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ASPECTS OF OPTIMAL MULTIPERIOD LIFE INSURANCE
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ASPECTS OF OPTIMAL MULTIPERIOD LIFE INSURANCE

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ABSTRACT

A multiperiod model is used in analyzing aspects of whole life and term insurance contracts within the context of the lifetime consumption-investment problem. Not only the life status, but the state of health of the consumer is also considered in examining optimal insurance purchases. Particular attention is given to the policy loan and guaranteed reinsurability options of the whole life insurance contract. When viewed in this context, whole life insurance, term insurance, and savings are shown likely to coexist in an optimal consumption-investment plan rather than act as substitutes for each other. In this paper it is shown how the bounds of the insurance pricing parameters can be derived to ensure that both kinds of insurance will continue to be sought by rational consumers, while avoiding adverse selection and lapsation.

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Consider the following problem. An individual wishes to protect his dependents against the loss of income that would result in the event of his premature death. Single-period term and whole life insurance are available, as well as a one-period bond. If the individual provides for his heirs through investment in bonds, he is unlikely to have sufficient funds, after providing for current consumption, to cover very much of the potential (lifetime) income loss that would be incurred were he to suddenly die, especially if he has many expected earning years ahead. However, if he does not die before retirement, he perhaps is able to accumulate a sizable amount of savings for consumption at that time or for bequest.

Suppose instead he purchases single-period (nonrenewable) term insurance, and invests the difference between the term and whole life premium in a bond. If the insured survives the first period, he may wish to continue some (presumably diminished) level of insurance in force. This he may do, provided that he is sufficiently healthy to qualify for a new policy. If not, his heirs could be left financially unprotected. In the event that the individual survives until retirement, he could have some funds available either for consumption or bequest from his program of “investing the difference.”

Now suppose that instead of term insurance, the insured had purchased a whole life policy. He could continue this insurance throughout his life, without regard to his future health and insurability, while surrender cash and policy loan values would accumulate over time, providing funds for his retirement or sooner. Alternatively, if he remains healthy through the first period, he could replace his policy with term insurance or with another, perhaps less costly whole life policy. The advantages to the initial whole life purchase are clear. Among the drawbacks are that the patterns of desired insurance coverage and savings accumulation may not coincide with those attainable strictly through the purchase of whole life policies. The consumer may be underinsured initially (perhaps due to policy cost) and overinsured in subsequent periods, and his savings levels over time may not be at optimal levels.

To remedy these drawbacks, the insured could purchase a smaller whole life policy and, during the crucial early earning years, supplement his insurance coverage with a term policy. The desired pattern of accumulated physical wealth could then be sought through the growing whole life policy cash values, modified by additional savings or policy loans. Although oversimplified, this line of reasoning contains the rudiments of logic so often used by

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1 There is little loss of generality in considering only these two forms of life insurance, as virtually all other forms are comprised of term and whole life insurance, in some combination, in some instances supplemented with a savings or investment program.

2 As time passes the amount of remaining human capital subject to loss diminishes, and so, presumably, does the amount of life insurance needed to protect against this loss.

3 Other alternatives, including policy types that encompass various combinations of these three elements in a unified package, are ignored here, but not in practice. One important example is renewable term insurance. Some varieties enable an individual to renew annually or every five to ten years without medical reexamination, although there is usually an age beyond which this option expires.
insurance salespeople in their presentations to potential policyowners. Term insurance and savings are thus put forth as complements to the purchase of “permanent” (i.e., whole life) insurance, not as replacements or substitutes for it.

Turning to the situation from the point of view of the insurer, if the consumer purchases only nonrenewable term insurance, the actuarial problem is quite simple. At the beginning of each period, the health of individuals applying for insurance is monitored and the premiums set accordingly. The challenge for the insurer lies with the whole life policy. The difficulty arises from asymmetric information between the insurer and insured. At the outset of a policy contract, information regarding the health status of the individual is obtained from the same third party for both the insurer and insured, and can be considered symmetric. However, beginning with the second period, the insured is in a much better position of knowing whether his health is sufficiently good to enable him to obtain a new policy. If his health has deteriorated, the insured is more likely to retain in force his whole life policy than if his health is fine. In the latter case, the individual could seek a new, lower cost policy or could surrender the policy for other reasons, leaving the first insurer with a problem of adverse selection. In anticipation of this adverse selection, the insurer could charge higher level premium rates to cover higher expected insurance claims, leading to even lower initial demand and exacerbating the problem of adverse selection. Alternatively, the insurer could add features to the permanent policy that would elicit greater persistency by healthy policyholders. One such option is the policy loan. Another way of improving the persistency of individuals is to arrange the pattern of savings accumulation so that it becomes more attractive as time passes. This could be achieved through high front-end charges, or implemented by offering no cash loan or surrender values for the first year or two, which effectively amounts to the same thing as charging an entrance fee. Indeed, these incentives are precisely what insurers provide, and are increasingly important as alternatives to the reinsurability concern are becoming available (e.g., group term, multi-year term, annual renewable term, etc.).

In this paper a decision-theoretic approach is taken in examining aspects of multiperiod life insurance contracts from both the consumers’ and insurance companies’ viewpoints. The analysis is undertaken in a three-period (four time-points) model where the consumer is assumed to have two earning periods and one retirement period. The three-period model provides a framework sufficiently rich for examining aspects of multiperiod life insurance contracts that have received only casual treatment in the literature. Special attention is paid to the relationships between policy surrender and insurability, interest rates and policy loans, and health and persistency. Certain features of commonly available whole life contracts are found to be crucial to maintaining an insurable population for insurance companies, and the commonly stated axiom of “Buy term and invest the difference” is shown to be capable of leading consumers to suboptimal behavior. Moreover, when examined in this richer context where various options of whole life insurance are explicitly taken into account, term insurance and savings emerge as complementary rather than substitute products to whole life insurance. Devolving from this study is an insurance pricing model which can be used in showing how

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4 One element frequently used in sales presentations, but dismissed by actuaries and economists, is the attraction of “locking in low premium rates forever” by purchasing level premium whole life instead of term. Insurability concerns and policy loan provisions aside, it is easy to demonstrate how one could do as well (or better, depending on future interest rates) by purchasing term and investing the difference, using the accumulated savings to help pay the high premiums in later policy years.

5 A term policy could be sought if reinsurability diminishes as a concern. Furthermore, new insurers offering either type of policy would only need to charge rates based on the mortality of currently healthy individuals, whereas the first carrier would have embedded in its rates charges for both healthy and unhealthy individuals. An increase in interest rates or improvement in the general mortality experience could also enable the term insurer to lower rates for second-period term policy applicants.

6 The time 0 and time 3 elements occur only once, but the time 1 and time 2 decision points can be considered to occur repeatedly. By allowing for this repeated occurrence, the theoretical model presented later has added realism. However, for analytical tractability and ease of presentation, the ensuing discussion is restricted to the four time-points.
premiums and policy benefits can be configured to lead consumers to desire both kinds of life insurance, while providing for ample rates of return to the insurer.

In addition to the usual assumptions employed in the economic literature dealing with life insurance (e.g., uncertain lifetime, individual endowments of inheritable and noninheritable wealth, consumer state-dependent utilities for consumption and bequest, existence of savings and insurance instruments), two elements are added that are key to the results obtained. First, not only is there uncertainty associated with the length of life, there is also uncertainty about the future state of health of those individuals who are alive. This stochastic element is important in creating a demand for multiperiod insurance contracts, where concern over future insurability is met. Second, interest rates are allowed to fluctuate over time, adding potential value to the policy loan provision. When combined with cash value accumulations that are judiciously patterned by company actuaries, the whole life policy can become an attractive complement to a program of savings combined with term insurance.

This paper is organized into five sections. The first section contains a brief review of some of the salient economic literature and a discussion of aspects of the problem to be investigated. This section is followed by an elaboration and development of a theoretical model that forms the basis for the remainder of the paper. In section three, dynamic programming and numerical analysis techniques are used to derive consumer demand equilibria. Next, comparative statics are presented. The final section contains a summary and outlines directions for further research.

I. Related Literature

Among the important advances in financial economics in recent years is the development of multiperiod investment theory. In the wake of this development have come numerous studies focusing on the structure of optimal multiperiod contracts in the presence of informational asymmetries (e.g., Townsend [1982]). Various types of multiperiod contracts have been the subjects of investigation. To give a few examples, Arnott [1982] studied the structure of multiperiod employment contracts with incomplete insurance markets, Merton [1981] viewed the role of social security as a means for efficient inter-generational risk bearing in an economy where human capital is not tradable, and Venezia and Levy [1982] derived optimal multiperiod automobile insurance contracts.

The ubiquitous multiperiod life insurance policy has also been a subject of theoretical inquiry. Typically the approach taken has been to posit the life insurance policy as a series of contracts of the single-period or instantaneous term variety (e.g., Yaari [1965], Hakansson [1969], Richard [1974]). This approach has meant that some of the complexities associated with the ever popular whole life policy have escaped formal analysis within the context of multiperiod consumption-investment models. The degree of abstraction customarily employed in these models has rendered the whole life policy indistinguishable from a constructible linear combination of single-period term insurance and a savings program of some sort (Richard [1974]), with the latter usually adjudged superior to the former (e.g., Fischer [1973], Moffitt [1979], and Winter [1981]). Fischer explicitly considered a multiperiod term insurance contract and concluded that it either presents the individual with arbitrage possibilities, or makes no difference to his welfare. Moffitt has extended the analysis to the whole life policy and reached a similar conclusion, under well specified conditions. Winter [1981], who took into account the option of withdrawal, also concluded that the fair cash value whole life policy will not be purchased in equilibrium. When Fortune [1973] introduced (with a two-period, three time-point model) a policy featuring some aspects usually associated with a whole life policy (e.g., maturity or surrender cash value, level premiums, and (implicitly)

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2 During most of the past three decades, approximately half of new individual (ordinary) life insurance in force sold has been of the whole life variety (see Life Insurance Fact Book, various issues).
guaranteed reinsurability), he was criticized by Klein [1975] and others on the grounds that his whole life policy was little more than a combination of pure insurance and savings, and therefore dominated by a strategy of purchasing single-period term insurance and investing the difference.

More recently it has been stressed in the academic literature that the whole life policy is really a package of options and not simply a linear combination of single-period term insurance and a savings program (e.g., Smith [1982], Walden [1985] and Outreville [1981]). Hence, the notion that term and whole life forms of insurance are substitute goods has been called into question. It should be emphasized here that the aforementioned analyses are correct within the frameworks they propose. It is only by adding new dimensions to the framework that the multiperiod life insurance policy can be more fully analyzed. This is what we propose to undertake in this paper.

The model set forth in the following section extends the work of Peter Fortune and Stanley Fischer by allowing for an analysis of perhaps the two most important options of a whole life contract — the policy loan option and guaranteed reinsurance option — within the context of a multiperiod consumption-investment framework. Of course, basic features of the whole life policy, such as level premiums and surrender cash values, are also incorporated into the analysis. Other options that are often part of the contract (see Smith [1982] for a list) are not modeled here.

II. Model Development

The consumer is hypothesized to possess state-dependent utility for consumption, $U(C)$, and bequest, $V(B)$. The underlying functional form of consumer utility across all states is assumed to be of the generalized logarithmic class. This class was chosen due to its many attractive theoretical properties (see Rubinstein [1976, 1977] and Hakansson [1979]) as well as its consistency with observed empirical facts (see, for example, Brown and Gibbons [1984] and Grauer and Hakansson [1982]).

The model proposed here is comprised of three periods (four points in time): two productive earning periods, and a retirement period covered by pension earnings and savings, after which survivors are assumed to reach the end of their lifetimes. At the beginning of the first period, the consumer is endowed with wealth and earns a certain income, which sum to $Y_0$. This total wealth is then divided among consumption, $C_0$, whole life insurance $W$ at a fixed premium rate of $\pi$ per unit of $W$, term life insurance $T_0$ at a first period rate of $\pi_0$, and savings, $S_0$ (which may be positive or negative — i.e., borrowing), which are placed in a one-period bond (or taken in a single-period loan) that will return (or cost) $R_0S_0$ at the end of the first period. Whenever a new insurance policy is desired, the state of health of the consumer is assessed by a medical expert and reported to both the insurer and the consumer. The premium rates stated above are based upon a clear bill of health.

At the end of the first period, the consumer could be in three states of health: good (insurable), bad ( uninsurable via new policies), or dead, with probabilities $p_1g$, $p_1b$, and $p_1d$, respectively, where the number on the subscript denotes end of the first period and the letter

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9 While the insurance industry has often maintained that this distinction should be made, the academic literature has been slow to incorporate this view into formal modeling, perhaps due to the much greater analytical complexities introduced by so doing.

10 The value of a policy loan option has been studied independently of such a framework by Bykerk and Thompson [1979] and several others. Additionally, the notion of guaranteed reinsurance without incurring a changing premium is present in the work by Venezia and Levy [1982], although there the focus is on the optimal timing of claims. In the case of life insurance, the timing issue is moot, as only a single claim is made for a given policy.

11 Although logarithmic utility possesses theoretical properties that are unsurpassed for many multiperiod situations, experimentation was also conducted for our 30-state model using quadratic and exponential utility forms. The results, which can be viewed as approximations for the preferred (logarithmic) functional form tended to confirm those attained here based on the logarithmic utility.
denotes health status. If the insured is dead, no additional income is earned. However, the estate receives bequest $B_3$, comprised of proceeds from any life insurance policies in force, $W$ and $T_0$, and savings with interest, $R_\sigma S_0$. If the consumer survives the first period but is in a state of poor health, he is ineligible for new term life insurance, but is assumed to maintain in force whatever amount of whole life insurance had been earlier purchased by making an additional premium payment of $\pi W$. The consumer has available for consumption reduced wages $Y_{1a}$, if any, as well as $R_\sigma S_0$ plus the proceeds from any policy loan undertaken, $Q_1 W$. The latter is assumed to be incurred whenever the second-period (after-tax) market rate of interest exceeds that charged on policy loans (after tax). This loan can be used as a low cost source of funds for current consumption or for additional investment at the favorable, higher market interest rate. Second-period consumption and saving in the event of poor health will depend, in part, upon investment opportunities, captured in the model by (one plus) the interest rate, $R_{1l}$ ($l$ for low rate) or $R_{1h}$ ($h$ for high rate). Subscripts are placed on the interest, savings, consumption and bequest variables throughout to indicate point in time $(0, 1, 2, 3)$.

If the individual retains good health until the start of the second period, he may choose whether to purchase a new term policy at the revised rates for newly issued policies, $\pi_{1a}$ or $\pi_{1h}$. These rates will reflect the current interest rate levels. $R_{1a}$ or $R_{1l}$ as well as any updated information about the mortality experience obtained from period 1. The individual can either surrender or continue his whole life policy. If he elects to maintain in force his whole life insurance, he may choose to obtain a policy loan either for consumption or investment if the market rates of interest are high. (If they are low, he would do better to get additional funds through a loan at the low market rates.) In any case, the individual will again need to allocate his available funds from earned income ($Y_{1g}$), physical capital ($R_\sigma S_0$), and any policy loans ($Q_1 W$) among current consumption, insurance and savings.

Any individuals who were uninsurable at the start of the second period are assumed not to survive beyond the end of the period. At this time, their estates are settled, with any remaining funds going toward their bequests. If a policy loan had been taken out earlier, the loan plus interest ($Q_1 W$) is subtracted from the proceeds of the whole life policy.

Individuals who started the second period in good health may either be alive or dead (a or d) at the start of the third period. If they are not alive, their estates are settled along similar lines to that described earlier, where the amount available for their bequest will depend, in part, upon their prior decisions about what kind and how much insurance to purchase. If alive, they once again are faced with the decision of allocating their retirement income and wealth among consumption, savings and whole life insurance (provided that the policy has been continued). The option of policy surrender at this late date was not modeled explicitly. This is because during the last period, the amount available for a policy loan should equal the present value of a sure payment of $W$ one period hence, else arbitrage.

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12 In an extended version of this model, part of first period income could be expended for a policy rider that provided for a disability income supplement in the second period in the event of bad health. The disability income was then added to the reduced wages, savings accumulations and policy loans, thereby expanding the sources of money available for second-period consumption. Additionally, unlike in the present version, it was assumed that the insured could possibly survive beyond the second period, even if his health rendered him uninsurable. Furthermore, it allowed for first-period income to vary not only with health status, but also with evolving interest rates. These extensions were dropped from the present version to achieve greater simplicity. They were not deemed essential elements of the model for examining the attractiveness of whole life insurance. The extended model also permitted purchase of high cost term insurance if health was poor. When the assumption was later added that a consumer in poor health would only survive one more period, such a policy became actuarially indistinguishable from saving and so this feature was also deleted.

13 The health status, if alive, is not considered in this period. Since, whether in good or bad health, the model operates under the assumption that all individuals will not survive beyond the end of the third period, and since retirement income is received irrespective of future earning capacity, the variables of the model can not distinguish between good and bad health, although the state-dependent utility function could make this distinction. For simplicity, this consideration was not included.
opportunities present themselves. If higher, the loan could be taken and later repaid at the bargain rates. If lower, it would be better to get any desired funds through an outside loan, as the cash values are accumulating at more favorable rates than those available outside. In neither case would the policy be surrendered.

At the end of the retirement period, all available funds net of any loan repayments become the bequest available for heirs. The amount of these funds will depend upon prior decisions, and is influenced by the level of interest rates during the final period. If Figure 1 on the following page, the cash flows and decision variables associated with each of the thirty states described above are presented. To assist in deciphering the diagram, the notation is reviewed below.

Definitions:

\( Y_t \): Wages (after-tax) received by individual at time \( t \). When \( t = 0 \), \( Y \) includes accumulated physical wealth.

\( S_t \): Savings undertaken or (if negative) loans incurred at time \( t \).

\( C_t \): Consumption at time \( t \).

\( W \): Face amount of whole life insurance policy.

\( T_t \): Face amount of term life insurance policy at time \( t \).

\( B_t \): Amount of bequest at time \( t \).

\( R_t \): One plus the after-tax market rate of interest for saving or borrowing (assumed to be equal) at time \( t \).

\( Q_t \): The proportion of the face amount of the whole life policy available for policy loan at time \( t \).

\( G \): One plus the after-tax rate of interest charged on policy loans (assumed to be fixed throughout life of policy).\(^{14}\)

\( p_t \): Probability of occurrence of stated event at time \( t \).

\( \pi \): The whole life insurance premium rate, per dollar of insurance in force. This periodic premium rate is fixed throughout the life of the policy.

\( \pi_0 \): The term insurance premium rate, per dollar of insurance in force during the first period.

\( \pi_1 \): The term insurance premium rate, per dollar of insurance in force during the second period, available only to individuals who have retained their good health.

Some additional subscripts are attached to certain variables to clarify the state paths over time of these variables, or indicate present state. Where additional subscripts are present, they refer to states of consumer health, interest rate levels, and consumer actions as follows:

\( g, b, d, a \): \( g \) and \( b \) refer to state of consumer health at time \( t = 1 \), where \( g \) indicates good, and \( b \) indicates bad (hence, uninsurable); \( d \) indicates dead and \( a \) indicates alive.

\( h, l \): Market interest rate is high or low.

\( s, p, c \): Refers to consumer choice at time \( t = 1 \) to surrender his policy, take out a policy loan, or continue keeping his insurance in force without taking out a policy loan, respectively.

As was mentioned above, utility is assumed to be of the generalized logarithmic class, such that \( U(C_t) = \ln(C_t - \gamma_t) \) and \( U(B_t) = \ln(B_t - \beta_t) \), where \( \gamma_t \) and \( \beta_t \) can be interpreted as the minimum tolerable standard of living and minimum acceptable bequest provision, respectively, for period \( t \) and state \( \cdot, \cdot \). As Rubinstein has pointed out, this assumption of

\(^{14}\) The majority of states in the U.S. have recently enacted legislation permitting the insurers to charge variable policy loan rates tied to market conditions. Even so, many policies continue to feature fixed policy loan rates of interest. The advent of variable policy loan rates could decrease the value of the policy loan option, a consideration which might reduce the attractiveness of the whole life policy, other things equal.
generalized logarithmic utility is not as restrictive as it might at first seem. While it is true that all consumers are required by this functional form to have decreasing absolute risk aversion (the theoretically preferred kind), the heterogeneity of the model permits consumers to have separate taste parameters \( \gamma_l \) and \( \beta_l \) for their consumption and bequest preference functions for each state and at each point in time, and tolerates quite diverse attitudes of proportional risk aversion. \( \gamma \) (and/or \( \beta \)) may be positive, zero or negative, implying decreasing, constant or increasing proportional risk aversion, respectively. Presumably, in the context of this model, \( \gamma \) must be positive to provide for at least some consumption, but \( \beta \) could be either zero or positive. The higher the \( \gamma \) (and/or \( \beta \)), the more risk averse the consumer. Furthermore, each utility is influenced by a separate state-dependent risk preference parameter \( \rho \) as well as a time-preference factor \( \rho_{t} \). Thus, the model used here accommodates a tremendous variety of individual time and state preference forms. (To simplify notation and computation, without any loss of generality, the state probabilities \( \rho_{t} \) can be combined with the state-dependent risk preference parameter \( \rho \) into a single parameter \( \rho_{t} \).)

The problem for the consumer, then, is to maximize expected utility \( \Omega \) of consumption and bequest across all periods and states by selecting appropriate levels of consumption, insurance, and investment, while making the optimal choice at time 1: surrender policy, persist with policy and incur a policy loan, or persist without incurring a policy loan. The formal problem is presented in the equations that follow.

\[
\Omega = U(C_0) + \rho_1 \left[ \rho_{1gh} U(C_{1gh}) + \rho_{1gl} U(C_{1gl}) + \rho_{1bh} U(C_{1bhp}) + \rho_{1bl} U(C_{1blc}) + \rho_{1d} V(B_1) \right] \\
+ \rho_1 \rho_2 \left[ \rho_{2hh} U(C_{2hh}) + \rho_{2hl} U(C_{2hl}) + \rho_{2lh} U(C_{2lh}) + \rho_{2ll} U(C_{2ll}) \right] \\
+ \rho_{2ghd} V(B_{2gh}) + \rho_{2gld} V(B_{2gld}) + \rho_{2bhd} V(B_{2bhd}) + \rho_{2bld} V(B_{2bld}) \\
+ \rho_1 \rho_2 \rho_3 \left[ \rho_{3hh} V(B_{3hh}) + \rho_{3hl} V(B_{3hl}) + \rho_{3lh} V(B_{3lh}) + \rho_{3ll} V(B_{3ll}) \right] \\
= \ln(C_0 - \gamma_0) + \rho_1 \left[ \rho_{1gh} \ln(C_{1gh} - \gamma_1) + \rho_{1gl} \ln(C_{1gl} - \gamma_1) + \rho_{1bh} \ln(C_{1bhp} - \gamma_1) \right] \\
+ \rho_1 \rho_2 \left[ \rho_{2hh} \ln(C_{2hh} - \gamma_2) + \rho_{2hl} \ln(C_{2hl} - \gamma_2) + \rho_{2lh} \ln(C_{2lh} - \gamma_2) + \rho_{2ll} \ln(C_{2ll} - \gamma_2) \right] \\
+ \rho_{2ghd} \ln(B_{2gh} - \beta_2) + \rho_{2gld} \ln(B_{2gld} - \beta_2) + \rho_{2bhd} \ln(B_{2bhd} - \beta_2) + \rho_{2bld} \ln(B_{2bld} - \beta_2) \\
+ \rho_1 \rho_2 \rho_3 \left[ \rho_{3hh} \ln(B_{3hh} - \beta_3hh) + \rho_{3hl} \ln(B_{3hl} - \beta_3hl) + \rho_{3lh} \ln(B_{3lh} - \beta_3lh) + \rho_{3ll} \ln(B_{3ll} - \beta_3ll) \right]
\]

It will be observed that there are fewer arguments in the multiperiod utility function than there are states shown in Figure 1. This is because beginning with the second period, two branches of the decision tree will be dominated by their alternatives; in particular, the consumer in good health will choose either to surrender or take out a policy loan if interest rates are high in the second period, or either continue with his policy or surrender it if interest rates are low in the second period. The branch of the decision tree that will be followed depends upon which is associated with higher utility for the consumer. Thus, our first task was to compute the utilities associated with each branch and eliminate those branches with
Figure 1: States of Nature

FIRST EARNING PERIOD          SECOND EARNING PERIOD          RETIREMENT PERIOD
lower utility over which the consumer could exercise control. As one might expect, the two branches entailing policy surrender are only dominated by those entailing persistence if the insurer has priced his policies within certain bounds. Thus, this stage becomes important to the insurer that wishes to avoid the problem of adverse selection mentioned earlier.

The consumer then maximizes his expected utility subject to the solvency constraints given in the boxes of Figure 1.

III. Consumer Demand Equilibria

To solve a maximization problem of this complexity, a natural technique would seem to be dynamic programming. Unfortunately, this technique revealed that an algebraic solution was implausible. Consequently, we were compelled to rely on numerical analysis to gain insights into the dynamics of the decision variables over time. These numerical methods were employed to solve the equations for widely ranging values. In selecting these ranges of values, we substituted for several of the variables varying plausible values, where the values chosen adhered to certain known relationships. For example, it was assumed that the probability of death increased over time for a healthy individual, that retirement income was lower than income during working years, that \( \pi > \pi_0 < \pi_1 \), that \( Q_1 < Q_2 < 1 \), and that \( G < R_\pi \). While both \( G \) and \( R_\pi \) > 1. A further technical requirement arising from the use of logarithmic utility was that \( C > \gamma \) and \( B > \delta \), to ensure that utility is always defined. Another constraint imposed was that both term and whole life insurance were priced at or above their actuarially fair cost. In this paper, we report the results for only a small fraction of the many values used in experiments. The details of the inputs for these numerical analyses are provided in Appendix A.

When the values for the above parameters were input, whole life demand was found to be positive for a wide range of values, as was the demand for term insurance. As one would expect in light of the way the model was set up, where term insurance and savings could not be linearly combined to substitute perfectly for whole life insurance, there were many levels of feasible parameter values for which there was a simultaneous, positive demand for both types of insurance. This contrasts with other studies, in which only one form of insurance would appear in an optimal portfolio of a rational individual. When loading factors became excessive on the premiums of one type of insurance relative to the other, however, it was not uncommon for an optimal portfolio to include only one kind of insurance.

In Table 1, the equilibrium values of consumption, bequest, savings, and insurance are reported for two cases: actuarially fair whole life and term insurance premiums, and standard loaded premiums for both kinds of insurance. The loading percentages of 27.5 and 110 for whole life and term, were chosen to approximate markup averages found by Babbel and Staking [1983], who sampled a moderate number of insurance companies. Details are given in

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15 To solve the maximization problem, we began with the last period and worked backwards using the procedure of backwards optimization. By the time the beginning of the second period was reached, the solutions were becoming very complex and increasingly difficult to obtain. A computer program called "MACSYMA" was enlisted in the attempt to obtain results and ensure accuracy. This program enables a user to solve algebraic and calculus problems symbolically, and is especially adept at symbolically solving systems of simultaneous equations. Additional nonnegativity constraints on some of the decision variables were not imposed at this stage because the problem had already reached the practical limits of the computer. Rather, solutions were obtained and then experimentation was done with the parameters to determine what levels would result in the nonnegativity criteria being met, without imposing the constraints.

Although it was possible to find closed-form analytical solutions for each of the decision variables for each of the last three time-points, the nature of the results rendered the solutions of little immediate practical value. Solution values for the coefficients of interest each required between 60 to 80 lines of algebraic notation to state. Clearly, further analysis as to whether the solution values for \( T_0, T_1 \), and \( W \) were globally nonnegative was practically prohibitive. Worse, when attempting to solve the problem for the first time-point, the model involved taking roots of equations of the fifth order and higher, for which no mathematical solutions have yet been devised.
Appendix A. The table is based on period interval lengths of fifteen years each, where the initial age of the insured is thirty-five years. Beginning income/wealth is $20,000. To reflect productivity growth over time, income was assumed to grow at two percent per year, in real terms, until the second period, provided that the insured remains healthy. If the individual becomes unhealthly and uninsurable, less second-period income is assumed to be received; in particular, such income is assumed to grow only at the rate of inflation. Interest rates range from five to slightly in excess of eight percent, where the probabilities of attaining either level are given in the appendix.

Table 1

<table>
<thead>
<tr>
<th>EQUILIBRIUM VALUES: Consumption, Bequests, Savings, Insurance</th>
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</thead>
<tbody>
<tr>
<td>(Values below are given in terms of thousands of dollars)</td>
</tr>
<tr>
<td>W</td>
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<tr>
<td>C₀</td>
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<td>C₁₂php</td>
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<tr>
<td>C₁₂glc</td>
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<tr>
<td>C₂₂lhc</td>
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<tr>
<td>C₂₂lci</td>
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</tbody>
</table>

In examining the results reported in Table 1, several items are worthy of note. First, it is observed that when insurance was priced either on an actuarially fair basis or with substantial loadings, both kinds of insurance figured in the optimal portfolio. However, total insurance in force declined from $26,968 (= $20,758 + $6,210) to $15,823 (= $5,780 + $10,043) when premiums reflected their loading factors. It is also interesting that second-period term insurance demand was negligible, whether it was priced with or without a loading factor. It should be recalled from Figure 1 that second-period term insurance only protects the insured against the loss of retirement income, and not against the loss of income from a productive earning period. The retirement income was set in real terms at one-third of the level of the highest earning period in this model.

First-period saving was negative, where borrowings were used to support consumption and insurance purchases during the first period of low earnings. Another item of interest is that when premiums are loaded, more term insurance is purchased, although less whole life is sought. Accompanying this decline in whole life insurance (which contains a large element of savings) is a decrease in first-period borrowing, and an increase in outside savings for the majority of states of the world. First-period bequests are drastically lowered in the markup case, while second-period bequests are affected only moderately; third-period bequests are almost unaffected by the higher insurance premiums, a result of the increased levels of savings. Consumption levels are almost unaffected by the increased premiums in the cases reported here.

IV. Comparative Statics and Simulations

While it was impossible to obtain algebraic solutions for those comparative statics that relate to first-period choices, we have done a substantial number of numerical simulations and can glean from them the likely direction of shifts in the levels of consumption, savings,
bequests, and insurance purchases associated with changing parameters. In Table 2 we report the direction of shifts in these decision variables under changing premiums, risk aversion, incomes, and preference for bequest relative to consumption. The arrows shown relate to changes based on initially fair insurance premiums. In all cases the directions of shifts were maintained when based on standard premium loadings, unless otherwise noted by an asterisk, in which cases the shifts went in the opposite directions. A single dagger indicates that the initial shift (for the fair premium case) was negligible, whereas a double dagger indicates that the shift under loaded premiums was negligible, meaning a change of less than one-tenth of one percent.

Table 2

<table>
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<tr>
<th>Variable</th>
<th>( \pi )</th>
<th>( \pi_0 )</th>
<th>( \gamma )</th>
<th>( \beta )</th>
<th>( Y_0 )</th>
<th>( Y_1 )</th>
<th>( Y_2 )</th>
<th>( Y_{1,2} )</th>
<th>( Y )</th>
<th>( Y_{1,b} )</th>
<th>( \rho_d )</th>
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Of particular note is that in almost every case where the direction of shifts changed between fair and loaded premiums, at least one and often both of these shifts were of extremely small amounts. The major exceptions are term insurance purchases, that switch
directions when the minimum living standard is raised or the preference for bequest is raised, under fair and loaded premiums. The only other cases where the reversed directions in the comparative statics were nonnegligible involved the influence of increased second-period income (time-point 1) on first- and second-period savings.

In reviewing the comparative statics, several other findings should be highlighted. Because the focus of this study is on life insurance, we direct our remarks to that subset of our findings. Although it has been shown that whole life and term insurance coexist together with savings in an optimal investment portfolio, whether premiums are fair or unfair, there is a limited degree of substitutability between the insurance products. For example, if only the whole life premium \( x \) rises, whole life purchases \( W \) decline while first-period term purchases \( T_0 \) and savings \( S_0 \) rise. If, instead, term rates \( q_0 \) increase, whole life purchases rise while term sales, together with first-period savings, decline.

Another item of interest is that when the insured increases his risk aversion for bequest states wealth, \( \beta \), more whole life and less term insurance are purchased. This observation underscores the different properties of these kinds of insurance, where whole life provides income for all bequest states, while also supplementing savings available for use in every state, whereas term provides only for bequests in states that begin with a healthy individual.

The influence of changing income on insurance demand is mixed, but to be expected. If endowed wealth \( Y_0 \) rises relative to future income, whole life purchases diminish but term insurance increases. This is because endowed wealth is inheritable in every bequest state and, in a sense, serves as a substitute for whole life insurance. If only second period income \( Y_1 \) rises, demand for both kinds of insurance protection rises, but if only retirement income \( Y_2 \) is augmented, demand for whole life rises while term falls. Clearly term insurance \( T_0 \) has no role in protecting against the loss of retirement income, and as the latter becomes more important relative to other income, the natural insurance vehicle was whole life.

The comparative statics shown in Table 2 relate, strictly speaking, only to small departures from the standard fair and loaded premium cases showcased here, but a large number of other, wide-ranging parameter values has also been tried with similar results. Thus, the model seems to be reasonably robust in isolating the effect of changing exogenous and taste parameters on the decision variables. The reader is directed to Appendix B, which shows the consumer's insurance and savings patterns under a wide array of circumstances.

VI. Summary

The major conclusion of this study is that by using a richer model, as was done here, one can show how rational consumers would desire to purchase both whole life and term insurance under various sets of feasible pricing systems and economic environments. This contrasts with the results of prior economic models, which do not show both term and whole life insurance coexisting in an optimum portfolio. This difference arises from the introduction of stochastic health status and interest rates into the model, which together impute value to some of the options that are part of the whole life insurance product. A further and perhaps more fruitful use of the model derived here is in illustrating how demand would shift among these two products, along with desire for savings, by changing some of the cash value accumulation patterns, premium levels, income, initial wealth, interest rates, and risk-preference parameters. At certain prices, the demand for insurance of one type or another disappears altogether.

It should be reemphasized that many of the options usually associated with whole life policies were not included as part of the model put forth here, and these other options could only help add to the attractiveness of the whole life policy, other things equal. Finally, while the results were achieved using a class of utility functions that is theoretically attractive, and fulfills most of the rationality requirements that economists feel are important, they nonetheless are valid only within that class of utility functions, and have empirical counterparts only to the extent that consumer preferences can indeed be accurately reflected by utility functions and that these same consumers act in a manner consistent with them.
Many further extensions of the basic model have already been tested, as alluded to in various places throughout this manuscript. However, we can suggest several extensions that have not yet been undertaken. The inclusion of uncertain future policy dividends, like those provided by participating whole life policies, would be an interesting addition. Likewise, the introduction of variable policy loan rates would contribute to our understanding of this, relatively recent practice. Also, the allowance for partial policy loans is something not considered here. The nature of our model was such that partial loans would never rationally enter the model, although in practice such partial loans are often observed. Moreover, some of the techniques for taking full advantage of tax provisions through a prescribed pattern of policy loans was not modeled. Nonetheless, the model proposed takes some important steps towards enhancing our ability to analyze more fully various types of insurance within a consumption-investment framework.

REFERENCES


Brown, David P. and Michael R. Gibbons, 1984, A simple econometric approach for utility-based asset pricing models. research paper #685R, Graduate School of Business, Stanford University.


Rubinstein, Mark, 1977, The strong case for the generalized logarithmic utility model as the premier model of financial markets. *Financial Decision Making Under Uncertainty*, edited by H. Levy and M. Sarnat, Academic Press. (This paper is an extended version of that listed directly above and contains many results only alluded to in the shorter version.)


APPENDIX A: Technical Notes

1. Basic Assumptions
(a) Real annual growth rate: \( g = 0.02 \).
(b) Annual inflation rates: \( i_d = 0.03 \); \( i_h = 0.06 \).
(c) Annual interest rates: \( r_d = (1+g)(1+i_d)-1 = 0.0506 \); \( r_h = (1+g)(1+i_h)-1 = 0.0812 \). For simplicity, we set \( \gamma_0 = r_d \) in the computational program.
(d) Probability of higher inflation (and thus, higher interest rates): \( p_{1h} = .3 \); \( p_{2h} = .2 \).
(e) Planning period: ages 35, 50, 65, and 80 at time points 0, 1, 2, and 3, respectively. Starting age is 35 years old; interval length between periods, \( T \), is 15 years.
(f) Incomes, base case:

\[
Y_0 = 20,000
\]
\[
Y_1 = \begin{bmatrix}
Y_{1g} = Y_0((1+g)(1+i_d))^T \\
Y_{1b} = Y_0(1+i_0)^T \\
Y_{2h} = Y_{1g}(1+i_h)^T/3 \\
Y_{2f} = Y_{1g}(1+i_d)^T/3
\end{bmatrix}
\]

It is apparent from the above equations that if the insured remains in good health until the end of the first period, his income will grow in real terms along with the economy at rate \( g \). However, if health deteriorates so that the insured becomes uninsurable through new policies, the assumption is that income only grows at the rate of inflation. “Bad health” does not necessarily mean “disabled,” but that the productivity of the individual, in his weakened state, is unable to keep pace with the economic growth. This latter assumption is quite arbitrary and unnecessary; the model only requires that income in states of poor health be lower than that of healthy states. In Appendix B, the sensitivity of insurance purchases to changes in income associated with the uninsurable states is analyzed.

The above equations also entail the assumption that when the individual approaches retirement, pension income is only one-third the level of that received during the second productive period, in real terms. This income is supplemented by savings and insurance to provide for the desired relative levels of retirement consumption and bequest.

(g) Probability of bad health, \( p_{1b} \), is compatible with the mortality table:

\[
(1) \quad p_{1b} = p_{1g}p_{2d} = \frac{\eta_0 - \eta_2}{\eta_0}
\]
\[
(2) \quad p_{1b} = 1 - p_{1b} - p_{1d}
\]
\[
(3) \quad p_{1d} = \frac{\eta_0 - \eta_1}{\eta_0},
\]

where \( \eta_0 \), \( \eta_1 \), and \( \eta_2 \) represent the number of people alive, at time points 0, 1, and 2, respectively. We assume that half of the people who die in period 2 were in bad health in period 1:

\[
(4) \quad p_{1b} = p_{1g}p_{2d}.
\]

Because \( \eta_0 \), \( \eta_1 \), and \( \eta_2 \) are given in the mortality table, we can solve (1), (2), (3) and (4) for \( p_{1b}, p_{1g}, p_{1d} \) and \( p_{2d} \).

(h) Minimum living standard (\( \gamma \)) and minimum bequest (\( \beta \)):

\[
\gamma_0 = \frac{1}{2} Y_0 = 10,000;
\]
\[
\gamma_1 = \gamma_0(1+i_0)^T
\]
\begin{align*}
\gamma_{2h} & = \gamma_1 (1 + i_{1h})^T \\
\gamma_{2l} & = \gamma_1 (1 + i_{1l})^T \\
\beta_1 & = \frac{1}{2} \gamma_0 \\
\beta_{2h} & = \beta_1 (1 + i_{1h})^T \\
\beta_{2l} & = \beta_1 (1 + i_{1l})^T \\
\beta_{3h,h} & = \beta_{2h} (1 + i_{2h})^T \\
\beta_{3h,l} & = \beta_{2h} (1 + i_{2l})^T \\
\beta_{3l,h} & = \beta_{2l} (1 + i_{2h})^T \\
\beta_{3l,l} & = \beta_{2l} (1 + i_{2l})^T \\
\end{align*}

While the model is very general, the computations presented in this paper are based on a simple premise — the minimum acceptable living standard and bequest in nominal terms are adjusted to offset changes in the cost of living. One can argue plausibly that the bequest function ought to reflect a greater degree of risk aversion than the utility-of-consumption function since an individual may be more risk averse on behalf of his heirs than on his own behalf (Fischer [1973], p. 133, fn. 5). However, the heirs may realistically supplemental sources of income, and that because the insured is likely to expend some of the income for personal consumption, his absence would release these funds for his heirs. Hence, our base cases show \( \gamma > \beta \).

(i) Annual rate of time preference: 4%. That is:
\[
\rho_1 = 1/(1.04)^{15}, \\
\rho_2 = 1/(1.04)^{30}, \quad \rho_3 = 1/(1.04)^{45}.
\]

All other \( \rho \) values of eq. (2) reflect the state probabilities, but do not include any unequal preference weightings for the base case, although in the comparative statics section, we show how decisions are influenced by changing preferences (via \( \rho_d \)) for bequest wealth relative to consumption wealth. One possible explanation for a heavier weight on bequest states relative to consumption states is provided by Mussa and Fischer (see Fischer [1973], p. 147, fn. 25).

2. Fair Premiums and Reserves:

The reader is referred to Winter [1981] for a more thorough discussion of the computation and implications of actuarially fair premiums and reserves. In this paper, traditional actuarial practice is adapted for the four time-points model employed here. The fair premium, \( \pi \), and reserve, \( Q \), of whole life insurance are calculated using the following table. Here \( \eta_0, \eta_1, \) and \( \eta_2, \eta_3 \) are taken from the mortality table, and \( \delta = (1 + r_0)^T \). \( \eta_3 \) is assumed to be zero; in other words, we assume that the maximum age attainable is 80 years.

<table>
<thead>
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<th>Period</th>
<th>0 (35)</th>
<th>1 (50)</th>
<th>2 (65)</th>
<th>3 (80)</th>
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<tbody>
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<td>( \eta_0 )</td>
<td>( \eta_1 )</td>
<td>( \eta_2 )</td>
<td>( \eta_3 = 0 )</td>
</tr>
<tr>
<td>Revenue</td>
<td>( \eta_0 \pi )</td>
<td>( \eta_1 \pi )</td>
<td>( \eta_2 \pi )</td>
<td>0</td>
</tr>
<tr>
<td>Benefit</td>
<td>0</td>
<td>( \eta_0 - \eta_1 )</td>
<td>( \eta_1 - \eta_2 )</td>
<td>( \eta_2 - \eta_3 = \eta_2 )</td>
</tr>
<tr>
<td>Total Reserve</td>
<td>( Z_0 = \eta_0 \pi )</td>
<td>( Z_1 = Z_0 \delta + \eta_1 \pi - (\eta_0 - \eta_1) )</td>
<td>( Z_2 = Z_1 \delta + \eta_2 \pi - (\eta_1 - \eta_2) )</td>
<td>( Z_3 = Z_2 \delta - \eta_2 )</td>
</tr>
<tr>
<td>Reserve/p.c.</td>
<td>( Q_0 = \frac{Z_0}{\eta_0} )</td>
<td>( Q_1 = \frac{Z_1}{\eta_1} )</td>
<td>( Q_2 = \frac{Z_2}{\eta_2} )</td>
<td>.</td>
</tr>
</tbody>
</table>
The fair premium $\pi$ is such that $Z_1 = 0$. Fair reserves can be obtained by using the fair premiums and formulae given in the table:

\[
Q_1 = \pi + \frac{\delta}{1 - p_{1d}} \pi - \frac{p_{1d}}{1 - p_{1d}}
\]

\[
Q_2 = \pi + \frac{\delta}{1 - p_{2d}} Q_1 - \frac{p_{2d}}{1 - p_{2d}}
\]

Fair premiums for term life insurance are defined as follows:

\[
\pi_0 = \frac{\eta_0 - \eta_1}{\eta_0 (1 + r_0)^F}
\]

\[
\pi_{1l} = \frac{\eta_1 - \eta_2}{\eta_1 (1 + r_1)^F}
\]

\[
\pi_{1h} = \frac{\eta_1 - \eta_2}{\eta_1 (1 + r_1)^F}
\]

## 3 Algorithm

A program called the Gradient Projection Method was employed to solve numerically our optimization problems. To apply this method to our problem, $\infty$ must be approximated by some finite number, due to the nature of our logarithmic utility function. We approximated $\infty$ by 100; hence, $ln(0) = -100$, and $\frac{d}{dx} \ln x \bigg|_{x=0} = 100$ for computational purposes. We also approximated any number less than $10^{-8}$ by 0; a search stops when $\|d(x)\|^2 \leq 10^{-8}$, where $d(x)$ is a projected gradient vector at point $x$ on the constraint hyperplane.
APPENDIX B: Comparative Statics

In this appendix we present a more detailed look at the influence of changing parameter values on the purchases of insurance and savings instruments. While the direction of change in these purchases was shown for increasing parameter values at two points (actuarially fair and standard markup cases) in Table 2, the figures presented here show the speed of change over a much wider range of parameter values. The figures are presented in pairs, where the figures on the left show the influence of changing parameter values for the actuarially fair premium case, whereas the adjacent figures to their right relate to the case where premiums incorporate a standard loading factor (27.5 percent for whole life and 110 percent for single-period term insurance).16

The vertical axes are in units of thousands of dollars. The horizontal axes are in terms of insurance rates per dollar of insurance in force for the first four figures, units of thousands of dollars for the next four, and percentage change for the ensuing twelve figures. The final two figures have a horizontal axis that shows the subjective weights placed on the bequest states relative to the consumption states.

In comparing the two cases for each variable, it should be noted that the vertical and sometimes horizontal axes are not shown in comparable scale, in order to capture better the rich dynamics of insurance and savings demand. Considerably larger diagrams would be required to demonstrate these dynamics on comparable scales.

In the first pair of diagrams we observe in the influence of changing whole life premiums. Note, in the fair case, how precipitously the sales of whole life insurance fall as its premium reflects increasing profit loadings. As these sales fall, term sales increase and savings (borrowing) levels also increase (decrease). In the left diagram, term insurance premiums are held constant at their actuarially fair value. No whole life at all is demanded once its markup exceeds 16 percent (corresponding to a rate of .114 per dollar of insurance in force), as long as term insurance at actuarially fair rates is available. In the diagram to its right, we observe demand for the insurance and savings products under typical premium markups. In this case, the term insurance markup is held constant at 110 percent (or a rate of .065 per dollar of insurance in force), while the load on whole life varies from 12 to 41 percent (corresponding to rates of .11 to .134 per dollar of insurance in force). Interestingly, this diagram shows that with term insurance premiums incorporating such a high loading factor, term sales do not begin until the whole life premium loading exceeds 18.4 percent (or until the rate exceeds .116 per dollar of insurance in force). Whole life purchases drop to zero when its markup reaches 41 percent (for a rate of .134 per dollar of insurance).

16 The exact amount of the loading factor was obtained by approximating the average loading factors reported in Babbel and Staking [1983] over the three decades given in that study for both types of life insurance. The term insurance averaged a 120% markup (see Table 3, column 2 of their study), but declined in recent years, so a 110% markup was used as a starting point. The whole life markup was based on both nonparticipating and participating whole life averages over the 30-year period (see Tables 1 and 2, columns 2). The combined markup during the overall period was 21% (assuming equal weights), but because the trend in recent years was upward, we used a 27.5% loading factor for our basic markup case.
The second set of diagrams show insurance and savings levels when the term premium varies, while the whole life premium remains either at its actuarially fair rate (left) or at a standard markup (right). We note how term purchases dry up when term rates rise to .035 per dollar (corresponding to a markup of 12.9 percent), as long as whole life is available at fair rates. In the unfair case, whole life purchases do not begin to appear until term rates exceed .055 per dollar, a markup of 77 percent. On the other hand, demand for term insurance evaporates as its rate approaches .085, a markup of 174 percent.

In the figures that follow, no additional commentary is provided. Having established a pattern for interpretation, further comparisons are left to the reader. While these figures relate only to a given set of assumptions, results for alternative sets of assumptions are available from the authors, upon request.