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BY
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Tax Effects of Production and Finance

by

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Abstract

This paper develops a production-capital structure equilibrium in which the firm's optimal real and financial decisions are simultaneously determined in an environment with production uncertainty and with corporate and progressive personal taxes. The firm's choice of both a production and capital structure plan leads to an equilibrium in which the firm's optimal level of debt-related (i.e., interest) and nondebt-related (e.g., depreciation) tax shields are endogenously determined, thus generalizing the work of DeAngelo and Masulis [DM, 1980]. When comparative statics are performed on the equilibrium conditions, however, the DM prediction of an inverse relationship between financial leverage and the level of nondebt-related tax shields is not upheld. We find instead that the relationship between financial leverage and nondebt-related tax shields is highly complex and nonmonotonic. Our theoretical findings thus provide a possible explanation for why the DM hypothesis has received relatively little support from recent empirical studies designed to test the determinants of capital structure.
1. Introduction

It is customary in finance and economics to treat the real and financial decisions of the firm as separate and independent issues despite an apparent linkage between production and finance. The economic theory of the firm, for example, has developed by fixing or ignoring its financial policies, while the financial theory of the firm has developed by fixing its investment, input mix and output policies. This treatment is particularly evident in the literature on the tax effects of corporate finance, where equilibria in the financial markets are determined without regard for the tax effects on factor inputs and the corresponding output in the real sector. Notable examples include the pathbreaking papers by Modigliani and Miller [MM, 1963] and Miller[1977], and the subsequent literature which attempts to generalize their results.

Although the assumed independence of production and finance has often times allowed financial economists to obtain intuitively appealing results, an issue still remains as to whether simultaneous optimization in both the real and financial sectors of the economy will generate new insights regarding firm behavior, or perhaps overturn or modify existing results. An early attempt to link the financial policy of the firm with the firm's real decisions was by Hite [1977], who modifies the MM[1963] tax-adjusted valuation model by incorporating production. Using the single-period CAPM and the tax framework of MM[1963], Hite formulates a model in which he examines the impact of financial leverage on the optimal levels of capital and labor employed by the firm. He shows, under a restrictive set of assumptions, that an increase in financial leverage will cause an unambiguous decline in the "user cost of capital", while leaving the "user cost of labor" unchanged, thus leading to an increase in the optimal capital intensity of the firm. Hite's analysis,
however, places implicit limits on the amount of debt financing the firm can obtain by assuming away default, and ignores the effect of personal taxes on the relative cost of debt financing. These assumptions implicitly make the firm's pretax cost of debt financing independent of its capital structure and, hence, provides for a net tax advantage to financial leverage (at the margin) in equilibrium.

In some sense, one can also view the DeAngelo and Masulis [1980] analysis of capital structure as a partial recognition of the interactions between the real and financial decisions of the firm. Under the Miller [1977] tax environment and risk-neutrality, they demonstrate that the existence of investment-related tax shields, such as depreciation or investment tax credits, is sufficient to overturn the limited leverage irrelevance theorem of Miller. Contrary to Hite, the DeAngelo-Masulis model predicts a negative relationship between the level of available investment-related tax shield substitutes for debt (a proxy for capital intensity) and financial leverage. Unfortunately, however, the DeAngelo-Masulis analysis falls short of fully incorporating the productive side of the economy, since the nondebt-related tax shields are exogenous to their model.

The purpose of this paper is to develop a unified theory of the firm that permits simultaneous behavior in both the real and financial sectors of the economy. We incorporate risky debt, risk-aversion and progressive personal taxation into an environment in which firms are allowed to make simultaneous adjustments in their production and capital structure plans. The production-capital structure equilibrium also endogenizes the determination of the firm's optimal nondebt-related tax shields, thus generalizing the work of DeAngelo and Masulis [1980]. We show, among other things, that when production (i.e., factor inputs) and capital structure choices are made simultaneously, the
DeAngelo-Masulis hypothesis of an inverse relationship between the level of investment-related tax shields and financial leverage is no longer upheld.

The paper is organized as follows. Section 2 develops a two-date, state preference model of individual consumption-investment choice and presents the basic problem. In Section 3 an initial portfolio equilibrium is established for a given set of production and financing decisions for firms. Section 4 allows firms to make changes in their production and capital structure plans so as to maximize the welfare of ex ante shareholders. Given equal access to the financial markets for all investors, we demonstrate that initial shareholders will unanimously support the production and capital structure plans that maximize firm value. The necessary conditions for an optimal production and finance clearly demonstrate that the real and financial decisions of the firm are related and, hence, must be made simultaneously. However, given the firm's optimal production decisions, our analysis implies an interior optimal capital structure for the firm at the point where the net tax subsidy on corporate debt is driven to zero at the margin.

In Section 5 comparative statics are performed on the equilibrium conditions, yielding several propositions. We find that financial leverage and capital intensity, or, equivalently, nondebt-related tax shields (e.g., depreciation), are in general not strictly inversely related when the real and financial decisions of the firm are made simultaneously, contrary to the proposed hypothesis of DeAngelo and Masulis [1980]. Instead, we show that the general relationship between financial leverage and capital (or labor) intensity is highly complex and is neither strictly increasing nor strictly decreasing. Section 6 concludes the paper.
2. The Model

The production-financing decision of the firm is analyzed within the context of a two-date, state-preference model of individual consumption-investment choice under production uncertainty and with a capital market that permits risk sharing, although incompletely. Following Diamond [1967], uncertainty is assumed to enter the production functions of individual firms in a multiplicative fashion through a random variable reflecting the state of nature. Thus, the output of the \( j^{th} \) firm, if state \( \theta \) occurs, is a function of the firm's capital \( (K^j) \) and labor \( (L^j) \) intensities (which are current decision variables) and the state of nature. Formally, for an arbitrary firm \( j \),

\[
X^j(\theta) = f^j(\theta)g^j(K^j,L^j) \quad j = 1, \ldots, J
\]

(1)

Under multiplicative uncertainty, a change in the input variables, capital or labor, alters the scale of output but not the pattern of output across the states of nature.

Each individual \( i \) begins the first period with an initial endowment of cash, \( Y_{i1} \), and initial ownership shares in each firm \( j \), \( \alpha_{i1}^j \). Firms determine the size of their capital and labor inputs prior to the start of the first period and finance the capital expenditures \( K^j \) by selling corporate debt and equity claims to the end of period output. Given the firm's production and financing decisions, the total amount of funds that investors are willing to pay for the securities of firm \( j \) may exceed the capital investment \( K^j \), and any surplus is assumed to be distributed tax-free to the initial owners (those with \( \alpha_{i1}^j > 0 \)). Letting \( B^j \) and \( S^j \) represent the total market value of firm
j's debt and equity securities, respectively, the value of individual i's first-period wealth endowment is

\[ W_{1i} = Y_{1i} + \sum_j a_j^i (S_j^i + B_j^i - K_j^i) \]  \hspace{1cm} (2)

where \( S_j^i + B_j^i - K_j^i \) is the net present value of corporation j's capital budget.

Individuals allocate their initial wealth endowments \( W_{1i} \) in the first period between current consumption \( C_{1i} \) and investment in a portfolio of securities. Investment may take the form of either (risky and riskless) fully taxable corporate bonds, \( B_j^i \), or (risky and riskless) tax-exempt corporate equity, \( S_j^i \).\(^3\) Letting \( w_i^j \) and \( a_i^j \) represent the fractions of the total market value of firm j's bonds and stock, respectively, that are demanded by individual i, the individual's first-period budget constraint can be written as

\[ W_{1i} = \sum_j w_i^j B_j^i + \sum_j a_i^j S_j^i + C_{1i} \]  \hspace{1cm} (3)

In the second period, a state of nature is revealed and individuals consume the sum of their (exogenously specified) wage income, \( Y_{2i}(\theta) \), and investment income, \( I_{2i}(\theta) \), net of personal taxes. Investment income is composed of the dollar payoffs from the individual's holdings of corporate debt and equity securities, the returns from which are determined by firm cash flows. Because of the limited liability nature of corporate equity, the pretax dollar return to the bondholders of firm j, in state \( \theta \), will be\(^4\)

\[ Y_{j}^{\pi}(\theta) \equiv \min[R_j^i B_j^i, X_j^{\pi}(\theta) - \omega L_j^i + (1-\delta)K_j^i] \]  \hspace{1cm} (4)
where
\[ R^j = \text{one plus the promised rate of interest} \]
\[ \omega = \text{per unit wage rate} \]
\[ \delta = \text{economic rate of capital depreciation} \]

Since the payments made to corporate debt are fully deductible at the corporate level in computing taxable income, while payments on equity are not, the total state \( \theta \) dollar return to the equityholders of firm \( j \), after corporate taxes, will be

\[ Y^j_e(\theta) = \max\{0, X^j(\theta) - \omega L^j(\theta) + (1 - \delta) K^j - R^j B^j - \max\{0, X^j(\theta) - \omega L^j(\theta) - \gamma K^j - R^j B^j\}t^C\} \]  

(5)

where
\[ t^C = \text{corporate tax rate} \]
\[ \gamma = \text{depreciation rate used for tax purposes} \]

Individual \( i \)'s total second-period, state \( \theta \), investment income can thus be expressed as

\[ I_{2i}(\theta) = \sum_j u^j_{2i} Y^j_b(\theta) + \sum_j \alpha^j_{2i} Y^j_e(\theta) \]  

(6)

Note that in specifying \( Y^j_b(\theta) \) and \( Y^j_e(\theta) \), two assumptions about the nature of the corporate tax code are implicitly being made. First, it is implicitly assumed that both principal and interest are deductible in computing corporate taxes. This assumption, although not necessary, is made in order to derive expressions that are comparable to Miller [1977], DeAngelo and Masulis [1980], Taggart [1980] and others, who either make the same assumption or operate in the infinite horizon case. Second, in writing the state \( \theta \) corporate tax liability in (5) as
FIGURE 1
PRE-PERSONAL-TAX RETURNS TO BONDHOLDERS \((Y^j_{b}(\theta))\),
STOCKHOLDERS \((Y^j_{e}(\theta))\), GOVERNMENT TAX
AUTHORITY \((T^j(\theta))\) AND LABOR \((\omega L^j)\)

\[ x^j(\theta_1) = R^j_{B} + \omega L^j - (1-\delta)K^j \]
\[ x^j(\theta_2) = R^j_{B} + \omega L^j + \gamma K^j \]
T^J(\theta) \equiv \max[0, X^J(\theta) - \omega L^J - \gamma K^J - R^J_B J] t^C \tag{7}

it is assumed that the corporate tax schedule is asymmetric in its treatment of gains and losses. Gains are assumed to be taxed proportionately at the rate \( t^C \), while tax losses are assumed to provide a zero offset.\textsuperscript{5} This asymmetric treatment of gains and losses is consistent with the tax environment of DeAngelo and Masulis [1980], where oversheielding of corporate income results in the partial redundancy of tax shields.

Figure 1 illustrates how the pretax output of firm \( j \), \( X^J(\theta) \), is distributed to its four classes of claimants: (i) labor \( (\omega L^J) \), (ii) bondholders \( (Y^J_B(\theta)) \), (iii) stockholders \( (Y^J_e(\theta)) \), and (iv) the tax authority \( (T^J(\theta)) \). Because labor has priority over all of the firm's claimants, and because the firm is assumed to earn its wage expense with certainty, the payment to labor is riskless and is given by the horizontal line through \( \omega L^J \). Bondholders, on the other hand, hold a fixed claim which may be risky. The second period payment to bondholders is given by relation (4) and is represented by the concave function \( Y^J_B(\theta) \) in Figure 1, where \( X^J(\theta_1) \) is the boundary point of bankruptcy such that

\[
X^J(\theta) + (1-\delta)K^J - \omega L^J \leq R^J_B J \quad WX^J(\theta) \leq X^J(\theta_1)
\]

Since labor is always paid in full, the minimum payment that can possibly be made to bondholders occurs when \( X^J(\theta)=0 \) and is equal to the difference between the salvage value of capital, \( (1-\delta)K^J \), and the wage expense, \( \omega L^J \).

The equityholders of the firm hold a residual claim on the firm's assets and, therefore, receive nothing in the event of bankruptcy. As the firm's
pretax output level rises above the bankruptcy point \( X^J(\Theta_1) \), however, the payoff to shareholders increases continuously. Initially the increase is one for one with increases in \( X^J(\Theta) \), reflecting the fact that corporate taxes are zero over the range of nonpositive taxable corporate income levels, \( 0 \leq X^J(\Theta) \leq X^J(\Theta_2) \), where \( X^J(\Theta_2) \) represents the boundary point below which the firm has redundant tax shields. That is,

\[
X^J(\Theta) \leq R^J B^J + \omega L^J + \gamma K^J \quad \forall X^J(\Theta) \leq X^J(\Theta_2)
\]

Beyond \( X^J(\Theta_2) \), the firm is liable for paying corporate taxes equal to the corporate tax rate, \( t^C \), times the firm's taxable income level, \( X^J(\Theta) - R^J B^J - \omega L^J - \gamma K^J \). This tax liability is calculated in relation (7) and is given by the convex function \( T^J(\Theta) \) in Figure 1. Since the corporate tax liability increases linearly at a rate of \( t^C \) for output levels greater than \( X^J(\Theta_2) \), the payment to equityholders will also increase linearly over this range, but at the rate of \( (1-t^C) \). The equity stream just described is given by relation (5) and is depicted as the convex-concave function labeled \( Y^J e(\Theta) \) in Figure 1.

In the second period, each individual is assumed to pay personal taxes on his combined wage and bond income according to a progressive marginal tax schedule \( t(\pi) \). The total state \( \Theta \) tax liability for individual \( i \), therefore, is determined by integrating over the marginal tax schedule up to the level of state \( \Theta \) taxable income \( \pi^i(\Theta) \). Formally, for an arbitrary individual \( i \), the tax liability is

\[
T(\pi^i(\Theta)) = \int_0^{\pi^i(\Theta)} t(\pi) d\pi \quad (8)
\]
where

\[ T(\pi_i(\Theta)) \equiv \text{total tax liability of individual } i \text{ in state } \Theta. \]

\[ \pi_i(\Theta) \equiv \text{individual } i\text{'s state } \Theta \text{ taxable income} \]

\[ = Y_{2i}(\Theta) + \sum_j u_i^j b_j(\Theta) \]

The state \( \Theta \) consumption level of the \( i^{th} \) individual can thus be written as

\[ C_{2i}(\Theta) = Y_{2i}(\Theta) + \sum_j u_i^j b_j(\Theta) + \sum_j q_i^j e_j(\Theta) - T(\pi_i(\Theta)) \]  \hspace{1cm} (9)

3. An Initial Portfolio Equilibrium

In this section an initial portfolio equilibrium will be established, assuming that the production (input-output) and financing decisions of firms have already been made and are, therefore, fixed. Once the important characteristics of a portfolio equilibrium have been derived and discussed, firms will then be allowed to alter both their production and financing decisions in order to maximize shareholder welfare, and a new equilibrium will then be established. This will be the subject of the next section.

Given the production and financing decisions of all firms in the economy, each individual is assumed to behave so as to maximize his expected utility of two-date consumption by choosing the optimal level of current consumption, \( C_{1i} \), and the proper portfolio of securities to provide for future consumption. The formal problem for an arbitrary individual \( i \) can be expressed as
\[
\max_{j, \mu_i, \alpha_i} E_i(U_i) = \int_0^\delta U_i(C_{1i}, C_{2i}(\theta)) h_i(\theta) d\theta
\]  \hspace{1cm} (10)

subject to the constraints provided in (3) and (9) and where

\[ h_i(\theta) \equiv \text{individual } i\text{'s subjective probability density function over the}\]
\[ \text{state space } \theta, \theta \in [\underline{\theta}, \bar{\theta}]. \]

Substituting the budget constraints (3) and (9) directly into the individual's expected utility function for \( C_{1i} \) and \( C_{2i}(\theta) \), respectively, the first-order necessary conditions for an optimal portfolio can be expressed as follows.

\[
B^j = \int_0^\delta \rho_i(\theta) Y^j_d(\theta)(1-t(\pi_i(\theta))) d\theta \quad j = 1, \ldots, J
\]  \hspace{1cm} (11)

\[
S^j = \int_0^\delta \rho_i(\theta) Y^j_e(\theta) d\theta \quad j = 1, \ldots, J
\]  \hspace{1cm} (12)

where

\[ \rho_i(\theta) \equiv \text{individual } i\text{'s marginal rate of substitution between current}\]
\[ \text{consumption and state } \theta \text{ contingent future consumption.}\]

\[ = [U_{i2}(\theta)h_i(\theta)]/E_i(U_i) \]

\[ t(\pi_i(\theta)) \equiv \text{individual } i\text{'s state } \theta \text{ marginal personal tax rate.}\]

\[ = \frac{dT(\pi_i(\theta))}{d\pi} \]
The first-order conditions (11) and (12) imply that, at a portfolio equilibrium, all individuals will be in unanimous agreement about the marginal value of the overall return stream generated by each of the traded assets in the economy. This unanimous agreement about valuation is a direct consequence of the implicit assumption that all investors have unrestricted access to the financial markets and, therefore, can tax arbitrage freely. Under market incompleteness, however, there is no guarantee that marginal rates of substitution, \( \rho_i(\theta) \), or marginal personal tax rates, \( t(\pi_i(\theta)) \), will be driven to equality across all individuals. Thus, personal evaluations of the separate state-contingent payoffs generated by an asset (i.e., the separate components on the right-hand sides of (11) and (12)) may differ across individuals.

A portfolio equilibrium will be defined as a set of security demands \( (w_i^j, q_i^j) \), \( j = 1, \ldots, J \), for each individual \( i \) and a set of security market values \( (B^j, S^j) \), \( j = 1, \ldots, J \), such that (i) the set of security demands solves each individual's maximization problem when facing the set of market prices and (ii) all markets clear, \( \sum_i w_i^j = \sum_i q_i^j = 1 \), \( j = 1, \ldots, J \).

4. A Production-Capital Structure Equilibrium

The portfolio equilibrium of the previous section may be altered if we allow firms to adjust their production (input-output) and financing decisions. A production-capital structure equilibrium, therefore, will be defined as a set of production and capital structure plans for each firm \( j \), such that at a portfolio equilibrium relative to these production and financing plans, no further changes would be preferred by any firm's shareholders.
To investigate the personal preferences of investors for capital structure and production changes, we can simply differentiate each individual's expected utility function with respect to $B^j$, $K^j$, and $L^j$, holding all other variables constant. This yields $^{10,11}$

$$\frac{\partial E_i(U_{i1})}{\partial B^j} = E_i(U_{i1}) \left\{ (\bar{a}_i^j - u_i^j) \frac{\partial b^j}{\partial B^j} + (\bar{a}_i^j - a_i^j) \frac{\partial S^j}{\partial B^j} \right\}$$

$$+ \int_0^\infty \left[ u_{i2}(\theta) \frac{\partial Y_j(\theta)}{\partial B^j} (1 - t_i(\theta)) + a_i^j \frac{\partial Y_j(\theta)}{\partial B^j} \right] h_i(\theta) d\theta = 0$$

$$j = 1, \ldots, J$$

$$\frac{\partial E_i(U_{i1})}{\partial K^j} = E_i(U_{i1}) \left\{ (\bar{a}_i^j - u_i^j) \frac{\partial b^j}{\partial K^j} + (\bar{a}_i^j - a_i^j) \frac{\partial S^j}{\partial K^j} - 1 \right\}$$

$$+ \int_0^\infty \left[ u_{i2}(\theta) \frac{\partial Y_j(\theta)}{\partial K^j} (1 - t_i(\theta)) + a_i^j \frac{\partial Y_j(\theta)}{\partial K^j} \right] h_i(\theta) d\theta = 0$$

$$j = 1, \ldots, J$$

$$\frac{\partial E_i(U_{i1})}{\partial L^j} = E_i(U_{i1}) \left\{ (\bar{a}_i^j - u_i^j) \frac{\partial b^j}{\partial L^j} + (\bar{a}_i^j - a_i^j) \frac{\partial S^j}{\partial L^j} \right\}$$

$$+ \int_0^\infty \left[ u_{i2}(\theta) \frac{\partial Y_j(\theta)}{\partial L^j} (1 - t_i(\theta)) + a_i^j \frac{\partial Y_j(\theta)}{\partial L^j} \right] h_i(\theta) d\theta = 0$$

$$j = 1, \ldots, J$$

The first-order conditions (13), (14) and (15) are derived under the usual Nash assumption that the individual behaves as if a change in the levels of $B^j$, $K^j$, or $L^j$, for firm $j$, does not affect the value of firm $m$'s $(m \neq j)$ bonds or stock so that $\frac{\partial g^m}{\partial \eta} = \frac{\partial s^m}{\partial \eta} = 0$, for all $m \neq j$. Moreover, it is assumed that all firms and individuals act as price-takers and behave competitively. Price-taking and competitive behavior is meant to imply that
individuals evaluate proposed capital structure or production revisions by using the market prices that correspond to the current capital structure and production plans at which the portfolio equilibrium in (11) and (12) are evaluated. It is easy to show that this strong competitiveness assumption is equivalent to holding $\rho_i(\theta)$ and $t_i(\theta)$ constant at their pre-revision portfolio equilibrium values when evaluating proposed capital structure or production revisions.

If we differentiate the equilibrium conditions (11) and (12) with respect to $B_j$, $K_j$ and $L_j$, holding $\rho_i(\theta)$ and $t_i(\theta)$ constant, and substitute the results into (13), (14) and (15), we obtain the following alternative forms of the necessary conditions for optimal financial leverage and factor input choices.

$$E_i(U_{i1}) \bar{a}_1^j \left[ \frac{\partial S_j}{\partial B_j} + 1 \right] = 0 \quad j=1,\ldots,J$$

$$E_i(U_{i1}) \bar{a}_1^j \left[ \frac{\partial S_j}{\partial K_j} + \frac{\partial S_j}{\partial L_j} - 1 \right] = 0 \quad j=1,\ldots,J$$

$$E_i(U_{i1}) \bar{a}_1^j \left[ \frac{\partial S_j}{\partial L_j} \right] = 0 \quad j=1,\ldots,J$$

where

$$\bar{a}_n^j = \int_0^\theta \rho_1(\theta) \frac{\partial Y_j^j(\theta)}{\partial \bar{a}_n} (1-t_i(\theta)) d\theta \quad n=B_j, K_j, L_j$$

$$\bar{S}_n^j = \int_0^\theta \rho_1(\theta) \frac{\partial Y_j^j(\theta)}{\partial \bar{S}_n} d\theta \quad n=B_j, K_j, L_j$$
The major implication of the optimality conditions (16), (17), and (18) is immediate. Individuals who are not ex ante shareholders (those with $\tilde{a}_i^j = 0$) are unanimously indifferent to capital structure and production revisions, while those individuals who are initial shareholders (those with $\tilde{a}_i^j > 0$) unanimously prefer production and capital structure changes that are perceived to increase the net market value of the firm. This result has two additional implications. First, it demonstrates that ex ante unanimity can still survive under personal taxation and market incompleteness, provided that investors have equal access to the financial markets and do not face restrictions on tax arbitrage.\(^1\) Second, it demonstrates the equivalence between ex ante welfare maximization and perceived net market value maximization, and hence, demonstrates that the validity of the Fisher Separation Theorem extends even to markets characterized by tax imperfections. These results have been previously derived by Dammon [1984].

At this point we wish to explore the implications of the marginal preference functions in (16)-(18) in order to establish the interrelationship between the firm's productive and financial decisions. To do so we need to expand upon the marginal valuation equations (19) and (20) by investigating the marginal payoff vectors $\frac{\partial V^j_0}{\partial \pi n}$ and $\frac{\partial V^j_0}{\partial \pi n}$, $n=B^j, K^j, L^j$. Appendix A derives the functional form of these partial derivatives. Substituting the results from Appendix A into (19) and (20), and making a subsequent substitution of (19) and (20) into the marginal preference functions (16)-(18), yields the following optimality conditions.\(^1\)
\[
[R^J + \sum_{j=1}^{N} B_j] \{ \int_{\theta_2}^{\theta} \rho_1(\theta) [t^C - t_1(\theta)] d\theta - \int_{\theta_1}^{\theta} \rho_1(\theta) t_1(\theta) d\theta \} = 0
\]  
(21)

\[
\left[ [R^J + \sum_{j=1}^{N} B_j] \right] \left[ \sum_{j=1}^{N} B_j \right] \{ \int_{\theta_2}^{\theta} \rho_1(\theta) [t^C - t_1(\theta)] d\theta - \int_{\theta_1}^{\theta} \rho_1(\theta) t_1(\theta) d\theta \} 
+ \sum_{j=1}^{N} g_j \left[ \int_{\theta_1}^{\theta} \rho_1(\theta) (1-t_1(\theta)) d\theta + \int_{\theta_1}^{\theta} \rho_1(\theta) f^J(\theta) d\theta + \int_{\theta_1}^{\theta} \rho_1(\theta) f^J(\theta)(1-t^C) d\theta \right] 
+ (1-\bar{g})(1-t_1(\theta)) d\theta + \int_{\theta_1}^{\theta} \rho_1(\theta) d\theta + t^C \int_{\theta_2}^{\theta} \rho_1(\theta) d\theta - 1 = 0
\]  
(22)

\[
\left[ [R^J + \sum_{j=1}^{N} B_j] \right] \left[ \sum_{j=1}^{N} B_j \right] \{ \int_{\theta_2}^{\theta} \rho_1(\theta) [t^C - t_1(\theta)] d\theta - \int_{\theta_1}^{\theta} \rho_1(\theta) t_1(\theta) d\theta \} 
+ \sum_{j=1}^{N} g_j \left[ \int_{\theta_1}^{\theta} \rho_1(\theta) (1-t_1(\theta)) d\theta + \int_{\theta_1}^{\theta} \rho_1(\theta) f^J(\theta) d\theta + \int_{\theta_1}^{\theta} \rho_1(\theta) f^J(\theta)(1-t^C) d\theta \right] 
- \omega \left[ \int_{\theta_1}^{\theta} \rho_1(\theta) (1-t_1(\theta)) d\theta + \int_{\theta_1}^{\theta} \rho_1(\theta) d\theta + \int_{\theta_1}^{\theta} \rho_1(\theta)(1-t^C) d\theta \right] = 0
\]  
(23)

Relation (21) implies that the optimal capital structure plan for firm \( j \) will satisfy the following condition.

\[
\int_{\theta_2}^{\theta} \rho_1(\theta) t^C d\theta = \int_{\theta_1}^{\theta} \rho_1(\theta) t_1(\theta) d\theta
\]  
(24)

Since the boundary state of default (\( \theta_1 \)) and the boundary state in which the firm's tax liability is zero (\( \theta_2 \)) both depend upon the levels of \( B^J, K^J \) and \( L^J \), the real and financial decisions of the firm are inseparable and must be made simultaneously. Given a set of real decisions (\( K^J \) and \( L^J \)), however, relation (24) implies an interior optimal capital structure for the firm. The leverage decision is irrelevant if, and only if, relation (24) holds for all
feasible choices of $B^j$. However, because $\theta_1$ and $\theta_2$ are both affected by the value of $B^j$, relation (24) cannot hold for any arbitrary choice of $B^j$, so that at least some capital structure plans are strictly preferred to others.

Substituting the equilibrium condition (21), which implies that the net tax subsidy associated with corporate debt financing is driven to zero at the margin in equilibrium, into the optimality conditions (22) and (23), allows us to describe the optimal capital and labor input choices as follows.

\[
g^j_k = \frac{1-(1-\delta)\left\{ \int_0^{\theta_1} \rho_1(\theta)(1-t_1(\theta))d\theta + \int_0^{\theta_2} \rho_1(\theta)d\theta \right\} - t^c \gamma \int_0^{\theta_2} \rho_1(\theta)d\theta}{\int_0^{\theta_1} \rho_1(\theta)f^j(\theta)(1-t_1(\theta))d\theta + \int_0^{\theta_2} \rho_1(\theta)f^j(\theta)d\theta + \int_0^{\theta_2} \rho_1(\theta)f^j(\theta)(1-t^c)d\theta}
\]

\[
g^j_l = \frac{\omega\left\{ \int_0^{\theta_1} \rho_1(\theta)(1-t_1(\theta))d\theta + \int_0^{\theta_2} \rho_1(\theta)d\theta + \int_0^{\theta_2} \rho_1(\theta)(1-t^c)d\theta \right\}}{\int_0^{\theta_1} \rho_1(\theta)f^j(\theta)(1-t_1(\theta))d\theta + \int_0^{\theta_2} \rho_1(\theta)f^j(\theta)d\theta + \int_0^{\theta_2} \rho_1(\theta)f^j(\theta)(1-t^c)d\theta}
\]

Relations (25) and (26) are the uncertainty counterparts of the well-known optimality conditions for factor input usage in a world of certainty. That is, the optimal level of the factor input (capital or labor) equates the marginal product of the input with the ratio of the marginal cost of the input to the marginal revenue of the output. The marginal cost of capital (the numerator of (25)) is composed of the current one dollar cost of acquiring the marginal unit of capital, less the present value of the after-tax salvage value of capital $(1-\delta)$ (which accrues to bondholders if the firm defaults and
to equityholders if the firm is solvent), less the present value of the corporate tax savings due to the added depreciation expense $\gamma$ (which occurs only when the firm has taxable income). The marginal cost of labor (the numerator of (26)) is calculated as the present value of the after-tax wage expense (which is paid by the bondholders in the event of bankruptcy and is utilized as a corporate tax deduction only when the firm has positive taxable income). Interpreting $f^J(\omega)$ as the price per unit of output in state $\omega$, the marginal revenue of an additional unit of output (the denominator of (25) and (26)) is equal to the present value of the risky second-period price (which is paid to bondholders in the event of default, to equityholders when the firm is solvent, and is subject to corporate taxation only when the firm has positive taxable income). Once again, however, the production decisions of the firm cannot be made separately from the financing decision because of the effect that financial leverage has on the marginal costs of capital and labor and on the marginal revenue of the risky output. Thus, the optimal values of $B^J, K^J$ and $L^J$ must be made simultaneously, such that at a production-capital structure equilibrium, relations (24), (25), and (26) hold simultaneously.

The equilibrium conditions (24), (25), and (26) generalize the results of a number of earlier research efforts, including those of Miller [1977], Dammon [1984], Hite [1977], and DeAngelo and Masulis [1980]. The Miller equilibrium, for example, is obtained as a special case of relation (24). In the Miller framework, investment is fixed, capital markets are complete, interest on debt is the only legitimate corporate tax deduction, and personal tax rates are assumed to be constant, independent of taxable income, and exogenously specified. Under these assumptions, relation (24) yields:

$$\left[t^C-t^L\right] \int_{\theta_1}^{\bar{\theta}} p(\theta) d\theta = 0 \quad (24')$$
where \( p(\theta) \) is the current market price of a pure claim on one unit of state \( \theta \) consumption and \( t_i \) is the tax rate of the marginal investor in corporate bonds. As suggested by Miller, supply adjustments by the corporate sector will lead to an equilibrium in which the marginal investor in corporate bonds is in the corporate tax bracket (i.e., \( t^c = t_i \)). Thus, in equilibrium, relation \( (24') \) will be identically zero for all capital structure choices, implying that leverage is inconsequential to the value of the firm.

Dammon [1984] extends the capital structure problem by formulating a model which endogenizes the determination of investors' marginal tax rates, allows for unrestricted tax arbitrage, and assumes that, except for taxes, markets are perfect. Under these conditions, relation \( (24) \) becomes:

\[
\int_{\bar{\theta}}^{\overline{\theta}} \rho_i(\theta) [t^c - t_i(\theta)] d\theta = 0 \tag{24''}
\]

which implies that the firm will maximize its market value by choosing that level of debt such that the division of states (into those in which the firm is solvent, \( [\theta_1, \bar{\theta}] \), and those in which it is insolvent, \( [\overline{\theta}, \theta_1] \)) drives the marginal net tax subsidy associated with corporate debt financing to zero. Thus, contrary to the results of Miller [1977], relation \( (24'') \) implies the existence of an unique interior optimal capital structure for each individual firm.

DeAngelo and Masulis [1980] consider an environment similar to that of Miller [1977], but with risk-neutral investors and (exogenously specified) tax shield substitutes for debt, such as depreciation deductions or investment tax credits. Defining \( p_d \) and \( p_e \) as the current market prices per unit of before-personal-tax expected cash flow to debt and equity, risk-neutral valuation
implies \( p(\theta) = P_e h(\theta) \) and \( p(\theta)(1-t_i) = P_d h(\theta) \) for all \( \theta \), where \( p(\theta) \) and \( t_i \) were previously defined below expression (24') and where \( h(\theta) \) is the density function of \( \theta \). Substituting these conditions into (24) yields:

\[
\left[ P_d - P_e (1-t^C) \right] \int_{\theta_1}^{\theta_2} h(\theta) d\theta + \left[ P_d - P_e \right] \int_{\theta_1}^{\theta_2} h(\theta) d\theta = 0
\]  

(24'')

which duplicates equation (2) in DeAngelo and Masulis [1980] when no maximum limit is placed on the amount of tax credits that can be used to reduce the firm's tax liability. In the absence of corporate tax shield substitutes for debt, partial or total loss of the interest tax shield never occurs so that \( \theta_1 = \theta_2 \) and the first-order condition for an optimal capital structure collapses to the first term in (24''). In this situation, corporate supply adjustments will ensure that the equilibrium that is established satisfies the condition that \( P_d = P_e (1-t^C) \), thus re-establishing the leverage irrelevance result of Miller. In the presence of corporate tax shield substitutes for debt, however, the leverage decision is no longer irrelevant to the firm's shareholders, since it is impossible for expression (24'') to be identically zero for any arbitrary choice of \( B^j \).

Hite [1977] analyzes the impact of financial leverage on the real decisions of the firm in the context of the Modigliani and Miller [1963] tax framework. Hite recognizes that since the M-M leverage theorems are derived for a fixed set of real decisions, their results actually place lower limits on the value of financial leverage for a value maximizing firm. In Hite's model, debt is assumed to be free of default risk, personal taxes are zero, the corporate tax code provides for a full tax loss offset, interest (but not principal) on corporate debt is tax deductible, and any additions to the firm's capital stock is assumed to be financed with a constant fraction \( b \) of
corporate debt. Under these assumptions, we have
\[ \theta_1 = \theta_2 = \vartheta, \quad R^J = r^f, \quad \delta B^J / \delta K^J = b, \quad \text{and} \quad t_1(\theta) = 0 \ \text{for all} \ \theta. \]
Substituting these conditions into relations (21)-(23), recalculating expressions (25) and (26), and, finally, forming the ratio of (25) to (26), yields:

\[ \frac{g^j_K}{g^j_L} = \frac{(1-t^c) \delta + r^f - r^c t^c b}{u(1-t^c)} \]  

which is identical to Hite's equation (21). Hite refers to the numerator of 
(27) as the "user cost of capital" and to the denominator as the "user cost of 
labor."  

Two important points should be made about Hite's model in comparison to 
our model. First, note that under the assumptions of Hite, the "user cost of 
labor" is independent of both the real and financial decisions of the firm, 
whereas in the more general case of risky debt, the "user cost of labor" is 
directly affected by the firm's real and financial decisions (see relation 
(26)). Second, note that the "user cost of capital" in Hite's framework 
reflects the advantage of interest deductibility in computing corporate taxes, 
but that no such advantage is reflected in the "user cost of capital" of our 
model (see relation (25)). This is because in our more general framework, the 
tax advantage of interest at the corporate level is completely neutralized at 
the margin by the tax disadvantage of interest at the personal level. Thus, 
in the presence of both corporate and personal taxation, Hite's Proposition 2, 
that an increase in leverage (as measured by b) will increase the optimal 
stock of capital employed by the firm, will no longer hold. This is true even 
if debt is riskless, since it is no longer the case that the "user cost of 
capital" is strictly decreasing with increases in financial leverage. We shall
further explore the implications of these issues for the firm's optimal productive and financial decisions in the next section.

5. Comparative Statics

The preceding section provides conditions for an optimal production and financial plan. In this section, comparative statics will be performed on the equilibrium conditions to gain a better understanding of the general relationship that exists between the real and financial decisions of the firm in a world with corporate and progressive personal taxation. We begin by stating and proving a proposition first suggested by DeAngelo and Masulis [1980].

**Proposition 1:** Ceteris Paribus, increases in the corporate tax rate, $t^c$, or decreases in the tax accounting rate of depreciation, $\gamma$, will increase the firm's optimal level of financial leverage.

**Proof:** Let $B^*$ represent the firm's unique interior optimal level of debt financing which satisfies the optimality condition (21). Total differentiation of (21) with respect to $n$, where $n$ represents a dummy variable ($n=t^c, \gamma$), yields

$$
\frac{d}{dn} \left( \frac{\partial V_j}{\partial B_j} \right) = \frac{\partial^2 V_j}{(\partial B_j)^2} \frac{dB^*}{dn} + \frac{\partial^2 V_j}{\partial B_j \partial n} = 0
$$

which implies that

$$
\frac{dB^*}{dn} = -\frac{(\partial^2 V_j)}{(\partial B_j \partial n)} \div \left( \frac{(\partial^2 V_j)}{(\partial B_j)^2} \right)
$$
Assuming that the second-order condition \( \frac{\partial^2 \gamma_j}{\partial B^J \partial n} < 0 \) is satisfied, it follows immediately that

\[
\text{sign} \left( \frac{d\sigma^*}{dn} \right) = \text{sign} \left( \frac{\partial^2 \gamma_j}{\partial B^J \partial n} \right)
\]

Partial differentiating of (21) with respect to \( n \), using the first-order condition (24), yields

\[
\frac{\partial^2 \gamma_j}{\partial B^J \partial n} = \left[ R^J + \frac{\partial R^j}{\partial B^j} B^j \right] \left[ \rho_1(\Theta_1) t_1(\Theta_1) (d\Theta_1/dn) - [\rho_2(\Theta_2) t^c(\Theta_2) (d\Theta_2/dn)] \right]
\]

\[+ \left[ R^j + \frac{\partial R^j}{\partial B^j} B^j \right] \frac{dt^c}{dn} \int_{\Theta_2} \rho_1(\Theta) \ d\Theta \]

For \( n=t^c \), \( d\Theta_1/dt^c = d\Theta_2/dt^c = 0 \). Substituting these conditions into (28) yields

\[
\frac{\partial^2 \gamma_j}{\partial B^J \partial t^c} = \left[ R^J + \frac{\partial R^j}{\partial B^j} B^j \right] \int_{\Theta_2} \rho_1(\Theta) \ d\Theta > 0
\]

which implies that \( dB^*/dt^c > 0 \). For \( n=\gamma \), \( d\Theta_1/d\gamma = 0 \), \( d\Theta_2/d\gamma > 0 \) and \( dt^c/d\gamma = 0 \). Relation (28) then becomes

\[
\frac{\partial^2 \gamma_j}{\partial B^J \partial \gamma} = -\left[ R^J + \frac{\partial R^j}{\partial B^j} B^j \right] \rho_1(\Theta_2) t^c(d\Theta_2/d\gamma) < 0
\]

which implies that \( dB^*/d\gamma < 0 \).

The intuition behind Proposition 1 is obvious. If the corporate tax rate increases, the after-tax cost of debt financing, at the margin, will fall relative to the cost of equity financing. This provides an incentive for the firm to substitute debt for equity until the marginal condition in (24) is re-
established. On the other hand, if the accounting rate of depreciation $\gamma$ is increased, the probability that the interest tax shield will be redundant also increases, thus causing the after-tax cost of debt financing, at the margin, to increase relative to equity financing. This induces the firm to substitute equity for debt until the marginal condition in (24) is again satisfied.

We now turn our attention to the effect that changes in the corporate tax rate $t^c$ and tax accounting depreciation rate $\gamma$ have on the firm's optimal capital-labor ratio.

**Proposition 2:** *Ceteris Paribus,* an increase in the tax accounting rate of depreciation, $\gamma$, will unambiguously increase the firm's optimal capital-labor ratio. *Ceteris paribus,* an increase in the corporate tax rate, $t^c$, will lead to an increase (decrease) in the firm's optimal capital-labor ratio provided that $\beta/\phi < (>) \gamma/\omega$, where

$$
\beta \equiv 1 - (1-\delta)\int_0^\delta \rho_1(\theta)(1-1_t(\theta))d\theta + \int_\delta^1 \rho_1(\theta)d\theta - t^c \gamma \int_\delta^1 \rho_1(\theta)d\theta
$$

is the marginal "user cost of capital" and

$$
\phi \equiv \omega [\int_0^\delta \rho_1(\theta)(1-t^c(\theta))d\theta + \int_\delta^1 \rho_1(\theta)d\theta + \int_\delta^2 \rho_1(\theta)(1-t^c(\theta))d\theta]
$$

is the marginal "user cost of labor".

**Proof:** To prove Proposition 2, first form the ratio $g_j^j/g_j^L$ from relations (25) and (26) to yield

$$
g_j^k/g_j^L = \beta/\phi
$$

(29)
which implies that the optimal factor inputs, $K^*$ and $L^*$, will equate the ratio of marginal products to the ratio of factor prices. Consider now the impact of a change in the tax accounting rate of depreciation, $\gamma$, on the optimal ratio of marginal products.

\[
\frac{\partial (z_j^L/z_l^j)}{\partial \gamma} \quad = \quad \frac{\phi(\partial \beta/\partial \gamma) - \beta(\partial \phi/\partial \gamma)}{\phi^2}
\]

\[
= \frac{-t^C\left[ \int_0^\beta \phi_1(0)d\theta - [\gamma - \omega^S]\phi_2(0_2)(d\theta_2/d\gamma) \right]}{\phi^2} < 0
\]

Relation (30) suggests that as the fraction of capital which is allowed as a tax deduction increases, the "user cost of capital" falls relative to the "user cost of labor", enticing the firm to substitute capital for labor in the production process, and thus causing the optimal ratio of marginal products to decline. A closer look at relation (30), however, reveals that an increase in $\gamma$ affects the optimal capital-labor ratio in two ways. The first, and more dominating effect, is the reduction in the "user cost of capital" due to the additional tax savings that is generated by the increase in $\gamma$ when the firm is in a tax-paying position. The second effect stems from the fact that an increase in $\gamma$ reduces the probability of the firm ending up in a tax-paying position, thus causing both the "user cost of capital" and the "user cost of labor" to rise. Because the latter effect is secondary, it will be small relative to the first and, hence, the firm's optimal capital-labor ratio will rise following an increase in $\gamma$.

Consider now the impact of an increase in the corporate tax rate, $t^C$, on the optimal ratio of marginal products.
\[
\frac{\lambda(g^j_K / g^j_L)}{\phi^2} = \phi(\alpha \beta / \alpha t^c) - \phi(\lambda \phi / \alpha t^c)
\]

(31)

\[
= \left[ 8 \omega - \phi \gamma \right] \int_{\theta_2}^{\theta_1} \rho_i(\theta) d\theta
\]

\[
= \frac{\gamma}{\phi} \iff \frac{8}{\phi} \leq \frac{\gamma}{\omega}
\]

Since an increase in the corporate tax rate reduces both the "user cost of capital," \( \beta \), and the "user cost of labor," \( \phi \), the net effect on the optimal capital-labor ratio is ambiguous and depends upon the sign of \( [8/\phi - \gamma/\omega] \). The reduction in \( \beta \) following an increase in \( t^c \) is due to the tax deductibility of depreciation at the rate \( \gamma \), while the reduction in \( \phi \) is due to the tax deductibility of the wage expense \( \omega \). Relation (31) suggests that the firm will become more (less) capital intensive (as measured by its capital-labor ratio) if the reductions in \( \beta \) and \( \phi \) following an increase in the corporate tax rate leads to a reduction (an increase) in the ratio of factor prices. That is,

\[
\frac{d(K^j / L^j)^*}{dB^j} < 0 \iff \frac{8}{\phi} < \frac{\gamma}{\omega}
\]

We now turn our attention to the relationship between the firm's optimal capital and labor intensities and financial leverage. The next proposition provides us with a relative measure of this relationship.

**Proposition 3:** Ceteris paribus, increases in the firm's optimal financial leverage (due to an exogenous shift in an unidentified parameter), as measured by \( B^* \), will lead to an increase (decrease) in the firm's optimal capital-labor ratio provided that the following condition is satisfied.
Proof: See Appendix B

The intuition behind Proposition 3 is as follows. A change in the firm's optimal level of debt financing $B^*$ affects its optimal capital-labor ratio through its impact on relative factor prices. If the change in relative factor prices following a change in $B^*$ leads to a reduction (an increase) in the ratio of factor prices (as measured by $\delta/\phi$), then the firm's optimal capital-labor ratio will rise (fall). This, in fact, is exactly what is being determined in the above inequality where the ratio of changes in factor prices is being compared to the firm's current ratio of factor prices $\delta/\phi$.

To help clarify the interpretation of Proposition 3, consider an increase in the firm's optimal level of financial leverage $B^*$. The increase in $B^*$ results in an increase in the firm's probability of bankruptcy (i.e., $d\theta_1/dB^j > 0$) and reduces its probability of paying corporate taxes (i.e., $d\theta_2/dB^j > 0$). These changes in the boundary states $\theta_1$ and $\theta_2$ affect the firm's "user cost of capital" and "user cost of labor" through its impact on the after-tax cash flows generated by the residual value of capital, $(1-\delta)$, the depreciation deduction, $\gamma$, and the wage expense, $\omega$, in those boundary states.

The "user cost of capital," for example, will be higher following an increase in $B^*$ because of the reduced probability that the firm will be able to utilize $\gamma$ as a tax-deductible expense and because of the increased probability that the residual value of capital $(1-\delta)$ will accrue to bondholders rather than to equityholders and, hence, be subject to personal
taxation. This increase in the "user cost of capital" is given by the numerator of the middle term in the inequality in Proposition 3. The change in the "user cost of labor" following an increase in B* is ambiguous, however, and is given by the denominator of the middle term in the inequality in Proposition 3. The "user cost of labor" will be higher (lower) if the value of the tax deductibility of the wage expense at the corporate level in state $\theta_2$ (which is lost by the firm's shareholders following an increase in B*) is greater (less) than the value of the tax deductibility of the wage expense at the personal level in state $\theta_1$ (which is gained by the firm's bondholders following an increase in B*). For the special case in which $\rho_i(\theta_1) = \rho_i(\theta_2)$ and $(d\theta_1/dB^j) = (d\theta_2/dB^j)$, an increase (decrease) in the "user cost of labor" will require $t^C > (<) t_i(\theta_1)$. Proposition 3 suggests that if these changes in relative factor prices causes $\delta$ to rise (fall), the firm's optimal capital-labor ratio will decline (increase).

Proposition 3, although providing us with a relative measure of how the firm's capital intensity is affected by financial leverage, gives us no indication of how the optimal stock of capital K* employed by the firm responds to changes in financial leverage. The next proposition addresses this question, however, enabling us to more closely examine the existing theoretical and empirical studies relating to this issue.

**Proposition 4:** Ceteris paribus, increases in the firm's optimal financial leverage (due to an exogenous shift in an unidentified parameter), as measured by B*, will lead to an increase (decrease) in the firm's optimal stock of capital K*, provided that the following condition is satisfied.
\[ \rho_1(\theta_2) t^C(d\theta_2/DB^J) \left[ g_{K^J}^J(\theta_2) - \gamma - \frac{R^J}{\partial K^J} B^J \right] > < \]

\[ \rho_1(\theta_1) t_1(\theta_1)(d\theta_1/DB^J) \left[ g_{K^J}^J(\theta_1) + (1-\delta) - \frac{R^J}{\partial K^J} B^J \right] \]

**Proof:** See Appendix B

The interpretation of Proposition 4 is relatively straightforward. When the firm's optimal level of debt financing \( B^* \) is altered, the marginal benefits and costs associated with the capital input are affected because of the impact that financial leverage has on the probability of bankruptcy and on the probability of paying corporate taxes. If the change in \( B^* \) creates an inequality between the marginal revenue and marginal cost (i.e., the "user cost of capital") of the capital input, then the firm's use of the capital input will adjust until the equality between marginal revenue and marginal cost is re-established. Proposition 4 simply provides the appropriate cost-benefit comparison that is required to determine whether the firm's optimal use of the capital input will rise or fall following a change in \( B^* \).

To help us better understand the effect of a change in the firm's optimal level of financial leverage \( B^* \) on the firm's optimal use of capital, consider an exogenous shift in a variable that causes \( B^* \) to rise. This higher level of financial leverage increases the firm's probability of bankruptcy (i.e., \( d\theta_1/DB^J > 0 \)) and reduces its probability of paying corporate taxes (i.e., \( d\theta_2/DB^J > 0 \)), thus affecting the after-tax cash flows (to the firm's securityholders) generated by the marginal unit of capital in those new boundary states \( \theta_1 \) and \( \theta_2 \). The fact that the firm will no longer pay corporate taxes in state \( \theta_2 \) implies that the marginal unit of capital will generate an additional after-tax cash flow in that state amounting to
\[ t^c \{ g^j K^j (\theta_1) - \gamma - \frac{AR^j}{\partial K^j} bj \} \]

which has a current market value given by the left-hand side of the inequality in Proposition 4. Offsetting this positive impact on the capital intensity of the firm is the fact that the higher level of financial leverage means that the pretax cash flows generated by the firm in state \( \theta_1 \) will accrue to the firm's bondholders rather than to its stockholders and will, therefore, be subject to taxation at the personal level. The personal taxation of the firm's cash flows in state \( \theta_1 \) implies that the marginal unit of capital will generate a lower after-tax cash flow in that state than it had previously, with the reduction amounting to:

\[ \left( 1 - t_1 (\theta_1) \right) \left( g_K^j (\theta_1) + (1 - \delta) - \frac{AR^j}{\partial K^j} bj \right) \]

and having a current market value given by the right-hand side of the inequality in Proposition 4. The implications of Proposition 4 are now clear. The firm's optimal use of the capital input will increase (decrease) following a change in its optimal level of financial leverage if the current market value of the change in the after-tax cash flows associated with the marginal unit of capital is positive (negative).

Note that Proposition 4 conflicts with Hite's [1977] analogous theorem (Proposition 2), which states that an increase in financial leverage will lead to an unambiguous increase in the firm's optimal stock of capital. Hite's result, however, is a direct consequence of his assumptions of riskless debt, no personal taxes and full utilization of the depreciation deduction generated by the marginal unit of capital. Under these conditions, the "user cost of
capital" is strictly decreasing with the firm's use of financial leverage, thus leading to the increased use of the capital input as the firm increases its employment of debt.

Proposition 4 also conflicts with the hypothesis that "the use of leverage is negatively related to the magnitude of available investment related corporate tax shield substitutes for debt," as has been suggested by DeAngelo and Masulis [1980, 22]. Contrary to the claim of DeAngelo and Masulis, Proposition 4 suggests that the increased use of financial leverage may (over specific ranges) actually cause the firm to increase its use of the capital input and, hence, increase its nondebt related tax shields (i.e., depreciation). The conflicting results stem from the fact that DeAngelo and Masulis take the firm's investment related tax shields as given exogenously, whereas we are allowing the firm to make an optimal adjustment in its level of capital investment following changes in the firm's capital structure. Proposition 4 also has important implications for empirical studies that are designed to test for cross-sectional variations in corporate debt ratios. Since firms are likely to act (or attempt to act) optimally when making their (simultaneous) real and financial decisions, it need not be the case that those firms with higher investment related tax shields (controlling for differences in before-tax earnings) employ less debt in their capital structures.

For completeness, we now turn our attention to the relationship between the firm's optimal use of the labor input and financial leverage.

Proposition 5: *Ceteris paribus*, increases in the firm's financial leverage (due to an exogenous shift in an unidentified parameter), as measured by $B^*$, will lead to an increase (decrease) in the firm's optimal labor input $L^*$, provided that the following condition is satisfied.
Figure 2

Relationship Between Optimal Production and Finance
\[ \{ \alpha_1(\theta_2) \}^C (d \theta_2 / dB^j) \} [ g^j j (\theta_2) - \omega - \frac{\partial R^j}{\partial L^j} B^j ] > ( <) \]

\[ \{ \alpha_1(\theta_1) t_i (\theta_1, d \theta_1 / dB^j) \} [ g^j j (\theta_1) - \omega - \frac{\partial R^j}{\partial L^j} B^j ] \]

**Proof:** See Appendix B

The interpretation of Proposition 5 follows directly from the interpretation of Proposition 4. When the firm's optimal level of debt financing \( B^* \) is altered, the value of the marginal unit of labor is changed because of the impact that financial leverage has on the after-tax cash flows generated by the marginal unit of labor in the boundary state of bankruptcy \( \theta_1 \) and the boundary state of corporate taxability \( \theta_2 \). The market values of these changes in after-tax cash flows are given by the inequality in Proposition 5. This inequality implies that the firm's optimal use of the labor input will increase (decrease) following a change in \( B^* \) if the value of the net change in the after-tax cash flows associated with the marginal unit of labor is positive (negative).

Propositions 4 and 5 suggest that the relationship between capital and labor intensities and the firm's optimal level of financial leverage is highly complex. Figure 2 depicts one possibility. Since these relationships will differ across firms which utilize different production technologies, we cannot say a priori whether those firms which employ the higher levels of debt will be more or less capital intensive. Thus, in empirical testing of the relationship between financial leverage and the level of nondebt related tax shelters, the econometrician must be extremely careful to specify the proper
null hypothesis, realizing that once firm's are permitted to adjust their production decisions to changes in their capital structures the DeAngelo-Masulis' hypothesis is technically invalid.¹⁶

6. Conclusion

This paper has analyzed the interrelationship between the real and financial decisions of the firm under production uncertainty and in the presence of corporate and progressive personal taxation. The analysis proceeds from an establishment of an initial portfolio equilibrium with a fixed production and capital structure plan for each firm in the economy to a final equilibrium in which firms employ their optimal production and capital structure plans. Given equal access to the financial markets for all investors, we demonstrate that initial shareholders unanimously support the production and capital structure plans that maximize firm value. The necessary conditions for an optimal production and financial policy that are derived clearly show that the real and financial decisions of the firm are interrelated and, therefore, must be made simultaneously. Given the optimal production decisions, however, our analysis implies an unique interior optimal capital structure for the firm at the point where the net tax subsidy on corporate debt is driven to zero at the margin.

We then perform comparative statics on the equilibrium conditions in order to obtain a better understanding of the relationship between financial leverage and the optimal capital-labor intensity of the firm. We demonstrate that when the optimal production and capital structure decisions are simultaneously determined within a well-defined equilibrium model of the firm, the relationship between financial leverage and capital intensity, or,
equivalently, nondebt related tax shields (e.g., depreciation), need not be strictly negative as implied by the DeAngelo and Masulis [1980] analysis. Rather, we demonstrate that the relationship between financial leverage and capital (or labor) intensity can be highly complex and may not be either strictly increasing nor strictly decreasing (see Figure 2). We argued that this result has significant implications for those recent empirical studies that attempt to use the level of nondebt related tax shields to explain cross-sectional variations in financial leverage. Our theory suggests that such attempts are inappropriate unless the empiricist has prior knowledge of the relationship between the firm's real and financial decisions. Even then, however, empirical findings which reject the DeAngelo-Masulis hypothesis should not be surprising given the results of our model.

Future research extensions might include a multi-period investigation, where carry-back and carry-forwards are explicitly taken into account. Yet another extension might be to investigate the incomplete market effects of production and finance. For example, in attempting to help complete the financial markets, a firm may decide to split itself into less than perfectly correlated subfirms. This decision to split, although enriching the financial markets, comes at the expense of possibly losing the benefits of economies of scale by deviating from the firm's optimal size. The optimal trade-off, obviously, is the point at which the firm's value is maximized.
Footnotes

1. Although the model is developed under market incompleteness, the results of the paper extend to market completeness as well.

2. $f^j(\theta)$ can be thought of as representing the combined effect of price (demand) and production uncertainty. It is assumed that $g^j$ is twice continuously differentiable for all $j$, with $g^j_\theta > 0$, $g^j_\lambda > 0$, $g^j_{K} < 0$, $g^j_\lambda < 0$, $g^j_{KL} g^j_\lambda - (g^j_\lambda)^2 > 0$ and $g^j(0,L^j) = g^j(K^j,0) = 0$. In order to ensure an interior solution, it will also be assumed that $g^j_{KL}(K^j,L^j) \to +\infty$ as $K^j \to 0$ and $g^j_\lambda(K^j,L^j) \to +\infty$ as $L^j \to 0$.

3. Our results do not require equity to be strictly tax-exempt, but only that equity be preferentially taxed in relation to debt. Preferential taxation of equity can be justified on the grounds that a larger portion of equity's return is in the form of a capital gain which is taxed only when realized and then at a rate which is typically lower than the rate that applies to ordinary (and interest) income. The tax-exempt treatment of equity, however, simplifies our analysis by allowing us to ignore the tax-exempt securities that are issued by the government, such as municipal bonds, since these securities are redundant given the existence of riskless, tax-exempt, equity.

4. For simplicity, the wage expense incurred by the firm is assumed to be earned with certainty. That is, $x^j(\theta) + (1-\delta)K^j \geq wL^j$, for all $\theta$ and all feasible $(K^j,L^j)$ combinations. This assumption is inconsequential to our qualitative results.

5. Relation (7) also implies the absence of tax loss carrybacks and carryforwards, which cannot be adequately treated within a one-period model. We also abstract from the tax complications created by the difference between economic and accounting depreciation by treating the salvage value of capital as tax-exempt. This treatment of the salvage value does not alter our major conclusions however.
6. Relation (7) and Figure 1 clearly demonstrate that the government's tax claim, because of the asymmetry in the corporate tax schedule, replicates the payoff on a portfolio of \( t^c \) call options written on the firm's equity, with an exercise price of \( X^j(q_2) \). This option characteristic has been recognized and used by other authors to help explain a variety of financial phenomena. Green and Talmor [1983] use the option characteristics of the government's tax claim to analyze the effects of this option claim on the risk incentive investment choices of the firm. Madj and Myers [1983] have attempted to apply the concepts of option pricing to the valuation of the government's tax claim on risky assets when tax loss carrybacks and carryforwards are explicitly taken into account. Smith and Stulz [1984] use the option characteristics of the government's tax claim to rationalize corporate hedging. For a general discussion of the treatment of tax law asymmetries in the public finance literature, see Auerbach [1982] and [1983] and the references cited therein.

7. The endogenization of the personal tax liability has been previously considered by Dammon [1984] in analyzing the financing decision of the firm under both corporate and progressive personal taxation. He shows, under both complete and incomplete capital market settings and for a given firm investment plan, that the value of the firm is not independent of its capital structure, even though the capital markets are otherwise perfect, contrary to the previous results reported by Miller [1977] and Taggart [1980].

8. Unrestricted tax arbitrage does not destroy the existence of an equilibrium under progressive personal taxation as it would under proportional personal taxation as in Miller [1977]. For a discussion of this point see Dammon [1984].

9. In a capital market characterized by double completeness, marginal rates of substitution, \( \rho_1(\theta) \), and marginal personal tax rates, \( t(\pi_1(\theta)) \), would be driven to equality across all individuals at a portfolio
equilibrium, although these unique marginal rates of substitution and marginal personal tax rates may still differ across the states of nature \( \mathcal{O} \).

10. For notational convenience we shall suppress the dependence of the individual's marginal tax rate, \( t(\pi_i(\mathcal{O})) \), on the individual's taxable income level, \( \pi_i(\mathcal{O}) \), and instead write \( t_i(\mathcal{O}) \).

11. The partial derivatives \( \frac{\partial B^j}{\partial K^j} \) and \( \frac{\partial B^j}{\partial L^j} \) that appear in the first-order conditions (14) and (15), respectively, reflect the fact that \( B^j \) appears in the objective function of the maximization problem and is an implicit function of \( K^j \) and \( L^j \). To see why \( B^j \) is an implicit function of \( K^j \) and \( L^j \) consider the state contingent payoff to bondholders, \( Y^j_0(\mathcal{O}) \), defined in (4). From (4) it is clear that the payoff to bondholders in the event of bankruptcy depends upon the residual value of capital, \( (1-\delta)K^j \), and the wage expense, \( \omega L^j \), and, hence, \( K^j \) and \( L^j \) affect the market value of debt. Moreover, since changes in \( K^j \) and \( L^j \) will affect the probability of bankruptcy, the promised interest payment, \( R^j \), will also depend upon the values of \( K^j \) and \( L^j \).

12. Note that \textit{ex ante} unanimity will require the initial shareholders to evaluate the marginal impact of a capital structure or production revision on the firm's debt and equity values (i.e., equations (19) and (20)) identically. This will be the case only if the pretax marginal return vectors generated by a capital structure or production revision are spanned by the existing securities in the capital markets. Competitive firm behavior, however, will ensure that pretax spanning of the marginal return vectors is possible so that (19) and (20) are the same for all individuals and \textit{ex ante} unanimity goes through.

13. Since the numbering of the states of nature is arbitrary, we shall let them be ordered by the values of \( X^j(\mathcal{O}) \) such that \( X^j(\mathcal{O}) \leq \cdots \leq X^j(\mathcal{O}) \leq \cdots \leq X^j(\mathcal{O}) \), for \( \theta \in [\underline{\theta}, \bar{\theta}] \).
14. Dotan and Ravid [1984] recently attempt to extend the analysis of Hite [1977] by endogenizing the determination of the firm's nondebth-related tax shields and allowing the risky debt. They argue that the increased probability of realizing excessive tax shelters because of larger amounts of debt, will increase the "user cost of capital" and, hence, lower the optimal level of the capital input, thus providing additional support for the DeAngelo-Masulis hypothesis. The Dotan-Ravid analysis, however, in addition to making the unusual assumption that the firm's investment in capital is made prior to the realization of the output price, but that the optimal capital input usage is made after price is revealed, also abstracts from the equilibrium implications arising from personal taxation and ignores the impact of labor inputs.

15. DeAngelo and Masulis, of course, recognize that their hypothesis is based upon an exogenously given investment plan and that under more general conditions that allow for interactions between the investment and financing decisions their hypothesis may be invalid.

16. In a recent cross-sectional test of the determinants of capital structure, Bradley, Jarrell, and Kim [1984] find a significantly positive relationship between financial leverage and the level of nondebth tax shields. This finding is inconsistent with the DeAngelo and Masulis hypothesis, but is not inconsistent with the findings of our model. BJK (p. 875) suggest that this finding "raises doubts as to the validity of the DeAngelo and Masulis' argument that nondebth tax shields are substitutes (emphasis ours) for interest tax shields." We would argue, however, that this interpretation is extremely misleading because the validity of the DeAngelo-Masulis hypothesis requires a fixed firm investment plan, a condition which is surely violated in real world data. Once the interdependencies between the firm's investment and financing decisions are recognized and taken into account, the relationship between the level of nondebth tax shields and financial leverage becomes very complex (see Figure 2) and, on a subset of data, may exhibit positive, negative, or zero correlation even though nondebth tax shields and interest tax shields may still serve as substitutes.
In a related study, Mazzeo [1984] examines the behavior of firms during the period of time surrounding the maturity date of a debt issue and reports findings that suggest that the firm's decision to refinance with debt, with equity, or not at all is unrelated to the firm's level of nondebt-tax shields. This evidence is also not surprising given the interaction between production and finance and the results of our model.
This appendix provides the explicit expressions of the partial derivatives of $\gamma_b^j(\theta)$ and $\gamma_e^j(\theta)$ with respect to $B_J$, $K_J$, and $L_J$ that were used in deriving the optimality conditions (21)-(23). The partial derivatives are calculated from expressions (4) and (5), which define $\gamma_b^j(\theta)$ and $\gamma_e^j(\theta)$, respectively. They are:

\[
\frac{\partial \gamma_b^j(\theta)}{\partial B_J} = \begin{cases} 
0 & , R_J B_J \geq \lambda_1 \\
R_J^+ \frac{\partial r_J^j}{\partial B_J} B_J & , R_J B_J \leq \lambda_1 
\end{cases}
\tag{1A}
\]

\[
\frac{\partial \gamma_e^j(\theta)}{\partial B_J} = \begin{cases} 
0 & , R_J B_J \geq \lambda_1 \\
- [R_J^+ \frac{\partial r_J^j}{\partial B_J} B_J] & , \lambda_2 \leq R_J B_J \leq \lambda_1 \\
-(1-t^C) [R_J^+ \frac{\partial r_J^j}{\partial B_J} B_J] & , R_J B_J \leq \lambda_2 
\end{cases}
\tag{2A}
\]

\[
\frac{\partial \gamma_b^j(\theta)}{\partial K_J} = \begin{cases} 
0 & , R_J B_J \geq \lambda_1 \\
\frac{r_J(\theta) g_k^j}{\lambda_1} + (1-\delta) & , R_J B_J \leq \lambda_1 
\end{cases}
\tag{3A}
\]

\[
\frac{\partial \gamma_e^j(\theta)}{\partial K_J} = \begin{cases} 
0 & , R_J B_J \geq \lambda_1 \\
\frac{r_J(\theta) g_k^j}{\lambda_1} + (1-\delta) - \left[ R_J^+ \frac{\partial r_J^j}{\partial B_J} B_J \frac{\partial B_J}{\partial K_J} \right] + \frac{\partial r_J^j}{\partial K_J} B_J & , \lambda_2 \leq R_J B_J \leq \lambda_1 \\
\frac{r_J(\theta) g_k^j}{\lambda_1} + (1-t^C) + \frac{\partial r_J^j}{\partial K_J} B_J \right] (1-t^C) + t^C & , R_J B_J \leq \lambda_2 
\end{cases}
\tag{4A}
\]
\[
\frac{\partial y_j^i(\theta)}{\partial L^j} = \begin{cases} 
 r_j^i(\theta) g_L^j - \omega 
 & \text{if } R^j B^j \geq \lambda_1, \\
 [R^j + \frac{3R_j^j}{\partial B^j}] \frac{\partial B^j}{\partial L^j} + \frac{3R_j^j}{\partial L^j} B^j 
 & \text{if } R^j B^j \leq \lambda_1.
\end{cases}
\] (5a)

\[
\frac{\partial y_i^j(\theta)}{\partial L^j} = \begin{cases} 
 0 
 & \text{if } R^j B^j \geq \lambda_1, \\
 r_j^i(\theta) g_L^j - \omega - \left[ [R^j + \frac{3R_j^j}{\partial B^j}] \frac{\partial B^j}{\partial L^j} + \frac{3R_j^j}{\partial L^j} B^j \right] 
 & \text{if } \lambda_2 \leq R^j B^j \leq \lambda_1, \\
 r_j^i(\theta) g_L^j - \omega - \left[ [R^j + \frac{3R_j^j}{\partial B^j}] \frac{\partial B^j}{\partial L^j} + \frac{3R_j^j}{\partial L^j} B^j \right] (1-\tau_c) 
 & \text{if } R^j B^j \geq \lambda_2.
\end{cases}
\] (6a)

where

\[\lambda_1 = X^j(\theta_1) - \omega L^j + (1-\delta)K^j\]

is the point of bankruptcy and

\[\lambda_2 = X^j(\theta_2) - \omega L^j + \gamma K^j\]

is the point at which the firm begins to pay corporate taxes.
Appendix B: Proofs of Propositions

Proof of Proposition 3

To prove Proposition 3, we begin by forming the ratio of marginal products as given in (29). Differentiating (29) with respect to $B^j$ yields

$$\frac{\partial (a^j / g^j_L)}{\partial B^j} \bigg|_{B^*} = \frac{\phi(a^j / a^j_B) - \phi(a^j / a^j_B)}{\phi^2}$$

$$= \frac{[(1-\delta) + \omega \theta]}{\phi^2} \left[ \rho_i(\theta_1) t_i(\theta_1)(d\theta_1 / dB^j) \right]$$

$$+ \frac{[\gamma \phi - \omega \theta]}{\phi^2} \left[ \rho_i(\theta_2) t^c(d\theta_2 / dB^j) \right]$$

which implies that

$$\frac{a(K^j / L^j)^*}{aB^j} \bigg|_{B^*} \geq 0 \text{ iff } (1-\delta)[\rho_i(\theta_1) t_i(\theta_1)(d\theta_1 / dB^j)] + \gamma[\rho_i(\theta_2) t^c(d\theta_2 / dB^j)] \leq$$

$$\omega(\phi)[\rho_i(\theta_2) t^c(d\theta_2 / dB^j) - \rho_i(\theta_1) t_i(\theta_1)(d\theta_1 / dB^j)]$$

For increases in $B^*$, $d\theta_1 / dB^j > 0$ and $d\theta_2 / dB^j > 0$. Therefore, we can rewrite the above condition as

$$\frac{a(K^j / L^j)^*}{aB^j} \bigg|_{B^*} \geq 0 \text{ iff }$$

$$0 \leq \frac{[\rho_i(\theta_2) t^c(d\theta_2 / dB^j)] + (1-\delta)[\rho_i(\theta_1) t_i(\theta_1)(d\theta_1 / dB^j)]}{\omega[\rho_i(\theta_2) t^c(d\theta_2 / dB^j) - \rho_i(\theta_1) t_i(\theta_1)(d\theta_1 / dB^j)]} \leq \frac{\gamma \phi}{\phi^2}$$

For decreases in $B^*$, $d\theta_1 / dB^j < 0$ and $d\theta_2 / dB^j < 0$, and the inequalities in the above condition are reversed.
Proof of Proposition 4

To prove Proposition 4, we first totally differentiate condition (22), which $K^*$ satisfies, to yield:

$$\frac{d(aV^J/ak^J)}{dB^J} \bigg|_{B^*} = a^2v_j \frac{dk^*}{dB^J} + \frac{a^2v_j}{ak^J} \frac{a^2v_j}{aB^J}$$

Assuming that the second-order condition $\frac{a^2v_j}{(ak^J)^2} < 0$ is satisfied, it follows immediately that

$$\text{sign}\left(\frac{dk^*}{dB^J} \bigg|_{B^*}\right) = \text{sign}\left(\frac{a^2v_j}{ak^J} \frac{a^2v_j}{aB^J} \bigg|_{B^*}\right)$$

Partial differentiation of (22) with respect to $B^J$, using the fact that $B^*$ satisfies (24), yields

$$\frac{a^2v_j}{ak^J} \frac{a^2v_j}{aB^J} \bigg|_{B^*} = [\sigma_1(\theta_1) t_i(\theta_1) (d\sigma_1/dB^J)]\left[\frac{\sigma_j^R}{aB^J} - g^j_{RJ}(\theta_1) - (1-\delta)\right]$$

$$+ [\sigma_1(\theta_2) t_2(c) (d\sigma_2/dB^J)]\left[g^j_{KJ}(\theta_2) - \gamma - \frac{\sigma_j^R}{ak^J} B^J\right]$$

which implies that

$$\frac{dk^*}{dB^J} \bigg|_{B^*} > 0 \text{ iff } [\sigma_1(\theta_2) t_2(c) (d\sigma_2/dB^J)]\left[g^j_{KJ}(\theta_2) - \gamma - \frac{\sigma_j^R}{ak^J} B^J\right] > 0$$

$$[\sigma_1(\theta_1) t_i(\theta_1) (d\sigma_1/dB^J)]\left[g^j_{KJ}(\theta_1) + (1-\delta) - \frac{\sigma_j^R}{aB^J} B^J\right]$$
Proof of Proposition 5

To prove Proposition 5, we first totally differentiate condition (23), which $L^*$ satisfies, to yield:

$$\frac{d(aV^j/aL^j)}{dB^j} = \frac{a^2V^j}{aL^j} \frac{dL^*}{dB^j} + \frac{a^2V^j}{aL^j} \frac{dL^*}{aL^j} aB^j$$

Assuming that the second-order condition $\frac{a^2V^j}{(aL^j)^2} < 0$ is satisfied, it follows immediately that

$$\text{sign}(\frac{dL^*}{dB^j} B^*) = \text{sign}(\frac{a^2V^j}{aL^j} aB^j B^*)$$

Partial differentiation of (23) with respect to $B^j$, using the fact that $B^*$ satisfies (24), yields

$$\frac{a^2V^j}{aL^j} aB^j B^* = [\rho_i(\theta_1) t_i(\theta_1) (d\theta_1/dB^j)] \left[ \frac{aR^j}{aL^j} B^j + \omega - g^j f^j(\theta_1) \right]
$$

$$+ [\rho_i(\theta_2) t^c(d\theta_2/dB^j)] [g^j f^j(\theta_2) - \omega - \frac{aR^j}{aL^j} B^j]$$

which implies that

$$\frac{dL^*}{dB^j} B^* < 0 \iff [\rho_i(\theta_2) t^c(d\theta_2/dB^j)] [g^j f^j(\theta_2) - \omega - \frac{aR^j}{aL^j} B^j] > [\rho_i(\theta_1) t_i(\theta_1) (d\theta_1/dB^j)] [g^j f^j(\theta_1) - \omega - \frac{aR^j}{aL^j} B^j]$$
References


