfolio, $r^m$. As we saw in equation (7), the return on the market portfolio is a weighted average of the returns on the individual securities:

$$r^m_{t+1} = x_t' (r^d_{t+1} - r^d_{t+1}) + r^d_{t+1} \quad (A5)$$

We take the covariance of $r^i$ with (A5) to get our expression for the beta:

$$\beta^i_m = \frac{x_t' \text{Cov}(r^i, r^d_{t+1}) + \text{Cov}(r^i, r^d)}{\text{Var}(r^m)} \quad (A6)$$

This is equation (8) in the text; it shows how the portfolio shares enter into the beta.\textsuperscript{16}

An equation similar to (A4) holds for the rate of return on the numeraire asset. Taking the difference of the two equations gives us the equation for the expected relative return:
\[
\begin{align*}
Er^i_{t+1} - Er^d_{t+1} &= \beta^i_m (Er^m_{t+1} - r^o_t) - \beta^d_m (Er^m_{t+1} - r^o_t) \\
&= \frac{[x^i_t \text{ Cov}(r^i_i, r^m_{t+1}) + \text{ Cov}(r^i_i, r^d_{t+1})} - [x^d_t \text{ Cov}(r^d_i, r^m_{t+1}) + \text{ Cov}(r^d_i, r^d_{t+1})]}{\text{ Var}(r^m)} (Er^m_{t+1} - r^o_t) \\
&= \frac{x^i_t \text{ Cov}(r^i_i, r^m_{t+1}) + x^d_t \text{ Cov}(r^d_i, r^m_{t+1}) \frac{E(r^m_{t+1} - r^o_t)}{\text{ Var}(r^m_t)}}{\text{ Var}(r^m_t)} (A7)
\end{align*}
\]

We identify the coefficient of risk aversion as the overall market tradeoff between expected return and risk,

\[\rho = \frac{E_t(r^m_t - r^o_t)}{\text{ Var}(r^m_t)}. \quad (A8)\]

Substituting into (A7) gives us precisely equation (A2) in vector form, the expression for the inverted asset demand that we derived in Appendix 2.
APPENDIX 3

Using the assumption of normally-distributed (log) returns, the log likelihood function when no constraint is imposed on the coefficient matrix is

\[ L = - \frac{(G-1)T}{2} \log 2\pi - \frac{T}{2} \log |\Omega| - \frac{1}{2} \sum_{t=1}^{T} \varepsilon'_{t+1} \Omega^{-1} \varepsilon_{t+1}, \]

(6)

where we know from equation (4) that \( \varepsilon_{t+1} = (\tau_{t+1} - \tau_{d_{t+1}} - c - B^{-1} x_t). \)

The unconstrained MLE is simply the OLS estimates that we looked at in Table 1.

For the constrained MLE, we substitute \( \rho \Omega \) for \( B^{-1} \). \( \Omega \) now appears in the likelihood function in two ways. To maximize, we differentiate. The derivatives with respect to the coefficient of risk-aversion and the intercept term are easy:

\[ \partial L/\partial \rho = - \sum \varepsilon'_{t+1} \Omega^{-1} (\partial \varepsilon_{t+1}/\partial \rho) \]

\[ = - \sum \varepsilon'_{t+1} \Omega^{-1} (-\Omega x_t) \]

\[ = \sum \varepsilon'_{t+1} x_t \]

\[ \partial L/\partial c = - \sum \varepsilon'_{t+1} \Omega^{-1} (\partial \varepsilon_{t+1}/\partial c) \]

\[ = \sum \varepsilon'_{t+1} \Omega^{-1}. \]

The derivative with respect to the elements of the variance-covariance matrix is trickier. We use the two facts (from Theil (1971, pp. 31-32), equations (6-14) and (6-8), respectively):
\[
\frac{\partial \log |\Omega|}{\partial \Omega} = \Omega^{-1} \quad \text{and} \quad \frac{\partial (\varepsilon' \Omega^{-1} \varepsilon)}{\partial \Omega^{-1}} = \varepsilon \varepsilon'.
\]

\[\frac{\partial L}{\partial \Omega} = -\frac{T}{2} \Omega^{-1} - \frac{1}{2} \sum \left[ -\Omega^{-1} \varepsilon_{t+1} \varepsilon_{t+1}' \Omega^{-1} + (\Omega^{-1} + \Omega^{-1}) \varepsilon_{t+1} (\partial \varepsilon_{t+1} / \partial \Omega) \right]\]

where \( \partial \varepsilon_{t+1} / \partial \Omega = x_t' \). This much would be true for the derivatives with respect to any matrix "\( \Omega \)." The fact that \( \Omega \) is symmetric, and so contains only 15 independent parameters, means that all off-diagonal elements must be doubled. See the appendix to Frankel and Engel (1982). An equivalent approach is to work in terms of the 15 parameters in the Choleski factorization of \( \Omega \): the lower triangular \( S \), such that \( S'S = \Omega \). (This approach can be worthwhile if the MLE program is reluctant to converge when asked to work in terms of \( \Omega \). Note that \( \partial \Omega / \partial S = 2S \).

Setting the derivatives equal to zero gives first order conditions that characterize the MLE. However, due to nonlinearity they cannot be solved explicitly for the estimates of \( \rho \), \( c \), and \( \Omega \). The Berndt, Hall, Hall and Hausman (1974) algorithm uses the first derivatives to find the maximum of the likelihood function in non-linear models. For our problem, we modified a program written by Paul Ruud, based on this algorithm. As initial values for \( \hat{\Omega} \) in the iteration, we used the simple variance-covariance matrix of the relative rates of return, which had already been computed for the last row of Table 1.
APPENDIX 4
DATA

The main source for data on supplies of nine assets held by house-
holds was the Federal Reserve Board's Balance Sheets for the U.S. Economy
(October 1981) Table 702. This source was used in place of the Fed's Flow of
Funds Accounts, Assets and Liabilities Outstanding, to which it is closely
related, because only the Balance Sheets include data for tangible assets,
i.e. real estate and consumer durables (see page iii of the Flow of Funds
for an explanation). The variables used in the econometrics are shares of
wealth, the supply of the asset in question divided by the sum of all nine
asset supplies.

The asset supplies were taken from the Balance Sheets as follows. Real
estate is line 1 (total tangible assets) minus line 7 (consumer durables).18
Consumer durables is line 7.19 Open market paper is line 25. Short-term U.S.
government securities are line 20 [not available before 1951]. Deposits is
the sum of lines 13, checkable deposits and currency, 14, small time and
savings deposits, 15, money market fund shares, and 16, large time deposits.
Long-term federal debt is line 18 (U.S. government securities) minus line 20.
State and local debt is line 23. Private bonds are line 24 (corporate and
foreign bonds) plus line 26 (mortgages held).20 Finally, equities are
line 27 (corporate equities) plus line 32 (noncorporate business equity).21

For three of the asset supplies—long-term federal debt, state and local
bonds, and private bonds—the numbers represent book value and must be multi-
plied by some measure of current market prices to get the correct measure of
market value. The very large decline in prices of bonds over the postwar
period make this correction a crucial one. (Equities and tangible assets
are already measured at market value, while capital gains and losses are
irrelevant for the three short-term assets.) Measures of the current market bond prices are reported by *Standard and Poor's Trade and Security Statistics Security Price Index Record* (1982): page 235 for U.S. government bond prices, 233 for municipal bond prices, and 231 for high grade corporate bond prices. Standard and Poor's computes the price indexes from yield data, assuming a 3% coupon with 15 years to maturity for the federal bonds and a 4% coupon with 20 years to maturity for the other two. 22

Among the rates of return (all in log form for this paper), the two most problematical are those on real estate and durables, taken here as the percentage change in price indices reported in the *Economic Report of the President* 1982: the home purchase component of the CPI (p. 292) and the durable goods personal consumption expenditure component of the GNP deflator (p. 236). There exist better measures of house prices, and unpublished estimates of imputed service returns on housing and durables, but they are not available for the entire sample period. When the two tangibles are aggregated, we use real estate appreciation as the return.

and ASD 1980, table 25A. Alternatives such as the return on money market funds might be theoretically preferable but are not available for the early part of the sample period. Note that in aggregating non-interest paying money together with interest-paying accounts, we are assuming that the former performs an implicit liquidity service that brings its return up to the explicit return of the latter. When the three short-term assets are aggregated, we use the Treasury bill rate as the return.

Each of the long-term assets entails a yield plus capital gains. For each of the three kinds of bonds, capital gains are percentage change in the same bond prices from Standard and Poor's Trade and Securities Statistics that were discussed above. The yields are from the same source: respectively, the median yield to maturity of a number of government bonds restricted to those issues with more than ten years to maturity, p. 234, an arithmetic average of the yield to maturity of fifteen high grade municipal bonds, p. 232, and an average of the AAA Industrial and Utility bonds, p. 219. (The yields are also available from the Fed sources: BMS 1941-1970, table 12.12, ASD 1970-1979, table 22A and ASD 1980, table 25A.) For equities, capital gains are percentage change in Stanford and Poor's index of common stock prices from BMS 1941-1970, table 12.16, ASD 1970-1979, table 22A, and ASD 1980, table 26A. To capital gains we add the dividend price ratio on common stock, from BMS 1941-1970, table 12.19, ASD 1970-79, table 22A, and ASD 1980, table 25A.

The foregoing are all nominal returns. To convert to real returns when computing percentage returns on levels, we use the percentage change in the CPI, from the Economic Report of the President 1982. To be precise we divide one plus the nominal return by one plus the inflation rate. Subtracting the
inflation rate from the nominal return would give approximately the same answer, and when we computed real returns relative to the numeraire asset the two inflation rates would conveniently drop out, but this answer would differ from the correct one by a convexity term. When computing the percentage relative returns on logs, the inflation rates drop out in any case.

Absent from the calculations is any allowance for differences in tax treatment. In particular, the returns on state and local bonds, and to some extent on tangibles, are here understated relative to the other assets because they are tax-free. The unconstrained constant term that we allow for in the econometrics should capture most of this effect (and any other constant omitted factors such as the service return from tangibles, as well). But it would be desirable to compute after-tax real returns instead.
FOOTNOTES

1. Two common references are Black, Jensen and Scholes (1972) and Blume and Friend (1973):

2. See Roll (1977) and Ross (1978).

3. Previous studies with similar portfolios include Stambaugh (1982) and Norhaus and Durlauf (1982). Of course it would be desirable to disaggregate the assets further, and to include any missing assets such as human wealth, if the data were available.

4. The first objection above was that the assumption of constant expected returns is inconsistent with the seemingly undeniable empirical fact that real interest rates, for example, have been varying. One could conceivably argue that expected returns on all assets vary over time, but only in tandem with each other and with the aggregate market-portfolio, not relative to each other. But given the linearity of the security market line, equation (6) in the text, such an argument would require that the betas in the equation are constant over time. It is thus inconsistent with the proposition that the betas vary over time, a proposition which follows either from (1) variation in portfolio shares and thus in the numerator of the betas (equation (8)), which is the second objection above, or (2) variation in the variance of the market return, and thus in the denominator of the betas, which Merton (1980, p. 330) considers to have been empirically established "at almost any confidence level."

5. A few studies, such as Friend and Blume (1975), look at portfolios actually held by households. But they do not allow portfolios to change over time and they do not test whether the portfolios held are of the optimal type postulated by CAPM.
6. Tobin (1958). Tobin (1983) confirms that his original contribution "was intended primarily as a contribution to positive macroeconomics rather than to management science" (p. 236).

7. The choice to express returns relative to a numeraire is not restrictive. We could generalize (1) slightly to

\[ x_t = A + \beta E \begin{bmatrix} r_{t+1} \\ \vdots \\ d \\ r_{t+1} \end{bmatrix} \]

where \( B \) is \( G - 1 \) by \( G \). Then when we invert

\[ E \begin{bmatrix} r_{t+1} \\ \vdots \\ d \\ r_{t+1} \end{bmatrix} = -\beta^{-1}A + \beta^{-1}A, \]

we need only subtract the last row from each of the others to get an equation of the precise form as (2). In what follows we only use (2) anyway.

8. The validity of the technique depends on the assumption that the asset-demand function (1) holds exactly. If asset demands are determined by CAPM plus other factors, or if there are measurement errors in the data, the null hypothesis does not hold. The data are described in Appendix 4.

9. The test that the coefficients in a row are significantly different from zero is a test that the asset in question is not a perfect substitute for Treasury bills and other short-term assets. The 5 per cent critical level for the F statistic is 2.68.
10. The estimated log likelihood is given by equation (6) in Appendix 2, with the estimates \( \hat{\theta}_t \) and \( \hat{\Omega} \) substituted in for the true parameters. The last of the three terms is simply \(-\frac{(G-1)T}{2}\), because \( \sum \hat{\varepsilon}_t' \hat{\Omega}^{-1} \hat{\varepsilon}_t = \sum \text{tr} \hat{\varepsilon}_t' \hat{\Omega}^{-1} \hat{\varepsilon}_t = \text{tr} \sum \hat{\varepsilon}_t' \hat{\omega}_t \hat{\varepsilon}_t^{-1} = \text{tr} \hat{\Omega}^{-1} \hat{\omega}_t = T(G-1) \). (See G. S. Maddala, *Econometrics* (N.Y.: McGraw-Hill) 1977, p. 487 after equation C-50.) So the test statistic varies only with the determinant of \( \hat{\Omega} \). Under the zero-coefficient constraint, \( \hat{\Omega} \) is simply the variance-covariance matrix of the raw data, the relative rates of return. (We do allow for a non-zero constant term.) Unconstrained, \( \hat{\Omega} \) is the variance-covariance matrix of the residuals of the \( G-1 \) equations. Because the residuals are correlated across equations, \( T \log |\hat{\Omega}| \) is somewhat less than the sum of the logs of the \( G-1 \) individual equations' sums of squared residuals, and the log likelihood is correspondingly greater than the sum of the \( G-1 \) individual log likelihoods. (The 5% significance level for the \( \chi^2 \) test is 37.65).

11. The utility function will have a constant coefficient of relative risk-aversion if it is a power function:

\[
U(W) = \frac{1}{1-\rho} W^{1-\rho}.
\]

We could replace the last two assumptions with the single assumption of quadratic utility. But that assumption is unrealistic, and we will need to assume a normal distribution anyway in order to do our maximum likelihood estimation.

The solution to the one-period maximization problem considered here (Assumption 2) will give the same answer as the general intertemporal maximization problem if the utility function is further restricted to
the logarithmic form, the limiting case as $\rho$ goes to 1.0, or if expected returns in future periods are independent of the realization of this period's return. See Merton (1973, pp. 877-78), or Rubinstein (1976) and Fama (1970).

12. The derivation is relegated to the Appendix, not because of any degree of complexity, but rather because of its familiarity. Some similar formulations are Friend and Blume (1975, equation 5), Black (1976, equation 4), and Friedman and Roley (1979, equation 20').

13. One improvement has been the inclusion of more assets, as in the studies cited in footnote 3. Another improvement over the early "two-pass" tests has been the use of a likelihood ratio test statistic, as in Gibbons (1982) and Shanken (1983). The technique developed here shares both of these advantages.

14. The numbers in Table 2 might appear implausibly low and the numbers in Table 3 appear correspondingly high. For example an increase in the expected return on equities relative to other assets, from zero to a mere one per cent, would induce a shift of 48.2 per cent of the portfolio into equities. The own-return derivatives are even larger for the other assets. Note that this is an unavoidable property of the CAPM theory, not an artifact of the estimation technique. The variance-covariance matrix of the relative returns around their sample means is of a magnitude not much larger than our $\hat{\Sigma}$, so any estimate of $[\rho \Sigma]^{-1}$ is going to be of a magnitude not much smaller than the numbers in Table 3, unless people are far more risk-averse than is usually thought.
15. The flatness of the likelihood function with respect to $\rho$ means that, if we wanted to impose CAPM in order to estimate the degree of risk-aversion, we would not have much success. The asymptotic standard error of $\rho$ at the maximum is 156.9. An important check on the MLE program is to try it with $\rho$ constrained to zero. The log likelihood (and variance-covariance matrix) should be the same as those computed by a standard package, in our case TSP, for the bottom line of Table 1. (They are.)

16. Notice that the portfolio shares, $x_t$, break up the covariance with the market return, $\bar{\beta}_m^i$, into a weighted average of the covariances with the individual asset returns. For this reason the $x$'s can be thought of as the "Beta Breakers," so familiar to residents of the San Francisco Bay Area.

17. This is the same program used in Frankel and Engel (1982). An analytic solution was derived in Frankel (1982) for a problem that was the same but for the absence of an intercept term to be estimated.

18. An alternative here is to subtract lines 38 and 39, mortgages owed by households, viewing them as a liability that is institutionally tied to the real estate asset. One cannot explain otherwise households' decision to hold on net a negative quantity of mortgages on risk-return considerations, as the mortgage rate is higher than that on other bonds.

19. An alternative here is to subtract lines 40 and 41, consumer credit, viewing it as a liability that is tied to the durables asset, for the same reason as in the previous footnote.

20. An alternative here is to add in also lines 30 (life insurance reserves), 31 (pension fund reserves) and 34 (miscellaneous assets). These cannot be treated as separate assets because their rates of return are not
available, but it is desirable to have all forms of wealth included somewhere, and they fit into the category of private bonds better than anywhere else.

21. An alternative here is to subtract the difference of lines 44 and 33, representing net security credit, viewing it as a liability that is tied to the equity asset.

22. These same bond prices were reported in the Federal Reserve Board's Banking and Monetary Statistics 1941-1970. They have been discontinued apparently because the Capital Markets Section at the Federal Reserve Board feels that dispersion in the coupon rate and shifts in the term structure make the aggregation of all long-term bonds no longer possible. But some correction for the market price is clearly preferable to none.
REFERENCES


