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PORTFOLIO CHOICE IN
RESEARCH AND DEVELOPMENT

SUDIPTO BHATTACHARYA
AND
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PORTFOLIO CHOICE IN RESEARCH AND DEVELOPMENT

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Abstract

Possible distortions in the choice of a mix of research and development projects in the market, where competition is subject to “winner-take-all” payoff structure, are analyzed. Attention is devoted to choices of weights on alternative “knowledge bases” that affect the riskiness and correlation of research and development strategies, under varying assumptions about private or public access to these knowledge bases, and risk attitudes of competitors.
1. Introduction

In recent years, there has been much research effort devoted to understanding the performance of competitive markets in allocating resources for research and development (R & D) activity. Following the early work of Kamien and Schwartz [1972], Dasgupta and Stiglitz [1980a, b, and Loury [1979] on static (Nash) equilibrium choices, the literature has progressed to important issues in dynamic allocation of resources, e.g., Reinganaum [1981] and particularly Fudenberg, Gilbert, Stiglitz and Tirole [1983]. In most of this literature, however, the focus has been on the magnitude of (social and private) allocation of resources to rigidly specified "R & D technologies", in which the research payoffs accruing to different units are assumed to be either stochastically independent or deterministic (random) functions of resources allocated. Questions regarding the "choice of technique" for doing R & D, e.g., the allocation of research within a research unit to the pursuit of refining different prior knowledge bases, have not yet been seriously addressed (except partially in the recent work of Judd [1984]). In this paper, we provide an exploratory static equilibrium analysis of such portfolio choices within research units, which determine the riskiness of and correlation among the payoffs accruing to competing entities endogenously, and compare these to the choices that would be socially optimal.

Our work is motivated by anecdotal, personal, and codified observations in the history of science regarding the common occurrence of multiple discoveries or inventions of very similar ideas and applications, often clustered within very short spans of time. Such occurrences have been attributed to the existence of commonalities in the "knowledge bases" (e.g., the range of ideas and projects that are likely to succeed in solving a problem) available to different researchers, particularly in the pure sciences. Even in the context of commercial applications of applied technology, it has been noted by Nelson [1982] that a considerable portion of the implications for future development embedded in the current best-practice technologies - whose detailed properties may indeed be proprietary and protected by patents - is often in the public domain. Nelson attributes the existence of such common knowledge pools to (a) basic research done in universities, (b) movement of and professional discussions among scientific personnel of different research units, and (c) firm's
incentives to build "research reputations" in order to attract personnel or capital.\textsuperscript{1} The recent work of Spence [1984] on "spillovers" across units in R & D could also be envisioned as arising from related phenomena. However, Spence's focus is on the implications of such spillovers for the privately optimal magnitude of R & D activity having deterministic payoffs, whereas we analyze the implications of ex ante commonality in knowledge bases on the kind of research activity - with stochastic payoffs having endogenized joint distributions across units - that is undertaken.

There are several potential reasons for divergences between private and socially optimal choices of R & D activity. Private producer's surplus obtainable from a (patentable) invention may be lower than the full social surplus generated by it (Arrow [1962]), thus leading to underinvestment in R & D. Spillover in R & D benefits, as modeled by Spence [1984], also reduce private benefits relative to the social benefit. However, this is ameliorated by the fact that private competing units care differentially about winning the research "race" - whereas society is only interested in some unit succeeding - and thus there is (given benefits to R & D) either excessive investment relative to what is socially optimal with a given number of units, or an excessive number of units in a free-entry environment, e.g., Loury [1979] and Dasgupta and Stiglitz [1980b]. In recent dynamic models, e.g., Fudenberg-Gilbert- Stiglitz-Tiroli [1983], "preemptive" behavior by an entrenched unit has been shown to lead (possibly) to excessive investment. In a recent extension of such analyses, Judd [1984] has shown that there may be a tendency for competing firms to overinvest in a Poisson "leapfrogging" technology, relative to one of stochastic but continuous progress. In a stochastic environment, differences in risk-attitudes between society and R & D units, arising from differential access to diversification in capital markets, can also result in divergences between privately and socially optimal choices.

In this paper we choose to analyze a model that eschews all the above-mentioned complications, except for the differential private incentive to win the research race. Specifically, we postulate a static portfolio choice environment in which a given number of R & D units can allocate a given level of resources to alternative research avenues. The social payoff, given the realized private research outcomes, equals the maximum of the private

\textsuperscript{1} The latter motive has been modeled by Bhattacharya and Ritter [1983].
payoffs. The winner of the R & D "race," the unit generating the highest realizable social payoff, obtains precisely the social payoff, and losers obtain none - this is the strictest interpretation of patent protection (and pricing ability). Society and R & D units have the same risk attitudes (utility functions), and no unit has a lead or other dynamic issues to consider. We are able to show that there may nevertheless be important, and characterizable, qualitative divergences between privately and socially optimal choices. Moreover, the nature and existence of these divergences, particularly in ex ante symmetric environments (across units), are shown to depend in important ways on (i) the private or public nature of different knowledge bases, (ii) the distributional characteristics of the stochastic payoffs from these bases and (iii) on the (common) risk attitudes. The first point sets this study apart from much of the extant literature on R & D literature, particularly Judd [1984], and the second makes for important qualitative differences with agency problems arising from "option-type" payoff schemes (see Merton [1973]), which lead quite generally to private preference for greater risk in the sense of Hadar and Russell [1971].

The organization of this paper is as follows. In section 2, we briefly set out the general structure of the problem. The following sections analyze a variety of more special models. Section 3 considers a model where all knowledge bases are publicly available. Section 4 forms the bulk of the paper - here some knowledge bases are public while others are privately available to some firms. Typically firm’s strategies affect both the riskiness of and correlation between their R & D outcomes. To develop intuition for the results in such cases, we also consider in Section 4 a case where strategies affect only the correlation, and in Section 5 one where strategies affect only the riskiness of each firm’s R & D outcome. Section 6 offers a brief summary of the results.

Before proceeding to the detailed analysis, the basic welfare-theoretic “stance” of our study bears clarification. The justification of the “winner take-all” feature lies in a number of informational problems. Investment in R & D by any firm is not always publicly observable; further the choice of technique made by any firm is also unobservable, as a social planner is rarely aware of the existence of and detailed properties of different knowledge bases available to any competitor (owing partly to lack of expertise). These unobservabilities imply that to provide firms with incentives to invest (in a 0-1 fashion, say) in R & D, they have to be rewarded on the basis of their performance (for example in a patent race.
the time taken to discover or invent a well-specified product or process). These rewards are usually provided by the patenting mechanism, where the firm earliest to "discover" is rewarded monopoly rights to the product while unsuccessful firms obtain nothing. This "winner take-all" feature results because of an additional public unobservability: the quantitative magnitude of a loser's R & D performance (i.e., how far behind the winner he was) can be easily exaggerated by the loser, given knowledge of the nature of the discovery made public by the patenting mechanism. Hence losers' rewards must be independent of their (unobservable) performance. Loser's (fixed) rewards are then set at zero for a variety of reasons: (a) they generate maximum incentives for investment in R & D (though this may generate a tendency toward overinvestment), (b) the public unobservability of the number or identities of firms involved in the patent race ex ante implies that with positive loser payoffs many parties would step forward and claim to be losers, and (c) if there is an ex post social budget balance condition of rewards not exceeding the social benefit from the discovery, it will be violated for low realizations of the latter, or cause the winner's payoff to fall below the loser's payoff (in which case the winner would have no incentive to announce his discovery).

Our aim is then to compare the competitive outcome resulting from a winner-take-all reward system, with the first-best social optimum that would have been attainable in the absence of any of the above incentive problems. To make this comparison we have to specify the risk-attitudes of the planner as well as of the competing firms. To concentrate purely on the distortion created by the reward structure, we assume for most part that the planner and competing firms have identical risk attitudes. This can be justified (approximately) in an economy where agents have diversified holdings across firms (which still act noncooperatively) with utility functions satisfying the conditions for aggregation (Wilson [1968]) given risksharing arrangements for social payoffs across firms which are optimal (except for the risks involved in the patent race in question, assumed small relative to

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2 This is in an idealized sense of course. The essential feature we are focusing on is that the winner of a patent race obtains a discontinuously higher reward than others, no matter how far behind the latter were.

3 For instance, publication of the solution to a problem in a professional journal could lead all unsuccessful researchers involved to claim they were only slightly behind the winner. In the pharmaceutical industry where R & D involves searching alternative molecular combinations to obtain a particular drug, patenting the "winning" combination effectively makes public all information relevant to the discovery (see Grabowski and Vernon [1982]).
overall risk borne). Alternatively our results can be viewed as providing characterizations of socially optimal and market equilibrium outcomes under varying assumptions about risk attitudes of the planner, and of competing firms respectively: this provides insights into the additional distortion produced when firm and planner risk attitudes differ. It should be noted however that we are abstracting from an explicit characterization of the moral hazard problem with respect to R & D investment per se (by assuming a given number of firms investing on a given scale in R & D, and concentrating solely on questions of choice of technique). The reason is that some of these issues have been addressed earlier in the literature. A complete and realistic analysis would synthesize and study the interaction between the different qualitative issues, to produce a quantifiable model of distortions and socially optimal interventions (along the lines of Spence [1984]). We leave that task for future research.


There are two research units (or firms) labeled $i = 1, 2$. Firm $i$ chooses a policy $a_i \in A_i$, when $A_i$ is a compact subset of the real line. This results in a research project which is a real valued random variable $\hat{z}_i(a_i)$ whose range, assumed nonnegative, we write as $X_i(a_i)$. The economic interpretation is that $X_i$ is the realized social surplus from $i$'s research. The joint distribution of $\hat{z}_1(a_1)$ and $\hat{z}_2(a_2)$ has the probability density function $g(x_1, x_2; a_1, a_2)$.

Write $a_{-i}$ (resp. $x_{-i}$) as the label of the policy (resp. research outcome) pursued (resp. obtained) by the research unit other than $i$. Now define:

$$f_i(x_i; a_i) \equiv \int_{X_{-i}(a_{-i})} g(x_i, x_{-i}; a_i, a_{-i})dx_{-i}$$

as the marginal distribution of $\hat{z}_i$. What we have assumed in (1) is:

$$\frac{\partial f_i}{\partial a_{-i}} = 0.$$  

That is, there are no technical "interferences" among the R & D processes of the research units. Next, denote by $m_i(a_i)$ and $\sigma^2_i(a_i)$ the mean and variance, respectively, of $x_i$ and by $\rho(a_1, a_2)$ the correlation coefficient between $x_1$ and $x_2$, all of which exist and are finite.
Our earlier distinction between private and public knowledge pools can now be made precise. Two research projects can be said to draw greatly on the public knowledge pool if their correlation coefficient is large. In sections 3 and 4 we will in fact assume:

\[ \exists \bar{a}_1 \epsilon A_1 \text{ and } \bar{a}_2 \epsilon A_2 \text{ such that } \rho(\bar{a}_1, \bar{a}_2) = 1. \] (3)

There may of course be many commonly known research projects. Condition (3) merely says that there is at least one such project.

An extreme special case of this is:

\[ \forall a_i, \exists a_{-i} \text{ such that } \rho(a_i, a_{-i}) = 1. \] (4)

Condition (4) says that the entire knowledge pool is public. A model with this property is the subject of Section 3. As a matter of fact not all knowledge is public. There is a certain amount of prior R & D which is firm-specific. In general a research unit can tap from both private and public knowledge pools. This is formalized by:

\[ \exists \hat{a}_i \epsilon A_i \text{ such that } \forall a_{-i} \epsilon A_{-i}, \rho(\hat{a}_i, a_{-i}) < 1. \] (5)

In condition (5), \( z_i(\hat{a}_i) \) is a research strategy available to \( i \) which draws to some extent on the private knowledge pool of \( i \). In Sections 4 and 5 we will assume (5).

An extreme special case of (5) is:

\[ \forall a_1 \epsilon A_1 \text{ and } \forall a_2 \epsilon A_2 \rho(a_1, a_2) = 0 \] (6)

that is, the public knowledge pool is empty. It is (6) which has dominated the literature on patent races; we shall consider it in Section 5 and study the way competing firms make choices about riskiness of R & D strategy.

We now come to payoffs. We suppose costs and benefits are separable. Let \( C_i(a_i) \) be the cost to \( i \) of pursuing \( a_i \). If \((x_1, x_2)\) is the pair of outcomes then \( \hat{U}(x_1, x_2) \epsilon R_+ \) is the measure of social benefits. We take it that

\[ \hat{U}(x_1, x_2) = U(\max\{x_1, x_2\}) \] (7)

for some monotone, concave utility function \( U(\cdot) : R_+ \rightarrow R_+ \).
The welfare criterion is expected net social benefit. Thus society’s problem is:

$$\max_{a_1 \in A_1, a_2 \in A_2} W(a_1, a_2) = \int \int U(\max(z_1, z_2)) g(z_1, z_2; a_1, a_2) dz_1 dz_2 - C_1(a_1) - C_2(a_2).$$

(8)

Let \((a_1^*, a_2^*)\) be a solution of (8).

We now come to the market environment. It makes sense to identify social costs with private costs, so that the difference will rest entirely on the benefit side. Consider the private benefit to \(i\) when \(z_i\) and \(z_{-i}\) are the realizations. Each research unit is concerned with its expected net private benefit. To formalize the winner take-all feature of the market we suppose that there are real valued increasing functions \(I_i(z_i)\) and \(J_i(z_i)\), with the property \(I_i(z_i) > J_i(z_i) > 0\) for all \(z_i > 0\) such that the private payoff \(P_i(z_i, z_{-i})\) of firm \(i\) satisfies:

$$P_i(z_i, z_{-i}) = \begin{cases} I_i(z_i) & \text{if } z_i > z_{-i} \\ J_i(z_i) & \text{if } z_i = z_{-i} \\ 0 & \text{if } z_i < z_{-i} \end{cases}$$

(9a)

The research units (or firms) are engaged in a noncooperative game in the market and by hypothesis it is one of complete information. Thus a market equilibrium \((a_1^*, a_2^*)\) will be a Nash equilibrium, i.e., satisfy the condition

$$a_i^* = \arg \max_{a_i \in A_i} \left[ \int_{X_i(a_{-i})} \int_{X_i(a_i)} P_i(z_i, z_{-i}) g(z_i, z_{-i}; a_i^*, a_{-i}) dz_i dz_{-i} - C_i(a_i) \right]$$

$$= \arg \max_{a_i \in A_i} V_i(a_i, a_{-i}^*) \text{ for } i = 1, 2$$

(9b)

In what follows we will compare \((a_1^*, a_2^*)\) with \((a_1^*, a_2^*)\) and also the corresponding stochastic properties of \(\tilde{x}_1\) and \(\tilde{x}_2\). To make the problem tractable, a number of simplifying assumptions are imposed. The game is assumed to be symmetric, and that all concerned parties (including the planner) have the same attitude to risk. The latter implies in particular (along with the assumption that a winner is awarded the entire social surplus resulting from the discovery) that:

$$U(z) = I_i(z) \text{ for all } z, \text{ and } i$$

(9c)

In the case of a tie, either the entire discovery is awarded to one of the two firms chosen randomly (the case of nonsplit patents) or divided equally between them (the case of split
patents). It will generally turn out that the nature of equilibria depends importantly on the way ties are resolved. The existence of residual uncertainty in the R & D process not explicitly modeled here would generally imply the case of nonsplit patents is more natural and realistic. Accordingly for most part of the exposition, we shall be considering this case.

Finally, we will generally ignore the possibility that different strategies entail different costs, and concentrate entirely on the benefits side. The reason is partly due to our wish to focus specially on the distortion provided by the reward system (which impacts only on the benefits), and partly because in many circumstances no natural assumptions about costs suggest themselves. However many of the results pertaining to the (social or private) desirability of extreme specialization (i.e., different firms pursuing entirely different knowledge bases) will obviously carry over to the case where there are fixed (and no variable) costs of using a particular knowledge base.

3. A Model with Purely Public Knowledge Bases

Consider the case where the choice of techniques \((a_1, a_2)\) affects the R & D outcomes \((\hat{x}_1, \hat{x}_2)\) as follows

\[
\hat{x}_1 = K + a_1 \tilde{\eta}_1 + (1 - a_1) \tilde{\eta}_2 \tag{10a}
\]

\[
\hat{x}_2 = K + a_2 \tilde{\eta}_1 + (1 - a_2) \tilde{\eta}_2 \tag{10b}
\]

\[
a_1, a_2 \in [0, 1], \quad K > 0 \tag{10c}
\]

where \(\tilde{\eta}_1\) and \(\tilde{\eta}_2\) are independent random variables representing benefits obtainable from two different knowledge bases available to both firms. The supports of \(\tilde{\eta}_1\) and \(\tilde{\eta}_2\) are included in \([-b, \infty)\) where \(0 \leq b \leq K\), so \(\hat{x}_1\) and \(\hat{x}_2\) are always nonnegative.

(10) postulates that firms choose between alternative convex combinations of two publicly available research pools that affect the riskiness and correlation of their R & D performance. We concentrate on this case for its analytical simplicity, and also because the results generalize easily to the following class of situations:

\[
\hat{x}_i(a_i) = g(a_i \tilde{\eta}_1, (1 - a_i) \tilde{\eta}_2) \tag{10d}
\]
where \( g \) is increasing in each argument, and satisfies
\[
g(a_i \hat{\eta}_1, (1 - a_i) \hat{\eta}_2) \begin{cases} 
\leq a_i \hat{\eta}_1 + (1 - a_i) \hat{\eta}_2 & \text{for } a_i \in (0, 1) \\
= \hat{\eta}_1 & \text{for } a_i = 1 \\
= \hat{\eta}_2 & \text{for } a_i = 0
\end{cases}
\tag{10e}
\]
i.e., there are deadweight losses associated with mixing two different bases. A simple example of this is where \( g(a_i \hat{\eta}_1, (1 - a_i) \hat{\eta}_2) = \max[a_i \hat{\eta}_1, (1 - a_i) \hat{\eta}_2] \), since without loss of generality \( \hat{\eta}_1 \) and \( \hat{\eta}_2 \) are nonnegative. In this case \( \hat{\eta}_1 \) and \( \hat{\eta}_2 \) represent two alternative and completely noncomplementary research approaches.

We first characterize the nature of the social optimum.

**Proposition 1:** The social planner maximizing the expected utility of payoff, as in equation (8), will set \((a_1^*, a_2^*) = (1, 0)\) or, equivalently, \((a_1^*, a_2^*) = (0, 1)\), for all monotone increasing utility functions \(U(\cdot)\).

**Proof.** For any realization of \((\eta_1, \eta_2)\):
\[
\max(\eta_1, \eta_2) \geq a \eta_1 + (1 - a) \eta_2
\]
for any \(a \in [0, 1]\), with the inequality being strict if \(\eta_1 \neq \eta_2\) and \(a \in (0, 1)\). Hence the social payoff with \((a_1, a_2) = (1, 0)\) satisfies for realization \((\eta_1, \eta_2)\):
\[
\max[K + \eta_1, K + \eta_2] = K + \max(\eta_1, \eta_2)
\]
\[
\geq K + \max(a_1 \eta_1 + (1 - a_1) \eta_2, a_2 \eta_1 + (1 - a_2) \eta_2)
\]
strictly for \(\eta_1 \neq \eta_2\) and \((a_1, a_2)\) not equal to \((0, 1)\) or \((1, 0)\). Thus
\[
\max[K + \eta_1, K + \eta_2] > \max(a_1(\eta_1), a_2(\eta_2))
\]
for all \((a_1, a_2)\) not \((1, 0)\) or \((0, 1)\), and almost all \((\eta_1, \eta_2)\), which establishes the result.
Q.E.D.

We now consider the private Nash equilibrium set of \((a_1^*, a_2^*)\), under the further assumption that \(\eta_1, \eta_2\) are i.i.d., so that the potential motivation for interior \(a\)'s arises from risk-reduction considerations. Nevertheless, we have the following equivalence for any monotone increasing (private and social) utility function \(U(\cdot)\).
Proposition 2. Assume that \( \eta_1, \eta_2 \) are i.i.d., and that ties are resolved by randomizing the allocation of the social payoff with probability \( \frac{1}{2} \), i.e. nonsplit patents. Then \((a_1, a_2) = (1, 0) \) and \((0, 1) \) are the only Nash equilibria in pure strategies.

**Proof:**

Denoting the expected utility of firm \( i \) under \((a_1, a_2) \) by \( V_i(a_1, a_2) \), we have:

\[
V_1(a_1, a_2) \mid a_1 > a_2 = \int_{-b}^{\infty} \int_{-\infty}^{\infty} U[K + a_1 \eta_1 + (1 - a_1) \eta_2] dF_1(\eta_1) dF_2(\eta_2)
\]

\[
V_1(a_1, a_2) \mid a_1 < a_2 = \int_{-b}^{\infty} \int_{-b}^{\eta_2} U[K + a_1 \eta_1 + (1 - a_1) \eta_2] dF_1(\eta_1) dF_2(\eta_2)
\]

\[
V_1(a_1, a_2) \mid a_1 = a_2 = \frac{1}{2} \int_{-b}^{\infty} \int_{-b}^{\infty} U[K + a \eta_1 + (1 - a) \eta_2] dF_1(\eta_1) dF_2(\eta_2)
\]

Thus, the first derivatives satisfy the conditions:

\[
\frac{\partial V_1}{\partial a_1}(a_1, a_2) \mid a_1 > a_2 = \int_{-b}^{\infty} \int_{\eta_2}^{\infty} (\eta_1 - \eta_2) U'(K + a_1 \eta_1 + (1 - a_1) \eta_2) dF_1(\eta_1) dF_2(\eta_2) > 0
\]

\[
\frac{\partial V_1}{\partial a_1}(a_1, a_2) \mid a_1 < a_2 = \int_{-b}^{\infty} \int_{-b}^{\eta_2} (\eta_1 - \eta_2) U'(K + a_1 \eta_1 + (1 - a_1) \eta_2) dF_1(\eta_1) dF_2(\eta_2) < 0
\]

Hence,

\[
\begin{cases}
V_1(1, a_2) > V_1(a_1, a_2) \mid a_1 > a_2 & \text{for all } a_2 \epsilon [0, 1]
\end{cases}
\]

\[
\begin{cases}
V_1(0, a_2) > V_1(a_1, a_2) \mid a_1 < a_2
\end{cases}
\]

(11) implies the only possible pure-strategy Nash equilibria are \((1, 0) \), \((0, 1) \), as well as the set of symmetric tuples \((a, a) \) with \( a \epsilon [0, 1] \).

We first show that given \( \eta_1 \) and \( \eta_2 \) are i.i.d., \((1, 0) \) [and hence also \((0, 1) \)] is a Nash equilibrium. Since by (11), \( V_1(1, 0) > V_1(a, 0) \) for all \( a \epsilon (0, 1) \), we have to show \( V_1(1, 0) \geq V_1(0, 0) \). Of course, by the i.i.d. assumption on \((\eta_1, \eta_2)\), \( V_1(1, 0) = V_2(1, 0) \), and from the symmetry of the game, \( V_1(0, 0) = V_2(0, 0) \). Further, given nonsplit patents (and equality of private and social risk preferences), social welfare equals

\[
W(a_1, a_2) = V_1(a_1, a_2) + V_2(a_1, a_2) \text{ for all } (a_1, a_2) \epsilon [0, 1]^2
\]

so \( W(1, 0) = 2V_1(1, 0), W(0, 0) = 2V_1(0, 0) \).

Since \((1, 0) \) is a social optimum and \((0, 0) \) is not, \( W(1, 0) > W(0, 0) \), implying \( V_1(1, 0) > V_1(0, 0) \), for this i.i.d. case. By similar argument \( V_2(1, 0) > V_2(1, 1) \) and hence \((1, 0) \) is a Nash equilibrium. Analogous arguments establish that \((0, 1) \) is a Nash equilibrium.
It remains to show that no symmetric tuple \((a, a)\) can be a Nash equilibrium. Since 
\(W(1, 0) > W(a, a)\), we have \(V_1(1, 0) > V_1(a, a)\). But

\[
V_1(1, 0) = \int_{-\infty}^{\infty} \int_{\eta_2}^{\infty} U(K + \eta_1) dF_1(\eta_1) dF_2(\eta_2) = V_1(1, a) \mid_{a < 1}
\]

Hence \(V_1(1, a) \mid_{a < 1} > V_1(a, a)\) so for \(a < 1\), \((a, a)\) cannot be a Nash equilibrium. Also 
\(V_1(0, 1) > V_1(1, 1)\), hence \((1, 1)\) cannot be a Nash equilibrium either. Q.E.D.

Propositions 1 and 2 demonstrate (under the assumption \(\tilde{\eta}_1\) and \(\tilde{\eta}_2\) are identically
 distributed) equivalence between Nash equilibria and the social optimum: both involve
 extreme specialization, which implies both firms must be choosing their riskiest R & D
 strategy (i.e., concentrate all resources on one knowledge base) and these strategies are
 stochastically independent. Note that no assumption about risk attitudes has been made.
 The intuition for the result is that with all knowledge bases publicly available, the proba-
 bility of any particular firm "winning" is \(\frac{1}{2}\) and independent of strategies chosen. Since the
 only possible source of divergence of private and social incentives arises through the con-
 cern firms have about the identity of the winner (a concern not matched by the planner),
 private and socially optimal outcomes coincide.

Before concluding this Section, it is appropriate to describe how the result of Propo-
 sition 2 is affected if the split patent system is used, or if the i.i.d. assumption is dropped.
 With split patents (where half the social payoff is awarded to each firm given a tie) and
 risk-averse firms, it is possible that \((1, 1)\) is a Nash equilibrium while neither \((1, 0)\) or \((0, 1)\)
 are. The reason is that the corresponding payoffs become

\[
V_2(1, 1) \mid_{\text{split}} = \int_{-b}^{\infty} U\left(\frac{1}{2}(K + \eta_1)\right) dF_1(\eta_1)
\]

\[
V_2(1, 0) = \int_{-b}^{\infty} \int_{\eta_1}^{\infty} U(K + \eta_2) dF_2(\eta_2) dF_1(\eta_1)
\]

With sufficient risk-aversion it is possible that \(V_2(1, 1) \mid_{\text{split}} > V_2(1, 0)\), because at \((1,1)\)
both agents are tying with probability one and are thus insured against losing, whereas at
\((1, 0)\) each agent loses with probability \(\frac{1}{2}\). The source of the problem here is that a firm can
change its probability of winning by moving from a nonidentical to an identical strategy.

If one of the random variables \(\{\tilde{\eta}_1, \tilde{\eta}_2\}\) is riskier than the other, and both firms are
sufficiently risk-averse, then even with non-split patents it is possible that \((1, 1)\) is a Nash
equilibrium, while (1,0) and (0,1) are not equilibria. Consider the following example. \( \hat{\eta}_2 \)
is uniformly distributed on \([-b_2,b_2]\), \( \hat{\eta}_1 \) on \([-b_1,b_1]\) where \( b_2 > b_1 > 0 \). Also suppose
\( U(x) = \frac{1}{a} x^a \) where \( a < 1, a \neq 0 \). Then

\[
V_2(1,1) = \frac{(K + b_1)^{a+1} - (K - b_1)^{a+1}}{4ab_1(a + 1)}
\]

\[
V_2(1,0) = \frac{(K + b_2)^{a+1}}{2ab_2(a + 1)} - \frac{[(K + b_1)^{a+2} - (K - b_1)^{a+2}]}{4a\delta_1 b_2(a + 1)(a + 2)}
\]

Putting \( k = b_1 = 5, b_2 = 8 \) is can be checked that \( V_2(1,1) \) is less than \( V_2(1,0) \) with \( a = \frac{1}{2} \) and \(-\frac{1}{2}\), but greater with \( a = -5 \). A similar example could have been constructed by having
the means of \( \hat{\eta}_1 \) and \( \hat{\eta}_2 \) differ significantly. It may seem that “weaker” research projects
(i.e., in this case \( \hat{\eta}_2 \)) should be subsidized to overcome this problem, but this requires the
planner to monitor the research strategies chosen by different firms.  
Interestingly, in
the models with some private knowledge bases analyzed in following Sections, the non-
identical nature of two alternative knowledge bases that social optimality requires the two
firms to alternately specialize in does not by itself create any problem in terms of Nash
implementation.

4. Models with Private-cum-Public Knowledge Bases

Most of this section (i.e., the first two subsections) will study a model where firms’
strategies affect both the riskiness and correlation of their R & D outcomes. This model in
our view is the most realistic; however to obtain intuition for the results it helps to analyze
choices of riskiness and correlation separately as well. Accordingly the third subsection
considers a model involving correlation choices alone, while the following Section analyzes
a model with purely private knowledge bases, i.e., involving only choices of riskiness.

4.1 Risk cum Correlation Choices: Small Risk Aversion

Before introducing the specific model, we introduce a general result on sufficient condi-
tions for corner social optima (for a similar result see Nalebuff and Zeckhauser (1983)):

\footnote{This is possible when the two alternative research avenues are completely noncom-
plementary, so firm \( i \)'s R & D performance \( x_i \) is \( Max[a_i \hat{\eta}_1, (1 - a_i)\hat{\eta}_2] \). In this case the
planner may be able to monitor which particular research approach (\( \hat{\eta}_1 \), or \( \hat{\eta}_2 \)) generated
the “discovery” and the winner’s prize may be enhanced if the discovery resulted from the
“weaker” research avenue.}
Proposition 3

Let \((\bar{\eta}_1, \ldots, \bar{\eta}_\ell)\) be a fundamental set of random variables or knowledge bases, with the joint distribution function \(L(\eta_1, \ldots, \eta_\ell)\). Suppose the R & D performance of firm \(i\) is given by

\[ x_i = \phi_i(\bar{\eta}_1, \ldots, \bar{\eta}_\ell; a_i) \]

where \(a_i \in A_i\), a compact rectangle in \(R^{\ell_i}\), \(\ell_i \leq \ell\) and \(\phi_i\) is a function from \(R^{\ell_i} \times A_i\) to \(R^+\). Then if (i) each \(\phi_i\) is convex in \(a_i\),

(ii) the social planner is risk-neutral, i.e. \(U(x) = x\),

a corner of \(A_1 \times A_2\) is socially optimal.

Proof:

Since the social planner maximizes

\[ W(a_1, a_2) = \int \max\{\phi_1(\eta_1, \ldots, \eta_\ell; a_1), \phi_2(\eta_1, \ldots, \eta_\ell; a_2)\} dL(\eta_1, \ldots, \eta_\ell) \]

which is a convex function of \((a_1, a_2)\) under (i) and (ii), the optimal choice is a corner of \(A_1 \times A_2\) Q.E.D.

Proposition 3 applies to the following model (as well as the one in the previous Section):

\[ \tilde{z}_i = K + a_i \bar{\theta} + (1 - a_i) \bar{\eta}_i, \quad a_i \in [0, 1] \] (12)

where \(\bar{\theta}, \bar{\eta}_1, \bar{\eta}_2\) are mutually independent random variables (with supports contained in \([\bar{\theta}, \bar{\delta}]\) where \(K + \bar{\delta} \geq 0\)) with finite means and variances. In (12), there is a publicly available knowledge base \(\bar{\theta}\), and firm \(i\) has private access to \(\bar{\eta}_i\). Choices of \(a_i\) affect both the riskiness of \(\tilde{z}_i\), and the correlation between \(\tilde{z}_1\) and \(\tilde{z}_2\). As in the model with purely public knowledge bases, the results of this subsection generalize to the case where \(\tilde{z}_i = g(a_i \bar{\theta}, (1 - a_i) \bar{\eta}_i)\), and \(g\) satisfies (10e).

By Proposition 3, the social optimum for a risk-neutral planner is a corner of \([0, 1]^2\), i.e., lies within the set \(\{(0, 0), (0, 1), (1, 0), (1, 1)\}\). Of these \((1, 1)\) involves perfect correlation while the other three lead to independence between \(\tilde{z}_1\) and \(\tilde{z}_2\). It is thus important to go further and isolate which of these corners are social optima.
Proposition 4

Suppose the planner is risk-neutral, and (12) holds where \( \bar{\eta}_1, \bar{\eta}_2 \) are identically distributed with the distribution function \( F(\cdot) \) with density \( f(\cdot) \), and \( \theta \) has distribution function \( G(\cdot) \) with density \( g(\cdot) \). Then \((0,0)\) is socially optimal if and only if

\[
R_1 \equiv \int_{\bar{\eta}_1}^{\bar{\eta}_2} [G(\eta) - F(\eta)]F(\eta) d\eta \geq 0
\] (13)

and \((1,0)\) and \((0,1)\) are socially optimal if and only if \( R_1 \leq 0 \). No other strategy tuple is socially optimal.

Proof:

Given Proposition 3 it is easy to check that the only candidate optima are \((0,0),(0,1),(1,0),(1,1)\). The last of these can be ruled out by noting that the social payoff distribution generated by \((1,1)\) is dominated by \((0,1)\) and \((1,0)\), along the lines of Proposition 1 above. To choose between \((0,0)\) and \((1,0)/(0,1)\), notice that the social expected payoff is given by

\[
W(0,0) = \int_{\bar{\eta}_1}^{\bar{\eta}_2} \{\eta_2 F(\eta_2) + \int_{\eta_2}^{\bar{\eta}_2} \eta_1 dF(\eta_1)\} dF(\eta_2) + K
\]

which, in integrating the outer integral by parts, gives

\[
W(0,0) = K + \left[ \{\eta_2 F(\eta_2) + \int_{\eta_2}^{\bar{\eta}_2} \eta_1 dF(\eta_1)\} F(\eta_2) \right]_{\eta_2=\bar{\eta}_1}^{\eta_2=\bar{\eta}_2}
\]

\[
- \int_{\bar{\eta}_1}^{\bar{\eta}_2} F(\eta_2)\{\eta_2 f(\eta_2) + F(\eta_2) - \eta_2 f(\eta_2)\} d\eta_2
\]

\[
= K + \bar{\eta}_1 - \int_{\bar{\eta}_1}^{\bar{\eta}_2} F^2(\eta_2) d\eta_2
\] (14a)

Similarly, we have

\[
W(0,1) = \int_{\bar{\eta}_1}^{\bar{\eta}_2} \left[ \int_{\theta}^{\bar{\theta}} (K + \eta_1)f(\eta_1) d\eta_1 + (K + \theta) \int_{\bar{\eta}_1}^{\bar{\theta}} f(\eta_1) d\eta_1 \right] g(\theta) d\theta
\]

\[
= K + \bar{\eta}_1 - \int_{\bar{\eta}_1}^{\bar{\eta}_2} G(\theta) F(\theta) d\theta
\] (14b)

16
where (14b) has also been obtained by integration by parts. On comparing (14a) and (14b),
the result is immediate. Q.E.D.

**Remark:** Consider the special case in which \( \tilde{\eta}_i \) and \( \tilde{\theta} \) have identical means. In that case,
the criterion in (13a) can be interpreted as follows. If \( \tilde{\theta} \) is riskier than \( \tilde{\eta}_i \), i.e., worse in the
second degree stochastic dominance (SSD) sense, then it follows, for example from Hadar
and Russell [1971], that
\[
\int_{\tilde{b}}^{\tilde{b}} [G(\epsilon) - F(\epsilon)] d\epsilon \geq 0 \tag{13b}
\]
for all \( b \geq \tilde{b} \), and strictly for some \( b \in (\tilde{b}, \tilde{b}) \) for strictly riskier \( \tilde{\theta} \). It follows from (13b) that
\[
R_2 = \int_{\tilde{b}}^{\tilde{b}} \int_{\tilde{b}}^{\tilde{b}} [G(\epsilon) - F(\epsilon)] d\epsilon f(b) db \geq 0 \tag{13c}
\]

Integrating the outer integral by parts in (13c), we obtain that
\[
R_2 = \int_{\tilde{b}}^{\tilde{b}} [G(\epsilon) - F(\epsilon)] d\epsilon - \int_{\tilde{b}}^{\tilde{b}} [G(b) - F(b)] F(b) db \geq 0 \tag{13d}
\]
By the assumption of equality of the means of \((\tilde{\eta}_i, \tilde{\theta})\), the first integral in (13d) is zero, and
thus, on comparing \( R_2 \) with \( R_1 \) in Proposition 4, we see that \((0, 1)/(1, 0)\) is preferred to
\((0, 0)\) by a risk-neutral social planner when \( \tilde{\theta} \) is riskier than \( \tilde{\eta}_i \), and vice-versa. The theory
of option valuation (Merton [1973]) clearly suggests such a result.

We now look at the private Nash equilibrium \((a_1^*, a_2^*)\) for model (12) above. The private
objective functions \( V_i(a_1, a_2) \), given the utility function \( U(\cdot) \) for payoffs, and assuming a
regime of randomized patents given ties, can be written as follows:
\[
V_1(a_1, a_2) = \int_{\tilde{b}}^{\tilde{b}} \int_{\tilde{b}}^{\tilde{b}} \int_{Z_1(a_1, a_2, \eta_2, \theta)} U[K + a_1 \theta + (1 - a_1) \eta_1] f(\eta_1) d\eta_1 f(\eta_2) d\eta_2 g(\theta) d\theta \tag{15}
\]
for \((0 \leq a_1 < 1, 0 \leq a_2 \leq 1)\), where
\[
Z_1(a_1, a_2, \eta_2, \theta) = \frac{(1 - a_2) \eta_2 + (a_2 - a_1) \theta}{(1 - a_1)} \tag{16}
\]
and \( \eta_1 \geq Z_1 \) implies that \( \eta_1 \geq Z_1 \).

The dependence of \( Z_1 \) on \( a_1 \) and \( a_2 \) indicates how (in contrast to the common access
model) the probability of any firm winning the patent depends on the research techniques,
chosen. Thus in this model a potential divergence between social and private incentives (at the margin) appears, making it less likely that social optima can be implemented by the patent mechanism as generally as in the common access model.

Similarly,

\[ V_1(a_1 = 1, a_2) = \int_{\eta_2}^5 \int_{\eta_2}^5 U[K + \theta]g(\theta)d\theta f(\eta_2)d\eta_2 \]  

for \(0 \leq a_2 < 1\), and

\[ V_1(1, 1) = \int_{\frac{K}{2}}^5 \frac{1}{2} U[K + \theta]g(\theta)d\theta \]  

with analogous definitions for \(V_2(a_1, a_2)\). In this subsection, we shall be restricting our analysis to cases of "small" risk-aversion, i.e., where the social objective function \(W(a_1, a_2)\) is maximized by choosing a corner of \([0, 1]^2\), given the same utility function \(U(\cdot)\) for social payoffs, e.g., \(U(x) = x\) as in Proposition 3 above. Our first result deals with the case where the private and public knowledge bases are identically distributed.

**Proposition 5:** Assume that

(I) the random variables \((\tilde{\eta}_i, \tilde{\theta})\) are i.i.d.

(II) the social objective function

\[ W(a_1, a_2) = V_1(a_1, a_2) + V_2(a_1, a_2) \]  

is maximized at \([(0, 0)/(0, 1)/(1, 0)]\)

(III) the densities \(\{f(\eta), g(\theta)\}\) are symmetric around the mean.

Further define

\[ \tilde{r}(a_i) \equiv a_i \tilde{\theta} + (1 - a_i)\tilde{\eta}_i \]  

with the distribution function \(H_{a_i}(\cdot)\). \(H_{a_i}(\cdot)\) is less risky than \(F(\cdot) = G(\cdot)\) by Theorem 8 in Hadar and Russell [1971], strictly for \(a_i \in (0, 1)\). Assume also that:

(IV) the distribution functions \(H_{a_i}(\cdot)\) satisfy the single-crossing property with respect to the distribution functions \(F(\cdot)/G(\cdot)\), i.e., there exists \(d(a_i) \in (K + \frac{b}{5}, K + 5)\) such that

\[ H_{a_i}(x) \leq F(x) \equiv G(x) \text{ for } x \in [K + \frac{b}{5}, d(a_i)] \]  

\[ H_{a_i}(x) \geq F(x) \equiv G(x) \text{ for } x \in [d(a_i), K + 5] \]
with strict inequalities for some set of $x$ values with positive measure, given $a_i \in (0, 1)$. Then the set of pure-strategy private Nash equilibria $(a_1^*, a_2^*)$ defined in equation (9b) above, is $\{(0,0), (0, 1), (1, 0)\}$: i.e., identical to the set of social optima.

**Proof:** It is straightforward to show that $\hat{r}(a)$ also has a symmetric density function $h_a(r)$, and thus the crossing point $d(a_i)$ in (19a,b) is independent of $a_i$, and equals the common means of $\hat{\theta}$ and $\hat{\eta_i}$. Given this, we first show that for any $a \in (0, 1)$

$$V_1(0, a) > V_1(0, 0) = V_1(1, 0) = V_1(0, 1) \quad (20a)$$

The last two equalities are obvious from assumption (1), and the first is shown as follows. Using (15a) and (16), one can alternatively express

$$V_1(0, a) = \int h_a(r) \left\{ \int \{U[K + \eta_1]f(\eta_1)d\eta_1 \} \right\} dr \quad (20b)$$

which, on interchanging the order of integration over $\{r, \eta_1\}$ can be written as

$$V_1(0, a) = \int U[K + \eta_1]H_a(\eta_1)f(\eta_1)d\eta_1 \quad (20c)$$

where $H_a(\cdot)$ is the distribution function for $\hat{r}(a)$ which satisfies the following property, given (19a,b) and the symmetry assumptions on the densities. For all symmetric pairs $\{\eta_1^+, \eta_1^-\}$ around the mean $\bar{\eta}$ of $\hat{\eta}$, i.e., $\{\eta_1^+ - \bar{\eta} - \eta_1^- = \eta \geq 0\}$ there exists $\delta_a(\eta) \geq 0$ such that

$$H_a(\eta_1^+) - \delta_a(\eta) = H_a(\eta_1^+) = F(\eta_1^+) \quad (20d)$$

$$H_a(\eta_1^-) + \delta_a(\eta) = H_a(\eta_1^-) = F(\eta_1^-) \quad (20e)$$

and $\delta_a(\eta) > 0$ given $a \in (0, 1)$.

Substituting (20d,e) in (20c), and keeping in mind the symmetry of $f(\eta_1)$ around $\eta_1 = \bar{\eta}$, the first inequality in (20a) follows immediately.

Using (17), and the fact that $W(0, 0) > W(0, 1)$ it then follows that

$$V_2(0, a) \leq V_2(0, 0) \quad (20f)$$

with strict inequality for $a \in (0, 1)$. Thus $a_2^* \in (0, 1)$ constitute best responses to $a_1 = 0$. It is straightforward to show, using only the i.i.d. assumption about $\{\hat{\eta}_i, \hat{\theta}\}$ and along the
lines of Proposition 2 that \( a_1^* = 0 \) constitutes the best response to \( a_2^* = 1 \). We have thus shown that \( [(0, 0), (0, 0), (1, 0)] \) are Nash equilibria.

It is also simple to show that there are no other pure-strategy Nash equilibria. Suppose otherwise; i.e., that there is a Nash equilibrium \( (a_1, a_2) \) with \( a_1 \in (0, 1) \), without loss of generality. This implies that

\[
V_1(0, 0) \leq V_1(0, a_2) \leq V_1(a_1, a_2)
\]

\[
V_2(0, 0) < V_2(a_1, 0) \leq V_2(a_1, a_2)
\]

which contradicts the assumption that \( (0, 0) \) is a maximizer of

\[
W(a_1, a_2) = V_1(a_1, a_2) + V_2(a_1, a_2) \quad \text{Q.E.D.}
\]

Remark: The symmetry and single-crossing assumptions (III and IV) satisfied by the normal and uniform distributions, for example, are quite critical. In their absence, there may be divergences between private Nash equilibria and social optima even with risk-neutral preferences. With \( U(x) = x \), the private first-order condition equation (15a) is given, using (16), as

\[
\frac{\partial V_1(a_1 = 0, a_2 = 0)}{\partial a_1} = \int_{\hat{\eta}_2}^{\hat{\eta}_2} \int_{\hat{\theta}}^{\hat{\theta}} (\hat{\eta}_2 - \eta_2 f(\eta_2) d\eta_2 g(\theta) d\theta
\]

\[
- \int_{\hat{\eta}_2}^{\hat{\eta}_2} \int_{\hat{\theta}}^{\hat{\theta}} (\eta_2 - \eta_2 (K + \eta_2) f^2(\eta_2) d\eta_2 g(\theta) d\theta
\]  

(21a)

The first term on the right-hand side of (21a) - which is the only term in the social first-order condition \( \frac{\partial W(0, 0)}{\partial a_1} \) - is clearly negative given the i.i.d. assumption on \( \{\eta_1, \delta\} \), using elementary properties of conditional expectations. In the second term, writing \( (\eta_2 - \theta)(K + \eta_2) = (\eta_2^2 - \eta_2 \theta - K \theta + K \eta_2) \) and assuming without loss of generality that \( \eta_2 \) and \( \delta \) have zero means, it is easily seen that all terms except

\[
D \equiv - \int_{\hat{\eta}_2}^{\hat{\eta}_2} K \eta_2 f^2(\eta_2) d\eta_2
\]

(21b)

contribute to a negative derivative in (21a). For symmetric \( f(\eta_2) \) around the zero mean, the "divergence term" \( D \) in (21b) is zero. But if \( f(\eta_2) \) is skewed to the left, e.g., for a lognormal distribution, then \( D > 0 \), and for sufficiently large \( K \) this will make \( \frac{\partial V_1(0, 0)}{\partial a_1} > 0 \),
thus removing $(0,0)$ as a Nash equilibrium. In a similar fashion \((0,1)\) or \((1,0)\) may also not be Nash equilibria, because \(a_2 = 1\) may no longer be the best response to \(a_1 = 0\).

The following observation reinforces the view that divergences between social optima and Nash equilibria arise (in cases of asymmetric distributions) only with sufficient left-skewness (rather than right-skewness). If the density function \(f\) for \(\hat{\eta}_i\) and \(\hat{\eta}\) is nondecreasing everywhere, then also the result of Proposition 5 goes through. This is because integrating (20c) by parts yields

\[
V_1(0, a) = U(K + h)f(h)E(\hat{\eta}_i) - \int_h^\delta \left[U'(K + \eta)f(\eta) + U(K + \eta)f'(\eta)\right]I_a(\eta) \, d\eta \tag{20g}
\]

where \(I_a(\eta)\) is defined to be \(\int_h^\delta H_a(\eta_1) \, d\eta_1\). By second order dominance of \(\hat{r}(a)\) over \(\hat{r}(0)\) for any \(a\) in \((0,1)\) it follows that \(I_a(\eta) < I_0(\eta)\) for all \(\eta\) in \([h, \delta]\). Hence \(f' \geq 0\) implies

\[
V_1(0, a) > V_1(0, 0) \quad \text{for any} \quad a \quad \text{in} \quad (0,1)
\]

which implies the result of Proposition 5. Thus in this extreme situation of right-skewness characterized by nondecreasing density, social optima and Nash equilibrium coincide (even without the single-crossing condition).

The intuition for the equivalence of social optima and Nash equilibria under the conditions postulated in Proposition 5 can be explained in two steps. (a) If one firm 2, say, chooses its private knowledge base exclusively (i.e., \(a_2 = 0\)) the outcomes of the two firm's R & D will be uncorrelated, no matter what firm 1 does. Thus firm 1's strategy affects only the riskiness of its own R & D outcome, and under regularity conditions (III) and (IV) prefers the riskiest strategy possible, i.e., a corner strategy. The reason for the latter is (roughly speaking) that a riskier R & D strategy increases the probability of winning large discoveries, and reduces that of winning small discoveries symmetrically. This explains why corner strategies form an equilibrium under (III) and (IV). However when the distribution of R & D outcomes from any knowledge base is skewed to the left, the effect of choosing a riskier strategy (uncorrelated with the strategy of the other firm) is to reduce the probability of winning small discoveries more than proportionate to the increase in probability of winning large discoveries. With sufficient left-skewness an increase in risk is undesirable, and it is optimal for a firm to choose an interior strategy in order to reduce risk. These
distributional effects, which arise from the discontinuity of the private payoff function at \( x_1 = x_2 \) and the effect of strategy \( (a_i) \) choice on the probability of suffering this discontinuity, is what sets the "agency" problem in our model apart from the usual option pricing analysis, e.g. Merton (1973). (b) It remains to explain why no pair of R & D strategies involving some correlation between outcomes of the two firms, can be an equilibrium. The essential reason is that each firm can veto correlation by switching to its private knowledge base, and guarantee for itself at least the payoff it receives at the social optimum. Thus if an equilibrium were to involve some correlation, it must generate gains for both firms (relative to the social optimum). But this is not possible, since the social payoff (the sum of individual payoffs) is maximized at strategy pairs involving no correlation. Another way of viewing this is that private and social incentives differ only through the effect of a firms choice of technique on the probability of its winning. Since correlated R & D strategies are socially suboptimal, they can be advantageous to a firm only through the effect on the probability of that firm winning. This is equivalent to an opposite effect on the probability of the other firm losing; thus correlation must prove disadvantageous to the other firm. The latter firm would then do better for itself by switching off the correlation.

We now consider the case where \( \tilde{\eta}_i \) and \( \tilde{\theta} \) are not assumed to be identically distributed, but only mutually independent. However, without loss of generality we can assume that all three random variables have equal means of zero.

**Proposition 6a:** Assume that \( \{\tilde{\eta}_1, \tilde{\eta}_2\} \) are i.i.d. and \( \tilde{\eta}_i \) is less risky than \( \tilde{\theta} \) in the second-degree stochastic dominance sense (Hadar and Russell [1971]), (thus) that \( [(0,1),(1,0)] \) maximize the social objective function \( W(a_1,a_2) \) given the (for example, risk-neutral) utility function \( U(\cdot) \). Assume also that the means of \( \tilde{\theta} \) and \( \tilde{\eta}_i \) equal 0, that \( g(\theta) \) and \( f(\eta) \) are symmetric, and that \( \tilde{r}(a_i) \) in (18) - which is less risky than \( \tilde{\theta} \) - has a distribution function \( H_{a_i}(\cdot) \) that satisfies the single-crossing property of equations (19a,b) with respect to \( G(\theta) \). Then \( [(0,1),(1,0)] \) are the only pure-strategy Nash equilibria (with nonsplit patents).

**Proof:** To show that \( [(1,0),(0,1)] \) are Nash equilibria, we proceed as follows.

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5 With split patents, the set of pure strategy Nash equilibria depend on the sign of \([V(1,1) - V_i(0,1)]\). (1,0) and (0,1) are equilibria if and only if this is nonpositive, and (1,1) is an equilibrium if and only if it is nonnegative (this is possible with sufficient risk-aversion, but not with risk-neutrality). There exist no other pure strategy equilibria.
Step I. To show that \( a_2 = 0 \) is the (unique) best response to \( a_1 = 1 \), we first note that

\[
V_2(1, 0) > V_2(1, \alpha) \quad \text{for all } \alpha \in (0, 1) 
\]  

(22)

along the lines of the proof of Proposition 5. To show that

\[
\int_{\hat{\theta}}^{\tilde{\theta}} U(K + \eta_2) f(\eta_2) G(\eta_2) d\eta_2 = V_2(1, 0) > V_2(1, 1) = \frac{1}{2} \int_{\hat{\theta}}^{\tilde{\theta}} U(K + \theta) g(\theta) d\theta 
\]  

(23)

we use the facts that

\[
(i) \quad 2 \int_{\hat{\theta}}^{\tilde{\theta}} U(K + \eta_2) f(\eta_2) G(\eta_2) d\eta_2 > \int_{\hat{\theta}}^{\tilde{\theta}} U(K + \eta_2) f(\eta_2) d\eta_2 
\]

(22b)

because of the symmetry of \( g(\theta) \) around \( \theta = 0 \) (i.e., for \( \eta > 0, [2G(\eta) - 1] = [1 - 2G(-\eta)] > 0 \)), as well as the symmetry of \( f(\eta_2) \) around \( \eta_2 = 0 \), and also that

\[
(ii) \quad \int_{\hat{\theta}}^{\tilde{\theta}} U(K + \eta_2) f(\eta_2) d\eta_2 \geq \int_{\hat{\theta}}^{\tilde{\theta}} U(K + \theta) g(\theta) d\theta 
\]

for all increasing and (weakly) concave \( U(\cdot) \) by second-degree stochastic dominance of \( \hat{\eta}_2 \) over \( \tilde{\theta} \).

Step II. To show that \( a_1 = 1 \) is the (unique) best response to \( a_2 = 0 \), one uses reasoning identical to that in Proposition 2 to show that

\[
V_2(a, 0) > V_2(1, 0) \quad \text{for all } a \in [0, 1) 
\]  

(22c)

and thus, given that \( (1, 0) \) maximizes \([V_1(a_1, a_2) + V_2(a_1, a_2)]\)

\[
V_1(a, 0) < V_1(1, 0) 
\]

(22d)

The proof uses the facts that \( (i) \tilde{\tilde{\theta}}(a) = a\tilde{\theta} + (1 - a)\tilde{\eta}_1 \) is strictly less risky than \( \tilde{\theta} \) for \( a \in [0, 1) \), (ii) that the corresponding distribution function \( H_a(\cdot) \) satisfies the single-crossing property with respect to \( G(\cdot) = H_1(\cdot) \), and (iii) that both the densities \( h_a(\cdot) \) and \( g(\cdot) \) are symmetric around 0.

Step III. To show that there is no other pure strategy Nash equilibrium, it suffices to show that no \((a_1, a_2) \in (0, 1)^2 \) can be an equilibrium, given that 0 and 1 are unique best
responses to each other. Suppose the contrary, i.e., that \((a_1, a_2)\) with \(a_1 \notin \{0, 1\}, a_2 \notin \{0, 1\}\) is a Nash equilibrium. That implies

\[
V_2(0, 1) = V_2(a_1, 1) \leq V_2(a_1, a_2)
\]

\[
V_1(0, 1) < V_1(0, a_2) \leq V_1(a_1, a_2)
\]

using (22c), which contradicts the social optimality of \((0, 1)\). Q.E.D.

Essentially similar arguments also establish the following result.

**Proposition 6b.**

Suppose \(\tilde{\eta}_1\) and \(\tilde{\eta}_2\) are i.i.d and riskier than \(\tilde{\delta}\), but have symmetric densities with zero mean, and that the (common) social and private utility function \(U(\cdot)\) is such that \((0, 0)\) is the unique social optimum. Assume also that the random variable \(\tilde{\eta}(a)\) in (18) - which is now less risky than \(\tilde{\eta}_i\) - has the distribution function \(H_3(\cdot)\) satisfying the single-crossing property (19a,b) with respect to \(F(\cdot) = H_4(\cdot)\). Then \((0, 0)\) is the unique pure strategy Nash equilibrium with nonsplit patents.

Proposition 6a and 6b demonstrate how the equivalence of Nash equilibria and social optima described in Proposition 5 extends to the case where the private and public knowledge bases differ in their riskiness. This is an interesting contrast with the pure public knowledge model. The essential reason is that unlike that model, here a firm cannot switch to the knowledge base pursued by the other firm (thus guaranteeing a tie) if the latter has private access to it.

**4.2 Risk and Correlation Choices: Large Risk-aversion**

Consider the social objective function:

\[
\max_{(a_1, a_2) \in \{0, 1\}^2} W(a_1, a_2) = V_1(a_1, a_2) + V_2(a_1, a_2)
\]

\[
= \int_{-b}^{b} \int_{-b}^{b} \left\{ \int_{-b}^{b} U[K + a_1 \theta + (1 - a_1) \eta_1] f(\eta_1) d\eta_1 \right\} d\theta
\]

24
\[ \mathcal{F}(Z_1) \mathcal{U} \{ K + a_2 \theta + (1 - a_2) \eta_2 \} \mathcal{f}(\eta_2) d\eta_2 g(\theta) d\theta \]  

(23)

where \( Z(\cdot) \) is defined in equation (16) above. It is clear (and demonstrated below) that for sufficiently risk-averse \( \mathcal{U}(\cdot) \), the social optimum \( (a_1^S, a_2^S) \) will be interior, in order to reduce the riskiness of the social payoff through diversification among knowledge bases. Unlike the model of section 3, diversification between \( \eta_1 \) and \( \theta \) through \( a_1 \) does not result in any correlation with \( z_2 \) if \( a_2 = 0 \), for example, and thus the results in Proposition 2 do not apply. Unfortunately, a general characterization of the optima in (23) has not been obtained. It is not clear, for example, that an interior optimum involves \( a_1^S = a_2^S \), i.e., symmetry, even with i.i.d. distributions for \( \{ \tilde{\eta}_i, \tilde{\theta} \} \). That would be the case if \( \mathcal{W}(a_1, a_2 = c - a_1) \) is concave in \( a_1 \) - we have partial results suggesting that this is true if \( \mathcal{U}(\cdot) \) is sufficiently risk-averse.

Difficult problems also arise in characterizing Nash equilibria. If \( (a_1, a_2) \) are interior, then the technique of proof used for symmetric single-crossing distributions in Proposition 5 breaks down. For example, it is now the conditional distribution function of \( (\tilde{z}_1(a_1) \mid \tilde{z}_2 = z_2) \) whose properties need to satisfy conditions analogous to equations (19a,b). With correlation between \( (\tilde{z}_1, \tilde{z}_2) \), however, there are now “mean” effects as well as “risk” effects in the shifts of these conditional distributions even if the unconditional distributions of \( \{ \tilde{\eta}_i, \tilde{\theta} \} \) are assumed to be symmetric, e.g., uniform.

In what follows, we provide an illustrative set of calculations and results, contrasting symmetric social optima with symmetric Nash equilibria, for the special case of uniformly distributed i.i.d \( \{ \tilde{\eta}_2, \tilde{\theta} \} \). Even there we can only show for the special case of quadratic utility that the natural intuition to be derived from Proposition 5 survives, i.e., that Nash equilibria involve more risk \( (a_1^* < a_2^S) \) than in the social optimum. We conjecture, but have not proved, that similar results will hold with more general utility functions given normal distributions for the underlying random variables, the reason being that conditional distributions of \( (\tilde{z}_i(a_i) \mid \tilde{z}_j = z_j) \) would continue to inherit the symmetry properties utilized in Proposition 5. However, the impact of conditioning on conditional means and variances (which are functions of \( a_i \) given \( a_j \neq 0 \)) makes it non-trivial to obtain this extension of Proposition 5. The following result describes the nature of symmetric social optima.

**Proposition 7.** Assume that \( \{ \tilde{\eta}_i, \tilde{\theta} \} \) are i.i.d. with zero mean and densities symmetric around zero. Then the symmetric social optimum in (23) is characterized by \( a_1^S = a_2^S = \)
\( a \in [0, \frac{1}{2}) \).

**Proof:** From (23) and (16) we obtain, on defining \( \bar{W}(a, a) \equiv \bar{W}(a) \), that

\[
\bar{W}'(a \mid a < 1) = 2 \int_{\bar{\theta}}^{\bar{\theta}} \int_{\bar{\eta}_2}^{\bar{\eta}_2} (\theta - \eta_1) U'[K + a\theta + (1 - a)\eta_1] f(\eta_1) d\eta_1 f(\eta_2) d\eta_2 g(\theta) d\theta \quad (24a)
\]

with second derivative

\[
\bar{W}''(a \mid a < 1) = 2 \int_{\bar{\theta}}^{\bar{\theta}} \int_{\bar{\eta}_2}^{\bar{\eta}_2} (\theta - \eta_1)^2 U''[K + a\theta + (1 - a)\eta_1] f(\eta_1) d\eta_1 f(\eta_2) d\eta_2 g(\theta) d\theta \quad (24b)
\]

\[
\leq 0 \quad \text{since} \quad U''(\cdot) \leq 0
\]

It is also clear that

\[
\bar{W}(0) > \bar{W}(1) \quad (24c)
\]

from the i.i.d. assumption on \( \bar{\eta}_i, \bar{\theta} \) implying first-order stochastic dominance of the social payoff at \((0,0)\) over that at \((1,1)\). Now examine

\[
\bar{W}'(a = \frac{1}{2}) = 2 \int_{\bar{\theta}}^{\bar{\theta}} \int_{\bar{\eta}_2}^{\bar{\eta}_2} (\theta - \eta_1) U'[K + \frac{1}{2} \eta_1 + \frac{1}{2} \theta] g(\eta_1) d\eta_1 f(\eta_2) d\eta_2 g(\theta) d\theta \quad (24d)
\]

It is clear that if the innermost integral were untruncated, then the integral in (24d) would be zero by symmetry. Owing to the truncation, given equiprobable permutations of \( \{\theta, \eta_1\} \) with \( |\theta - \eta_1| = C > 0 \) it is the case that if \((\theta, \eta_1)\) with \( \theta > \eta_1 \) is included in the integral in (24d), then so is its permutation with \( \theta, \eta_1 \) “switched”, but the converse is not true with strictly positive probability. Hence \( \bar{W}'(a = \frac{1}{2}) < 0 \) in (24d), thus proving the result given (24b). Q.E.D.

**Example** Let

\[
U(z) = \frac{z^\beta}{\beta} \quad \beta \leq 1, \quad \beta \neq 0 \quad (25a)
\]

\[
k = 1, \quad \bar{b} = -\bar{b} = b \in (0, 1), \quad \{\bar{\eta}_1, \bar{\theta}\} \sim \text{uniform} \quad [-b, b] \quad (25b)
\]

For which, using (24a) we obtain

\[
\bar{W}'(0) = \frac{-1}{4b^2} (1 + b)^{\beta + 1} \left( \frac{2b}{(\beta + 1)} - \frac{(1 + b)}{(\beta + 1)(\beta + 2)} \right)
\]
\begin{equation}
+(1 + b)^\beta \left\{ \frac{(1 + b)}{\beta(\beta + 1)} - \frac{2b}{\beta} \right\}
\end{equation}

It is easy to verify that (i) \( \tilde{W}^\prime(0) \leq 0 \) for \( \beta \geq -1 \), and (ii) \( \tilde{W}^\prime(0) > 0 \) for \( \beta \) sufficiently close to -2. Thus for relative risk-aversion levels less than 2, \( (0,0) \) is the symmetric social optimum, whereas for relative risk-aversion level around 3, that is no longer the case, i.e., an interior symmetric \((a,a)\) dominates \((0,0)\), and thus \((0,1)/(1,0)\) as well, given the i.i.d. assumption on \( \{ \tilde{\eta}_i, \tilde{\vartheta}_i \} \).

The following result summarizes our findings on symmetric Nash equilibria with uniformly distributed \( \{ \tilde{\eta}_1, \tilde{\vartheta} \} \), for strictly concave \( U(\cdot) \) such that the social optimum with the same utility function \( U(\cdot) \) is interior.

**Proposition 8:** Assume \( \{ \tilde{\eta}_i, \tilde{\vartheta} \} \) are uniformly distributed on \([-\breve{a}, \breve{a}]\) and that the symmetric social optimum is \( (\bar{a}^S_1 = \bar{a}^S_2 = \bar{a}^S \in (0, \frac{1}{2}) ) \). Then

(a) \( (\bar{a}^S, \bar{a}^S) \) is not a Nash equilibrium, and neither is \( (a, a) \) for any \( a \) satisfying \( a^S < a \leq \frac{1}{2} \).

(b) There exists \( a^*_A \in [0, a^S) \) such that \( (a^*_A, a^*_A) \) satisfies the local first and second order conditions for a Nash equilibrium.

(c) There exists \( a^*_B \in (\frac{1}{2}, 1) \) such that \( (a^*_B, a^*_B) \) satisfies the local first-order conditions for a Nash equilibrium.

(d) If \( U(\cdot) \) is quadratic, there is no \( a > a^S \) such that \( (a, a) \) is a Nash equilibrium.

**Proof:** Consider the private first-order condition, using (15a) and (16), that

\[
\frac{\partial V_k(a, a^*)}{\partial a} \bigg|_{a=a^*<1} = \frac{1}{8b^3} \int_{-b}^{b} \int_{-b}^{b} \int_{\eta_1}^{b} (\theta - \eta_1) U'[K + a^*\theta + (1-a^*)\eta_1] d\eta_1 d\eta_2 d\theta
\]

\[
+ \frac{1}{8b^3} \int_{-b}^{b} \int_{-b}^{b} \int_{\eta_1}^{b} (\theta - \eta_2) U[K + a^*\theta + (1-a^*)\eta_2] d\eta_2 d\theta
\]

\[
= \frac{1}{2} \tilde{W}'(a^*) + \frac{1}{8b^3} Q(a^*)
\]

where

\[
Q(a) \equiv \int_{-b}^{b} \int_{-b}^{b} \frac{(\theta - \eta)}{(1-a)} U[K + a\theta + (1-a)\eta] d\eta d\theta
\]

\[
(26a)
\]

\[
(26b)
\]

\[
(26c)
\]
Note that the first term in (26b) represents the effect on firm 1's payoff conditional on winning, and is proportional to the effect of varying the symmetric strategy choice on the planner's welfare. The second term is the source of divergence between the planner and firm 1's interests: it picks up the effect of varying firm 1's strategy on the probability of its winning. Given the result of Proposition 7, it suffices to demonstrate the following properties for $Q(a)$:

(i) $Q(a) < 0$ for all $a < \frac{1}{2}$

(ii) $Q\left(\frac{1}{2}\right) = 0$

(iii) $Q(a) > 0$ and $Q'(a) > 0$ for all $a > \frac{1}{2}$, and

(iv) $\lim_{a \to 1} Q(a) = +\infty$

Result (a) follows from (i) and (ii) and the fact that $a^S < \frac{1}{2}$, $W'$ is concave and $W'(\frac{1}{2}) < 0$. That there exists $a^*_A < a^S$ and $a^*_B > \frac{1}{2}$ satisfying the local first-order conditions for a Nash equilibrium follows from (i) - (iv), the continuity of $W'(\cdot)$ and $Q(\cdot)$ and the fact that $W'(\cdot)$ is bounded over $[0, 1]$.

To prove (i) - (iv), note that given any pair $(\theta, \eta)$ and its equiprobable permutation, the pair with a positive value for $(\theta - \eta)$ is given a lower (resp. higher) weight $U[K + a\theta + (1-a)\eta]$ when $a$ is less (resp. greater) than $\frac{1}{2}$. Hence $Q$ is negative for $a < \frac{1}{2}$, zero at $a = \frac{1}{2}$ and positive for $a > \frac{1}{2}$. Further

$$Q'(a) \big|_{a<1} = \frac{1}{(1-a)} \int_{-\theta}^{\theta} \int_{-\eta}^{\eta} (\theta - \eta)^2 U'[K + a\theta + (1-a)\eta] d\theta d\eta + \frac{1}{(1-a)} Q(a)$$

which is positive whenever $Q(a)$ is, e.g., when $a > \frac{1}{2}$. (iv) follows from the boundedness of $(1-a)Q(a)$ as $a$ approaches 1.

To complete the proof of (b) we need to show that there exists $a^*_A < a^S$ such that $(a^*_A, a^*_A)$ satisfies the local first and second-order conditions for a Nash equilibrium. Now either $[\frac{1}{2} W'(a) + \frac{1}{8\theta^3} Q(a)] \leq 0$ for all $a \in [0, a^S)$ in which case $(0, 0)$ satisfies all the local conditions for a Nash equilibrium, or $[\frac{1}{2} W'(a) + \frac{1}{8\theta^3} Q(a)]$ is positive for some $a \in [0, a^S]$. In the latter case, continuity of $W'(\cdot)$ and $Q(\cdot)$ implies the existence of $a^*_A \in [0, a^S)$ such that $[\frac{1}{2} W'(a^*_A) + \frac{1}{8\theta^3} Q(a^*_A)] = 0$ and $[\frac{1}{2} W''(a^*_A) + \frac{1}{8\theta^3} Q'(a^*_A)] \leq 0$. It is easy to check that

$$\frac{\partial^2 V}{\partial a^2}(a, a^*) \big|_{a=a^*<1} = \frac{1}{2} W''(a^*) + \frac{1}{8\theta^3} [Q'(a^*) + \frac{1}{(1-a^*)} Q(a^*)]$$  \hspace{1cm} (27)
Since $Q(a^*_A) < 0$, the result follows.

To prove (d), note that when utility is quadratic, $U''(\cdot)$ is a constant and this implies $[\frac{1}{2}W'(a) + \frac{1}{8a_2}Q(a)]$ is convex over $(\frac{1}{2}, 1)$. Since $[\frac{1}{2}W'(a) + \frac{1}{8a_2}Q(a)]$ is negative at $a = \frac{1}{2}$ and zero at any $a^*_B > \frac{1}{2}$ where the first order conditions for a Nash equilibrium are satisfied, it follows that $\frac{1}{2}W'(a^*_B) + \frac{1}{8a_2}Q''(a^*_B) > 0$. Since $Q(a) > 0$ for all $a > \frac{1}{2}$, (27) is positive when $a^* = a^*_B$, i.e., $a^*_B$ is a locally utility-minimizing response to the other firms' strategy $a^*_B$. Thus $(a^*_B, a^*_B)$ cannot be a Nash equilibrium. Q.E.B.

Thus under the special circumstances of Proposition 8(d) there may be a symmetric equilibrium involving smaller correlation (and higher risk) than at the symmetric social optimum, but there cannot exist one with the same or higher correlation. Part of the problem with analyzing the large risk-aversion case is that interior social optima cannot be implemented as an equilibrium (because of the effect of strategies on the probability of winning that always "bites" in the interior), and no direct global arguments for a Nash equilibrium can be presented as in the small risk-aversion case. On the other hand possible nonconcavities of payoff functions make it difficult to rely on local conditions for equilibrium. Nevertheless it does become clear that with large (common) risk aversion, the winner-take-all feature can produce a genuine distortion relative to the first-best. Further if there is a divergence of risk-attitudes (i.e., firms are sufficiently risk-averse, while the planner is risk-neutral), the social optimum (involving maximum risk and zero correlation), may not be implementable. What is somewhat surprising is that large-risk aversion may cause equilibria to be involved with less than socially optimal correlation (if the planner is equally risk-averse), i.e., with a smaller weight on the publicly available knowledge base $\tilde{\theta}$. To understand why this might be the case, it is useful to separate the risk effect from the correlation effect and analyze them separately.

4.3 Pure Correlation Effects

To isolate pure correlation effects, where changes in strategies do not affect the marginal distribution of either firm's R & D performance, consider the case where the knowledge bases $\tilde{\theta}$, $\tilde{\eta}_1$ and $\tilde{\eta}_2$ are independent, each with zero mean (and symmetric distributions if
\( K > 0 \):
\[
\tilde{z}_i = K + a_i^{\frac{1}{2}} \tilde{\rho} + (1 - a_i)^{\frac{1}{2}} \tilde{\eta}_i, \quad a_i \in [0, 1] \quad \text{for} \quad i = 1, 2
\]  
(28)

Assume that the only way the joint distribution of \( \tilde{z}_1 \) and \( \tilde{z}_2 \) depends on \( a_1 \) and \( a_2 \) is through their product \( a_1 a_2 \), which is the square of their correlation. The case where \( \tilde{\eta} \) and \( \tilde{\eta}_i \) are normally distributed is an example of this. However, if \( \tilde{z}_i \) becomes negative, it is natural to assume "free disposal", i.e., whenever \( \tilde{z} < 0 \) the winner can dispose of the "discovery" and obtain a payoff of 0, equal to the loser's payoff. Hence setting \( U(0) = 0 \) without loss of generality,
\[
U(z_i) = 0 \quad \text{for all} \quad z_i < 0
\]
(29)

The result for this model can now be stated.

**Proposition 9**

Consider the model described by (28) and (29). As long as \( U(\cdot) \) is monotonically nondecreasing and \( E_\theta[U(K + \theta)] \) is finite, the social optimum requires at least one of the firms to choose \( a_i = 0 \), i.e., the situation of zero correlation. This is the unique pure strategy Nash equilibrium outcome (under non-split patents).

**Proof**

With non-split patents, social welfare can be written as the sum of ex ante welfare \( \tilde{V}_1 \) and \( \tilde{V}_2 \) of the two firms where
\[
\tilde{V}_1(a_1, a_2) = \int_{-\infty}^{\infty} U(z_1) \text{Prob}(\tilde{z}_2 \leq z_1 \mid \tilde{z}_1 = z_1) dF_N(z_1)
\]
(30)
where \( F_N(\cdot) \) denotes the marginal distribution function of \( \tilde{z}_1 \) independent of \( a_1 \) and \( a_2 \) by assumption. Now the distribution of \( \tilde{z}_2 \) conditional on \( \tilde{z}_1 = z_1 \) and (hence) \( \tilde{V}_1 \) depend on \( (a_1, a_2) \) only through \( \sqrt{a_1 a_2} \), the correlation between \( \tilde{z}_1 \) and \( \tilde{z}_2 \); similarly for firm 2's payoff and social welfare \( \tilde{W} \). This and the symmetry of the game implies \( \tilde{V}_1 = \tilde{V}_2 = \frac{1}{2} \tilde{W} \). Let \( V(\rho) \) denote the welfare of a representative firm when the correlation between \( \tilde{z}_1 \) and \( \tilde{z}_2 \) is \( \rho \). To show that the social optimum involves putting \( \rho = 0 \), we shall demonstrate the stronger result that \( V \) and hence social welfare is strictly decreasing in \( \rho \).

First note that the social payoff when \( \rho = 1 \) is first-order-stochastically dominated by the payoff corresponding to any \( \rho < 1 \). So take any value of \( \rho \in (0, 1) \). Since \( V \) depends only on the product of \( a_1 \) and \( a_2 \), without loss of generality put \( a_1 = 1 \) and \( a_2 = \rho \). Given any
realization \((\theta, \eta_1, \eta_2)\) firm 1 wins if and only if \(\theta > \ell(\rho)\eta_2\), where \(\ell(\rho) \equiv (1 - \rho^{\frac{1}{2}})^{-1}(1 - \rho)^{\frac{1}{2}}\) is strictly increasing in \(\rho\) over \((0,1)\).

Hence the ex ante welfare of firm 1 is (for \(\rho < 1\)):

\[
\hat{V}_1(1, \rho) \equiv V(\rho) = \int_{-\infty}^{\infty} \int_{\ell(\rho)\eta_2}^{\infty} U(K + \theta) dF_N(\theta) dF(\eta_2)
\]

Integrating (31) by parts and using finiteness of \(E_\theta[U(K + \theta)]\),

\[
V(\rho) = \int_{-\infty}^{\infty} U(K + \eta_3) f_N(\eta_3) F\left(\frac{\eta_3}{\ell(\rho)}\right) d\eta_3 \quad \text{where} \quad \eta_3 \equiv \eta_2 \ell(\rho)
\]

implying

\[
V'(\rho) = \frac{-\ell'(\rho)}{[\ell(\rho)]^2} \int_{-\infty}^{\infty} U(K + \eta_3) f\left(\frac{\eta_3}{\ell(\rho)}\right) \eta_3 f_N(\eta_3) d\eta_3
\]

which is negative, using \(U(x) \geq 0\) for all \(x\) by (29) and \(\ell'(\rho) > 0\).

That the situation of zero correlation is the unique pure strategy equilibrium outcome is obvious, because the fact that \(V(\rho)\) is decreasing in \(\rho\) and \(\rho = \sqrt{a_1 a_2}\) implies each firm has a dominant strategy of choosing \(a_i = 0\). The set of pure strategy Nash equilibria consists of \((a_1 = 0, 0 \leq a_2 \leq 1)\) and \((0 < a_1 \leq 1, a_2 = 0)\). Q.E.D.

Thus in this model less correlation leads to higher social and individual welfare irrespective of risk-attitudes. Further, no distortions are introduced by the winner-take-all reward scheme; in fact social and individual welfare functions are coincident. The social desirability of smaller levels of correlation probably stems from the fact that conditional on any realization of \(\tilde{z}_2 = z_2\), a lower correlation increases the conditional variance of \(\tilde{z}_1\) given \(\tilde{z}_2 = z_2\) and thus increases the probability that (for given conditional mean) the realization of \(\tilde{z}_1\) exceeds \(z_2\), and hence so also does the social payoff. However the intuition is not altogether clear, because of two complicating factors: (a) conditional on \(\tilde{z}_1\) exceeding \(z_2\) and \(\tilde{z}_2 = z_2\), a higher variance of \(\tilde{z}_1\) and thus also of social payoff is undesirable, under risk-aversion, and (b) when \(z_2\) is positive the conditional mean of \(\tilde{z}_1\) given \(\tilde{z}_2 = z_2\) is reduced whenever the correlation is reduced, and this can outweigh the effect of a higher conditional variance.

A better source of intuition is the fact that the planners welfare is correlation-averse in the sense of Epstein and Tanny (1980), i.e.,

\[
U(\max(x_1, x_2)) - U(\max(x_1, y_2)) \leq U(\max(y_1, x_2)) - U(\max(y_1, y_2))
\]
whenever $(x_1 - y_1)(x_2 - y_2) > 0$, with strict inequality for some pairs $(x_1, x_2)(y_1, y_2)$ satisfying the latter condition. Since for a bivariate normal distribution an increase in the Pearsonian correlation coefficient corresponds to an increase in correlation in the Epstein-Tanny sense, it follows that independence is optimal from the planner's point of view. In fact this approach suggests a way of generalizing the result of Proposition 9 to a model where the marginal distributions of $z_i$ are independent of $(a_1, a_2)$, but their joint distribution depends on $c(a_1, a_2)$, a continuous, symmetric and increasing real valued function defined on $[0, 1]^2$. Given any pair $(a_1^*, a_2^*)$, $(a_1, a_2)$ of strategies, where $c(a_1^*, a_2^*) > c(a_1, a_2)$, $z_1$ and $z_2$ are more correlated in the Epstein-Tanny sense under $(a_1^*, a_2^*)$ than under $(a_1, a_2)$; if $a_1 = a_2 = 0(a_1 = a_2 = 1)$, $z_1$ and $z_2$ are independent (perfectly and positively correlated). Then independent research is both socially optimal and the unique Nash equilibrium outcome.


In this section we isolate the role of pure risk effects, and abstract from mean or correlation effects. An appropriate model will clearly require that no knowledge base is publicly available. Assume that firm $i$ has a range $[0, 1]$ of possible strategies, and strategy $a_i$ leads to a stochastic R & D outcome $z_i(a_i)$ whose mean is independent of $a_i$ but $a^* > a$ implies $z_i(a^*)$ is less risky than $z_i(a)$ in the sense of second-order stochastic dominance. Further, irrespective of $a_1$ and $a_2$, $z_1$ and $z_2$ are distributed independently. To compare results of such a model with the model of the previous section, we need to impose similar regularity conditions (i.e., symmetry and single-crossing property) on the distributional characteristics of $z_i$ (for similar reasons).

**Proposition 10**

In the model described above:

(a) The social optimum for a risk-neutral planner is $(0,0)$, i.e., involving maximum riskiness for each firm.

(b) If (i) firms have utility functions $U(\cdot)$ and a planner with identical utility function has $(0,0)$ as an optimum, and (ii) $z_i$ is distributed symmetrically around its mean, and the

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6 See also Hadar and Russell (1974) and Levy and Paroush (1974).


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distribution functions corresponding to any pair of strategies \((a_i^*, a_i)\) for firm \(i\) intersect only once, then \((0,0)\) is the unique pure strategy Nash equilibrium.

(c) If (i) firms have utility functions \(U(\cdot)\) and a planner with identical utility function has \((a_1^S, a_2^S) \neq (0,0)\) as an optimum, and (ii) \(\hat{z}_i\) has the same distributional properties as in b(ii) above, then no pure-strategy pair \((a_1, a_2)\) for the two firms with \(a_1 \geq a_1^S, a_2 \geq a_2^S\) can be an equilibrium. If \(a_1^S = a_2^S = a^S > 0\), and the distribution of \(\hat{z}_i\) is continuously differentiable in \(a_i\), there exists \(a^* \in [0, a^S]\) such that \((a^*, a^*)\) satisfies the local first-order conditions for an equilibrium.

The proof of Proposition 10 is similar to proofs in Sections 4.1 and 4.2, and thus omitted. Interior social optima may arise with sufficiently large risk-aversion, and in such cases there can be no equilibria with both firms choosing the same or lower levels of risk. Thus a distortion is introduced in cases of large risk aversion, just as in Section 4.2: there is a tendency toward excessive risk-taking. To examine this more precisely, consider the following example.

**Example**

Suppose that

\[
\hat{z}_i(a_i) = K + (1 - a_i)\hat{\eta}_i
\]

where \(a_i \in [0, \frac{1}{2}]\). \(\hat{\eta}_1\) and \(\hat{\eta}_2\) are identically and independently distributed with zero means on \([-b, b]\) where \(b > 0\) and \(K - b \geq 0\). Further \(\hat{\eta}_1\) and \(\hat{\eta}_2\) have densities that are symmetric around 0, so \(a_i^* > a_i\) implies \(\hat{z}_i(a_i^*)\) is less risky than \(\hat{z}_i(a_i)\). Then social welfare is

\[
W(a_1, a_2) = \int_{-b}^{b} [U(K + (1 - a_1)\eta_1)F(\frac{1 - a_1}{1 - a_2}\eta_1) + \int_{\frac{1 - a_1}{1 - a_2}\eta_1}^{b} U(K + (1 - a_2)\eta_2)dF(\eta_2)]dF(\eta_1)
\]

(33)

Using \(\bar{W}(a)\) to denote \(W(a, a)\), it is easy to check \(\bar{W}(\cdot)\) is strictly concave as is \(W(a_1, a_2)\) in \(a_1\) and \(a_2\) separately (as long as \(U(\cdot)\) is strictly concave). Also, using \(W^*(a, c)\) to denote \(W(a, c - a)\), \(W^*(a, c)\) is strictly concave in \(a\). Given the symmetry of the game, it follows that the social optimum is symmetric and unique, and is interior for sufficiently large risk-aversion.
The welfare of firm 1 is

$$V_1(a_1, a_2) = \int_{-\beta}^{\beta} U(K + (1 - a_1)\eta_1)F(\frac{1 - a_1}{1 - a_2} \eta_1) \ dF(\eta_1)$$

(34)

If the distribution of $\eta_1$ is unimodal, it can be checked that $V_1$ is strictly concave in $a_1$. Hence there always exists $a^* < a^S$ (whenever $a^S > 0$) such that $(a^*, a^*)$ is an equilibrium. However it does not seem possible to rule out the existence of asymmetric equilibria, including those where one firm chooses a strategy less risky than at $a^S$.

6. CONCLUSION

We can now summarize the main results of the paper. The major question posed was whether the winner-take-all feature associated with the patent system generated any significant distortions relative to what is socially optimal, in particular with respect to the degree of specialization of research approach and attendant effects on the riskiness of and correlation between the R & D performances of different competitors. And if so, what are the nature of these distortions and what factors do they depend on?

As for the first question, we were surprised to find a large class of situations where the winner take all reward system led to socially optimal outcomes. The existence of significant distortions requires both (a) the existence of some knowledge bases or approaches that are not commonly available to all competitors, and (b) R & D “technologies” wherein strategies chosen by competing firms have significant effects on the riskiness of their performance.\(^8\)

This conclusion tends to emerge from Sections 3 and 4.3. Even if both these conditions are met, extreme specialization (where competing firms choose polar research strategies and high risks) may be both socially optimal and the unique equilibrium outcome, provided all parties concerned are not too risk-averse and the distributional characteristics of different knowledge bases are well-behaved (i.e., satisfy symmetry and the single crossing condition described in Proposition 5). While these two conditions are not strictly necessary, some regularity conditions seem to be required to ensure firms prefer riskier distributions ceteris

\(^8\) (a) tends to suggest that at early stages of technological advancement where no competitor has had much previous “experience” the distortions produced by the winner-take-all feature are minimal - at this stage different competitors tend to specialize exclusively on alternative polar approaches and take large risks, which is precisely what society wants.
paribus to exploit the "option effect." Thus within the class of models (with at least some private knowledge and some risk effects) we have examined so far, significant distortions arise if and only if either the common level of risk aversion is large enough, or distributional characteristics of alternative research approaches are "irregular." Alternatively they could arise from disparities in risk attitudes between competing firms and the social planner: equilibrium would then involve levels of specialization and risk-taking that are too small from the social point of view.

The precise nature of distortions produced by the winner-take-all system seems to be driven by the nature of risk effects of R & D strategy choices. Competing firms typically (with well-behaved distributions) tend towards excessive risk-taking, when planner and firm risk-attitudes coincide. This arises from the option effect: i.e., the effect of strategy choice of a particular firm on the probability of its winning the patent, something the social planner does not care about. This divergence between social and private incentives obviously does not arise in situations where social optimality requires firms to choose their riskiest strategies, i.e., involves a "corner" solution. Thus, somewhat paradoxically, there is a tendency toward excessive risk-taking only when there is enough (common) risk-aversion.

For the nature of divergences between the equilibrium and socially optimal levels of correlation, the model of Section 4.3 suggests that correlation effects on their own do not lead to a distortion. In that model both social planner and competing firms prefer smaller levels of correlation, irrespective of what their risk-attitudes are. However the model of Section 4.2 suggests that where both risk and correlation effects are present, the (symmetric) equilibrium level of correlation tends to be lower than socially optimal. This can be explained by the tendency for firms to specialize on their private knowledge bases more than is socially optimal, arising from their preferences for riskiness. Thus no justification can be offered for the intuitive belief that the winner-take-all feature of the reward system encourages excessive imitation of research strategies by competing firms: the results suggest precisely the opposite in cases of large risk-aversion. The only situation where the numerous multiple or coincidental discoveries in the history of science and technology can be interpreted by the model as a result of "too much" correlation in equilibrium is where competing firms are very much more risk-averse than the social planner, and end up specializing less on their private knowledge bases than is socially optimal.
References


